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ELECTRONICS
and CIRCUIT
ANALYSIS
using MATLAB

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PREFACE

MATLAB is a numeric computation software for engineering and scientific calculations. MATLAB is increasingly being used by students, researchers, practicing engineers and technicians. The causes of MATLAB popularity are legion. Among them are its iterative mode of operation, built-in functions, simple programming, rich set of graphing facilities, possibilities for writing additional functions, and its extensive toolboxes.

The goals of writing this book are (1) to provide the reader with simple, easy, hands-on introduction to MATLAB; (2) to demonstrate the use of MATLAB for solving electronics problems; (3) to show the various ways MATLAB can be used to solve circuit analysis problems; and (4) to show the flexibility of MATLAB for solving general engineering and scientific problems.

Audience

The book can be used by students, professional engineers and technicians. The first part of the book can be used as a primer to MATLAB. It will be useful to all students and professionals who want a basic introduction to MATLAB. Parts 2 and 3 are for electrical and electrical engineering technology students and professionals who want to use MATLAB to explore the characteristics of semiconductor devices and the application of MATLAB for analysis and design of electrical and electronic circuits and systems.

Organization

The book is divided into three parts: Introduction to MATLAB, Circuit analysis applications using MATLAB, and electronics applications with MATLAB. It is recommended that the reader work through and experiment with the examples at a computer while reading Chapters 1, 2, and 3. The hands-on approach is one of the best ways of learning MATLAB.

Part II consists of Chapters 4 to 8. This part covers the applications of MATLAB in circuit analysis. The topics covered in Part II are dc analysis, transient analysis, alternating current analysis, and Fourier analysis. In addition, two-port networks are covered. I have briefly covered the underlying theory and concepts, not with the aim of writing a textbook on circuit analysis and electronics. Selected problems in circuit analysis have been solved using MATLAB.

Part III includes Chapters 9, 10, 11 and 12. The topics discussed in this part are diodes, semiconductor physics, operational amplifiers and transistor circuits. Application of MATLAB for problem solving in electronics is discussed. Extensive examples showing the use of MATLAB for solving problems in electronics are presented.

Each chapter has its own bibliography and exercises.

Text Diskette

Since the text contains a large number of examples that illustrate electronics and circuit analysis principles and applications with MATLAB, a diskette is included that contains all the examples in the book. The reader can run the examples without having to enter the commands. The examples can also be modified to suit the needs of the reader.

Acknowledgments

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I am grateful to Mr. Zhong You for typing and running some of the MATLAB programs in this book and I am also grateful to Mr. Carl Easton and Mr. Url Woods for drawing the circuit diagrams found in the text. I thank Ms. Debbie Hawkins and Cheryl Wright who typed several parts of this book. I am appreciative of Ms. Judith Hansen for her editing services. Special thanks go Ms. Nora Konopka, at CRC Press, who took an early interest in this book and offered me any assistance I needed to get it completed. I thank Ms. Mimi Williams, at CRC Press, for thoroughly proofreading the manuscript.

The questions and comments from electrical engineering students at Prairie View A&M University led to rewriting some sections of this work. Special thanks go to the students who used various drafts of this book and provided useful comments.

A final note of gratitude goes to my wife, Christine N. Okyere, who encouraged me to finish the book in record time. With equanimity and understanding, she stood by me during the endless hours I spent writing.

DEDICATION

Dedicated to my family members
Christine, John II and Angela
for
their unfailing love, support and encouragement

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CHAPTER ONE

MATLAB FUNDAMENTALS

MATLAB is a numeric computation software for engineering and scientific calculations. The name MATLAB stands for MATRIX LABORATORY. MATLAB is primarily a tool for matrix computations. It was developed by John Little and Cleve Moler of MathWorks, Inc. MATLAB was originally written to provide easy access to the matrix computation software packages LINPACK and EISPACK.

MATLAB is a high-level language whose basic data type is a matrix that does not require dimensioning. There is no compilation and linking as is done in high-level languages, such as C or FORTRAN. Computer solutions in MATLAB seem to be much quicker than those of a high-level language such as C or FORTRAN. All computations are performed in complex-valued double precision arithmetic to guarantee high accuracy.

MATLAB has a rich set of plotting capabilities. The graphics are integrated in MATLAB. Since MATLAB is also a programming environment, a user can extend the functional capabilities of MATLAB by writing new modules.

MATLAB has a large collection of toolboxes in a variety of domains. Some examples of MATLAB toolboxes are control system, signal processing, neural network, image processing, and system identification. The toolboxes consist of functions that can be used to perform computations in a specific domain.

1.1 MATLAB BASIC OPERATIONS

When MATLAB is invoked, the command window will display the prompt >>. MATLAB is then ready for entering data or executing commands. To quit MATLAB, type the command

exit or **quit**

MATLAB has on-line help. To see the list of MATLAB's help facility, type

help

The help command followed by a function name is used to obtain information on a specific MATLAB function. For example, to obtain information on the use of fast Fourier transform function, **fft**, one can type the command

help fft

The basic data object in MATLAB is a rectangular numerical matrix with real or complex elements. Scalars are thought of as a 1-by-1 matrix. Vectors are considered as matrices with a row or column. MATLAB has no dimension statement or type declarations. Storage of data and variables is allocated automatically once the data and variables are used.

MATLAB statements are normally of the form:

variable = expression

Expressions typed by the user are interpreted and immediately evaluated by the MATLAB system. If a MATLAB statement ends with a semicolon, MATLAB evaluates the statement but suppresses the display of the results. MATLAB is also capable of executing a number of commands that are stored in a file. This will be discussed in Section 1.6. A matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

may be entered as follows:

A = [1 2 3; 2 3 4; 3 4 5];

Note that the matrix entries must be surrounded by brackets [   ] with row elements separated by blanks or by commas. The end of each row, with the exception of the last row, is indicated by a semicolon. A matrix A can also be entered across three input lines as

A = [1 2 3
      2 3 4
      3 4 5];

In this case, the carriage returns replace the semicolons. A row vector B with four elements

B = [6 9 12 15 18]

can be entered in MATLAB as

B = [6 9 12 15 18];

or

B = [6 , 9,12,15,18]

For readability, it is better to use spaces rather than commas between the elements. The row vector B can be turned into a column vector by **transposition**, which is obtained by typing

C = B'

The above results in

C =

6
9
12
15
18

Other ways of entering the column vector C are

C = [6
9
12
15
18]

or

C = [6; 9; 12; 15; 18]

MATLAB is case sensitive in naming variables, commands and functions. Thus b and B are not the same variable. If you do not want MATLAB to be case sensitive, you can use the command

casesen off

To obtain the size of a specific variable, type **size ( )**. For example, to find the size of matrix A, you can execute the following command:

size(A)

The result will be a row vector with two entries. The first is the number of rows in A, the second the number of columns in A.

To find the list of variables that have been used in a MATLAB session, type the command

whos

There will be a display of variable names and dimensions. Table 1.1 shows the display of the variables that have been used so far in this book:

Table 1.1
Display of an output of whos command

Name	Size	Elements	Byte	Density	Complex
A	3 by 3	9	72	Full	No
B	1 by 5	5	40	Full	No
C	5 by 1	5	40	Full	No
ans	1 by 2	2	16	Full	No

The grand total is 21 elements using 168 bytes.

Table 1.2 shows additional MATLAB commands to get one started on MATLAB. Detailed descriptions and usages of the commands can be obtained from the MATLAB help facility or from MATLAB manuals.

Table 1.2
Some Basic MATLAB Commands

Command	Description
%	Comments. Everything appearing after % command is not executed.
demo	Access on-line demo programs
length	Length of a matrix
clear	Clears the variables or functions from workspace
clc	Clears the command window during a work session
clg	Clears graphic window
diary	Saves a session in a disk, possibly for printing at a later date

1.2   MATRIX OPERATIONS

The basic matrix operations are addition(+), subtraction(-), multiplication (*), and conjugate transpose(') of matrices. In addition to the above basic operations, MATLAB has two forms of matrix division: the left inverse operator \ or the right inverse operator /.

Matrices of the same dimension may be subtracted or added. Thus if E and F are entered in MATLAB as

        E = [7 2 3; 4 3 6; 8 1 5];

        F = [1 4 2; 6 7 5; 1 9 1];
and
        G = E - F
        H = E + F

then, matrices G and H will appear on the screen as

        G =
           6   -2   1
          -2   -4   1
           7   -8   4

        H =
            8   6   5
           10   10   11
            9   10   6

A scalar (1-by-1 matrix) may be added to or subtracted from a matrix. In this particular case, the scalar is added to or subtracted from all the elements of another matrix. For example,

        J = H + 1
gives
        J =
            9   7   6
           11   11   12
           10   11   7

Matrix multiplication is defined provided the inner dimensions of the two operands are the same. Thus, if X is an n-by-m matrix and Y is i-by-j matrix,

X*Y is defined provided m is equal to i. Since E and F are 3-by-3 matrices, the product

    Q = E*F

results as

    Q =
        22   69   27
        28   91   29
        19   84   26

Any matrix can be multiplied by a scalar. For example,

    2*Q
gives
    ans =
            44   138   54
            56   182   58
            38   168   52

Note that if a variable name and the "=" sign are omitted, a variable name **ans** is automatically created.

Matrix division can either be the left division operator \ or the right division operator /. The right division   a/b, for instance, is algebraically equivalent to $\dfrac{a}{b}$ while the left division a\b is algebraically equivalent to $\dfrac{b}{a}$.

If $Z*I = V$ and Z is non-singular, the left division, Z\V is equivalent to MATLAB expression

$$I = inv(Z)*V$$

where **inv** is the MATLAB function for obtaining the inverse of a matrix. The right division denoted by V/Z is equivalent to the MATLAB expression

$$I = V*inv(Z)$$

There are MATLAB functions that can be used to produce special matrices. Examples are given in Table 1.3.

Table 1.3
Some Utility Matrices

Function	Description
ones(n,m)	Produces n-by-m matrix with all the elements being unity
eye(n)	gives n-by-n identity matrix
zeros(n,m)	Produces n-by-m matrix of zeros
diag(A)	Produce a vector consisting of diagonal of a square matrix A

1.3   ARRAY OPERATIONS

Array operations refer to element-by-element arithmetic operations. Preceding the linear algebraic matrix operations, * / \ ' , by a period (**.**) indicates an array or element-by-element operation. Thus, the operators **.* , .\ , ./, .^** , represent element-by-element multiplication, left division, right division, and raising to the power, respectively. For addition and subtraction, the array and matrix operations are the same. Thus, + and .+ can be regarded as an array or matrix addition.

If A1 and B1 are matrices of the same dimensions, then A1.*B1 denotes an array whose elements are products of the corresponding elements of A1 and B1. Thus, if

        A1 = [2 7 6
              8 9 10];

        B1 = [6 4 3
              2 3 4];
then
        C1 = A1.*B1
results in

        C1 =
            12   28   18
            16   27   40

An array operation for left and right division also involves element-by-element operation. The expressions A1./B1 and A1.\B1 give the quotient of element-by-element division of matrices A1 and B1. The statement

D1 = A1./B1

gives the result

D1 =
  0.3333 1.7500 2.0000
  4.0000 3.0000 2.5000

and the statement

E1 = A1.\B1

gives

E1 =
  3.0000 0.5714 0.5000
  0.2500 0.3333 0.4000

The array operation of raising to the power is denoted by **.^**. The general statement will be of the form:

q = r1.^s1

If r1 and s1 are matrices of the same dimensions, then the result q is also a matrix of the same dimensions. For example, if

r1 = [7 3 5];

s1 = [2 4 3];

then

q1 = r1.^s1

gives the result

q1 =
  49 81 125

One of the operands can be scalar. For example,

  q2 = r1.^2

  q3 = (2).^s1

will give

  q2 =
   49 9 25

and

  q3 =
   4 16 8

Note that when one of the operands is scalar, the resulting matrix will have the same dimensions as the matrix operand.

1.4 COMPLEX NUMBERS

MATLAB allows operations involving complex numbers. Complex numbers are entered using function i or j. For example, a number $z = 2 + j2$ may be entered in MATLAB as

  z = 2+2*i

or

  z = 2+2*j

Also, a complex number za

$$za = 2\sqrt{2} \exp[(\pi / 4) j]$$

can be entered in MATLAB as

  za = 2*sqrt(2)*exp((pi/4)*j)

It should be noted that when complex numbers are entered as matrix elements within brackets, one should avoid any blank spaces. For example, $y = 3 + j4$ is represented in MATLAB as

y = 3+4*j

If spaces exist around the + sign, such as

u= 3 + 4*j

MATLAB considers it as two separate numbers, and y will not be equal to u.

If w is a complex matrix given as

$$w = \begin{bmatrix} 1 + j1 & 2 - j2 \\ 3 + j2 & 4 + j3 \end{bmatrix}$$

then we can represent it in MATLAB as

w = [1+j 2-2*j; 3+2*j 4+3*j]

which will produce the result

w =
        1.0000 + 1.0000i   2.0000 - 2.0000i
        3.0000 + 2.0000i   4.0000 + 3.0000i

If the entries in a matrix are complex, then the "prime" (') operator produces the conjugate transpose. Thus,

wp = w'
will produce

wp =
        1.0000 - 1.0000i   3.0000 - 2.0000i
        2.0000 + 2.0000i   4.0000 - 3.0000i

For the unconjugate transpose of a complex matrix, we can use the point transpose (.') command. For example,

wt = w.'

will yield

wt =
```
      1.0000 + 1.0000i   3.0000 + 2.0000i
      2.0000 - 2.0000i   4.0000 + 3.0000i
```

1.5   THE COLON SYMBOL (:)

The colon symbol (:) is one of the most important operators in MATLAB. It can be used (1) to create vectors and matrices, (2) to specify sub-matrices and vectors, and (3) to perform iterations. The statement

t1 = 1:6

will generate a row vector containing the numbers from 1 to 6 with unit increment. MATLAB produces the result

```
t1 =
      1   2   3   4   5   6
```

Non-unity, positive or negative increments, may be specified. For example, the statement

t2 = 3:-0.5:1

will result in

```
t2 =
      3.0000   2.5000   2.0000   1.5000   1.0000
```

The statement

t3 = [(0:2:10);(5:-0.2:4)]

will result in a 2-by-4 matrix

```
t3 =
           0    2.0000   4.0000   6.0000   8.0000  10.0000
      5.0000    4.8000   4.6000   4.4000   4.2000   4.0000
```

Other MATLAB functions for generating vectors are linspace and logspace. **Linspace** generates linearly evenly spaced vectors, while **logspace** generates

logarithmically evenly spaced vectors. The usage of these functions is of the form:

  linspace(i_value, f_value, np)

  logspace(i_value, f_value, np)

where

  i_value is the initial value

  f_value is the final value

  np is the total number of elements in the vector.

For example,

  t4 = linspace(2, 6, 8)

will generate the vector

  t4 =
    Columns 1 through 7

     2.0000   2.5714   3.1429   3.7143   4.2857   4.8571
     5.4286

    Column 8

     6.0000

Individual elements in a matrix can be referenced with subscripts inside parentheses. For example, t2(4) is the fourth element of vector t2. Also, for matrix t3, t3(2,3) denotes the entry in the second row and third column. Using the colon as one of the subscripts denotes all of the corresponding row or column. For example, t3(:,4) is the fourth column of matrix t3. Thus, the statement

  t5 = t3(:,4)
will give

  t5 =
     6.0000
     4.4000

Also, the statement t3(2,:) is the second row of matrix t3. That is the statement

t6 = t3(2,:)

will result in

t6 =
    5.0000   4.8000   4.6000   4.4000   4.2000   4.0000

If the colon exists as the only subscript, such as t3(:), the latter denotes the elements of matrix t3 strung out in a long column vector.   Thus, the statement

t7 = t3(:)
will result in

t7 =
    0
    5.0000
    2.0000
    4.8000
    4.0000
    4.6000
    6.0000
    4.4000
    8.0000
    4.2000
    10.0000
    4.0000

Example 1.1

The voltage, v, across a resistance is given as (Ohm's Law), $v = Ri$, where i is the current and R the resistance. The power dissipated in resistor R is given by the expression

$$P = Ri^2$$

If R = 10 Ohms and the current is increased from 0 to 10 A with increments of 2A, write a MATLAB program to generate a table of current, voltage and power dissipation.

Solution:

MATLAB Script

```
diary  ex1_1.dat
% diary causes output to be written into file ex1_1.dat
% Voltage and power calculation
R=10;     % Resistance value
i=(0:2:10); % Generate current values
v=i.*R;     % array multiplication to obtain voltage
p=(i.^2)*R; % power calculation
sol=[i v p]  % current, voltage and power values are printed
diary
% the last diary command turns off the diary state
```

MATLAB produces the following result:

```
sol =
    Columns 1 through 6

    0     2     4     6     8     10

    Columns 7 through 12

    0     20    40    60    80    100

    Columns 13 through 18

    0     40    160   360   640   1000
```

Columns 1 through 6 constitute the current values, columns 7 through 12 are the voltages, and columns 13 through 18 are the power dissipation values.

1.6   M-FILES

Normally, when single line commands are entered, MATLAB processes the commands immediately and displays the results. MATLAB is also capable of processing a sequence of commands that are stored in files with extension m. MATLAB files with extension m are called m-files. The latter are ASCII text files, and they are created with a text editor or word processor. To list m-files in the current directory on your disk, you can use the MATLAB command **what**. The MATLAB command, **type**, can be used to show the contents of a specified file. M-files can either be script files or function files. Both script and function files contain a sequence of commands. However, function files take arguments and return values.

1.6.1   Script files

Script files are especially useful for analysis and design problems that require long sequences of MATLAB commands. With script file written using a text editor or word processor, the file can be invoked by entering the name of the m-file, without the extension. Statements in a script file operate globally on the workspace data. Normally, when m-files are executing, the commands are not displayed on screen. The MATLAB echo command can be used to view m-files while they are executing. To illustrate the use of script file, a script file will be written to simplify the following complex valued expression z.

Example 1.2

Simplify the complex number z and express it both in rectangular and polar form.

$$z = \frac{(3+j4)(5+j2)(2\angle 60^0)}{(3+j6)(1+j2)}$$

Solution:

The following program shows the script file that was used to evaluate the complex number, z, and express the result in polar notation and rectangular form.

MATLAB Script

        diary ex1_2.dat

```
% Evaluation of Z
% the complex numbers are entered
Z1 = 3+4*j;
Z2 = 5+2*j;
theta = (60/180)*pi;   % angle in radians
Z3 = 2*exp(j*theta);
Z4 = 3+6*j;
Z5 = 1+2*j;
% Z_rect is complex number Z in rectangular form
disp('Z in rectangular form is');  % displays text inside brackets
Z_rect = Z1*Z2*Z3/(Z4+Z5);
Z_rect
Z_mag = abs (Z_rect);   % magnitude of Z
Z_angle = angle(Z_rect)*(180/pi);  % Angle in degrees
disp('complex number Z in polar form, mag, phase'); % displays text
%inside brackets
Z_polar = [Z_mag, Z_angle]
diary
```

The program is named ex1_2.m. It is included in the disk that accompanies this book. Execute it by typing ex1_2 in the MATLAB command window. Observe the result, which should be

Z in rectangular form is

Z_rect =
        1.9108 + 5.7095i

complex number Z in polar form (magnitude and phase) is

Z_polar =
        6.0208   71.4966

1.6.2   Function Files

Function files are m-files that are used to create new MATLAB functions. Variables defined and manipulated inside a function file are local to the function, and they do not operate globally on the workspace. However, arguments may be passed into and out of a function file.

The general form of a function file is

```
function variable(s) = function_name (arguments)
% help text in the  usage of the function
%
.
.
end
```

To illustrate the usage of function files and rules for writing m-file function, let us study the following two examples.

Example 1.3

Write a function file to solve the equivalent resistance of series connected resistors, R1, R2, R3, …, Rn.

Solution:

MATLAB Script

```
function req = equiv_sr(r)
% equiv_sr is a function program for obtaining
%          the equivalent resistance of series
%          connected resistors
% usage:  req = equiv_sr(r)
%            r is an input vector of length n
%            req is an output, the equivalent resistance(scalar)
%
n = length(r);   % number of resistors
req = sum (r);   % sum up all resistors
end
```

The above MATLAB script can be found in the function file equiv_sr.m, which is available on the disk that accompanies this book.

Suppose we want to find the equivalent resistance of the series connected resistors 10, 20, 15, 16 and 5 ohms. The following statements can be typed in the MATLAB command window to reference the function equiv_sr

```
a = [10 20 15 16 5];
Rseries = equiv_sr(a)
diary
```
The result obtained from MATLAB is

Rseries =
 66

Example 1.4

Write a MATLAB function to obtain the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

Solution:

MATLAB Script

```
function rt = rt_quad(coef)
%
% rt_quad  is a function for obtaining the roots of
%          of a quadratic equation
% usage: rt = rt_quad(coef)
%          coef is the coefficients a,b,c of the quadratic
%              equation ax*x + bx + c =0
%          rt are the roots, vector of length 2
% coefficient a, b, c are obtained from vector coef
a = coef(1);  b = coef(2);  c = coef(3);
int = b^2 - 4*a*c;
if int > 0
   srint = sqrt(int);
   x1= (-b + srint)/(2*a);
   x2= (-b - srint)/(2*a);
elseif int == 0
   x1= -b/(2*a);
   x2= x1;
elseif int < 0
   srint = sqrt(-int);
   p1 = -b/(2*a);
   p2 = srint/(2*a);
   x1 = p1+p2*j;
   x2 = p1-p2*j;
end
rt =[x1;
    x2];
end
```

The above MATLAB script can be found in the function file rt_quad.m, which is available on the disk that accompanies this book.

We can use m-file function, rt_quad, to find the roots of the following quadratic equations:

(a) $x^2 + 3x + 2 = 0$

(b) $x^2 + 2x + 1 = 0$

(c) $x^2 - 2x + 3 = 0$

The following statements, that can be found in the m-file ex1_4.m, can be used to obtain the roots:

```
diary ex1_4.dat
ca = [1 3 2];
ra = rt_quad(ca)
cb = [1 2 1];
rb = rt_quad(cb)
cc = [1 -2 3];
rc = rt_quad(cc)
diary
```

Type into the MATLAB command window the statement ex1_4 and observe the results. The following results will be obtained:

```
ra =
     -1
     -2

rb =
     -1
     -1

rc=
     1.0000 + 1.4142i
     1.0000 - 1.4142i
```

The following is a summary of the rules for writing MATLAB m-file functions:

(1)   The word, function, appears as the first word in a function file. This is followed by an output argument, an equal sign and the function name. The

arguments to the function follow the function name and are enclosed within parentheses.

(2)   The information that follows the function, beginning with the % sign, shows how the function is used and what arguments are passed. This information is displayed if help is requested for the function name.

(3)   MATLAB can accept multiple input arguments and multiple output arguments can be returned.

(4)   If a function is going to return more than one value, all the values should be returned as a vector in the function statement. For example,

> function [mean, variance] = data_in(x)

will return the mean and variance of a vector x. The mean and variance are computed with the function.

(5)   If a function has multiple input arguments, the function statement must list the input arguments. For example,

> function [mean, variance] = data(x,n)

will return mean and variance of a vector x of length n.

(6)   The last statement in the function file should be an "end" statement.

SELECTED BIBLIOGRAPHY

1.   MathWorks, Inc., *MATLAB, High-Performance Numeric Computation Software*, 1995.

2.   Biran, A. and Breiner, M., *MATLAB for Engineers*, Addison-Wesley, 1995.

3.   Etter, D.M., *Engineering Problem Solving with MATLAB*, 2nd Edition, Prentice Hall, 1997.

EXERCISES

1.1 The voltage across a discharging capacitor is

$$v(t) = 10(1 - e^{-0.2t})$$

Generate a table of voltage, $v(t)$, versus time, t, for $t = 0$ to 50 seconds with increment of 5 s.

1.2 Use MATLAB to evaluate the complex number

$$z = \frac{(3 + j6)(6 + j4)}{(2 + j1)j2} + 7 + j10$$

1.3 Write a function-file to obtain the dot product and the vector product of two vectors a and b. Use the function to evaluate the dot and vector products of vectors x and y, where $x = (1\ 5\ 6)$ and $y = (2\ 3\ 8)$.

1.4 Write a function-file that can be used to calculate the equivalent resistance of n parallel connected resistors. In general, the equivalent resistance of resistors R_1, R_2, R_3, ..., R_n is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ ...\ + \frac{1}{R_n}$$

1.5 The voltage V is given as $V = RI$, where R and I are resistance matrix and I current vector. Evaluate V given that

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 3 & 6 & 7 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

1.6 Use MATLAB to simplify the expression

$$y = 0.5 + j6 + 3.5e^{j0.6} + (3 + j6)e^{j0.3\pi}$$

1.7 Write a function file to evaluate n factorial (i.e. n!); where

$$n! = n(n-1)(n-2)..(2)(1)$$

Use the function to compute $\quad x = \dfrac{7!}{3!4!}$

1.8 For a triangle with sides of length a, b, and c, the area A is given as

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$\textit{where}$$
$$s = (a+b+c)/2$$

Write a function to compute the area given the sides of a triangle. Use the function to compute the area of triangles with the lengths: (a) 56, 27 and 43 (b) 5, 12 and 13.

Attia, John Okyere. "Plotting Commands."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER TWO

PLOTTING COMMANDS

2.1   GRAPH FUNCTIONS

MATLAB has built-in functions that allow one to generate bar charts, x-y, polar, contour and 3-D plots, and bar charts. MATLAB also allows one to give titles to graphs, label the x- and y-axes, and add a grid to graphs. In addition, there are commands for controlling the screen and scaling. Table 2.1 shows a list of MATLAB built-in graph functions. One can use MATLAB's help facility to get more information on the graph functions.

Table 2.1
Plotting Functions

FUNCTION	DESRIPTION
axis	freezes the axis limits
bar	plots bar chart
contour	performs contour plots
ginput	puts cross-hair input from mouse
grid	adds grid to a plot
gtext	does mouse positioned text
histogram	gives histogram bar graph
hold	holds plot (for overlaying other plots)
loglog	does log versus log plot
mesh	performs 3-D mesh plot
meshdom	domain for 3-D mesh plot
pause	wait between plots
plot	performs linear x-y plot
polar	performs polar plot
semilogx	does semilog x-y plot (x-axis logarithmic)
semilogy	does semilog x-y plot (y-axis logarithmic)
shg	shows graph screen
stairs	performs stair-step graph
text	positions text at a specified location on graph
title	used to put title on graph
xlabel	labels x-axis
ylabel	labels y-axis

2.2 X-Y PLOTS AND ANNOTATIONS

The plot command generates a linear x-y plot. There are three variations of the plot command.

> (a) **plot(x)**
>
> (b) **plot(x, y)**
>
> (c) **plot(x1, y1, x2, y2, x3, y3, ..., xn, yn)**

If x is a vector, the command

> plot(x)

will produce a linear plot of the elements in the vector x as a function of the index of the elements in x. MATLAB will connect the points by straight lines. If x is a matrix, each column will be plotted as a separate curve on the same graph. For example, if

> x = [0 3.7 6.1 6.4 5.8 3.9];

then, **plot(x)** results in the graph shown in Figure 2.1.

If x and y are vectors of the same length, then the command

> **plot(x, y)**

plots the elements of x (x-axis) versus the elements of y (y-axis). For example, the MATLAB commands

> t = 0:0.5:4;
> y = 6*exp(-2*t);
> plot(t,y)

will plot the function $y(t) = 6e^{-2t}$ at the following times: 0, 0.5, 1.0, …, 4 . The plot is shown in Figure 2.2.

To plot multiple curves on a single graph, one can use the plot command with multiple arguments, such as

> **plot(x1, y1, x2, y2, x3, y3, ..., xn, yn)**
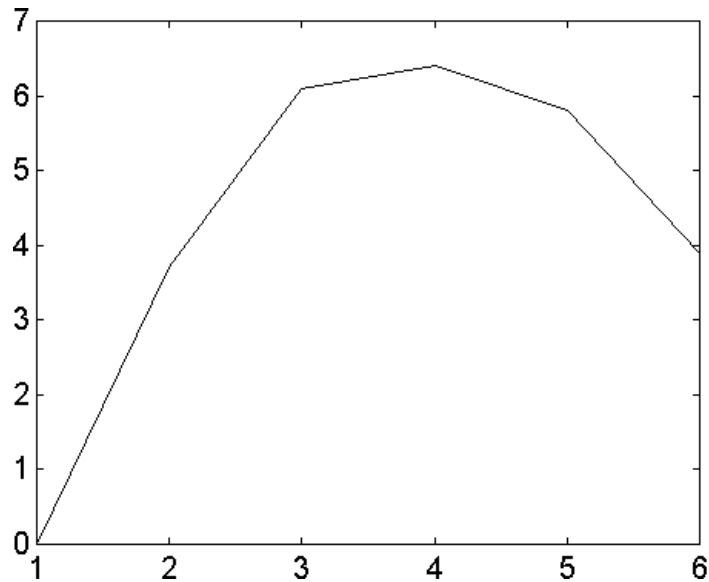
Figure 2.1 Graph of a Row Vector x

The variables x1, y1, x2, y2, etc., are pairs of vector. Each x-y pair is graphed, generating multiple lines on the plot. The above plot command allows vectors of different lengths to be displayed on the same graph. MATLAB automatically scales the plots. Also, the plot remains as the current plot until another plot is generated; in which case, the old plot is erased. The **hold** command holds the current plot on the screen, and inhibits erasure and rescaling. Subsequent plot commands will overplot on the original curves. The **hold** command remains in effect until the command is issued again.

When a graph is drawn, one can add a grid, a title, a label and x- and y-axes to the graph. The commands for grid, title, x-axis label, and y-axis label are **grid** (grid lines), **title** (graph title), **xlabel** (x-axis label), and **ylabel** (y-axis label), respectively. For example, Figure 2.2 can be titled, and axes labeled with the following commands:

```
t = 0:0.5:4;
y = 6*exp(-2*t);
plot(t, y)
title('Response of an RC circuit')
xlabel('time in seconds')
ylabel('voltage in volts')
grid
```

Figure 2.3 shows the graph of Figure 2.2 with title, x-axis, y-axis and grid added.
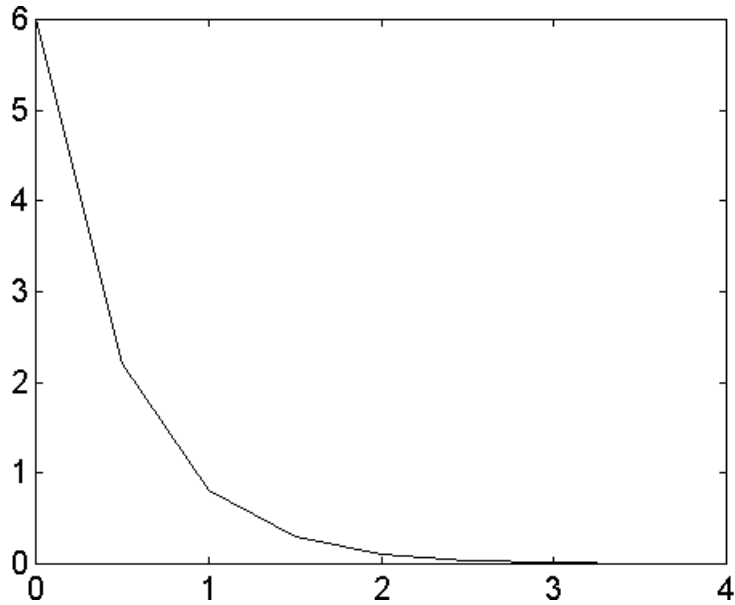
Figure 2.2   Graph of Two Vectors t and y

To write text on a graphic screen beginning at a point (x, y) on the graphic screen, one can use the command

text(x, y, 'text')

For example, the statement

text(2.0, 1.5, 'transient analysis')

will write the text, transient analysis, beginning at point (2.0,1.5). Multiple text commands can be used. For example, the statements

plot(a1,b1,a2,b2)
text(x1,y1,'voltage')
text(x2,y2,'power')

will provide texts for two curves: a1 versus b1 and a2 versus b2. The text will be at different locations on the screen provided x1 $\neq$ x2 or y1 $\neq$ y2.

If the default line-types used for graphing are not satisfactory, various symbols may be selected. For example:

plot(a1, b1, '*')

draws a curve, a1 versus b1, using star(*) symbols, while

plot(a1, b1, '*', a2, b2, '+')

uses a star(*) for the first curve and the plus(+) symbol for the second curve. Other print types are shown in Table 2.2.
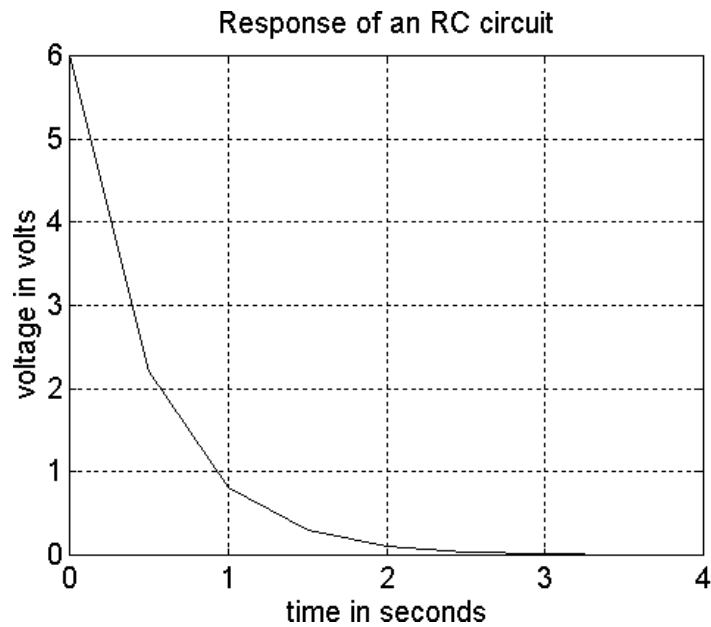
Figure 2.3 Graph of Voltage versus Time of a Response of an RLC Circuit

For systems that support color, the color of the graph may be specified using the statement:

plot(x, y, 'g')

implying, plot x versus y using green color. Line and mark style may be added to color type using the command

plot(x, y, '+w')

The above statement implies plot x versus y using white + marks. Other colors that can be used are shown in Table 2.3.

Table 2.2
Print Types

LINE-TYPES	INDICATORS	POINT TYPES	INDICATORS
solid	-	point	.
dash	--	plus	+
dotted	:	star	*
dashdot	-.	circle	o
		x-mark	x

Table 2.3
Symbols for Color Used in Plotting

COLOR	SYMBOL
red	r
green	g
blue	b
white	w
invisible	i

The argument of the plot command can be complex. If z is a complex vector, then plot(z) is equivalent to plot(real(z), imag(z)). The following example shows the use of the plot, title, xlabel, ylabel and text functions.

Example 2.1

For an R-L circuit, the voltage $v(t)$ and current $i(t)$ are given as

$$v(t) = 10\cos(377t)$$
$$i(t) = 5\cos(377t + 60^0)$$

Sketch $v(t)$ and $i(t)$ for $t = 0$ to 20 milliseconds.

Solution

MATLAB Script

```
% RL circuit
% current i(t) and  voltage v(t) are generated;  t is time
t = 0:1E-3:20E-3;  v  = 10*cos(377*t);
a_rad = (60*pi/180);  % angle in radians
i = 5*cos(377*t + a_rad);
plot(t,v,'*',t,i,'o')
title('Voltage and Current of an RL circuit')
xlabel('Sec')
ylabel('Voltage(V) and Current(mA)')
text(0.003, 1.5, 'v(t)');
text(0.009,2, 'i(t)')
```

Figure 2.4 shows the resulting graph. The file ex2_1.m is a script file for the solution of the problem.
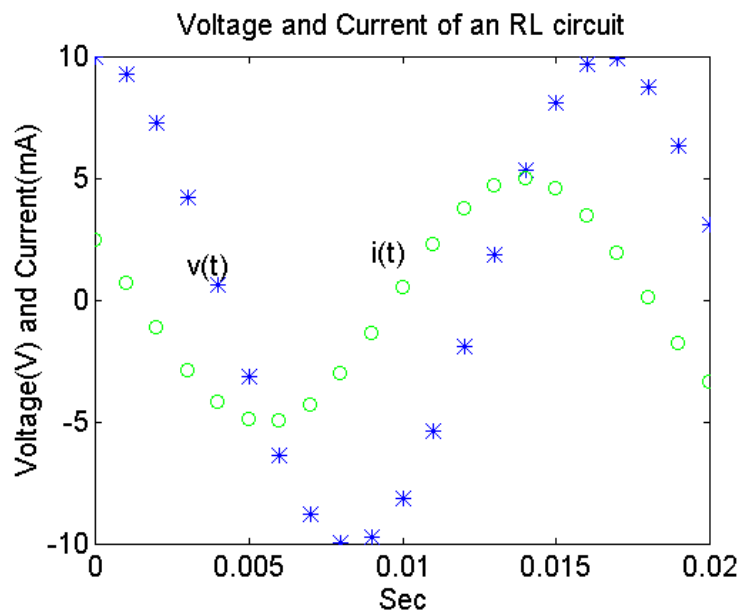
Figure 2.4 Plot of Voltage and Current of an RL Circuit under
Sinusoidal Steady State Conditions

2.3 LOGARITHMIC AND POLAR PLOTS

Logarithmic and semi-logarithmic plots can be generated using the commands **loglog, semilogx, and semilogy**. The use of the above plot commands is similar to those of the plot command discussed in the previous section. The description of these commands are as follows:

loglog(x, y) - generates a plot of $\log_{10}(x)$ versus $\log_{10}(y)$

semilogx(x, y) - generates a plot of $\log_{10}(x)$ versus linear axis of y

semilogy(x, y) - generates a plot of linear axis of x versus $\log_{10}(y)$

It should be noted that since the logarithm of negative numbers and zero does not exist, the data to be plotted on the semi-log axes or log-log axes should not contain zero or negative values.

Example 2.2

The gain versus frequency of a capacitively coupled amplifier is shown below. Draw a graph of gain versus frequency using a logarithmic scale for the frequency and a linear scale for the gain.

Frequency (Hz)	Gain (dB)	Frequency (Hz)	Gain (dB)
20	5	2000	34
40	10	5000	34
80	30	8000	34
100	32	10000	32
120	34	12000	30

Solution

MATLAB Script

```
% Bode plot for capacitively coupled amplifier
f = [20 40 80 100 120 2000 5000 8000 10000 ...
    12000 15000 20000];
g = [ 5 10 30 32 34 34 34 34 32 30 10 5];
semilogx(f, g)
```

```
title('Bode plot of an amplifier')
xlabel('Frequency in Hz')
ylabel('Gain in dB')
```

The plot is shown in Figure 2.5. The MATLAB script file is ex2_2.m.
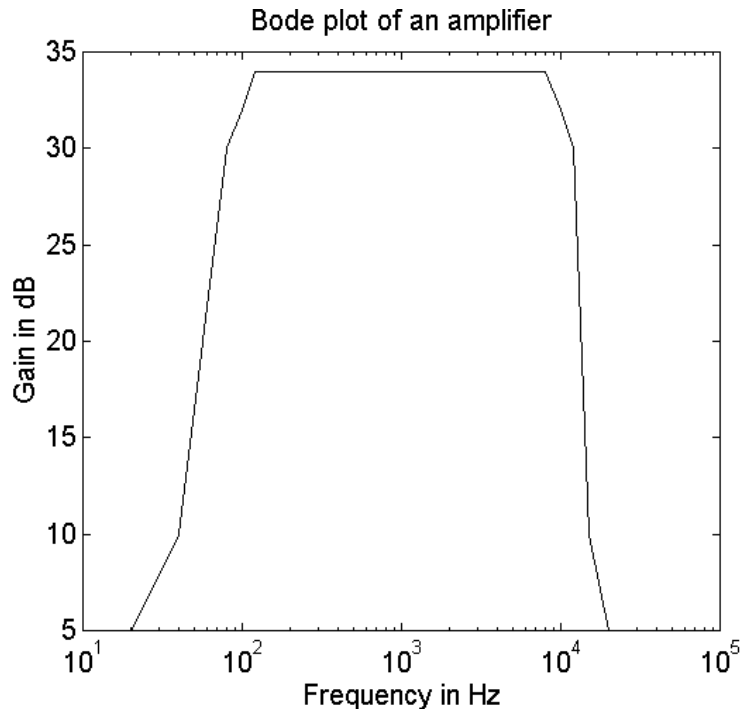
Figure 2.5 Plot of Gain versus Frequency of an Amplifier

A **polar** plot of an angle versus magnitude may be generated using the command

polar(theta, rho)

where,

theta and rho are vectors, with the theta being an angle in radians and rho being the magnitude.

When the grid command is issued after the polar plot command, polar grid lines will be drawn. The polar plot command is used in the following example.

Example 2.3

A complex number z can be represented as $z = re^{j\theta}$. The nth power of the complex number is given as $z^n = r^n e^{jn\theta}$. If $r = 1.2$ and $\theta = 10^0$, use the polar plot to plot $\left| z^n \right|$ versus $n\theta$ for $n = 1$ to $n = 36$.

Solution

MATLAB Script

```
% polar plot of z
r = 1.2;  theta = 10*pi/180;
angle = 0:theta:36*theta;  mag = r.^(angle/theta);
polar(angle,mag)
grid
title('Polar Plot')
```

The polar plot is shown in Figure 2.6.
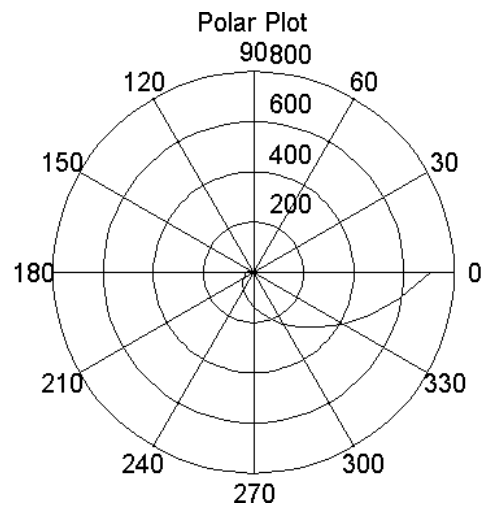
Figure 2.6 Polar Plot of $z = 1.2^n e^{j10n}$

2.4   SCREEN CONTROL

MATLAB has basically two display windows: a command window and a graph window. The hardware configuration an operator is using will either display both windows simultaneously or one at a time. The following commands can be used to select and clear the windows:

shg	-	shows graph window
any key	-	brings back command window
clc	-	clears command window
clg	-	clears graph window
home	-	home command cursor

The graph window can be partitioned into multiple windows. The **subplot** command allows one to split the graph window into two subdivisions or four subdivisions. Two sub-windows can be arranged either top or bottom or left or right. A four-window partition will have two sub-windows on top and two sub-windows on the bottom. The general form of the subplot command is

      subplot(i j k)

The digits i and j specify that the graph window is to be split into an i-by-j grid of smaller windows. The digit k specifies the kth window for the current plot. The sub-windows are numbered from left to right, top to bottom. For example,

```
%
x = -4:0.5:4;
y = x.^2; % square of x
z = x.^3; % cube of x
subplot(211), plot(x, y), title('square of x')
subplot(212), plot(x, z), title('cube of x')
```

will plot $y = x^2$ in the top half of the graph screen and $z = x^3$ will be plotted on the bottom half of the graph screen. The plots are shown in Figure 2.7.
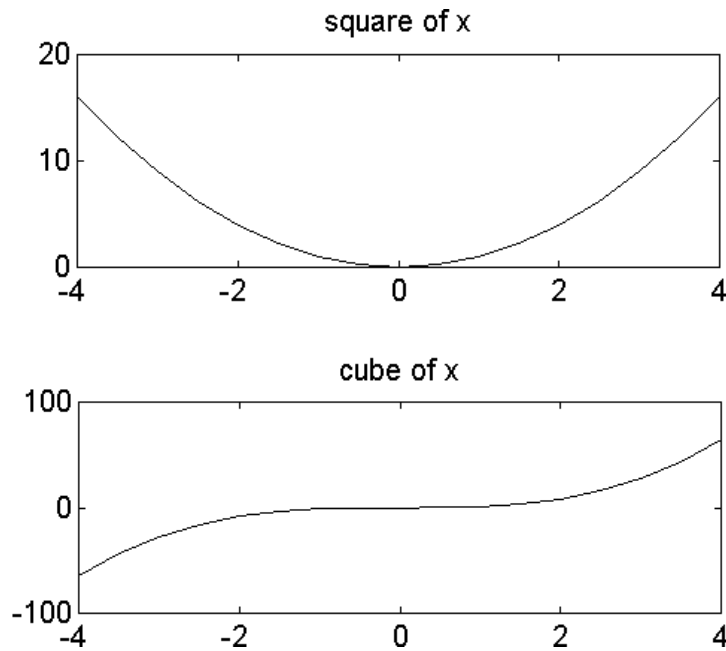
Figure 2.7 Plots of x^2 and x^3 using Subplot Commands.

The coordinates of points on the graph window can be obtained using the **ginput** command. There are two forms of the command:

[x y] = ginput

[x y] = ginput(n)

- [x y] = ginput command allows one to select an unlimited number of points from the graph window using a mouse or arrow keys. Pressing the return key terminates the input.

- [x y] = ginput(n) command allows the selection of n points from the graph window using a mouse or arrow keys. The points are stored in vectors x and y. Data points are entered by pressing a mouse button or any key on the keyboard (except return key). Pressing the return key terminates the input.

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1.   MathWorks, Inc, *MATLAB, High-Performance Numeric Computation Software*, 1995.

2.   Biran, A. and Breiner, M. *MATLAB for Engineers*, Addison-Wesley, 1995.

3.   Etter, D.M., *Engineering Problem Solving with MATLAB*, 2nd Edition, Prentice Hall, 1997.

EXERCISES

2.1   The repulsive coulomb force that exists between two protons in the nucleus of a conductor is given as

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

If $q_1 = q_2 = 1.6x10^{-19}$ C, and $\frac{1}{4\pi\varepsilon_0} = 8.99x10^9 \, Nm^2/C^2$,

sketch a graph of force versus radius r. Assume a radius from $1.0x10^{-15}$ to $1.0x10^{-14}$ m with increments of $2.0x10^{-15}$ m.

2.2   The current flowing through a drain of a field effect transistor during saturation is given as

$$i_{DS} = k(V_{GS} - V_t)^2$$

If $V_t = 1.0$ volt and $k = 2.5mA/V^2$, plot the current i_{DS} for the following values of V_{GS}: 1.5, 2.0, 2.5, ..., 5 V.

2.3   Plot the voltage across a parallel RLC circuit given as

$$v(t) = 5e^{2t} \sin(1000\pi t)$$

2.4   Obtain the polar plot of $z = r^{-n}e^{jn\theta}$ for $\theta = 15^0$ and $n = 1$ to 20.

2.5   The table below shows the grades of three examinations of ten students in a class.

STUDENT	EXAM #1	EXAM #2	EXAM #3
1	81	78	83
2	75	77	80
3	95	90	93
4	65	69	72
5	72	73	71
6	79	84	86
7	93	97	94
8	69	72	67
9	83	80	82
10	87	81	77

(a) Plot the results of each examination.

(b) Use MATLAB to calculate the mean and standard deviation of each examination.

2.6   A function $f(x)$ is given as

$$f(x) = x^4 + 3x^3 + 4x^2 + 2x + 6$$

(a) Plot $f(x)$ and
(b) Find the roots of $f(x)$

2.7   A message signal m(t) and the carrier signal $c(t)$ of a communication system are, respectively:

$$m(t) = 4\cos(120\pi t) + 2\cos(240\pi t)$$
$$c(t) = 10\cos(10{,}000\pi t)$$

A double-sideband suppressed carrier $s(t)$ is given as

$$s(t) = m(t)c(t)$$

Plot $m(t)$, $c(t)$ and $s(t)$ using the subplot command.

2.8   The voltage v and current I of a certain diode are related by the expression

$$i = I_S \exp[v / (nV_T)]$$

If $I_S = 1.0x10^{-14}$ A, $n = 2.0$ and $V_T = 26$ mV, plot the current versus voltage curve of the diode for diode voltage between 0 and 0.6 volts.

Attia, John Okyere. "Control Statements ."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER THREE

CONTROL STATEMENTS

3.1 FOR LOOPS

"**FOR**" loops allow a statement or group of statements to be repeated a fixed number of times. The general form of a for loop is

```
for index = expression
        statement group X
end
```

The expression is a matrix and the statement group X is repeated as many times as the number of elements in the columns of the expression matrix. The index takes on the elemental values in the matrix expression. Usually, the expression is something like

```
m:n  or m:i:n
```

where m is the beginning value, n the ending value, and i is the increment.

Suppose we would like to find the squares of all the integers starting from 1 to 100. We could use the following statements to solve the problem:

```
sum = 0;
for i = 1:100
   sum = sum + i^2;
end
sum
```

For loops can be nested, and it is recommended that the loop be indented for readability. Suppose we want to fill 10-by-20 matrix, b, with an element value equal to unity, the following statements can be used to perform the operation.

```
%
n = 10;            % number of rows
m = 20;            % number of columns
for i = 1:n
   for j = 1:m
      b(i,j) = 1;   % semicolon suppresses printing in the loop
   end
end
```

```
        b                    % display the result
        %
```

It is important to note that each for statement group must end with the word
end. The following program illustrates the use of a for loop.

Example 3.1

The horizontal displacement $x(t)$ and vertical displacement $y(t)$ are given
with respect to time, *t,* as

$$x(t) = 2t$$
$$y(t) = \sin(t)$$

For $t = 0$ to 10 ms, determine the values of $x(t)$ and $y(t)$. Use the values to
plot $x(t)$ versus $y(t)$.

Solution:

MATLAB Script

```
%
for i= 0:10
    x(i+1) = 2*i;
    y(i+1) = 2*sin(i);
end
plot(x,y)
```

Figure 3.1 shows the plots of $x(t)$ and $y(t)$.
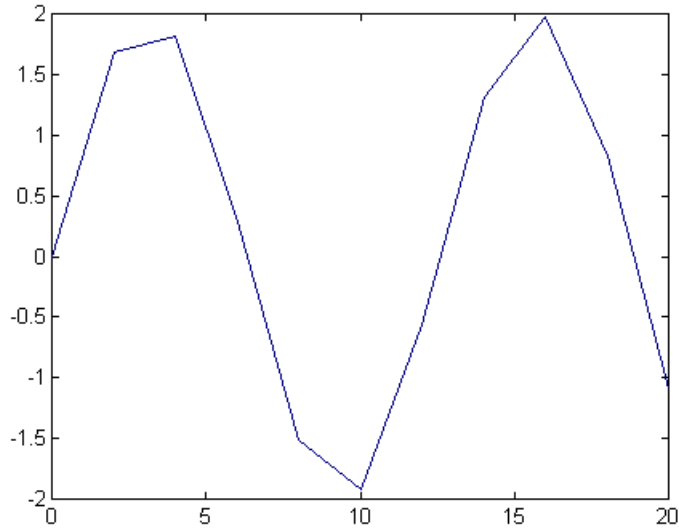
Figure 3.1 Plot of x versus y.

3.2   IF STATEMENTS

IF statements use relational or logical operations to determine what steps to perform in the solution of a problem. The relational operators in MATLAB for comparing two matrices of equal size are shown in Table 3.1.

Table 3.1
Relational Operators

RELATIONAL OPERATOR	MEANING
<	less than
<=	less than or equal
>	greater than
>=	greater than or equal
==	equal
~=	not equal

When any of the above relational operators are used, a comparison is done between the pairs of corresponding elements. The result is a matrix of ones and zeros, with one representing TRUE and zero FALSE. For example, if

a = [1 2 3 3 3 6];
b = [1 2 3 4 5 6];
a == *b*

The answer obtained is

ans =
    1   1   1   0   0   1

The 1s indicate the elements in vectors *a* and *b* that are the same and 0s are the ones that are different.

There are three logical operators in MATLAB. These are shown in Table 3.2.

Table 3.2
Logical Operators

LOGICAL OPERATOR SYMBOL	MEANING
&	and
!	or
~	not

Logical operators work element-wise and are usually used on 0-1 matrices, such as those generated by relational operators. The & and ! operators compare two matrices of equal dimensions. If A and B are 0-1 matrices, then A&B is another 0-1 matrix with ones representing TRUE and zeros FALSE. The NOT(~) operator is a unary operator. The expression ~C returns 1 where C is zero and 0 when C is nonzero.

There are several variations of the IF statement:

- simple if statement

- nested if statement

- if-else statement

- if-elseif statement

- if-elseif-else statement.

- The general form of the **simple if statement** is

    if logical expression 1
            statement group 1
    end

In the case of a simple if statement, if the logical expression1 is true, the statement group 1 is executed. However, if the logical expression is false, the statement group 1 is bypassed and the program control jumps to the statement that follows the end statement.

- The general form of a **nested if statement** is

    if logical expression 1
                    statement group 1
            if logical expression 2
                    statement group 2
            end
                    statement group 3
    end
    statement group 4

The program control is such that if expression 1 is true, then statement groups 1 and 3 are executed. If the logical expression 2 is also true, the statement groups 1 and 2 will be executed before executing statement group 3. If logical expression 1 is false, we jump to statement group 4 without executing statement groups 1, 2 and 3.

- The **if-else statement** allows one to execute one set of statements if a logical expression is true and a different set of statements if the logical statement is false. The general form of the if-else statement is

    if logical expression 1
                    statement group 1
            else
                    statement group 2
    end

In the above program segment, statement group 1 is executed if logical expression 1 is true. However, if logical expression 1 is false, statement group 2 is executed.

- **If-elseif statement** may be used to test various conditions before executing a set of statements. The general form of the if-elseif statement is

    if logical expression 1
        statement group1
    elseif logical expression 2
        statement group2
    elseif logical expression 3
        statement group 3
    elseif logical expression 4
        statement group 4
    end

A statement group is executed provided the logical expression above it is true. For example, if logical expression 1 is true, then statement group 1 is executed. If logical expression 1 is false and logical expression 2 is true, then statement group 2 will be executed. If logical expressions 1, 2 and 3 are false and logical expression 4 is true, then statement group 4 will be executed. If none of the logical expressions is true, then statement groups 1, 2, 3 and 4 will not be executed. Only three elseif statements are used in the above example. More elseif statements may be used if the application requires them.

- **If-elseif-else statement** provides a group of statements to be executed if other logical expressions are false. The general form of the if-elseif-else statement is

    if logical expression 1
        statement group1
    elseif logical expression 2
        statement group 2
    elseif logical expression 3
        statement group 3
    elseif logical expression 4
        statement group4
    else
        statement group 5
    end

The various logical expressions are tested. The one that is satisfied is executed. If the logical expressions 1, 2, 3 and 4 are false, then statement group 5 is executed. Example 3.2 shows the use of the if-elseif-else statement.

Example 3.2

A 3-bit A/D converter, with an analog input x and digital output y, is represented by the equation:

$$
\begin{aligned}
y &= 0 & & x < -2.5 \\
&= 1 & & -2.5 \leq x < -1.5 \\
&= 2 & & -1.5 \leq x < -0.5 \\
&= 3 & & -0.5 \leq x < 0.5 \\
&= 4 & & 0.5 \leq x < 1.5 \\
&= 5 & & 1.5 \leq x < 2.5 \\
&= 6 & & 2.5 \leq x < 3.5 \\
&= 7 & & x \geq 3.5
\end{aligned}
$$

Write a MATLAB program to convert analog signal x to digital signal y. Test the program by using an analog signal with the following amplitudes: -1.25, 2.57 and 6.0.

Solution

MATLAB Script

```
diary ex3_2.dat
%
y1 = bitatd_3(-1.25)
y2 = bitatd_3(2.57)
y3 = bitatd_3(6.0)
diary

function Y_dig = bitatd_3(X_analog)
%
% bitatd_3 is a function program for obtaining
%      the digital value given an input analog
%      signal
%
% usage:  Y_dig = bitatd_3(X_analog)
%      Y_dig is the digital number (in integer form)
```

```
%        X_analog is the analog input (in decimal form)
%
if X_analog < -2.5
   Y_dig = 0;
elseif X_analog >= -2.5 & X_analog < -1.5
   Y_dig = 1;
elseif X_analog >= -1.5 & X_analog < -0.5
   Y_dig = 2;
elseif X_analog >= -0.5 & X_analog < 0.5
   Y_dig = 3;
elseif X_analog >= 0.5 & X_analog < 1.5
   Y_dig = 4;
elseif X_analog >= 1.5 & X_analog < 2.5
   Y_dig = 5;
elseif X_analog >= 2.5 & X_analog < 3.5
   Y_dig = 6;
else
   Y_dig = 7;
end
Y_dig;
end
```

The function file, bitatd_3.m, is an m-file available in the disk that accompanies this book. In addition, the script file, ex3_2.m on the disk, can be used to perform this example. The results obtained, when the latter program is executed, are

y1 =
    2

y2 =
    6

y3 =
    7

3.3   WHILE LOOP

A **WHILE** loop allows one to repeat a group of statements as long as a specified condition is satisfied. The general form of the WHILE loop is

```
            while expression 1
                    statement group 1
            end
            statement group 2
```

When expression 1 is true, statement group 1 is executed. At the end of executing the statement group 1, the expression 1 is retested. If expression 1 is still true, the statement group 1 is again executed. However, if expression 1 is false, the program exits the while loop and executes statement group 2. The following example illustrates the use of the while loop.

Example 3.3

Determine the number of consecutive integer numbers which when added together will give a value equal to or just less than 210.

Solution

MATLAB Script

```
        diary ex3_3.dat
        % integer summation
        int = 1; int_sum = 0;
        max_val = 210;
        while int_sum < max_val
           int_sum = int_sum + int;
           int = int + 1;
        end
        last_int = int
        if int_sum == max_val
          num_int = int - 1
          tt_int_ct =  int_sum
         elseif int_sum  > max_val
           num_int = int - 1
           tt_int_ct = int_sum - last_int
          end
        end
        diary
```

The solution obtained will be

```
        last_int =
                 21
```

num_int =

$\quad\quad$ 20

tt_int_ct =

$\quad\quad$ 210

Thus, the number of integers starting from 1 that would add up to 210 is 20. That is,

$$1+2+3+4+...+20=210$$

3.4 INPUT/OUTPUT COMMANDS

MATLAB has commands for inputting information in the command window and outputting data. Examples of input/output commands are echo, input, pause, keyboard, break, error, display, format, and fprintf. Brief descriptions of these commands are shown in Table 3.3.

Table 3.3
Some Input/output Commands

COMMAND	DESCRIPTION
break	exits while or for loops
disp	displays text or matrix
echo	displays m-files during execution
error	displays error messages
format	switches output display to a particular format
fprintf	displays text and matrices and specifies format for printing values
input	allows user input
keyboard	invokes the keyboard as an m-file
pause	causes an m-file to stop executing. Pressing any key cause resumption of program execution.

Break

The **break** command may be used to terminate the execution of *for* and *while* loops. If the break command exits in an innermost part of a nested loop, the

break command will exit from that loop only. The break command is useful in exiting a loop when an error condition is detected.

Disp

The **disp** command displays a matrix without printing its name. It can also be used to display a text string. The general form of the disp command is

       disp(x)
       disp('text string')

disp(x) will display the matrix x. Another way of displaying matrix x is to type its name. This is not always desirable since the display will start with a leading "x = ".   **Disp('text string')**   will display the text string in quotes.   For example, the MATLAB statement

       disp('3-by-3 identity matrix')

will result in

       3-by-3 identity matrix
and
       disp(eye(3,3))

will result in

         1   0   0
         0   1   0
         0   0   1

Echo

The echo command can be used for debugging purposes. The **echo** command allows commands to be viewed as they execute. The echo can be enabled or disabled.

       echo on -   enables the echoing of commands
       echo off -   disables the echoing of commands
       echo -   by itself toggles the echo state

Error

The **error** command causes an error return from the m-files to the keyboard and displays a user written message. The general form of the command is

    error('message for display')

Consider the following MATLAB statements:

    x = input('Enter age of student');
    if x < 0
            error('wrong age was entered, try again')
    end
    x = input('Enter age of student')

For the above MATLAB statements, if the age is less that zero, the error message 'wrong age was entered, try again' will be displayed and the user will again be prompted for the correct age.

Format

The **format** controls the format of an output. Table 3.4 shows some formats available in MATLAB.

<div align="center">

Table 3.4
Format Displays

</div>

COMMAND	MEANING
format short	5 significant decimal digits
format long	15 significant digits
format short e	scientific notation with 5 significant digits
format long e	scientific notation with 15 significant digits
format hex	hexadecimal
format +	+ printed if value is positive, - if negative; space is skipped if value is zero

By default, MATLAB displays numbers in "short" format (5 significant digits). **Format compact** suppresses line-feeds that appear between matrix displays, thus allowing more lines of information to be seen on the screen. **For-**

mat loose reverts to the less compact display. Format compact and format loose do not affect the numeric format.

fprintf

The **fprintf** can be used to print both text and matrix values. The format for printing the matrix can be specified, and line feed can also be specified. The general form of this command is

      fprintf('text with format specification', matrices)

For example, the following statements

      cap = 1.0e-06;
      fprintf('The value of capacitance is %7.3e Farads\n', cap)

when executed will yield the output

      The value of capacitance is 1.000e-006 Farads

The format specifier %7.3e is used to show where the matrix value should be printed in the text. 7.3e indicates that the capacitance value should be printed with an exponential notation of 7 digits, three of which should be decimal digits. Other **format specifiers** are

      %f -   floating point
      %g -   signed decimal number in either %e or %f format,
              whichever is shorter

The text with format specification should end with **\n** to indicate the end of line. However, we can also use \n to get line feeds as represented by the following example:

      r1 = 1500;
      fprintf('resistance is \n%f Ohms \n', r1)

the output is

      resistance is
      1500.000000 Ohms

Input

The **input** command displays a user-written text string on the screen, waits for an input from the keyboard, and assigns the number entered on the keyboard as the value of a variable. For example, if one types the command

      r = input('Please enter the four resistor values');

when the above command is executed, the text string 'Please, enter the four resistor values' will be displayed on the terminal screen. The user can then type an expression such as

      [10 15 30 25];

The variable r will be assigned a vector [10 15 30 25]. If the user strikes the return key, without entering an input, an empty matrix will be assigned to r.

To return a string typed by a user as a text variable, the input command may take the form

      x = input('Enter string for prompt', 's')

For example, the command

      x = input('What is the title of your graph', 's')

when executed, will echo on the screen, 'What is the title of your graph.' The user can enter a string such as   'Voltage (mV) versus Current (mA).'

Keyboard

The **keyboard** command invokes the keyboard as an m-file. When the word **keyboard** is placed in an m-file, execution of the m-file stops when the word keyboard is encountered. MATLAB commands can then be entered. The keyboard mode is terminated by typing the word, "**return**" and pressing the return key. The keyboard command may be used to examine or change a variable or may be used as a tool for debugging m-files.

Pause

The **pause** command stops the execution of m-files. The execution of the m-file resumes upon pressing any key. The general forms of the pause command are

pause
pause(n)

Pause stops the execution of m-files until a key is pressed. **Pause(n)** stops the execution of m-files for n seconds before continuing. The pause command can be used to stop m-files temporarily when plotting commands are encountered during program execution. If pause is not used, the graphics are momentarily visible.

SELECTED BIBLIOGRAPHY

1.   MathWorks, Inc., *MATLAB, High-Performance Numeric Computation Software*, 1995.

2.   Biran, A. and Breiner, M., *MATLAB for Engineers*, Addison-Wesley, 1995.

3.   Etter, D.M., *Engineering Problem Solving with MATLAB*, 2nd Edition, Prentice Hall, 1997.

EXERCISES

3.1   Write a MATLAB program to add all the even numbers from 0 to 100.

3.2   Add all the terms in the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

until the sum exceeds 1.995. Print out the sum and the number of terms needed to just exceed the sum of 1.995.

3.3   The Fibonacci sequence is given as

1 1 2 3 5 8 13 21 34 …

Write a MATLAB program to generate the Fibonacci sequence up to the twelfth term. Print out the results.

3.4   The table below shows the final course grade and its corresponding relevant letter grade.

LETTER GRADE	FINAL COURSE GRADE
A	$90 < \text{grade} \leq 100$
B	$80 < \text{grade} \leq 90$
C	$70 < \text{grade} \leq 80$
D	$60 < \text{grade} \leq 70$
F	$\text{grade} \leq 60$

For the course grades: 70, 85, 90, 97, 50, 60, 71, 83, 91, 86, 77, 45, 67, 88, 64, 79, 75, 92, and 69

(a) Determine the number of students who attained the grade of A
 and F.
(b) What are the mean grade and the standard deviation?

3.5   Write a script file to evaluate y[1], y[2], y[3] and y[4] for the difference equation:

$$y[n] = 2\,y[n-1] - y[n-2] + x[n]$$

for $n \geq 0$. Assume that $x[n] = 1$ for $n \geq 0$, $y[-2] = 2$ and $y[-1] = 1$.

3.6   The equivalent impedance of a circuit is given as

$$Z_{eq}(jw) = 100 + jwL + \frac{1}{jwC}$$

If $L = 4$ H and $C = 1$ μF,

(a) Plot $\left| Z_{eq}(jw) \right|$ versus w.   (b) What is the minimum impedance?

(c) With what frequency does the minimum impedance occur?

Attia, John Okyere. "DC Analysis."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER FOUR

DC ANALYSIS

4.1   NODAL ANALYSIS

Kirchhoff's current law states that for any electrical circuit, the algebraic sum of all the currents at any node in the circuit equals zero. In nodal analysis, if there are n nodes in a circuit, and we select a reference node, the other nodes can be numbered from V_1 through V_{n-1}. With one node selected as the reference node, there will be n-1 independent equations. If we assume that the admittance between nodes i and j is given as Y_{ij}, we can write the nodal equations:

$$Y_{11}\,V_1 + Y_{12}\,V_2 + \ldots \qquad + Y_{1m}\,V_m = \sum\,I_1$$

$$Y_{21}\,V_1 + Y_{22}\,V_2 + \ldots \qquad + Y_{2m}\,V_m = \sum\,I_2$$

$$Y_{m1}\,V_1 + Y_{m2}\,V_2 + \ldots \qquad + Y_{mm}\,V_m = \sum\,I_m$$

(4.1)

where

$m = n - 1$

V_1, V_2 and V_m are voltages from nodes 1, 2 and so on ..., n with respect to the reference node.

$\sum\,I_x$ is the algebraic sum of current sources at node x.

Equation (4.1) can be expressed in matrix form as

$$[Y][V] = [I]$$

(4.2)

The solution of the above equation is

$$[V] = [Y]^{-1}[I]$$

(4.3)

where

$[Y]^{-1}$ is an inverse of $[Y]$.

In MATLAB, we can compute [V] by using the command

$$V = inv(Y) * I \tag{4.4}$$

where

$inv(Y)$ is the inverse of matrix Y

The matrix left and right divisions can also be used to obtain the nodal voltages. The following MATLAB commands can be used to find the matrix [V]

$$V = {I \big/ Y} \tag{4.5}$$

or

$$V = Y \setminus I \tag{4.6}$$

The solutions obtained from Equations (4.4) to (4.6) will be the same, provided the system is not ill-conditioned. The following two examples illustrate the use of MATLAB for solving nodal voltages of electrical circuits.

Example 4.1

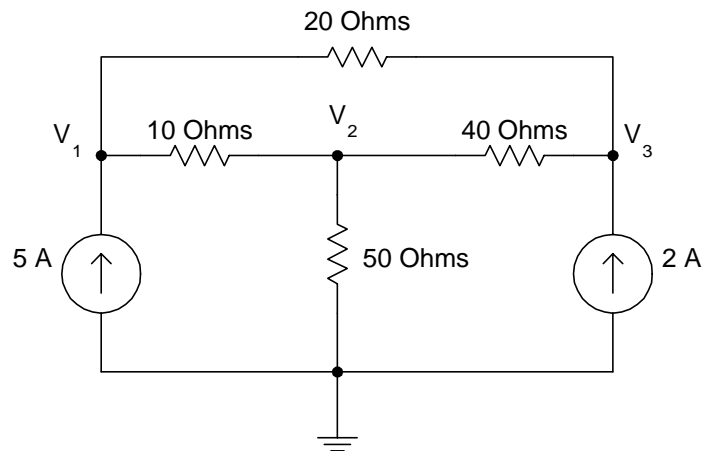For the circuit shown below, find the nodal voltages V_1, V_2 and V_3.

Figure 4.1 Circuit with Nodal Voltages

Solution

Using KCL and assuming that the currents leaving a node are positive, we have

For node 1,

$$\frac{V_1 - V_2}{10} + \frac{V_1 - V_3}{20} - 5 = 0$$

i.e.,

$$0.15V_1 - 0.1V_2 - 0.05V_3 = 5 \qquad (4.7)$$

At node 2,

$$\frac{V_2 - V_1}{10} + \frac{V_2}{50} + \frac{V_2 - V_3}{40} = 0$$

i.e.,

$$-0.1V_1 + 0.145V_2 - 0.025V_3 = 0 \qquad (4.8)$$

At node 3,

$$\frac{V_3 - V_1}{20} + \frac{V_3 - V_2}{40} - 2 = 0$$

i.e.,

$$-0.05V_1 - 0.025V_2 + 0.075V_3 = 2 \qquad (4.9)$$

In matrix form, we have

$$\begin{bmatrix} 0.15 & -0.1 & -0.05 \\ -0.1 & 0.145 & -0.025 \\ -0.05 & -0.025 & 0.075 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \qquad (4.10)$$

The MATLAB program for solving the nodal voltages is

MATLAB Script

```
diary ex4_1.dat
% program computes the nodal voltages
```

```
% given the admittance matrix Y and current vector I
% Y is the admittance matrix and I is the current vector
% initialize matrix y and vector I using YV=I form
Y = [ 0.15   -0.1    -0.05;
     -0.1     0.145  -0.025;
     -0.05   -0.025   0.075];
I = [5;
     0;
     2];
% solve for the voltage
fprintf('Nodal voltages V1, V2 and V3 are \n')
v = inv(Y)*I
diary
```

The results obtained from MATLAB are

Nodal voltages V1, V2 and V3,

v =
    404.2857
    350.0000
    412.8571

Example 4.2:

Find the nodal voltages of the circuit shown below.
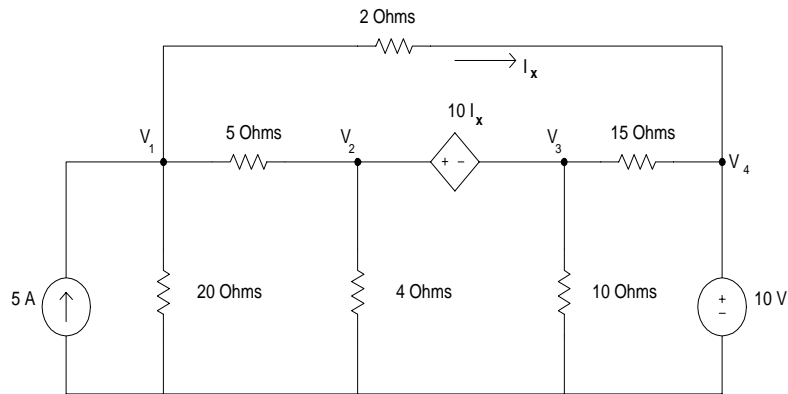
Figure 4.2 Circuit with Dependent and Independent Sources

Solution

Using KCL and the convention that currents leaving a node is positive, we have

At node 1

$$\frac{V_1}{20} + \frac{V_1 - V_2}{5} + \frac{V_1 - V_4}{2} - 5 = 0$$

Simplifying, we get

$$0.75V_1 - 0.2V_2 - 0.5V_4 = 5 \qquad (4.11)$$

At node 2,

$$V_2 - V_3 = 10I_X$$

But

$$I_X = \frac{(V_1 - V_4)}{2}$$

Thus

$$V_2 - V_3 = \frac{10(V_1 - V_4)}{2}$$

Simplifying, we get

$$-5V_1 + V_2 - V_3 + 5V_4 = 0 \qquad (4.12)$$

From supernodes 2 and 3, we have

$$\frac{V_3}{10} + \frac{V_2 - V_1}{5} + \frac{V_2}{4} + \frac{V_3 - V_4}{15} = 0$$

Simplifying, we get

$$-0.2V_1 + 0.45V_2 + 0.1667V_3 - 0.06667V_4 = 0 \qquad (4.13)$$

At node 4, we have

$$V_4 = 10 \qquad\qquad (4.14)$$

In matrix form, equations (4.11) to (4.14) become

$$\begin{bmatrix} 0.75 & -0.2 & 0 & -0.5 \\ -5 & 1 & -1 & 5 \\ -0.2 & 0.45 & 0.1667 & -0.06667 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 10 \end{bmatrix} \qquad (4.15)$$

The MATLAB program for solving the nodal voltages is

MATLAB Script

```
diary ex4_2.dat
% this program computes the nodal voltages
% given the admittance matrix Y and current vector I
% Y is the admittance matrix
% I is the current vector
% initialize the matrix y and vector I using YV=I

Y = [0.75  -0.2  0   -0.5;
      -5    1  -1     5;
     -0.2  0.45  0.166666667  -0.0666666667;
      0    0  0     1];

% current vector is entered as a transpose of row vector
I = [5   0   0   10]';

% solve for nodal voltage
fprintf('Nodal voltages V1,V2,V3,V4 are \n')
V = inv(Y)*I
diary
```

We obtain the following results.

Nodal voltages V1,V2,V3,V4 are

$$V = \begin{matrix} 18.1107 \\ 17.9153 \\ -22.6384 \\ 10.0000 \end{matrix}$$

4.2   LOOP ANALYSIS

Loop analysis is a method for obtaining loop currents. The technique uses Kirchoff voltage law (KVL) to write a set of independent simultaneous equations. The Kirchoff voltage law states that the algebraic sum of all the voltages around any closed path in a circuit equals zero.

In loop analysis, we want to obtain current from a set of simultaneous equations. The latter equations are easily set up if the circuit can be drawn in planar fashion. This implies that a set of simultaneous equations can be obtained if the circuit can be redrawn without crossovers.

For a planar circuit with n-meshes, the KVL can be used to write equations for each mesh that does not contain a dependent or independent current source. Using KVL and writing equations for each mesh, the resulting equations will have the general form:

$$Z_{11}I_1 + Z_{12}I_2 + Z_{13}I_3 + \quad ... \quad Z_{1n}I_n = \sum V_1$$

$$Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 + \quad ... \quad Z_{2n}I_n = \sum V_2$$

$$Z_{n1}I_1 + Z_{n2}I_2 + Z_{n3}I_3 + \quad ... \quad Z_{nn}I_n = \sum V_n$$

$$(4.16)$$

where

$I_1, I_2, ... I_n$ are the unknown currents for meshes 1 through n.

$Z_{11}, Z_{22}, ..., Z_{nn}$ are the impedance for each mesh through which individual current flows.

$Z_{ij}, \quad j \# i$ denote mutual impedance.

$\sum V_x$ is the algebraic sum of the voltage sources in mesh x.

Equation (4.16) can be expressed in matrix form as

$$[Z][I] = [V] \tag{4.17}$$

where

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \ldots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & \ldots & Z_{2n} \\ Z_{31} & Z_{32} & Z_{33} & \ldots & Z_{3n} \\ .. & .. & . & \ldots & .. \\ Z_{n1} & Z_{n2} & Z_{n3} & \ldots & Z_{nn} \end{bmatrix}$$

$$I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ . \\ I_n \end{bmatrix}$$

and

$$V = \begin{bmatrix} \sum V_1 \\ \sum V_2 \\ \sum V_3 \\ .. \\ \sum V_n \end{bmatrix}$$

The solution to Equation (4.17) is

$$[I] = [Z]^{-1}[V] \tag{4.18}$$

In MATLAB, we can compute [I] by using the command

$$I = inv(Z) * V \tag{4.19}$$

where

$inv(Z)$ is the inverse of the matrix Z

The matrix left and right divisions can also be used to obtain the loop currents. Thus, the current I can be obtained by the MATLAB commands

$$I = V\!\!\diagup\!\!Z \qquad\qquad (4.20)$$

or

$$I = Z \setminus V \qquad\qquad (4.21)$$

As mentioned earlier, Equations (4.19) to (4.21) will give the same results, provided the circuit is not ill-conditioned. The following examples illustrate the use of MATLAB for loop analysis.

Example 4.3

Use the mesh analysis to find the current flowing through the resistor R_B. In addition, find the power supplied by the 10-volt voltage source.
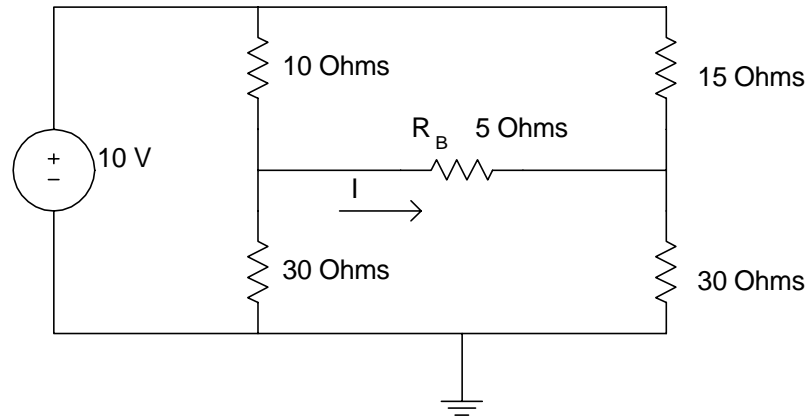
Figure 4.3a Bridge Circuit

Solution

Using loop analysis and designating the loop currents as I_1, I_2, I_3, we obtain the following figure.
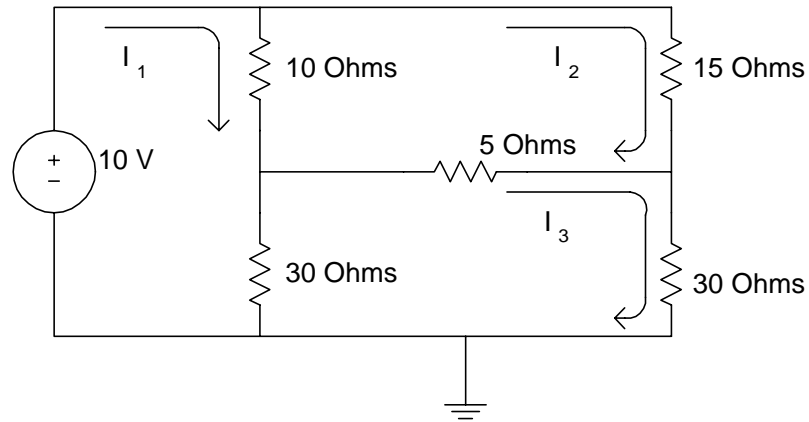
Figure 4.3b   Bridge Circuit with Loop Currents

Note that $I = I_3 - I_2$ and power supplied by the source is $P = 10I_1$

The loop equations are

Loop 1,

$$10(I_1 - I_2) + 30(I_1 - I_3) - 10 = 0$$

$$40I_1 - 10I_2 - 30I_3 = 10 \qquad (4.22)$$

Loop 2,

$$10(I_2 - I_1) + 15I_2 + 5(I_2 - I_3) = 0$$

$$-10I_1 + 30I_2 - 5I_3 = 0 \qquad (4.23)$$

Loop 3,

$$30(I_3 - I_1) + 5(I_3 - I_2) + 30I_3 = 0$$

$$-30I_1 - 5I_2 + 65I_3 = 0 \qquad (4.24)$$

In matrix form, Equations (4.22) and (4.23) become

$$\begin{bmatrix} 40 & -10 & -30 \\ -10 & 30 & -5 \\ -30 & -5 & 65 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \tag{4.25}$$

The MATLAB program for solving the loop currents I_1, I_2, I_3, the current I and the power supplied by the 10-volt source is

MATLAB Script

```
diary ex4_3.dat
% this program determines the current
% flowing in a resistor RB and power supplied  by source
% it computes the loop currents given the impedance
% matrix Z and voltage vector V
% Z is the impedance matrix
% V is the voltage matrix
% initialize the matrix Z and vector V

Z = [40   -10   -30;
     -10    30    -5;
     -30    -5    65];

V = [10  0  0]';

% solve for the loop currents
I = inv(Z)*V;
% current through RB is calculated
IRB = I(3) - I(2);
fprintf('the current through R is %8.3f Amps \n',IRB)
% the power supplied by source is calculated
PS = I(1)*10;
fprintf('the power supplied by 10V source is %8.4f watts \n',PS)
diary
```

MATLAB answers are

```
the current through R is    0.037 Amps
the power supplied by 10V source is   4.7531 watts
```

Circuits with dependent voltage sources can be analyzed in a manner similar to that of example 4.3. Example 4.4 illustrates the use of KVL and MATLAB to solve loop currents.

Example 4.4

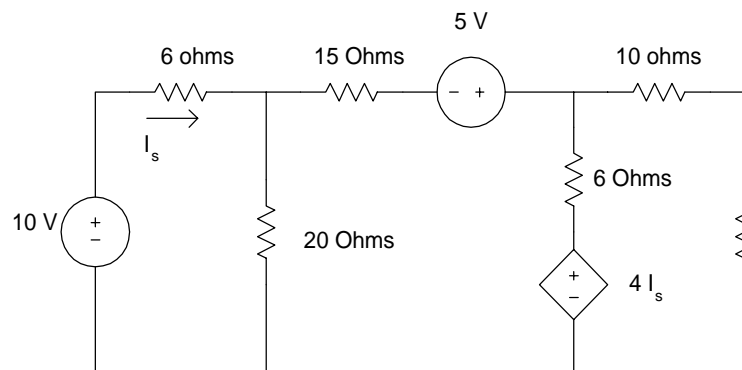Find the power dissipated by the 8 Ohm resistor and the current supplied by the 10-volt source.

Figure 4.4a Circuit for Example 4.4

Solution

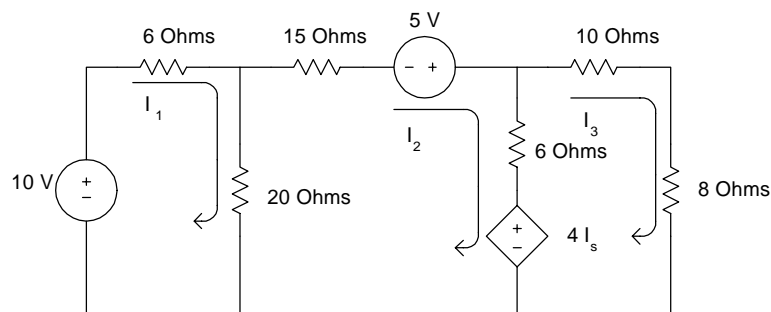Using loop analysis and denoting the loop currents as I_1, I_2 and I_3, the circuit can be redrawn as

Figure 4.4b   Figure 4.4 with Loop Currents

By inspection,

$$I_S = I_1 \tag{4.26}$$

For loop 1,

$$-10 + 6I_1 + 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 10 \tag{4.27}$$

For loop 2,

$$15I_2 - 5 + 6(I_2 - I_3) + 4I_S + 20(I_2 - I_1) = 0$$

Using Equation (4.26), the above expression simplifies to

$$-16I_1 + 41I_2 - 6_3 I = 5 \tag{4.28}$$

For loop 3,

$$10I_3 + 8I_3 - 4I_S + 6(I_3 - I_2) = 0$$

Using Equation (4.26), the above expression simplifies to

$$-4I_1 - 6I_2 + 24I_3 = 0 \tag{4.29}$$

Equations (4.25) to (4.27) can be expressed in matrix form as

$$\begin{bmatrix} 26 & -20 & 0 \\ -16 & 41 & -6 \\ -4 & -6 & 24 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \tag{4.30}$$

The power dissipated by the 8 Ohm resistor is

$$P = RI_3^2 = 8I_3^2$$

The current supplied by the source is $I_S = I_1$

A MATLAB program for obtaining the power dissipated by the 8 Ohm resistor and the current supplied by the source is shown below

MATLAB Script

```
diary ex4_4.dat
% This program determines the power dissipated by
% 8 ohm resistor and current supplied by the
% 10V source
%
% the program computes the loop currents, given
% the impedance matrix Z and voltage vector V
%
% Z is the impedance matrix
% V is the voltage vector
% initialize the matrix Z and vector V of equation
% ZI=V

Z = [26   -20    0;
      -16    40   -6;
       -4    -6   24];
V = [10   5   0]';

% solve for loop currents
I = inv(Z)*V;
% the power dissipation in 8 ohm resistor is P
P = 8*I(3)^2;
% print out the results
fprintf('Power dissipated in 8 ohm resistor is %8.2f Watts\n',P)
fprintf('Current in 10V source is %8.2f Amps\n',I(1))
diary
```

MATLAB results are

```
Power dissipated in 8 ohm resistor is     0.42  Watts
Current in 10V source is     0.72 Amps
```

For circuits that contain both current and voltage sources, irrespective of whether they are dependent sources, both KVL and KVL can be used to obtain equations that can be solved using MATLAB. Example 4.5 illustrates one such circuit.

Example 4.5

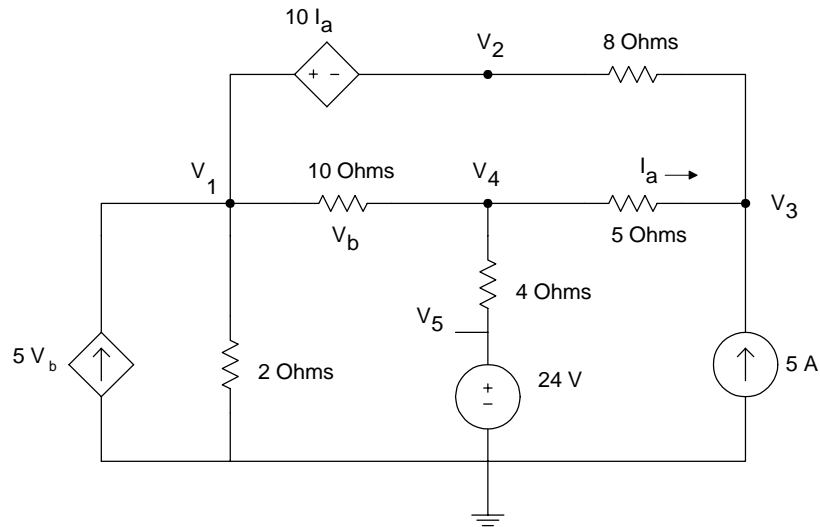Find the nodal voltages in the circuit, i.e., $V_1, V_2, \ldots, V_5$

Figure 4.5 Circuit for Example 4.5

Solution

By inspection,

$$V_b = V_1 - V_4 \tag{4.31}$$

Using Ohm's Law

$$I_a = \frac{V_4 - V_3}{5} \tag{4.32}$$

Using KCL at node 1, and supernode 1-2, we get

$$\frac{V_1}{2} + \frac{V_1 - V_4}{10} - 5V_b + \frac{V_2 - V_3}{8} = 0 \tag{4.33}$$

Using Equation (4.31), Equation (4.33) simplifies to

$$-4.4V_1 + 0.125V_2 - 0.125V_3 + 4.9V_4 = 0 \qquad (4.34)$$

Using KCL at node 4, we have

$$\frac{V_4 - V_5}{4} + \frac{V_4 - V_3}{5} + \frac{V_4 - V_1}{10} = 10$$

This simplifies to

$$-0.1V_1 - 0.2V_3 + 0.55V_4 - 0.25V_5 = 0 \qquad (4.35)$$

Using KCL at node 3, we get

$$\frac{V_3 - V_4}{5} + \frac{V_3 - V_2}{8} - 5 = 0$$

which simplifies to

$$-0.125V_2 + 0.325V_3 - 0.2V_4 = 5 \qquad (4.36)$$

Using KVL for loop 1, we have

$$-10I_a + V_b + 5I_a + 8(I_a + 5) = 0 \qquad (4.37)$$

Using Equations (4.31) and (4.32), Equation (4.37) becomes

$$-10I_a + V_b + 5I_a + 8I_a + 40 = 0$$

i.e.,

$$3I_a + V_b = -40$$

Using Equation (4.32), the above expression simplifies to

$$3\frac{V_4 - V_3}{5} + V_1 - V_4 = -40$$

Simplifying the above expression, we get

$$V_1 - 0.6V_3 - 0.4V_4 = -40 \qquad (4.38)$$

By inspection

$$V_S = 24 \qquad (4.39)$$

Using Equations (4.34), (4.35), (4.36), (4.38) and (4.39), we get the matrix equation

$$
\begin{bmatrix}
-4.4 & 0.125 & -0.125 & 4.9 & 0 \\
-0.1 & -0.2 & 0 & 0.55 & -0.25 \\
0 & -0.125 & 0.325 & -0.2 & 0 \\
1 & 0 & -0.6 & -0.4 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
5 \\
-40 \\
24
\end{bmatrix}
\qquad (4.40)
$$

The MATLAB program for obtaining the nodal voltages is shown below.

MATLAB Script

```
diary ex4_5.dat
% Program determines the nodal voltages
%        given an admittance matrix Y and  current vector I
% Initialize matrix Y and the current vector I of
%        matrix equation Y V = I
Y = [-4.4  0.125  -0.125  4.9  0;
   -0.1  0     -0.2   0.55 -0.25;
    0   -0.125  0.325 -0.2  0;
    1   0     -0.6   -0.4  0;
    0   0      0      0    1];
I = [0  0  5  -40  24]';
%  Solve for the nodal voltages
fprintf('Nodal voltages V(1), V(2), .. V(5) are \n')
V = inv(Y)*I; diary
```

The results obtained from MATLAB are

Nodal voltages V(1), V(2), ... V(5) are

V =
    117.4792
    299.7708
    193.9375
    102.7917
     24.0000

4.3   MAXIMUM POWER TRANSFER

Assume that we have a voltage source V_S with resistance R_S connected to a load R_L. The circuit is shown in Figure 4.6.
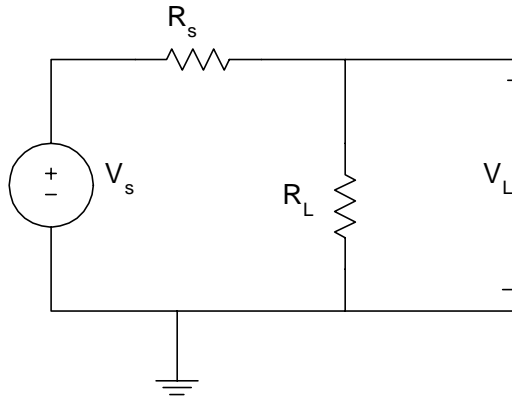

Figure 4.6   Circuit for Obtaining Maximum Power Dissipation

The voltage across the Load R_L is given as

$$V_L = \frac{V_s R_L}{R_s + R_L}$$

The power dissipated by the load R_L is given as

$$P_L = \frac{V_L^2}{R_L} = \frac{V_s^2 R_L}{(R_s + R_L)^2} \tag{4.41}$$

The value of R_L that dissipates the maximum power is obtained by differentiating P_L with respect to R_L, and equating the derivative to zero. That is,

$$\frac{dP_L}{dR_L} = \frac{(R_s + R_L)^2 V_S - V_s^2 R_L (2)(R_s + R_L)}{(R_s + R_L)^4}$$
$$\frac{dP_L}{dR_L} = 0 \tag{4.42}$$

Simplifying the above we get

$$(R_s + R_L) - 2R_L = 0$$

i.e.,

$$R_L = R_S \tag{4.43}$$

Thus, for a resistive network, the maximum power is supplied to a load provided the load resistance is equal to the source resistance. When $R_L = 0$, the voltage across and power dissipated by R_L are zero. On the other hand, when R_L approaches infinity, the voltage across the load is maximum, but the power dissipation is zero. MATLAB can be used to observe the voltage across and power dissipation of the load as functions of load resistance value. Example 4.6 shows the use of MATLAB to plot the voltage and display the power dissipation of a resistive circuit.

Before presenting an example on the maximum power transfer theorem, let us discuss the MATLAB functions **diff** and **find**.

4.3.1 MATLAB Diff and Find Functions

Numerical differentiation can be obtained using the backward difference expression

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{4.44}$$

or by the forward difference expression

$$f'(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \tag{4.45}$$

The derivative of $f(x)$ can be obtained by using the MATLAB **diff** function as

$$f'(x) \cong diff(f)./diff(x). \tag{4.46}$$

If f is a row or column vector

$$f = [f(1) \quad f(2) \quad ... \quad f(n)]$$

then **diff(f)** returns a vector of difference between adjacent elements

$$diff(f) = [f(2) - f(1) \quad f(3) - f(2) \quad ... \quad f(n) - f(n-1)]$$
(4.47)

The output vector $diff(f)$ will be one element less than the input vector f.

The **find** function determines the indices of the nonzero elements of a vector or matrix. The statement

$$B = find(f)$$
(4.48)

will return the indices of the vector f that are nonzero. For example, to obtain the points where a change in sign occurs, the statement

$$Pt\_change = find(product < 0)$$
(4.49)

will show the indices of the locations in *product* that are negative.

The **diff** and **find** are used in the following example to find the value of resistance at which the maximum power transfer occurs.

Example 4.6

In Figure 4.7, as R_L varies from 0 to 50KΩ, plot the power dissipated by the load. Verify that the maximum power dissipation by the load occurs when R_L is 10 KΩ.
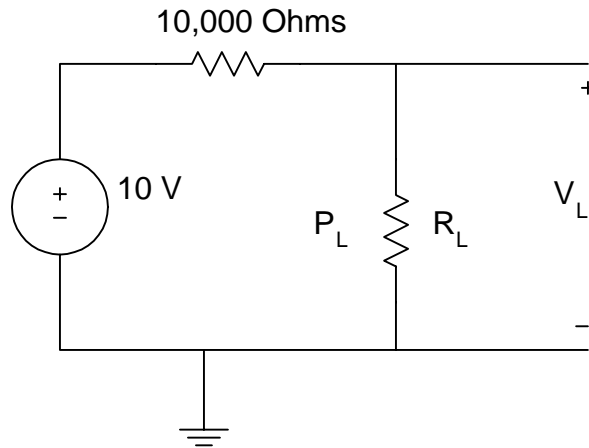
Figure 4.7 Resistive Circuit for Example 4.6

Solution

MATLAB Script

```
% maximum power transfer
% vs is the supply voltage
% rs is the supply resistance
% rl is the load resistance
% vl is the voltage across the load
% pl is the power dissipated by the load
vs = 10;   rs = 10e3;
rl = 0:1e3:50e3;
k = length(rl); % components in vector rl
% Power dissipation calculation
for i=1:k
  pl(i) = ((vs/(rs+rl(i)))^2)*rl(i);
end
% Derivative of power is calculated using backward difference
dp = diff(pl)./diff(rl);
rld = rl(2:length(rl)); % length of rld is 1 less than that of rl
% Determination of critical points of derivative of power
prod = dp(1:length(dp) - 1).*dp(2:length(dp));
crit_pt = rld(find(prod < 0));
max_power = max(pl); % maximum power is calculated
% print out results
```

```
fprintf('Maximum power occurs at %8.2f Ohms\n',crit_pt)
fprintf('Maximum power dissipation is %8.4f Watts\n', max_power)
% Plot power versus load
plot(rl,pl,'+')
title('Power delivered to load')
xlabel('load resistance in Ohms')
ylabel('power in watts')
```

The results obtained from MATLAB are

```
Maximum power occurs at 10000.00 Ohms
Maximum power dissipation is   0.0025 Watts
```

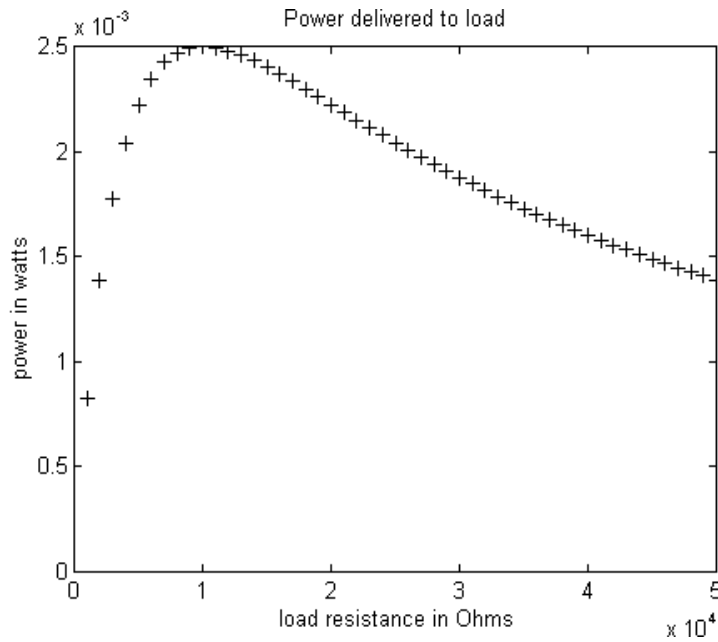The plot of the power dissipation obtained from MATLAB is shown in Figure 4.8.

Figure 4.8 Power delivered to load

SELECTED BIBLIOGRAPHY

1.   MathWorks, Inc., *MATLAB, High-Performance Numeric Computation Software*, 1995.

2.   Etter, D.M., *Engineering Problem Solving with MATLAB*, 2nd Edition, Prentice Hall, 1997.

3.   Gottling, J.G., *Matrix Analysis of Circuits Using MATLAB*, Prentice Hall, 1995.

4.   Johnson, D.E., Johnson, J.R. and Hilburn, J.L., *Electric Circuit Analysis*, 3rd Edition, Prentice Hall, 1997.

5.   Dorf, R.C. and Svoboda, J.A., *Introduction to Electric Circuits*, 3rd Edition, John Wiley & Sons, 1996.

EXERCISES

4.1   Use loop analysis to write equations for the circuit shown in Figure P4.1. Determine the current I using MATLAB.
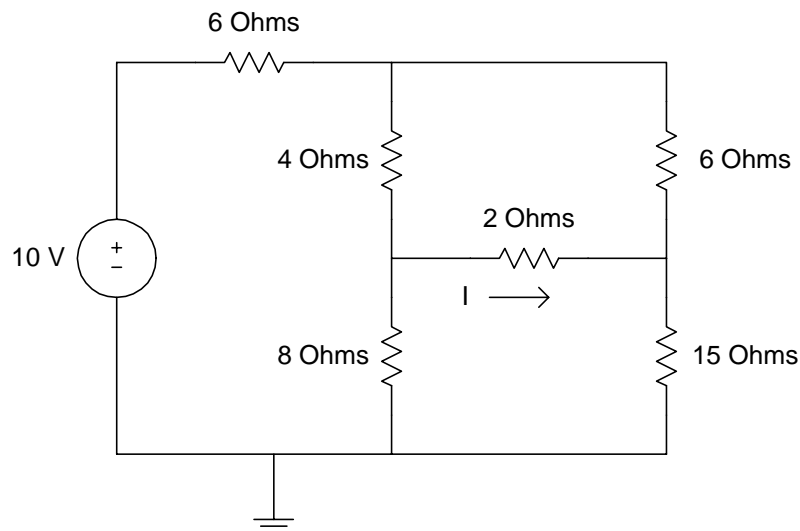
Figure P4.1   Circuit for Exercise 4.1

4.2   Use nodal analysis to solve for the nodal voltages for the circuit
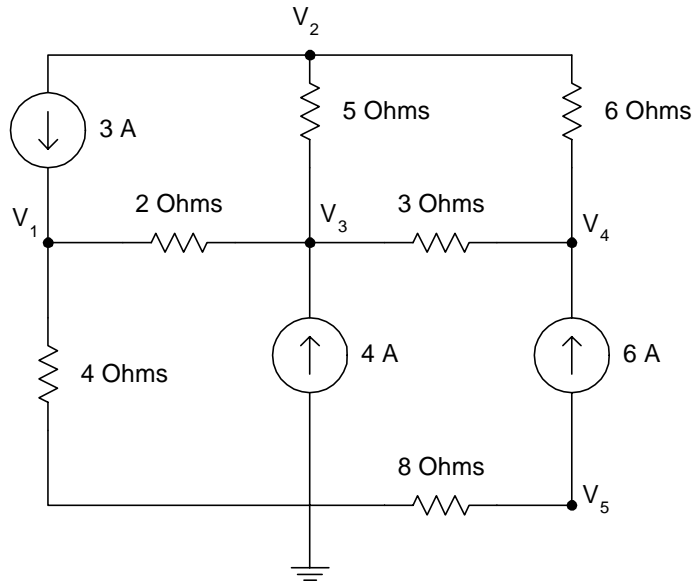shown in Figure P4.2. Solve the equations using MATLAB.

Figure P4.2   Circuit for Exercise 4.2

4.3   Find the power dissipated by the 4Ω resistor and the voltage V_1.
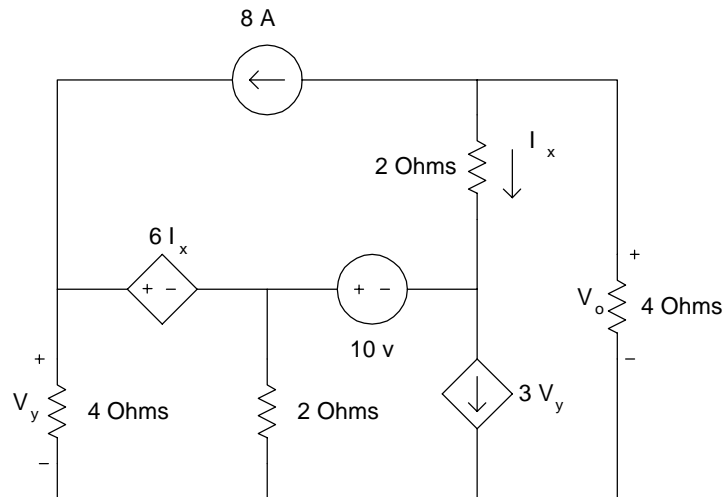

Figure P4.3   Circuit for Exercise 4.3

4.4   Using both loop and nodal analysis, find the power delivered by a
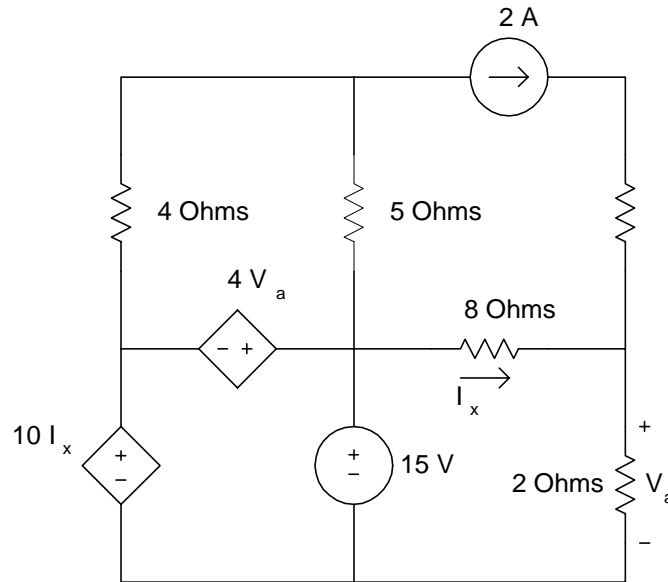15V source.

Figure P4.4   Circuit for Exercise 4.4

4.5   As R_L varies from 0 to 12 in increments of 2Ω, calculate the power
dissipated by R_L.   Plot the power dissipation with respect to the
variation in R_L.   What is the maximum power dissipated by R_L ?
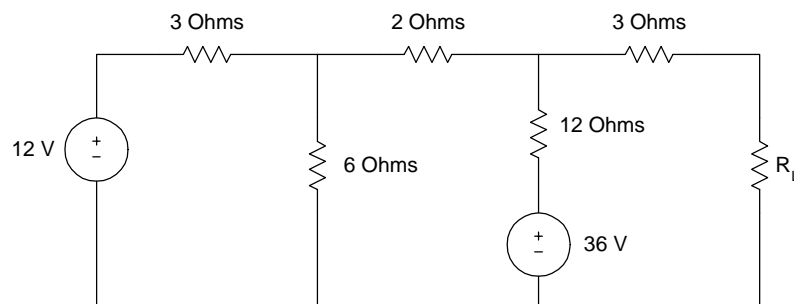What is the value of R_L needed for maximum power dissipation?

Figure P4.5   Circuit for Exercise 4.5

4.6   Using loop analysis and MATLAB, find the loop currents. What is the power supplied by the source?
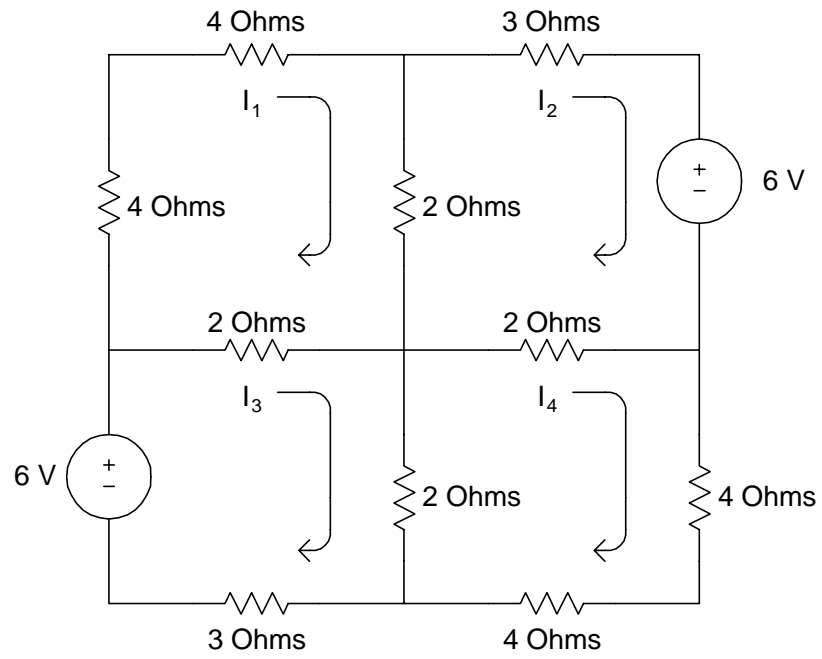

Figure P4.6   Circuit for Exercise 4.6

Attia, John Okyere. "Transient Analysis."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

5.1 RC NETWORK

Considering the RC Network shown in Figure 5.1, we can use KCL to write Equation (5.1).
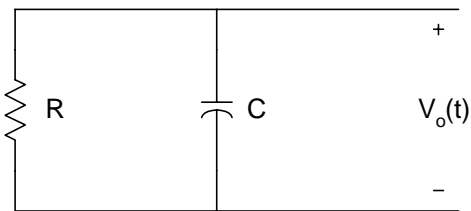

Figure 5.1 Source-free RC Network

$$C\frac{dv_o(t)}{dt} + \frac{v_o(t)}{R} = 0 \qquad (5.1)$$

i.e.,

$$\frac{dv_o(t)}{dt} + \frac{v_o(t)}{CR} = 0$$

If V_m is the initial voltage across the capacitor, then the solution to Equation (5.1) is

$$v_0(t) = V_m e^{-\left(\frac{t}{CR}\right)} \qquad (5.2)$$

where

CR is the time constant

Equation (5.2) represents the voltage across a discharging capacitor. To obtain the voltage across a charging capacitor, let us consider Figure 5.2.
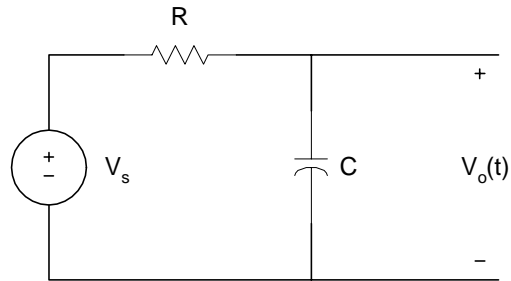
Figure 5.2 Charging of a Capacitor

Using KCL, we get

$$C\frac{dv_o(t)}{dt} + \frac{v_o(t) - V_s}{R} = 0 \tag{5.3}$$

If the capacitor is initially uncharged, that is $v_0(t) = 0$ at $t = 0$, the solution to Equation (5.3) is given as

$$v_0(t) = V_S\left(1 - e^{-\left(\frac{t}{CR}\right)}\right) \tag{5.4}$$

Examples 5.1 and 5.2 illustrate the use of MATLAB for solving problems related to RC Network.

Example 5.1

Assume that for Figure 5.2 $C = 10$ μF, use MATLAB to plot the voltage across the capacitor if R is equal to (a) 1.0 kΩ, (b) 10 kΩ and (c) 0.1 kΩ.

Solution

MATLAB Script

```
% Charging of an RC circuit
%
c = 10e-6;
r1 = 1e3;
```

```
tau1 = c*r1;
t = 0:0.002:0.05;
v1 = 10*(1-exp(-t/tau1));
r2 = 10e3;
tau2 = c*r2;
v2 = 10*(1-exp(-t/tau2));
r3 = .1e3;
tau3 = c*r3;
v3 = 10*(1-exp(-t/tau3));
plot(t,v1,'+',t,v2,'o', t,v3,'*')
axis([0 0.06 0 12])
title('Charging of a capacitor with three time constants')
xlabel('Time, s')
ylabel('Voltage across capacitor')
text(0.03, 5.0, '+ for R = 1 Kilohms')
text(0.03, 6.0, 'o for R = 10 Kilohms')
text(0.03, 7.0, '* for R = 0.1 Kilohms')
```
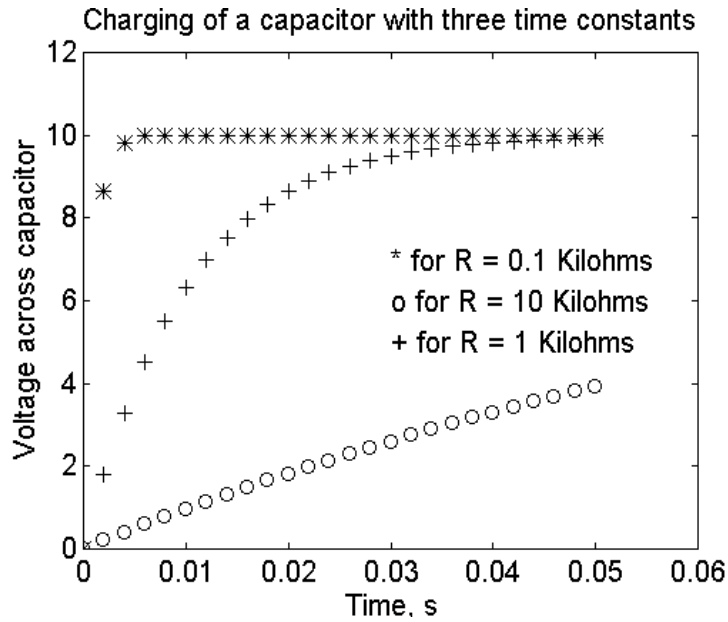
Figure 5.3 shows the charging curves.

Figure 5.3 Charging of Capacitor

From Figure 5.3, it can be seen that as the time constant is small, it takes a short time for the capacitor to charge up.

Example 5.2

For Figure 5.2, the input voltage is a rectangular pulse with an amplitude of 5V and a width of 0.5s.   If $C = 10$ µF, plot the output voltage, $v_0(t)$, for resistance R equal to (a) 1000 Ω, and (b) 10,000 Ω.   The plots should start from zero seconds and end at 1.5 seconds.

Solution

MATLAB Script

```
%  The problem will be solved using a function program rceval
function [v, t] = rceval(r, c)
% rceval is a function program for calculating
%       the output voltage given the values of
%        resistance and capacitance.
% usage [v, t] = rceval(r, c)
%       r is the resistance value(ohms)
%       c is the capacitance value(Farads)
%       v is the output voltage
%       t is the time corresponding to voltage v
tau  = r*c;
for i=1:50
   t(i) = i/100;
   v(i) = 5*(1-exp(-t(i)/tau));
end
vmax = v(50);

for i = 51:100
   t(i) = i/100;
   v(i) = vmax*exp(-t(i-50)/tau);
end
end

% The problem will be solved using function program
% rceval
% The output is obtained for the various resistances
c = 10.0e-6;
r1 = 2500;
```

```
[v1,t1] = rceval(r1,c);
r2 = 10000;
[v2,t2] = rceval(r2,c);

% plot the voltages
plot(t1,v1,'*w', t2,v2,'+w')
axis([0 1 0 6])
title('Response of an RC circuit to pulse input')
xlabel('Time, s')
ylabel('Voltage, V')
text(0.55,5.5,'* is for 2500 Ohms')
text(0.55,5.0, '+ is for 10000 Ohms')
```
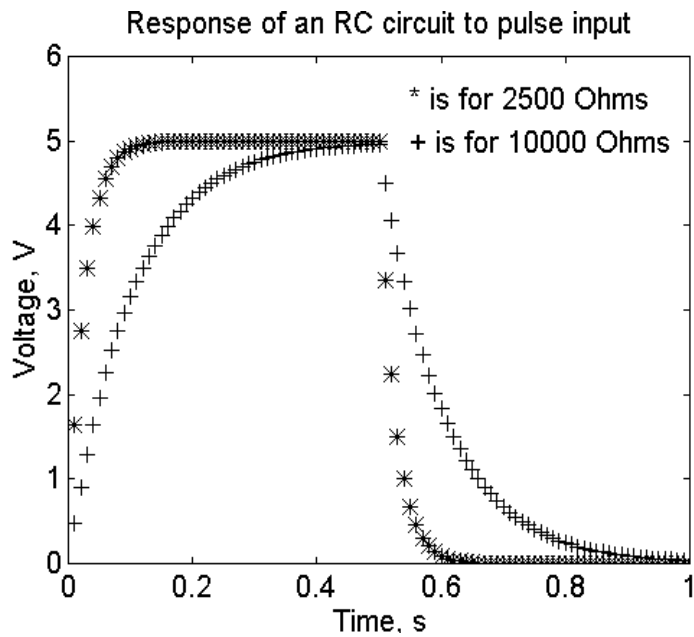
Figure 5.4 shows the charging and discharging curves.

Figure 5.4 Charging and Discharging of a Capacitor with Different Time Constants

5.2   RL NETWORK
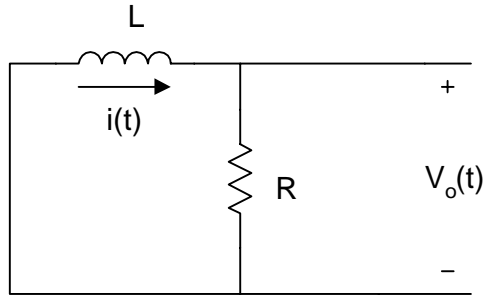
Consider the RL circuit shown in Figure 5.5.

Figure 5.5 Source-free RL Circuit

Using the KVL, we get

$$L\frac{di(t)}{dt} + Ri(t) = 0 \tag{5.5}$$

If the initial current flowing through the inductor is I_m, then the solution to Equation (5.5) is

$$i(t) = I_m e^{-\left(\frac{t}{\tau}\right)} \tag{5.6}$$

where

$$\tau = \frac{L}{R} \tag{5.7}$$

Equation (5.6) represents the current response of a source-free RL circuit with initial current I_m, and it represents the natural response of an RL circuit.

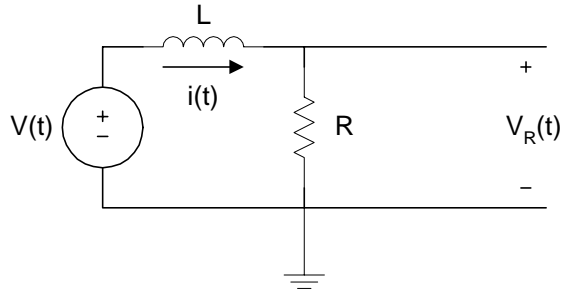Figure 5.6 is an RL circuit with source voltage $v(t) = V_S$.

Figure 5.6 RL Circuit with a Voltage Source

Using KVL, we get

$$L\frac{di(t)}{dt} + Ri(t) = V_S \tag{5.8}$$

If the initial current flowing through the series circuit is zero, the solution of Equation (5.8) is

$$i(t) = \frac{V_S}{R}\left(1 - e^{-\left(\frac{Rt}{L}\right)}\right) \tag{5.9}$$

The voltage across the resistor is

$$v_R(t) = Ri(t)$$
$$= V_S\left(1 - e^{-\left(\frac{Rt}{L}\right)}\right) \tag{5.10}$$

The voltage across the inductor is

$$v_L(t) = V_S - v_R(t)$$
$$= V_S e^{-\left(\frac{Rt}{L}\right)} \tag{5.11}$$

The following example illustrates the use of MATLAB for solving RL circuit problems.

Example 5.3

For the sequential circuit shown in Figure 5.7, the current flowing through the inductor is zero. At $t = 0$, the switch moved from position a to b, where it remained for 1 s. After the 1 s delay, the switch moved from position b to position c, where it remained indefinitely. Sketch the current flowing through the inductor versus time.
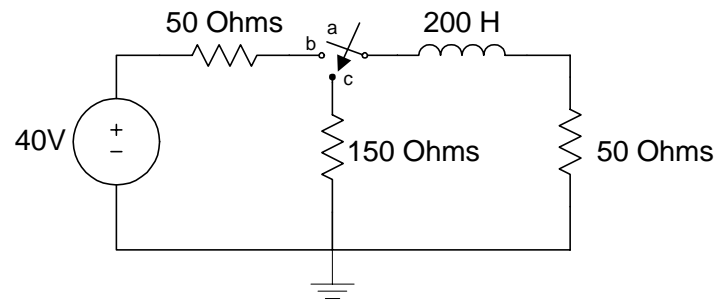

Figure 5.7   RL Circuit for Example 5.3

Solution

For $0 < t < 1$ s, we can use Equation (5.9) to find the current

$$i(t) = 0.4\left(1 - e^{-\left(\frac{t}{\tau_1}\right)}\right) \tag{5.12}$$

where

$$\tau_1 = L/R = 200/100 = 2 \text{ s}$$

At $t = 1$ s

$$i(t) = 0.4\left(1 - e^{-0.5}\right) \tag{5.13}$$
$$= I_{max}$$

For $t > 1$ s, we can use Equation (5.6) to obtain the current

$$i(t) = I_{max}e^{-\left(\frac{t-0.5}{\tau_2}\right)} \tag{5.14}$$

where

$$\tau_2 = \frac{L}{R_{eq2}} = \frac{200}{200} = 1 \ \text{s}$$

The MATLAB program for plotting $i(t)$ is shown below.

MATLAB Script

```
% Solution to Example 5.3
% tau1 is time constant when switch is at b
% tau2 is the time constant when the switch is in position c
%

tau1 = 200/100;

for k=1:20
  t(k) = k/20;
  i(k) = 0.4*(1-exp(-t(k)/tau1));
end

imax = i(20);
tau2 = 200/200;
for k = 21:120
  t(k) = k/20;
  i(k) = imax*exp(-t(k-20)/tau2);
end

% plot the current
plot(t,i,'o')
axis([0 6 0 0.18])
title('Current of an RL circuit')
xlabel('Time, s')
ylabel('Current, A')
```
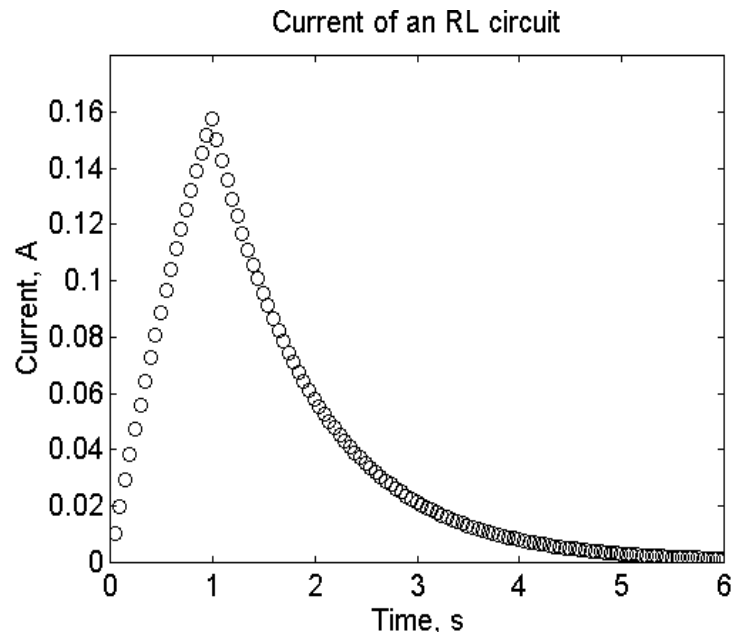
Figure 5.8 shows the current $i(t)$.

Figure 5.8 Current Flowing through Inductor

5.3   RLC CIRCUIT

For the series RLC circuit shown in Figure 5.9, we can use KVL to obtain
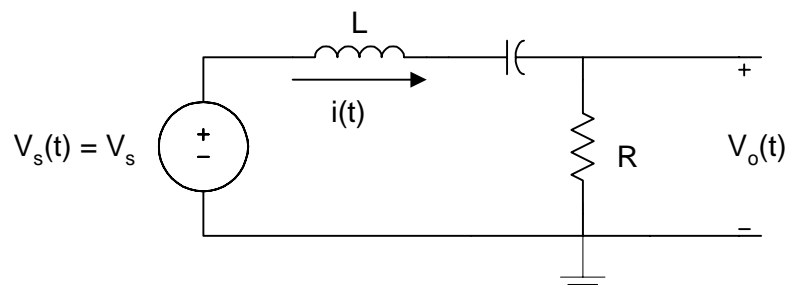the Equation (5.15).

Figure 5.9   Series RLC Circuit

$$v_S(t) = L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t) \qquad (5.15)$$

Differentiating the above expression, we get

$$\frac{dv_S(t)}{dt} = L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C}$$

i.e.,

$$\frac{1}{L}\frac{dv_S(t)}{dt} = \frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} \qquad (5.16)$$

The homogeneous solution can be found by making $v_S(t) = $ constant, thus

$$0 = \frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} \qquad (5.17)$$

The characteristic equation is

$$0 = \lambda^2 + a\lambda + b \qquad (5.18)$$

where

$$a = {R}\big/{L} \quad \text{and}$$

$$b = {1}\big/{LC}$$

The roots of the characteristic equation can be determined. If we assume that the roots are

$$\lambda = \alpha, \beta$$

then, the solution to the homogeneous solution is

$$i_h(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \qquad (5.19)$$

where

A_1 and A_2 are constants.

If $v_S(t)$ is a constant, then the forced solution will also be a constant and be given as

$$i_f(t) = A_3 \qquad\qquad (5.20)$$

The total solution is given as

$$i(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + A_3 \qquad\qquad (5.21)$$

where

A_1, A_2, and A_3 are obtained from initial conditions.

Example 5.4 illustrates the use of MATLAB for finding the roots of characteristic equations. The MATLAB function **roots,** described in Section 6.3.1, is used to obtain the roots of characteristic equations.

Example 5.4

For the series RLC circuit shown in Figure 5.9, If L = 10 H, R = 400 Ohms and C = 100μF, $v_S(t) = 0$, $i(0) = 4$ A and $\dfrac{di(0)}{dt} = 15$ A/s, find $i(t)$.

Solution

Since $v_S(t) = 0$, we use Equation (5.17) to get

$$0 = \frac{d^2 i(t)}{dt^2} + \frac{400}{10} \frac{di(t)}{dt} + 1000 i(t)$$

The characteristic equation is

$$0 = \lambda^2 + 40\lambda + 1000$$

The MATLAB function **roots** is used to obtain the roots of the characteristics equation.

MATLAB Script

```
p = [1 40 1000];
lambda = roots(p)

lambda =
        -20.0000 +24.4949i
        -20.0000 -24.4949i
```

Using the roots obtained from MATLAB, $i(t)$ is given as

$$i(t) = e^{-20t}(A_1 \cos(24.4949t) + A_2 \sin(24.4949t)$$

$$i(0) = e^{-0}(A_1 + A_2(0)) \qquad \Rightarrow A_1 = 4$$

$$\frac{di(t)}{dt} = -20e^{-20t}\left[A_1 \cos(24.4949t) + A_2 \sin(24.4949t)\right] +$$

$$e^{-20t}\left[-24.4949 A_1 \sin(24.4949t) + 24.4949 A_2 \cos(24.4949t)\right]$$

$$\frac{di(0)}{dt} = 24.4949 A_2 - 20A_1 = 15$$

Since $A_1 = 4$, $A_2 = 3.8784$

$$i(t) = e^{-20t}\left[4\cos(24.4949t) + 3.8784\sin(24.4949t)\right]$$

Perhaps the simplest way to obtain voltages and currents in an RLC circuit is to use Laplace transform. Table 5.1 shows Laplace transform pairs that are useful for solving RLC circuit problems.

From the RLC circuit, we write differential equations by using network analysis tools. The differential equations are converted into algebraic equations using the Laplace transform. The unknown current or voltage is then solved in the s-domain. By using an inverse Laplace transform, the solution can be expressed in the time domain. We will illustrate this method using Example 5.5

Table 5.1
Laplace Transform Pairs

	f (t)	**f(s)**	
1	1	$\dfrac{1}{s}$	s>0
2	t	$\dfrac{1}{s^2}$	s>0
3	t^n	$\dfrac{n!}{s^{n+1}}$	s>0
4	e^{-at}	$\dfrac{1}{s+a}$	s>a
5	te^{-at}	$\dfrac{1}{(s+a)^2}$	s>a
6	$\sin(wt)$	$\dfrac{w}{s^2+w^2}$	s>0
7	$\cos(wt)$	$\dfrac{s}{s^2+w^2}$	s>0
8	$e^{at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$	
9	$e^{at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$	
10	$\dfrac{df}{dt}$	$sF(s)-f(0^+)$	
11	$\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$	
12	$f(t-t_1)$	$e^{-t_1 s}F(s)$	

Example 5.5

The switch in Figure 5.10 has been opened for a long time. If the switch opens at t = 0, find the voltage $v(t)$. Assume that $R = 10\ \Omega$, $L = 1/32$ H, $C = 50\mu\text{F}$ and $I_S = 2\ A$.
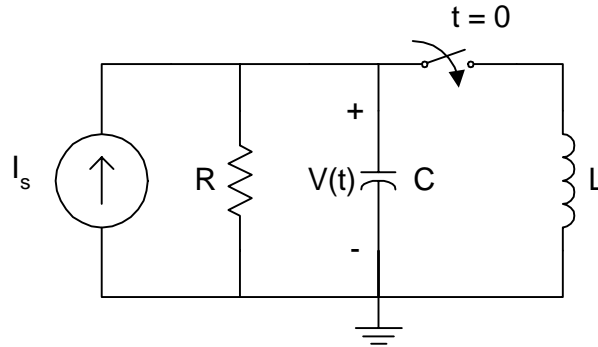

Figure 5.10 Circuit for Example 5.5

At t < 0, the voltage across the capacitor is

$$v_C(0) = (2)(10) = 20 \quad \text{V}$$

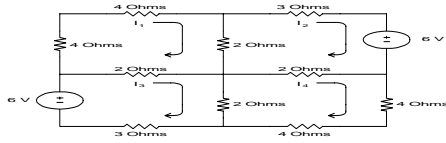In addition, the current flowing through the inductor

$$i_L(0) = 0$$

At t > 0, the switch closes and all the four elements of Figure 5.10 remain in parallel. Using KCL, we get

$$I_S = \frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0)$$

Taking the Laplace transform of the above expression, we get

$$\frac{I_S}{s} = \frac{V(s)}{R} + C[sV(s) - V_C(0)] + \frac{V(s)}{sL} + \frac{i_L(0)}{s}$$

Simplifying the above expression, we get

For $I_S = 2A$, $C = 50\mu$ F, $R = 10\Omega$, $L = 1/32$ H, $V(s)$ becomes

$$V(s) = \frac{40000 + 20s}{s^2 + 2000s + 64*10^4}$$

$$V(s) = \frac{40000 + 20s}{(s+1600)(s+400)} = \frac{A}{(s+1600)} + \frac{B}{(s+400)}$$

$$A = \underset{s \to -1600}{Lim} \quad V(s)(s+1600) \quad = \text{-6.67}$$

$$B = \underset{s \to -400}{Lim} \quad V(s)(s+400) \quad = 26.67$$

$$v(t) = -6.67e^{-1600t} + 26.67e^{-400t}$$

The plot of $v(t)$ is shown in Figure 5.13.

5.4   STATE VARIABLE APPROACH

Another method of finding the transient response of an RLC circuit is the state variable technique.   The later method   (i) can be used to analyze and synthesize control systems, (ii) can be applied to time-varying and nonlinear systems, (iii) is suitable for digital and computer solution and (iv) can be used to develop the general system characteristics.

A state of a system is a minimal set of variables chosen such that if their values are known at the time t, and all inputs are known for times greater than t_1, one can calculate the output of the system for times greater than t_1. In general, if we designate x as the state variable, u as the input, and y as the output of a system, we can express the input u and output y as

$$x(t) = Ax(t) + Bu(t) \qquad (5.22)$$

$$y(t) = Cx(t) + Du(t) \qquad (5.23)$$

where

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ u_n(t) \end{bmatrix} \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ y_n(t) \end{bmatrix}$$

and *A, B, C*, and *D* are matrices determined by constants of a system.

For example, consider a single-input and a single-output system described by the differential equation

$$\frac{d^4 y(t)}{dt^4} + 3\frac{d^3 y(t)}{dt^3} + 4\frac{d^2 y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 2y(t) = 6u(t)$$

$$(5.24)$$

We define the components of the state vector as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t)$$

$$x_3(t) = \frac{d^2 y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t)$$

$$x_4(t) = \frac{d^3 y(t)}{dt^3} = \frac{dx_3(t)}{dt} = \dot{x}_3(t)$$

$$x_5(t) = \frac{d^4 y(t)}{dt^4} = \frac{dx_4(t)}{dt} = \overset{\bullet}{x}_4(t) \tag{5.25}$$

Using Equations (5.24) and (5.25), we get

$$\overset{\bullet}{x}_4(t) = 6u(t) - 3x_4(t) - 4x_3(t) - 8x_2(t) - 2x_1(t) \tag{5.26}$$

From the Equations (5.25) and (5.26), we get

$$\begin{bmatrix} \overset{\bullet}{x}_1(t) \\ \overset{\bullet}{x}_2(t) \\ \overset{\bullet}{x}_3(t) \\ \overset{\bullet}{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -8 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} u(t) \tag{5.27}$$

or
$$\overset{\bullet}{x}(t) = Ax(t) + Bu(t) \tag{5.28}$$

where

$$\overset{\bullet}{x} = \begin{bmatrix} \overset{\bullet}{x}_1(t) \\ \overset{\bullet}{x}_2(t) \\ \overset{\bullet}{x}_3(t) \\ \overset{\bullet}{x}_4(t) \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -8 & -4 & -3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} \tag{5.29}$$

Since

$$y(t) = x_1(t)$$

we can express the output $y(t)$ in terms of the state $x(t)$ and input $u(t)$ as

$$y(t) = Cx(t) + Du(t) \tag{5.30}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad D = \begin{bmatrix} 0 \end{bmatrix} \tag{5.31}$$

In RLC circuits, if the voltage across a capacitor and the current flowing in an inductor are known at some initial time t, then the capacitor voltage and inductor current will allow the description of system behavior for all subsequent times. This suggests the following guidelines for the selection of acceptable state variables for RLC circuits:

1. Currents associated with inductors are state variables.

2. Voltages associated with capacitors are state variables.

3. Currents or voltages associated with resistors do not specify independent state variables.

4. When closed loops of capacitors or junctions of inductors exist in a circuit, the state variables chosen according to rules 1 and 2 are not independent.
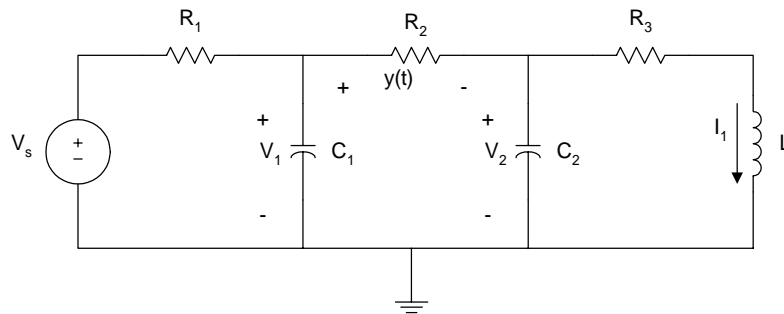
Consider the circuit shown in Figure 5.11.

Figure 5.11 Circuit for State Analysis

Using the above guidelines, we select the state variables to be V_1, V_2, and i_1.

Using nodal analysis, we have

$$C_1 \frac{dv_1(t)}{dt} + \frac{V_1 - V_s}{R_1} + \frac{V_1 - V_2}{R_2} = 0 \tag{5.32}$$

$$C_2 \frac{dv_2(t)}{dt} + \frac{V_2 - V_1}{R_2} + i_1 = 0 \tag{5.33}$$

Using loop analysis

$$V_2 = i_1 R_3 + L \frac{di_1(t)}{dt} \tag{5.34}$$

The output $y(t)$ is given as

$$y(t) = v_1(t) - v_2(t) \tag{5.35}$$

Simplifying Equations (5.32) to (5.34), we get

$$\frac{dv_1(t)}{dt} = -\left(\frac{1}{C_1 R_1} + \frac{1}{C_1 R_2}\right)V_1 + \frac{V_2}{C_1 R_2} + \frac{V_s}{C_1 R_1} \tag{5.36}$$

$$\frac{dv_2(t)}{dt} = \frac{V_1}{C_2 R_2} - \frac{V_2}{C_2 R_2} - \frac{i_1}{C_2} \tag{5.37}$$

$$\frac{di_1(t)}{dt} = \frac{V_2}{L} - \frac{R_3}{L} i_1 \tag{5.38}$$

Expressing the equations in matrix form, we get

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{i}_1 \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{1}{C_1 R_1} + \dfrac{1}{C_1 R_2}\right) & \dfrac{1}{C_1 R_2} & 0 \\ \dfrac{1}{C_2 R_2} & -\dfrac{1}{C_2 R_2} & -\dfrac{1}{C_2} \\ 0 & \dfrac{1}{L} & -\dfrac{R_3}{L} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C_1 R_1} \\ 0 \\ 0 \end{bmatrix} V_s \tag{5.39}$$

and the output is

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \dot{i}_1 \end{bmatrix} \tag{5.40}$$

MATLAB functions for solving ordinary differential equations are ODE functions. These are described in the following section.

5.4.1   MATLAB Ode Functions

MATLAB has two functions, **ode23 and ode45**, for computing numerical solutions to ordinary differential equations. The **ode23** function integrates a system of ordinary differential equations using second- and third-order Runge-Kutta formulas; the **ode45** function uses fourth- and fifth-order Runge-Kutta integration equations.

The general forms of the ode functions are

> [t,x] = ode23 (xprime, tstart, tfinal, xo, tol,trace)

>> or

> [t,x] = ode45 (xprime, tstart, tfinal, xo, tol, trace)

where

> xprime   is the name (in quotation marks) of the MATLAB function or m-file that contains the differential equations to be integrated. The function will compute the state derivative vector $\dot{x}(t)$ given the current time t, and state vector $x(t)$. The function must have 2 input arguments, scalar t (time) and column vector x (state) and the function returns the output argument $xdot, (\dot{x})$, a column vector of state derivatives

$$\dot{x}(t_1) = \frac{dx(t_1)}{dt}$$

tfinal   is the final time for the integration

xo   is a column vector of initial conditions

tol   is optional. It specifies the desired accuracy of the solution.

Let us illustrate the use of MATLAB ode functions with the following two examples.

Example 5.6

For Figure 5.2, $V_S = 10\text{V}$, $R = 10{,}000\ \Omega$, $C = 10\mu\text{F}$. Find the output voltage $v_0(t)$, between the interval 0 to 20 ms, assuming $v_0(0) = 0$ and by (a) using a numerical solution to the differential equation; and (b) analytical solution.

Solution

From Equation (5.3), we have

$$C\frac{dv_o(t)}{dt} + \frac{v_o(t) - V_s}{R} = 0$$

thus

$$\frac{dv_o(t)}{dt} = \frac{V_s}{CR} - \frac{v_o(t)}{CR} = 100 - 10v_0(t)$$

From Equation(5.4), the analytical solution is

$$v_0(t) = 10\left(1 - e^{-\left(\frac{t}{CR}\right)}\right)$$

MATLAB Script

```matlab
% Solution for first order differential equation
% the function diff1(t,y) is created to evaluate
% the differential equation
% Its m-file is diff1.m
%
% Transient analysis of RC circuit using ode
% function and analytical solution
% numerical solution using ode

t0 = 0;
tf = 20e-3;
xo = 0;  % initial conditions
[t, vo] = ode23('diff1',t0,tf,xo);

% the analytical solution given by Equation(5.4) is
vo_analy = 10*(1-exp(-10*t));

% plot two solutions
subplot(121)
plot(t,vo,'b')
title('State Variable Approach')
xlabel('Time, s'),ylabel('Capacitor Voltage, V'),grid
subplot(122)
plot(t,vo_analy,'b')
title('Analytical Approach')
xlabel('Time, s'),ylabel('Capacitor Voltage, V'),grid

%
function dy = diff1(t,y)
dy = 100 - 10*y;
end
```

Figure 5.12 shows the plot obtained using Equation (5.4) and that obtained from the MATLAB ode23 function. From the two plots, we can see that the two results are identical.
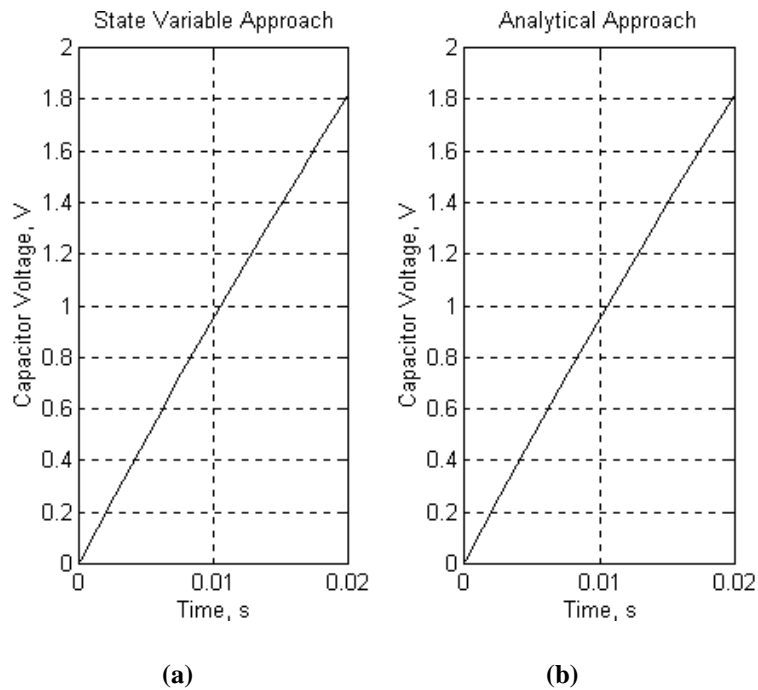
Figure 5.12 Output Voltage $v_0(t)$ Obtained from (a) State Variable Approach and (b) Analytical Method

Example 5.7

For Figure 5.10, if $R = 10\Omega$, $L = 1/32$ H, $C = 50\mu F$, use a numerical solution of the differential equation to solve $v(t)$. Compare the numerical solution to the analytical solution obtained from Example 5.5.

Solution

From Example 5.5, $v_C(0) = 20V$, $i_L(0) = 0$, and

$$L\frac{di_L(t)}{dt} = v_C(t)$$

$$C\frac{dv_C(t)}{dt} + i_L + \frac{v_C(t)}{R} - I_S = 0$$

Simplifying, we get

$$\frac{di_L(t)}{dt} = \frac{v_C(t)}{L}$$

$$\frac{dv_C(t)}{dt} = \frac{I_S}{C} - \frac{i_L(t)}{C} - \frac{v_C(t)}{RC}$$

Assuming that

$$x_1(t) = i_L(t)$$

$$x_2(t) = v_C(t)$$

We get

$$\overset{\bullet}{x}_1(t) = \frac{1}{L}x_2(t)$$

$$\overset{\bullet}{x}_2(t) = \frac{I_S}{C} - \frac{1}{C}x_1(t) - \frac{1}{RC}x_2(t)$$

We create function m-file containing the above differential equations.

MATLAB Script

```
%  Solution of second-order differential equation
%  The function diff2(x,y) is created to evaluate the diff. equation
%  the name of the m-file is diff2.m
%  the function is defined as:
%
function xdot = diff2(t,x)
is = 2;
c = 50e-6;  L = 1/32;  r = 10;
k1 = 1/c ;  %  1/C
k2 =  1/L ;  %  1/L
k3 = 1/(r*c);   %  1/RC

xdot(1) = k2*x(2);
xdot(2) = k1*is - k1*x(1) - k3*x(2);
end
```

To simulate the differential equation defined in diff2 in the interval $0 \le t \le 30$ ms, we note that

$$x_1(0) = i_L(0) = 0 \text{ V}$$

$$x_2(0) = v_C(0) = 20$$

Using the MATLAB ode23 function, we get

```
% solution of second-order differential equation
% the function diff2(x,y) is created to evaluate
% the differential equation
% the name of m-file is diff2.m
%
% Transient analysis of RLC circuit using ode function
% numerical solution

t0 = 0;
tf = 30e-3;
x0 = [0 20]; % Initial conditions
[t,x] = ode23('diff2',t0,tf,x0);

% Second column of matrix x represent capacitor voltage
subplot(211), plot(t,x(:,2))
xlabel('Time, s'), ylabel('Capacitor voltage, V')
text(0.01, 7, 'State Variable Approach')

% Transient analysis of RLC circuit from Example 5.5
t2 =0:1e-3:30e-3;
vt = -6.667*exp(-1600*t2) + 26.667*exp(-400*t2);
subplot(212), plot(t2,vt)
xlabel('Time, s'), ylabel('Capacitor voltage, V')
text(0.01, 4.5, 'Results from Example 5.5')
```
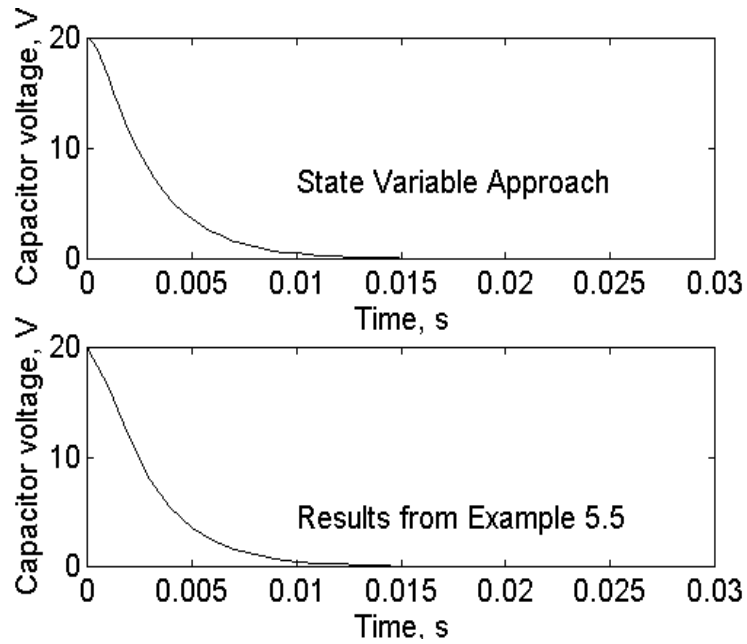
The plot is shown in Figure 5.13.

Figure 5.13 Capacitor Voltage $v_0(t)$ Obtained from Both State Variable Approach and Laplace Transform

The results from the state variable approach and those obtained from Example 5.5 are identical.

Example 5.8

For Figure 5.11, if $v_S(t) = 5u(t)$ where $u(t)$ is the unit step function and $R_1 = R_2 = R_3 = 10\ K\Omega$, $C_1 = C_2 = 5\ \mu F$, and $L = 10$ H, find and plot the voltage $v_0(t)$ within the intervals of 0 to 5 s.

Solution

Using the element values and Equations (5.36) to (5.38), we have

$$\frac{dv_1(t)}{dt} = -40v_1(t) + 20v_2(t) + 20V_s$$

$$\frac{dv_2(t)}{dt} = 20v_1(t) - 20v_2(t) - i_1(t)$$

$$\frac{di_1(t)}{dt} = 0.1v_2(t) - 1000i_1(t)$$

We create an m-file containing the above differential equations.

MATLAB Script

```
%
% solution of a set of first order differential equations
% the function diff3(t,v) is created to evaluate
% the differential equation
% the name of the m-file is diff3.m
%

function vdot = diff3(t,v)

vdot(1) = -40*v(1) + 20*v(2) + 20*5;
vdot(2) = 20*v(1) - 20*v(2) - v(3);
vdot(3) = 0.1*v(2) -1000*v(3);
end
```

To obtain the output voltage in the interval of $0 \le t \le 5$ s, we note that the output voltage

$$v_0(t) = v_1(t) - v_2(t)$$

Note that at t < 0, the step signal is zero so

$$v_0(0) = v_2(0) = i_1(0) = 0$$

Using ode45 we get

```
% solution of a set of first-order differential equations
% the function diff3(t,v) is created to evaluate
% the differential equation
% the name of the m-file is diff3.m
%
% Transient analysis of RLC circuit using state
```

% variable approach

t0 = 0;
tf = 2;
x0 = [0 0 0]; % initial conditions

[t,x] = ode23('diff3', t0, tf, x0);

tt = length(t);

for i = 1:tt
  vo(i) = x(i,1) - x(i,2);
end

plot(t, vo)
title('Transient analysis of RLC')
xlabel('Time, s'), ylabel('Output voltage')
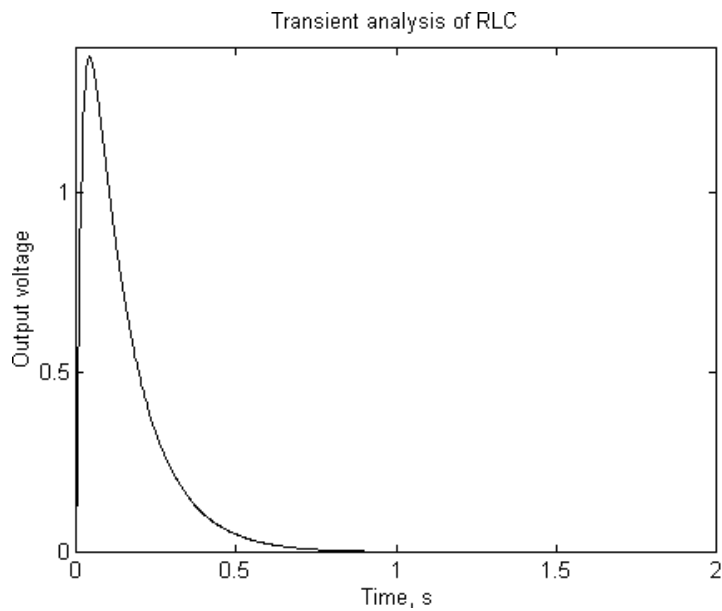
The plot of the output voltage is shown in Figure 5.14.

Figure 5.14 Output Voltage

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5.   Vlach, J.O., Network Theory and CAD, *IEEE Trans. on Education*, Vol. 36, No. 1, Feb. 1993, pp. 23 - 27.

6.   Meader, D. A., *Laplace Circuit Analysis and Active Filters*, Prentice Hall, New Jersey, 1991.

EXERCISES

5.1 If the switch is opened at $t = 0$, find $v_0(t)$. Plot $v_0(t)$ between the time interval $0 \le t \le 5$ s.
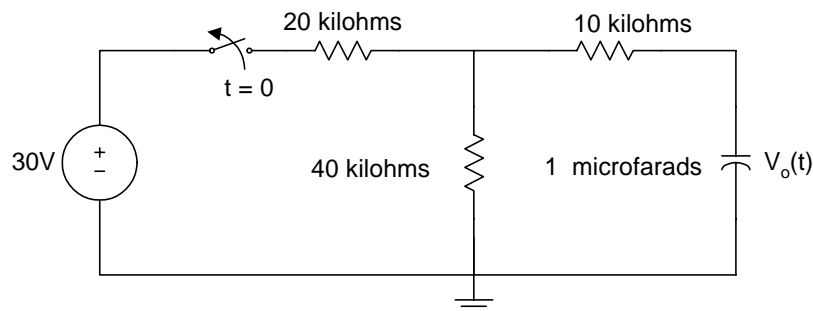

Figure P5.1 Figure for Exercise 5.1

5.2   The switch is close at t = 0; find $i(t)$ between the intervals 0 to 10 ms. The resistance values are in ohms.
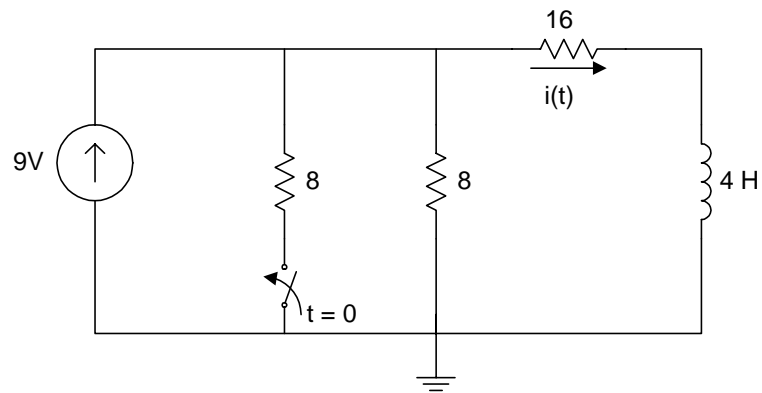

Figure P5.2 Figure for Exercise 5.2

5.3   For the series RLC circuit, the switch is closed at t = 0. The initial energy in the storage elements is zero. Use MATLAB to find $v_0(t)$.
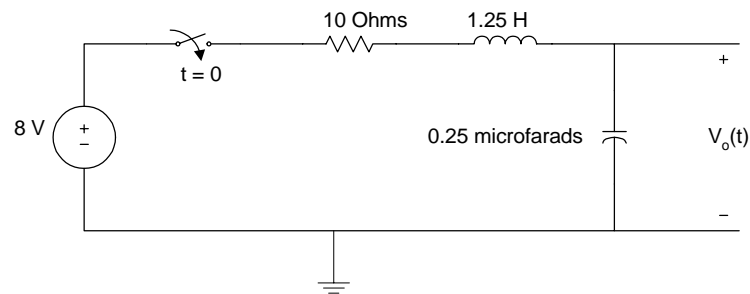

Figure P5.3 Circuit for Exercise 5.3

5.4   Use MATLAB to solve the following differential equation

$$\frac{d^3 y(t)}{dt^3} + 7\frac{d^2 y(t)}{dt^2} + 14\frac{dy(t)}{dt} + 12 y(t) = 10$$

with initial conditions

$$y(0) = 1, \qquad \frac{dy(0)}{dt} = 2, \qquad \frac{d^2 y(0)}{dt^2} = 5$$

Plot y(t) within the intervals of 0 and 10 s.

5.5 For Figure P5.5, if $V_S = 5u(t)$, determine the voltages $V_1(t)$, $V_2(t)$, $V_3(t)$ and $V_4(t)$ between the intervals of 0 to 20 s. Assume that the initial voltage across each capacitor is zero.
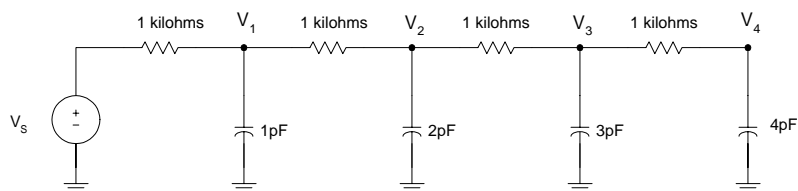

Figure P5.5 RC Network

5.6 For the differential equation

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6 y(t) = 3\sin(t) + 7\cos(t)$$

with initial conditions $y(0) = 4$ and $\frac{dy(0)}{dt} = -1$

(a) Determine $y(t)$ using Laplace transforms.

(b) Use MATLAB to determine $y(t)$.

(c) Sketch $y(t)$ obtained in parts (a) and (b).

(d) Compare the results obtained in part c.

Attia, John Okyere. "AC Analysis and Network Functions."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER SIX

AC ANALYSIS AND NETWORK FUNCTIONS

This chapter discusses sinusoidal steady state power calculations. Numerical integration is used to obtain the rms value, average power and quadrature power. Three-phase circuits are analyzed by converting the circuits into the frequency domain and by using the Kirchoff voltage and current laws. The unknown voltages and currents are solved using matrix techniques.

Given a network function or transfer function, MATLAB has functions that can be used to (i) obtain the poles and zeros, (ii) perform partial fraction expansion, and (iii) evaluate the transfer function at specific frequencies. Furthermore, the frequency response of networks can be obtained using a MATLAB function. These features of MATLAB are applied in this chapter.

6.1   STEADY STATE AC POWER

Figure 6.1 shows an impedance with voltage across it given by $v(t)$ and current through it $i(t)$.
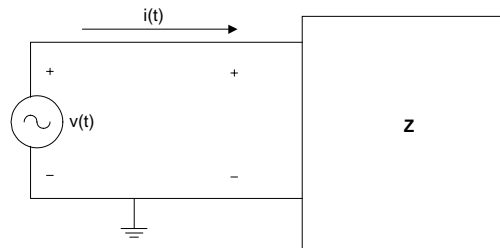

Figure 6.1 One-Port Network with Impedance Z

The instantaneous power $p(t)$ is

$$p(t) = v(t)i(t) \tag{6.1}$$

If $v(t)$ and $i(t)$ are periodic with period T, the rms or effective values of the voltage and current are

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} \qquad (6.2)$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \qquad (6.3)$$

where

V_{rms} is the rms value of $v(t)$

I_{rms} is the rms value of $i(t)$

The average power dissipated by the one-port network is

$$P = \frac{1}{T}\int_0^T v(t)i(t)dt \qquad (6.4)$$

The power factor, pf, is given as

$$pf = \frac{P}{V_{rms}I_{rms}} \qquad (6.5)$$

For the special case, where both the current $i(t)$ and voltage $v(t)$ are both sinusoidal, that is,

$$v(t) = V_m \cos(wt + \theta_V) \qquad (6.6)$$

and

$$i(t) = I_m \cos(wt + \theta_I) \qquad (6.7)$$

the rms value of the voltage $v(t)$ is

$$V_{rms} = \frac{V_m}{\sqrt{2}} \qquad (6.8)$$

and that of the current is

$$I_{rms} = \frac{I_m}{\sqrt{2}} \qquad (6.9)$$

The average power P is

$$P = V_{rms} I_{rms} \cos(\theta_V - \theta_I) \qquad (6.10)$$

The power factor, pf, is

$$pf = \cos(\theta_V - \theta_I) \qquad (6.11)$$

The reactive power Q is

$$Q = V_{rms} I_{rms} \sin(\theta_V - \theta_I) \qquad (6.12)$$

and the complex power, S, is

$$S = P + jQ \qquad (6.13)$$

$$S = V_{rms} I_{rms} \left[\cos(\theta_V - \theta_I) + j\sin(\theta_V - \theta_I) \right] \qquad (6.14)$$

Equations (6.2) to (6.4) involve the use of integration in the determination of the rms value and the average power. MATLAB has two functions, quad and quad8, for performing numerical function integration.

6.1.1 MATLAB Functions quad and quad8

The **quad** function uses an adaptive, recursive Simpson's rule. The **quad8** function uses an adaptive, recursive Newton Cutes 8 panel rule. The quad8 function is better than the quad at handling functions with "soft" singularities such as $\int \sqrt{x} dx$. Suppose we want to find q given as

$$q = \int_a^b funct(x)dx$$

The general forms of quad and quad8 functions that can be used to find q are

$$quad('\,funct'\,,a,b,tol,trace)$$

$$quad8('\,funct'\,,a,b,tol,trace)$$

where

funct	is a MATLAB function name (in quotes) that returns a vector of values of $f(x)$ for a given vector of input values x.
a	is the lower limit of integration.
b	is the upper limit of integration.
tol	is the tolerance limit set for stopping the iteration of the numerical integration. The iteration continues until the relative error is less than tol. The default value is 1.0e-3.
trace	allows the plot of a graph showing the process of the numerical integration. If the trace is nonzero, a graph is plotted. The default value is zero.

Example 6.1 shows the use of the quad function to perform alternating current power calculations.

Example 6.1

For Figure 6.1, if $v(t) = 10\cos(120\pi t + 30^0)$ and
$i(t) = 6\cos(120\pi t + 60^0)$. Determine the average power, rms value of $v(t)$ and the power factor using (a) analytical solution and (b) numerical solution.

Solution

MATLAB Script

```
diary  ex6_1.dat
% This program computes the average power, rms value and
% power factor using quad function. The analytical and
% numerical results are compared.
% numerical calculations
```

```
T = 2*pi/(120*pi); % period of the sin wave
a = 0; % lower limit of integration
b = T; % upper limit of integration
x = 0:0.02:1;
t = x.*b;
v_int = quad('voltage1', a, b);
v_rms = sqrt(v_int/b);  % rms of voltage
i_int = quad('current1',a,b);
i_rms = sqrt(i_int/b);  % rms of current

p_int = quad('inst_pr', a, b);
p_ave = p_int/b;    % average power
pf = p_ave/(i_rms*v_rms); % power factor
%
% analytical solution
%
p_ave_an = (60/2)*cos(30*pi/180);  % average power
v_rms_an = 10.0/sqrt(2);
pf_an = cos(30*pi/180);

% results are printed
fprintf('Average power, analytical %f \n Average power, numerical:
%f \n', p_ave_an,p_ave)
fprintf('rms voltage, analytical: %f \n rms voltage, numerical: %f \n',
v_rms_an, v_rms)
fprintf('power factor, analytical: %f \n power factor, numerical: %f \n',
pf_an, pf)
diary
```

The following functions are used in the above m-file:

```
function vsq = voltage1(t)
% voltage1  This function is used to
%        define the voltage function
vsq = (10*cos(120*pi*t + 60*pi/180)).^2;
end

function isq = current1(t)
% current1  This function is to define the current
%
isq = (6*cos(120*pi*t + 30.0*pi/180)).^2;
end
```

```
function pt = inst_pr(t)
% inst_pr   This function is used to define
%           instantaneous power obtained by multiplying
%           sinusoidal voltage and current
it = 6*cos(120*pi*t + 30.0*pi/180);
vt = 10*cos(120*pi*t + 60*pi/180);
pt = it.*vt;
end
```

The results obtained are

> Average power, analytical 25.980762
> Average power, numerical: 25.980762
> rms voltage, analytical: 7.071068
> rms voltage, numerical: 7.071076
> power factor, analytical: 0.866025
> power factor, numerical: 0.866023

From the results, it can be seen that the two techniques give almost the same answers.

6.2 SINGLE- AND THREE-PHASE AC CIRCUITS

Voltages and currents of a network can be obtained in the time domain. This normally involves solving differential equations. By transforming the differential equations into algebraic equations using phasors or complex frequency representation, the analysis can be simplified. For a voltage given by

$$v(t) = V_m e^{\sigma t} \cos(wt + \theta)$$

$$v(t) = \text{Re}\left[V_m e^{\sigma t} \cos(wt + \theta)\right] \tag{6.15}$$

the phasor is

$$V = V_m e^{j\theta} = V_m \angle \theta \tag{6.16}$$

and the complex frequency s is

$$s = \sigma + jw \qquad (6.17)$$

When the voltage is purely sinusoidal, that is

$$v_2(t) = V_{m2}\cos(wt + \theta_2) \qquad (6.18)$$

then the phasor

$$V_2 = V_{m2}e^{j\theta_2} = V_{m2}\angle\theta_2 \qquad (6.19)$$

and complex frequency is purely imaginary, that is,

$$s = jw \qquad (6.20)$$

To analyze circuits with sinusoidal excitations, we convert the circuits into the s-domain with $s = jw$. Network analysis laws, theorems, and rules are used to solve for unknown currents and voltages in the frequency domain. The solution is then converted into the time domain using inverse phasor transformation. For example, Figure 6.2 shows an RLC circuit in both the time and frequency domains.

(a)

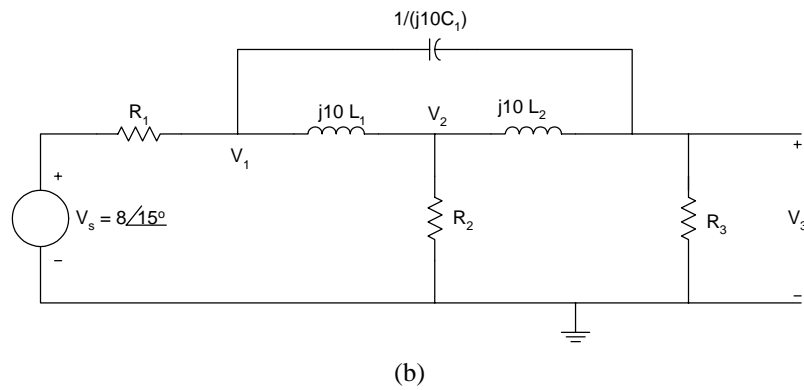1/(j10C₁)

R₁   j10 L₁   V₂   j10 L₂

V₁

V_s = 8∠15°   R₂   R₃   V₃

(b)
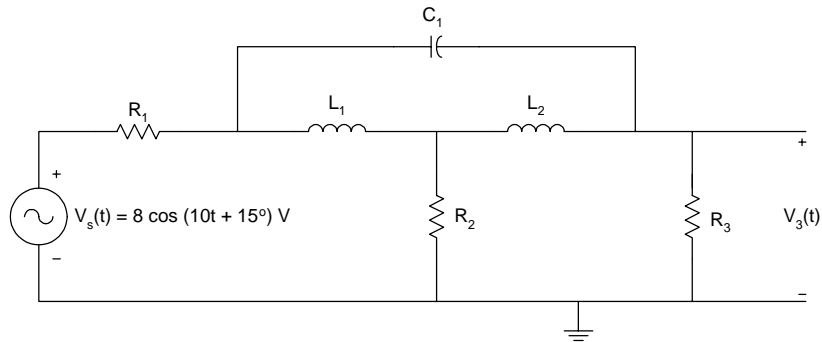
Figure 6.2 RLC Circuit with Sinusoidal Excitation (a) Time
Domain (b) Frequency Domain Equivalent

If the values of R_1, R_2, R_3, L_1, L_2 and C_1 are known, the voltage V_3 can
be obtained using circuit analysis tools. Suppose V_3 is

$$V_3 = V_{m3}\angle\theta_3,$$

then the time domain voltage V_3 *(t)* is

$$v_3(t) = V_{m3}\cos(wt + \theta_3)$$

The following two examples illustrate the use of MATLAB for solving one-
phase circuits.

Example 6.2

In Figure 6.2, if R_1 = 20 Ω, R_2 = 100Ω, R_3 = 50 Ω, and L_1 = 4 H, L_2 =
8 H and C_1 = 250μF, find $v_3(t)$ when $w = 10$ rad/s.

Solution

Using nodal analysis, we obtain the following equations.

At node 1,

$$\frac{V_1 - V_s}{R_1} + \frac{V_1 - V_2}{j10L_1} + \frac{V_1 - V_3}{1/(j10C_1)} = 0 \qquad (6.21)$$

At node 2,

$$\frac{V_2 - V_1}{j10L_1} + \frac{V_2}{R_2} + \frac{V_2 - V_3}{j10L_2} = 0 \qquad (6.22)$$

At node 3,

$$\frac{V_3}{R_3} + \frac{V_3 - V_2}{j10L_2} + \frac{V_3 - V_1}{1/(j10C_1)} = 0 \qquad (6.23)$$

Substituting the element values in the above three equations and simplifying, we get the matrix equation

$$\begin{bmatrix} 0.05 - j0.0225 & j0.025 & -j0.0025 \\ j0.025 & 0.01 - j0.0375 & j0.0125 \\ -j0.0025 & j0.0125 & 0.02 - j0.01 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0.4\angle 15^0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix can be written as

$$[Y][V] = [I].$$

We can compute the vector [v] using the MATLAB command

$$V = inv(Y) * I$$

where

$\quad inv(Y)$ is the inverse of the matrix $[Y]$.

A MATLAB program for solving V_3 is as follows:

MATLAB Script

```
diary ex6_2.dat
% This program computes the nodal voltage v3 of circuit Figure 6.2
```

```
% Y is the admittance matrix; % I is the current matrix
% V is the voltage vector

Y = [0.05-0.0225*j   0.025*j      -0.0025*j;
      0.025*j      0.01-0.0375*j  0.0125*j;
     -0.0025*j       0.0125*j      0.02-0.01*j];

c1 = 0.4*exp(pi*15*j/180);
I = [c1
    0
    0];  % current vector entered as column vector

V = inv(Y)*I;  % solve for nodal voltages
v3_abs = abs(V(3));
v3_ang = angle(V(3))*180/pi;

fprintf('voltage V3, magnitude: %f \n voltage V3, angle in degree:
%f', v3_abs, v3_ang)
diary
```

The following results are obtained:

> voltage V3, magnitude: 1.850409
> voltage V3, angle in degree: -72.453299

From the MATLAB results, the time domain voltage $v_3(t)$ is

$$v_3(t) = 1.85\cos(10t - 72.45^0) \text{ V}$$

Example 6.3

For the circuit shown in , find the current $i_1(t)$ and the voltage $v_C(t)$.
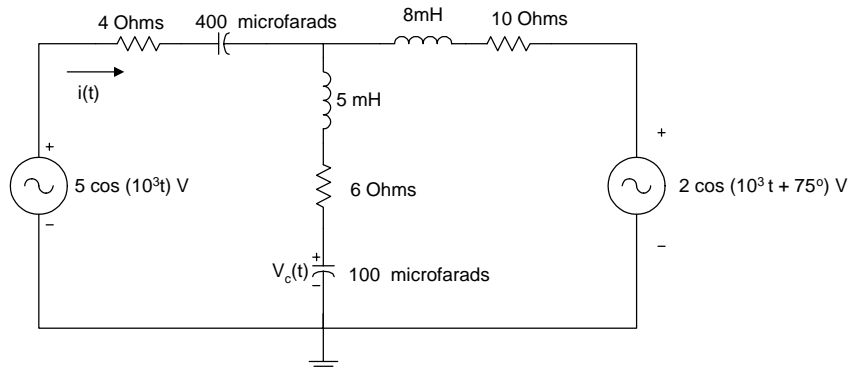
Figure 6.3 Circuit with Two Sources

Solution

Figure 6.3 is transformed into the frequency domain. The resulting circuit is shown in Figure 6.4. The impedances are in ohms.
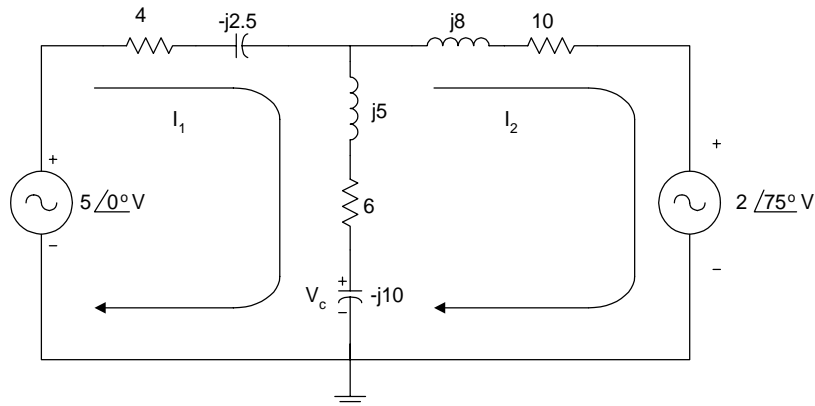

Figure 6.4 Frequency Domain Equivalent of Figure 6.3

Using loop analysis, we have

$$-5\angle 0^0 + (4 - j2.5)I_1 + (6 + j5 - j10)(I_1 - I_2) = 0 \quad (6.24)$$

$$(10 + j8)I_2 + 2\angle 75^0 + (6 + j5 - j10)(I_2 - I_1) = 0 \quad (6.25)$$

Simplifying, we have

$$(10 - j7.5)I_1 - (6 - j5)I_2 = 5\angle 0^0$$

$$-(6 - j5)I_1 + (16 + j3)I_2 = -2\angle 75^0$$

In matrix form, we obtain

$$\begin{bmatrix} 10 - j7.5 & -6 + j5 \\ -6 + j5 & 16 + j3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 5\angle 0^0 \\ -2\angle 75^0 \end{bmatrix}$$

The above matrix equation can be rewritten as

$$[Z][I] = [V].$$

We obtain the current vector $[I]$ using the MATLAB command

$$I = inv(Z) * V$$

where $inv(Z)$ is the inverse of the matrix $[Z]$.

The voltage V_C can be obtained as

$$V_C = (-j10)(I_1 - I_2)$$

A MATLAB program for determining I_1 and V_a is as follows:

MATLAB Script

```
diary ex6_3.dat
% This programs calculates the phasor current I1 and
% phasor voltage Va.
% Z is impedance matrix
% V is voltage vector
% I is current vector

Z = [10-7.5*j   -6+5*j;
    -6+5*j   16+3*j];

b = -2*exp(j*pi*75/180);
```

```
V = [5
      b];  % voltage vector in column form

I = inv(Z)*V; % solve for loop currents
i1 = I(1);
i2 = I(2);

Vc = -10*j*(i1 - i2);
i1_abs = abs(I(1));
i1_ang = angle(I(1))*180/pi;
Vc_abs = abs(Vc);
Vc_ang = angle(Vc)*180/pi;

%results are printed
fprintf('phasor current i1, magnitude: %f \n phasor current i1, angle in
degree: %f \n', i1_abs,i1_ang)
fprintf('phasor voltage Vc, magnitude: %f \n phasor voltage Vc, angle
in degree: %f \n',Vc_abs,Vc_ang)
diary
```

The following results were obtained:

> phasor current i1, magnitude: 0.387710
> phasor current i1, angle in degree: 15.019255
> phasor voltage Vc, magnitude: 4.218263
> phasor voltage Vc, angle in degree: -40.861691

The current $i_1(t)$ is

$$i_1(t) = 0.388\cos(10^3 t + 15.02^0) \quad \text{A}$$

and the voltage $v_C(t)$ is

$$v_C(t) = 4.21\cos(10^3 t - 40.86^0) \quad \text{V}$$

Power utility companies use three-phase circuits for the generation, transmission and distribution of large blocks of electrical power. The basic structure of a three-phase system consists of a three-phase voltage source connected to a three-phase load through transformers and transmission lines. The three-phase voltage source can be wye- or delta-connected. Also the three-phase load can be delta- or wye-connected. Figure 6.5 shows a 3-phase system with wye-connected source and wye-connected load.

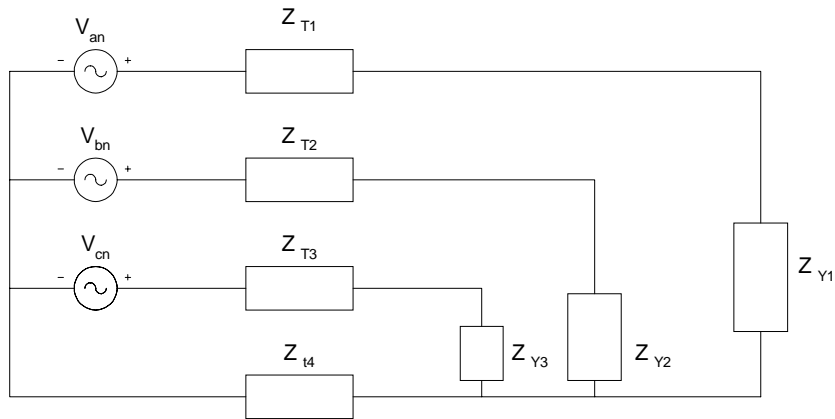**Figure 6.5** 3-phase System, Wye-connected Source and Wye-connected Load
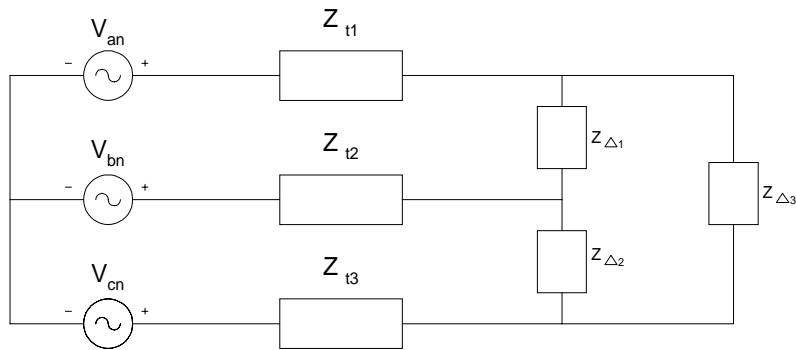

Figure 6.6 3-phase System, Wye-connected Source and Delta-connected Load

For a balanced abc system, the voltages V_{an}, V_{bn}, V_{cn} have the same magnitude and they are out of phase by 120^0. Specifically, for a balanced abc system, we have

$$V_{an} = V_P \angle 0^0$$
$$V_{bn} = V_P \angle -120^0 \qquad \qquad (6.26)$$
$$V_{cn} = V_P \angle 120^0$$

For cba system

$$V_{an} = V_P \angle 0^0$$
$$V_{bn} = V_P \angle 120^0 \tag{6.27}$$
$$V_{cn} = V_P \angle -120^0$$

The wye-connected load is balanced if

$$Z_{Y1} = Z_{Y2} = Z_{Y3} \tag{6.28}$$

Similarly, the delta-connected load is balanced if

$$Z_{\Delta 1} = Z_{\Delta 2} = Z_{\Delta 3} \tag{6.29}$$

We have a balanced three-phase system of Equations (6.26) to (6.29) that are satisfied with the additional condition

$$Z_{T1} = Z_{T2} = Z_{T3} \tag{6.30}$$

Analysis of balanced three-phase systems can easily be done by converting the three-phase system into an equivalent one-phase system and performing simple hand calculations. The method of symmetrical components can be used to analyze unbalanced three-phase systems. Another method that can be used to analyze three-phase systems is to use KVL and KCL. The unknown voltage or currents are solved using MATLAB. This is illustrated by the following example.

Example 6.4

In Figure 6.7, showing an unbalanced wye-wye system, find the phase voltages V_{AN}, V_{BN} and V_{CN}.

Solution

Using KVL, we can solve for I_1, I_2, I_3. From the figure, we have

$$110 \angle 0^0 = (1 + j1)I_1 + (5 + j12)I_1 \tag{6.31}$$

$$110\angle - 120^0 = (1 - j2)I_2 + (3 + j4)I_2 \qquad (6.32)$$

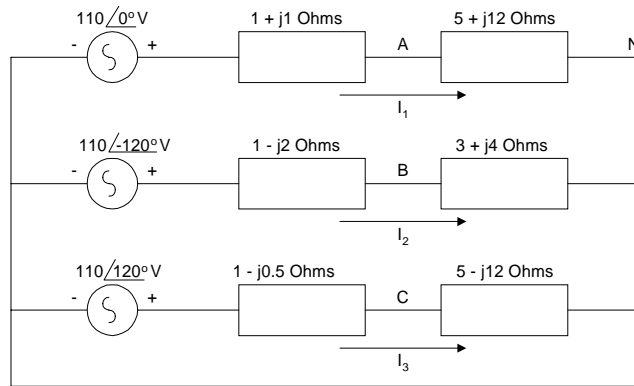$$110\angle 120^0 = (1 - j0.5)I_3 + (5 - j12)I_3 \qquad (6.33)$$

Figure 6.7   Unbalanced Three-phase System

Simplifying Equations (6.31), (6.32) and (6.33), we have

$$110\angle 0^0 = (6 + j13)I_1 \qquad (6.34)$$

$$110\angle - 120^0 = (4 + j2)I_2 \qquad (6.35)$$

$$110\angle 120^0 = (6 - j12.5)I_3 \qquad (6.36)$$

and expressing the above three equations in matrix form, we have

$$\begin{bmatrix} 6 + j13 & 0 & 0 \\ 0 & 4 + j2 & 0 \\ 0 & 0 & 6 - j12.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 110\angle 0^0 \\ 110\angle - 120^0 \\ 110\angle 120^0 \end{bmatrix}$$

The above matrix can be written as

$$[Z][I] = [V]$$

We obtain the vector $[I]$ using the MATLAB command

$$I = inv(Z)*V$$

The phase voltages can be obtained as

$$V_{AN} = (5 + j12)I_1$$

$$V_{BN} = (3 + j4)I_2$$

$$V_{CN} = (5 - j12)(I_3)$$

The MATLAB program for obtaining the phase voltages is

MATLAB Script

```
diary ex6_4.dat
% This program calculates the phasor voltage of an
% unbalanced three-phase system
% Z is impedance matrix
% V is voltage vector and
% I is current vector
Z = [6-13*j   0      0;
     0       4+2*j  0;
     0       0      6-12.5*j];
c2 = 110*exp(j*pi*(-120/180));
c3 = 110*exp(j*pi*(120/180));

V = [110; c2; c3]; % column voltage vector
I = inv(Z)*V;  % solve for loop currents
% calculate the phase voltages
%
Van = (5+12*j)*I(1);
Vbn = (3+4*j)*I(2);
Vcn = (5-12*j)*I(3);
Van_abs = abs(Van);
Van_ang = angle(Van)*180/pi;
Vbn_abs = abs(Vbn);
Vbn_ang = angle(Vbn)*180/pi;
Vcn_abs = abs(Vcn);
Vcn_ang = angle(Vcn)*180/pi;

% print out results
```

```
fprintf('phasor voltage Van,magnitude: %f \n phasor voltage Van, an-
gle in degree: %f \n', Van_abs, Van_ang)
fprintf('phasor voltage Vbn,magnitude: %f \n phasor voltage Vbn, an-
gle in degree: %f \n', Vbn_abs, Vbn_ang)
fprintf('phasor voltage Vcn,magnitude: %f \n phasor voltage Vcn, an-
gle in degree: %f \n', Vcn_abs, Vcn_ang)
diary
```

The following results were obtained:

```
phasor voltage Van,magnitude: 99.875532
phasor voltage Van, angle in degree: 132.604994
phasor voltage Vbn,magnitude: 122.983739
phasor voltage Vbn, angle in degree: -93.434949
phasor voltage Vcn,magnitude: 103.134238
phasor voltage Vcn, angle in degree: 116.978859
```

6.3   NETWORK CHARACTERISTICS

Figure 6.8 shows a linear network with input $x(t)$ and output $y(t)$. Its complex frequency representation is also shown.
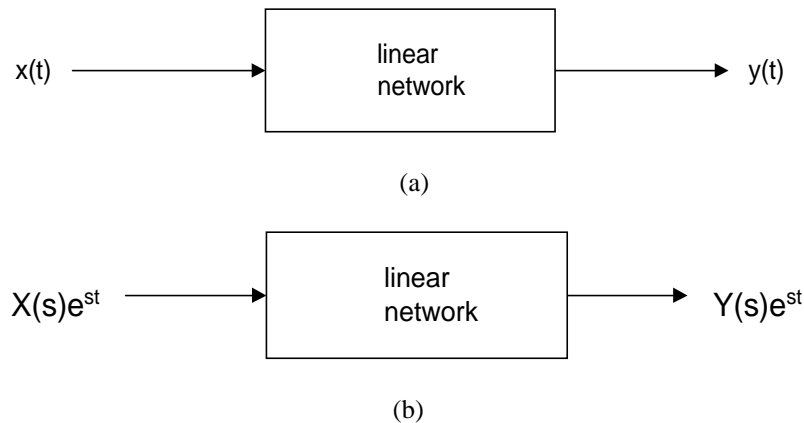

(a)

(b)

Figure 6.8 Linear Network Representation (a) Time Domain
(b) s- domain

In general, the input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

(6.37)

where $a_n, a_{n-1}, \ldots, a_0, b_m, b_{m-1}, \ldots b_0$ are real constants.

If $x(t) = X(s)e^{st}$, then the output must have the form $y(t) = Y(s)e^{st}$, where $X(s)$ and $Y(s)$ are phasor representations of $x(t)$ and $y(t)$. From equation (6.37), we have

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) e^{st} =$$

$$(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) X(s) e^{st}$$

(6.38)

and the network function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} \qquad (6.39)$$

The network function can be rewritten in factored form

$$H(s) = \frac{k(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \qquad (6.40)$$

where

k is a constant

$z_1, z_2, \ldots, z_m$ are zeros of the network function.

$p_1, p_2, \ldots, p_n$ are poles of the network function.

The network function can also be expanded using partial fractions as

$$H(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \quad \ldots \quad + \frac{r_n}{s - p_n} + k(s) \qquad (6.41)$$

6.3.1 MATLAB functions roots, residue and polyval

MATLAB has the function **roots** that can be used to obtain the poles and zeros of a network function. The MATLAB function **residue** can be used for partial fraction expansion.   Furthermore, the MATLAB function **polyval** can be used to evaluate the network function.

The MATLAB function **roots** determines the roots of a polynomial. The general form of the roots function is

$$r = roots(p) \qquad\qquad (6.42)$$

where

         p is a vector containing the coefficients of the polynomial in
               descending order
         r is a column vector containing the roots of the polynomials

For example, given the polynomial

$$f(x) = x^3 + 9x^2 + 23x + 15$$

the commands to compute and print out the roots of $f(x)$ are

```
p  = [1  9  23  15]
r  = roots (p)
```

and the values printed are

```
r  =
     -1.0000
     -3.0000
     -5.0000
```

Given the roots of a polynomial, we can obtain the coefficients of the polynomial by using the MATLAB function **poly**

Thus

$$S = poly ([-1 \quad -3 \quad -5]^1) \qquad\qquad (6.43)$$

will give a row vector s given as

```
S =
     1.0000          9.0000          23.0000          15.0000
```

The coefficients of S are the same as those of p.

The MATLAB function **polyval** is used for polynomial evaluation. The general form of polyval is

$$polyval(p, x) \qquad (6.44)$$

where

> p is a vector whose elements are the coefficients of a polynomial in
> descending powers
> $polyval(p, x)$ is the value of the polynomial evaluated at x

For example, to evaluate the polynomial

$$f(x) = x^3 - 3x^2 - 4x + 15$$

at $x = 2$, we use the command

```
p = [1 -3  -4   15];
polyval(p, 2)
```

Then we get

```
ans =
     3
```

The MATLAB function **residue** can be used to perform partial fraction expansion. Assuming $H(s)$ is the network function, since $H(s)$ may represent an improper fraction, we may express $H(s)$ as a mixed fraction

$$H(s) = \frac{B(s)}{A(s)} \qquad (6.45)$$

$$H(s) = \sum_{n=0}^{N} k_n s^n + \frac{N(s)}{D(s)} \tag{6.46}$$

where

$\dfrac{N(s)}{D(s)}$ is a proper fraction

From equations (6.41) and (6.46), we get

$$H(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \quad .... \quad + \frac{r_n}{s - p_n} + \sum_{n=0}^{N} k_n s^n \tag{6.47}$$

Given the coefficients of the numerator and denominator polynomials, the MATLAB **residue** function provides the values of r_1, r_2, ... r_n , p_1, p_2, ...p_n, an d k_1, k_2 , ...k_n . The general form of the residue function is

$$[r, p, k] = residue(num, den) \tag{6.48}$$

where

> num   is a row vector whose entries are the coefficients of the numerator polynomial in descending order

> den   is a row vector whose entries are the coefficient of the denominator polynomial in descending order

> r   is returned as a column vector

> p   (pole locations) is returned as a column vector

> k   (direct term) is returned as a row vector

The command

$$[num, den] = residue(r, p, k) \tag{6.49}$$

Converts the partial fraction expansion back to the polynomial ratio

$$H(s) = \frac{B(s)}{A(s)}$$

For example, given

$$H(s) = \frac{4s^4 + 3s^3 + 6s^2 + 10s + 20}{s^4 + 2s^3 + 5s^2 + 2s + 8} \qquad (6.50)$$

for the above network function, the following commands will perform partial fraction expansion

    num = [4 3 6 10 20];
    den = [1 2 5 2 8];
    [r, p, k] = residue(num, den)   (6.51)

and we shall get the following results

    r =
      -1.6970 + 3.0171i
      -1.6970 - 3.0171i
      -0.8030 - 0.9906i
      -0.8030 + 0.9906i

    p =
      -1.2629 + 1.7284i
      -1.2629 - 1.7284i
      0.2629 + 1.2949i
      0.2629 - 1.2949i

    k =
      4

The following two examples show how to use MATLAB function **roots** to find poles and zeros of circuits.

Example 6.5

For the circuit shown below, (a) Find the network function $H(s) = \dfrac{V_o(s)}{V_S(s)}$

(b) Find the poles and zeros of $H(s)$, and

(c) if $v_S(t) = 10e^{-3t}\cos(2t + 40^0)$, find $v_0(t)$.
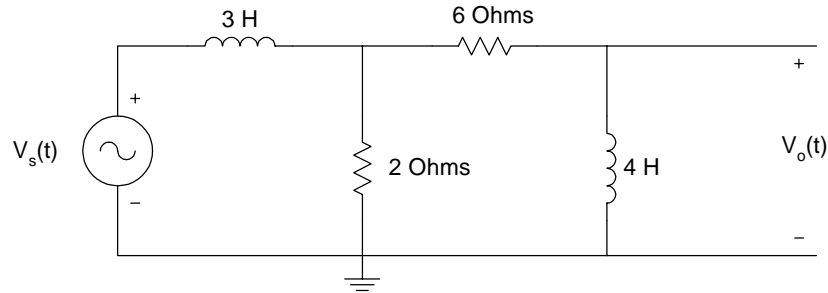
Figure 6.9 Circuit for Example 6.5
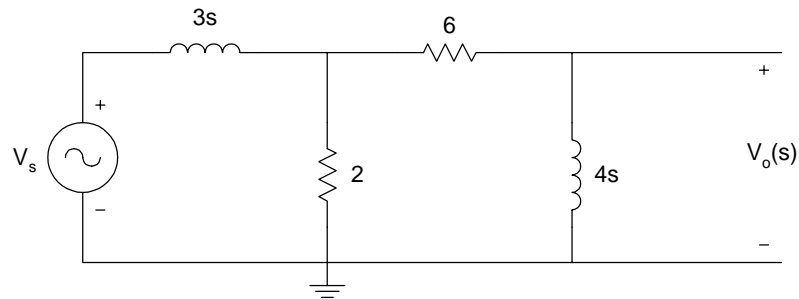
Solution

In the s-domain, the above figure becomes

Figure 6.10 S-domain Equivalent Circuit of Figure 6.9

$$\frac{V_0(s)}{V_S(s)} = \frac{V_0(s)}{V_X(s)}\frac{V_X(s)}{V_S(s)} = \frac{4s}{(6+4s)}\frac{[2\|(6+4s)]}{[(2\|(6+4s))+3s]}$$

Simplifying, we get

$$\frac{V_0(s)}{V_S(s)} = \frac{4s^2 + 6s}{6s^3 + 25s^2 + 30s + 9} \tag{6.52}$$

The phasor voltage $V_S = 10\angle 40^o$; $s = -3 + j2$

$$V_0(s) = (10\angle 40^o)H(s)\Big|_{s=-3+j2}$$

(b, c) MATLAB is used to find the poles, zeros and $v_0(t)$.

MATLAB Script

```
diary ex6_5.dat
% Program for poles and zeros
num = [4  6  0];
den = [6  25  30  9];
disp('the zeros are')
z = roots(num)
disp('the poles are')
p = roots(den)
% program to evaluate transfer function and
% find the output voltage
s1 = -3+2*j;
n1 = polyval(num,s1);
d1 = polyval(den,s1);
vo = 10.0*exp(j*pi*(40/180))*n1/d1;
vo_abs = abs(vo);
vo_ang = angle(vo)*180/pi;
% print magnitude and phase of output voltage
fprintf('phasor voltage vo, magnitude: %f \n phasor voltage vo, angle
in degrees: %f', vo_abs, vo_ang)
diary
```

MATLAB results are

Zeros
```
z =
    0
   -1.5000
```

Poles
```
p =
   -2.2153
   -1.5000
   -0.4514
```

phasor voltage vo, magnitude: 3.453492

phasor voltage vo, angle in degrees: -66.990823

From the results, the output voltage is given as

$$v(t) = 3.45e^{-3t} \cos(2t - 66.99^0)$$

Example 6.6

Find the inverse Laplace transform of

$$G(s) = \frac{10s^2 + 20s + 40}{s^3 + 12s^2 + 47s + 60}$$

Solution

MATLAB Script

```
diary ex6_6.dat
% MATLAB is used to do the partial fraction expansion
%
num = [10 20 40];
den = [1 12 47 60];

% we get the following results
[r, p, k] = residue(num,den)
diary
```

MATLAB results are

```
r =
      95.0000
    -120.0000
      35.0000

p =
     -5.0000
     -4.0000
     -3.0000

k =
```

[]

From the results, we get

$$G(s) = \frac{95}{s+5} - \frac{120}{s+4} + \frac{35}{s+3}$$

and the inverse Laplace transform is

$$g(t) = 35e^{-3t} - 120e^{-4t} + 95e^{-5t} \tag{6.53}$$

6.4 FREQUENCY RESPONSE

The general form of a transfer function of an analog circuit is given in Equation (6.39). It is repeated here.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0}$$

More specifically, for a second-order analog filter, the following transfer functions can be obtained:

(i) Lowpass

$$H_{LP}(s) = \frac{k_1}{s^2 + Bs + w_0^2} \tag{6.54}$$

(ii) Highpass

$$H_{HP}(s) = \frac{k_2 s^2}{s^2 + Bs + w_0^2} \tag{6.55}$$

(iii) Bandpass

$$H_{BP}(s) = \frac{k_3 s}{s^2 + Bs + w_0^2} \tag{6.56}$$

(iv) Bandreject

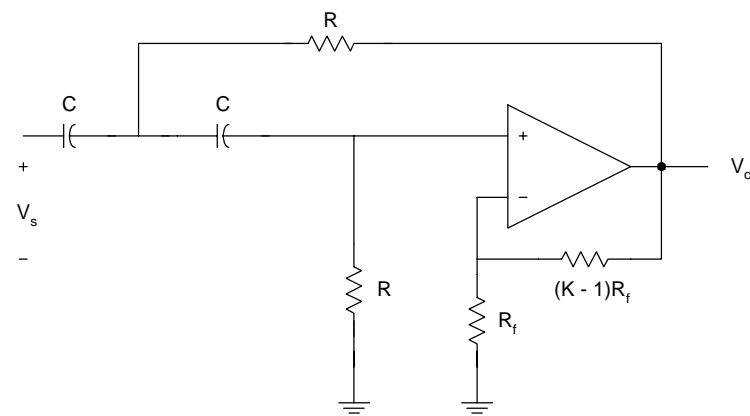$$H_{BR}(s) = \frac{k_4 s^2 + k_5}{s^2 + Bs + w_0^2} \qquad\qquad (6.57)$$

where

k_1, k_2, k_3, k_4, B and w_0 are constants

Figure 6.11 shows the circuit diagram of some filter sections.
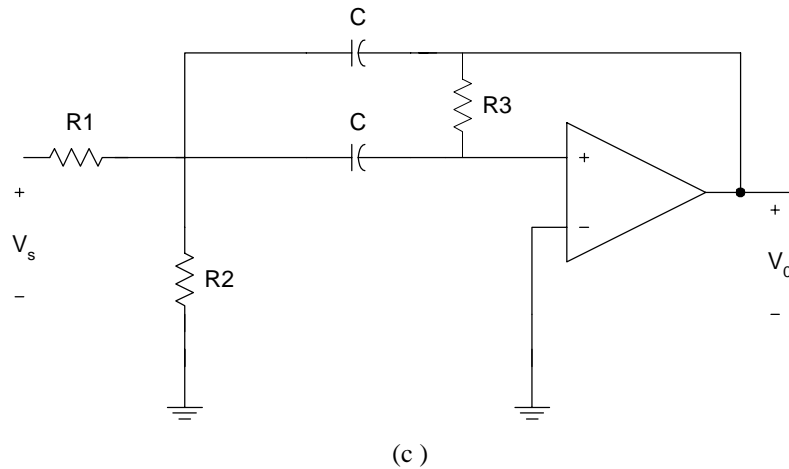

(a)

(b)

(c)
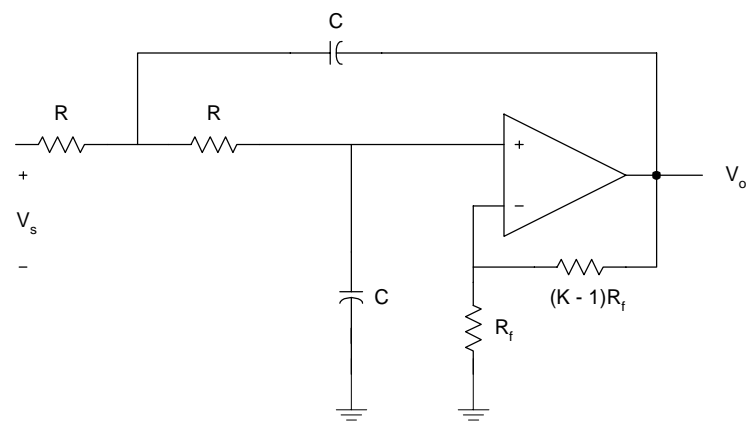
Figure 6.11 Active Filters (a) Lowpass, (b) Highpass and
(c) Bandpass

Frequency response is the response of a network to sinusoidal input signal. If we substitute $s = jw$ in the general network function, $H(s)$, we get

$$H(s)\Big|_{s=jw} = M(w)\angle\theta(w) \tag{6.58}$$

where

$$M(w) = \left|H(jw)\right| \tag{6.59}$$

and

$$\theta(w) = \angle H(jw) \tag{6.60}$$

The plot of $M(\omega)$ versus ω is the magnitude characteristics or response. Also, the plot of $\theta(w)$ versus ω is the phase response. The magnitude and phase characteristics can be obtained using MATLAB function **freqs**.

6.4.1 MATLAB function freqs

MATLAB function **freqs** is used to obtain the frequency response of transfer function $H(s)$. The general form of the frequency function is

$$hs = freqs(num, den, range) \tag{6.61}$$

where

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} \tag{6.62}$$

$$num = \begin{bmatrix} b_m & b_{m-1} & \dots & b_1 & b_0 \end{bmatrix} \tag{6.63}$$

$$den = \begin{bmatrix} a_n & a_{n-1} & \dots & a_1 & a_0 \end{bmatrix} \tag{6.64}$$

range   is range of frequencies for case

hs   is the frequency response (in complex number form)

Suppose we want to graph the frequency response of the transfer function given as

$$H(s) = \frac{2s^2 + 4}{s^2 + 4s + 16} \tag{6.65}$$

We can use the following commands to find the magnitude characteristics

```
num = [2 0 4];
den = [1 4 16];
w = logspace(-2, 4);
h = freqs(num, den, w);
f = w/(2*pi);
mag = 20*log10(abs(h));
semilogx(f, mag)
title('Magnitude Response')
xlabel('Frequency, Hz')
ylabel('Gain, dB')
```
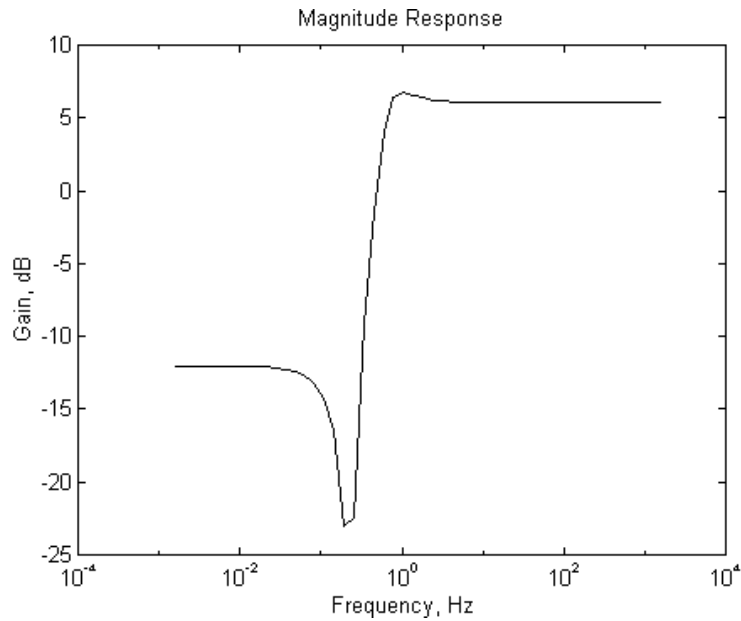
The frequency response is shown in Figure 6.12.

Figure 6.12 Magnitude Response of Equation (6.65)

The following example shows how to obtain and plot the frequency response of an RLC circuit.

Example 6.7

For the RLC circuit shown in Figure 6.13, (a) show that the transfer function is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s\dfrac{R}{L}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}} \qquad (6.66)$$

(b) If $L = 5$ H, $C = 1.12$ μF, and $R = 10000$ Ω, plot the frequency response.

(c) What happens when $R = 100$ Ω, but L and C remain unchanged?
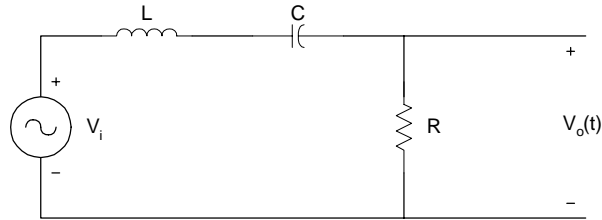
Figure 6.13 RLC Circuit

Solution

(a) In the frequency domain,

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{R}{R + sL + \dfrac{1}{sC}} = \frac{sCR}{s^2 LC + sCR + 1} \qquad (6.67)$$

which is

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{s\dfrac{R}{L}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}}$$

Parts (b) and (c) are solved using MATLAB.

MATLAB Script

```
% Frequency response of RLC filter
%
l = 5;
c = 1.25e-6;
r1 = 10000;
r2 = 100;

num1 = [r1/l 0];
den1 = [1 r1/l  1/(l*c)];

w = logspace(1,4);
h1 = freqs(num1,den1,w);
```

```
f = w/(2*pi);
mag1 = abs(h1);
phase1 = angle(h1)*180/pi;

num2 = [r2/l 0];
den2 = [1  r2/l  1/(l*c)];
h2 = freqs(num2,den2,w);
mag2 = abs(h2);
phase2 = angle(h2)*180/pi;

% Plot the response

subplot(221), loglog(f, mag1,'.')
title('magnitude response R=10K')
ylabel('magnitude')

subplot(222), loglog(f,mag2,'.')
title('magnitude response R=.1K')
ylabel('magnitude')

subplot(223), semilogx(f, phase1,'.')
title('phase response R=10K'),...
xlabel('Frequency, Hz'), ylabel('angle in degrees')

subplot(224), semilogx(f, phase2,'.')
title('phase response R=.1K'),...
xlabel('Frequency, Hz'), ylabel('angle in degrees')
```

The plots are shown in Figure 6.14. As the resistance is decreased from 10,000 to 100 Ohms, the bandwidth of the frequency response decreases and the quality factor of the circuit increases.
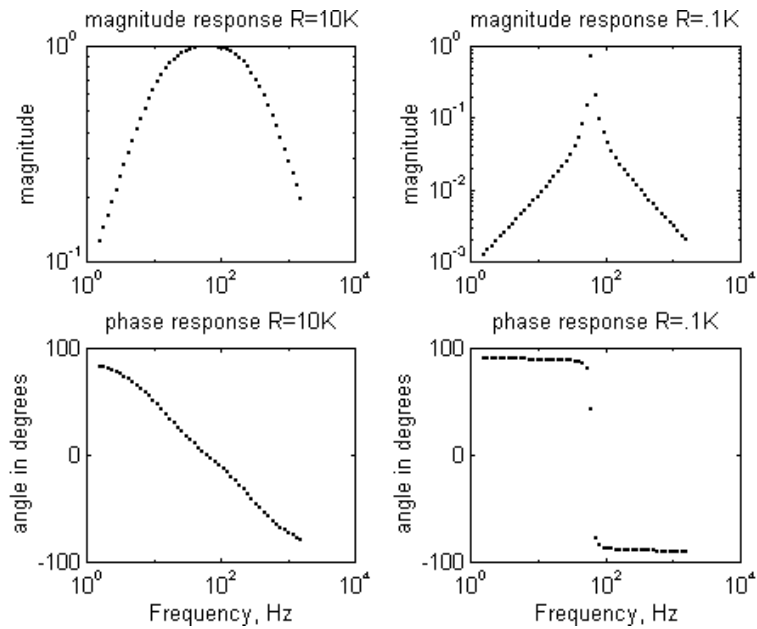
Figure 6.14 Frequency Response of an RLC Circuit

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2. Biran, A. and Breiner, M., *MATLAB for Engineers*, Addison-Wesley, 1995.

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4. Nilsson, J.W. , *Electric Circuits*, 3rd Edition, Addison-Wesley Publishing Company, 1990.

5. Vlach, J.O., *Network Theory and CAD*, IEEE Trans. on Education, Vol. 36, Feb. 1993, pp. 23-27.

6. Meader, D.A., *Laplace Circuit Analysis and Active Filters,*

Prentice Hall, New Jersey, 1991.

7.   Johnson, D. E. Johnson, J.R. and Hilburn, J.L., *Electric Circuit Analysis*, 3rd Edition, Prentice Hall, New Jersey, 1997.

EXERCISES

6.1   If $v(t)$ is periodic with one period of $v(t)$ given as

$$v(t) = 16(1 - e^{-6t}) \text{ V} \quad 0 \le t < 2 \text{ s}$$

(a) Use MATLAB to find the rms value of $v(t)$
(b) Obtain the rms value of $v(t)$ using analytical technique.
     Compare your result with that obtained in part (a).
(c) Find the power dissipated in the 4-ohm resistor when the
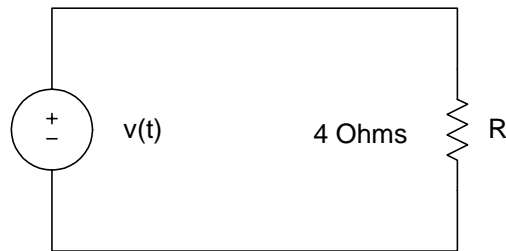   voltage $v(t)$ is applied across the 4-ohm resistor.

Figure P6.1   Resistive Circuit for part (c)

6.2   A balanced Y-Y positive sequence system has phase voltage of the
      source $V_{an} = 120\angle 0^0$ rms if the load impedance per phase is
      $(11 + j4.5) \,\Omega$, and the transmission line has an impedance per phase
      of $(1 + j0.5) \,\Omega$.

(a)   Use analytical techniques to find the magnitude of the line
           current, and the power delivered to the load.
(b)   Use MATLAB to solve for the line current and the power
           delivered to the load.

(c)   Compare the results of parts (a) and (b).

6.3   For the unbalanced 3-phase system shown in Figure P6.3, find the currents I_1, I_2, I_3 and hence I_{bB}. Assume that $Z_A = 10 + j5\,\Omega$, $Z_B = 15 + j7\,\Omega$ and $Z_C = 12 - j3\,\Omega$.
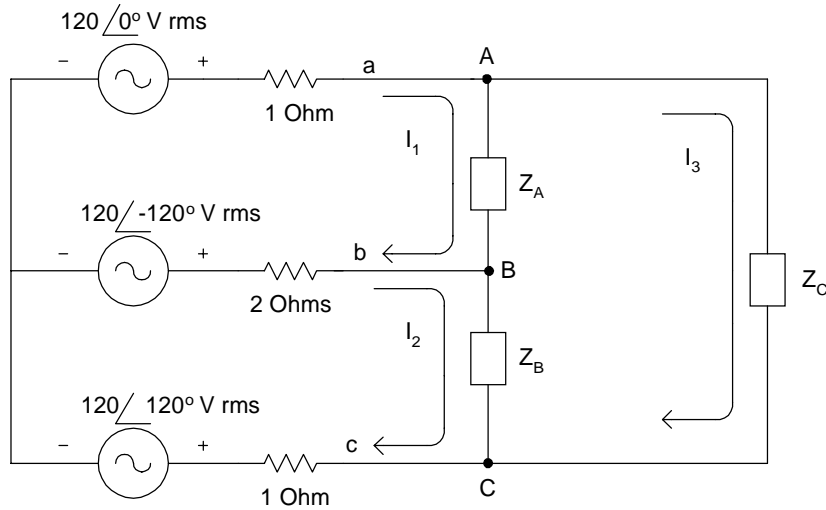

Figure P6.3 Unbalanced Three-phase System

6.4   For the system with network function

$$H(s) = \frac{s^3 + 4s^2 + 16s + 4}{s^4 + 20s^3 + 12s^2 + s + 10}$$

find the poles and zeros of $H(s)$.

6.5   Use MATLAB to determine the roots of the following polynomials. Plot the polynomial over the appropriate interval to verify the roots location.

(a) $f_1(x) = x^2 + 4x + 3$

(b) $f_2(x) = x^3 + 5x^2 + 9x + 5$

$$(c)\ f_3(x) = 2x^5 - 4x^4 - 12x^3 + 27x^2 + 8x - 16$$

6.6 If $\dfrac{V_o(s)}{V_i(s)} = \dfrac{20s}{15s^2 + 23s + 16}$,

find $v_0(t)$ given that $v_i(t) = 2.3e^{-2t}\cos(5t + 30^0)$.

6.7 For the circuit of Figure P6.7

(a) Find the transfer function $\dfrac{V_o(s)}{V_i(s)}$.

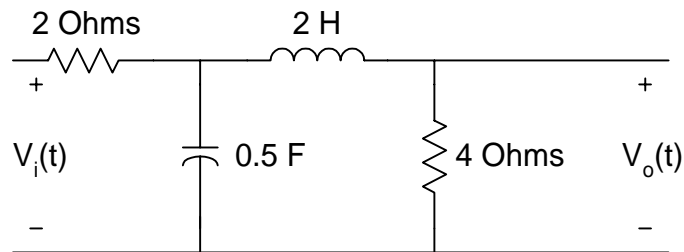(b) If $v_i(t) = 10e^{-5t}\cos(t + 10^0)$, find $v_0(t)$.

Figure P6.7 RLC Circuit

6.8 For Figure P6.8,

(a) Find the transfer function $H(s) = \dfrac{V_o(s)}{V_i(s)}$.

(b) Use MATLAB to plot the magnitude characteristics.
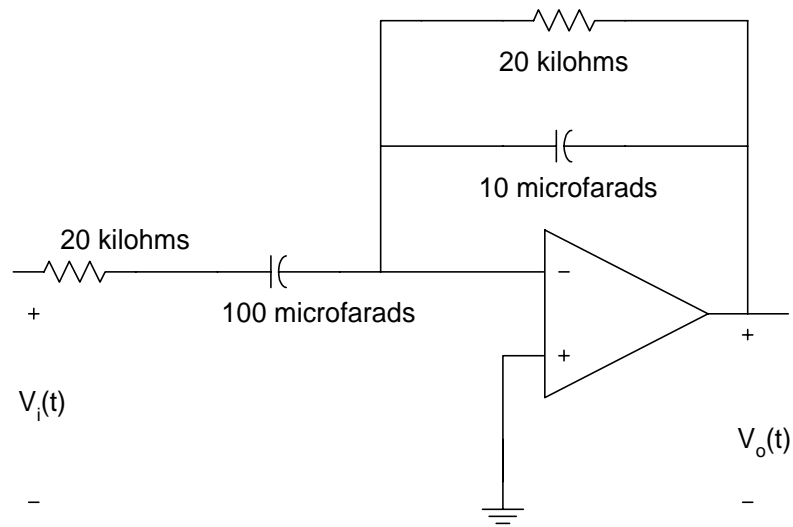
Figure P6.8 Simple Active Filter

Attia, John Okyere. "Two-Port Networks."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER SEVEN

TWO-PORT NETWORKS

This chapter discusses the application of MATLAB for analysis of two-port networks. The describing equations for the various two-port network representations are given. The use of MATLAB for solving problems involving parallel, series and cascaded two-port networks is shown. Example problems involving both passive and active circuits will be solved using MATLAB.

7.1 TWO-PORT NETWORK REPRESENTATIONS
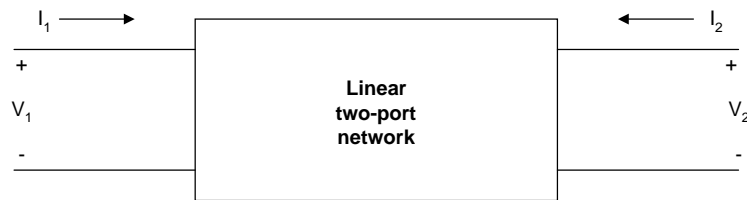
A general two-port network is shown in Figure 7.1.

Figure 7.1 General Two-Port Network

I_1 and V_1 are input current and voltage, respectively. Also, I_2 and V_2 are output current and voltage, respectively. It is assumed that the linear two-port circuit contains no independent sources of energy and that the circuit is initially at rest (no stored energy). Furthermore, any controlled sources within the linear two-port circuit cannot depend on variables that are outside the circuit.

7.1.1 z-parameters

A two-port network can be described by z-parameters as

$$V_1 = z_{11}I_1 + z_{12}I_2 \tag{7.1}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 \tag{7.2}$$

In matrix form, the above equation can be rewritten as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{7.3}$$

The z-parameter can be found as follows

$$z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0} \tag{7.4}$$

$$z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1 = 0} \tag{7.5}$$

$$z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0} \tag{7.6}$$

$$z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1 = 0} \tag{7.7}$$

The z-parameters are also called open-circuit impedance parameters since they are obtained as a ratio of voltage and current and the parameters are obtained by open-circuiting port 2 ($I_2 = 0$) or port1 ($I_1 = 0$). The following example shows a technique for finding the z-parameters of a simple circuit.

Example 7.1

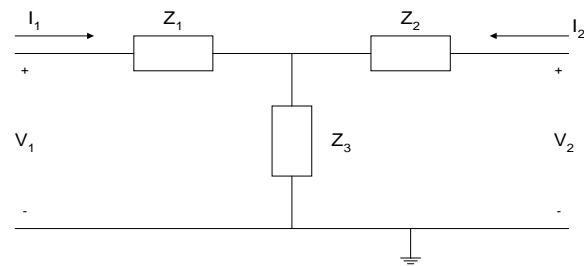For the T-network shown in Figure 7.2, find the z-parameters.

Figure 7.2 T-Network

Solution

Using KVL

$$V_1 = Z_1 I_1 + Z_3(I_1 + I_2) = (Z_1 + Z_3)I_1 + Z_3 I_2 \qquad (7.8)$$

$$V_2 = Z_2 I_2 + Z_3(I_1 + I_2) = (Z_3)I_1 + (Z_2 + Z_3)I_2 \qquad (7.9)$$

thus

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad (7.10)$$

and the z-parameters are

$$[Z] = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix} \qquad (7.11)$$

7.1.2 y-parameters

A two-port network can also be represented using y-parameters. The describing equations are

$$I_1 = y_{11}V_1 + y_{12}V_2 \qquad (7.12)$$

$$I_2 = y_{21}V_1 + y_{22}V_2 \qquad (7.13)$$

where

V_1 and V_2 are independent variables and
I_1 and I_2 are dependent variables.

In matrix form, the above equations can be rewritten as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (7.14)$$

The y-parameters can be found as follows:

$$y_{11} = \frac{I_1}{V_1}\bigg|_{V_2=0} \tag{7.15}$$

$$y_{12} = \frac{I_1}{V_2}\bigg|_{V_1=0} \tag{7.16}$$

$$y_{21} = \frac{I_2}{V_1}\bigg|_{V_2=0} \tag{7.17}$$

$$y_{22} = \frac{I_2}{V_2}\bigg|_{V_1=0} \tag{7.18}$$

The y-parameters are also called short-circuit admittance parameters. They are obtained as a ratio of current and voltage and the parameters are found by short-circuiting port 2 ($V_2 = 0$) or port 1 ($V_1 = 0$). The following two examples show how to obtain the y-parameters of simple circuits.

Example 7.2

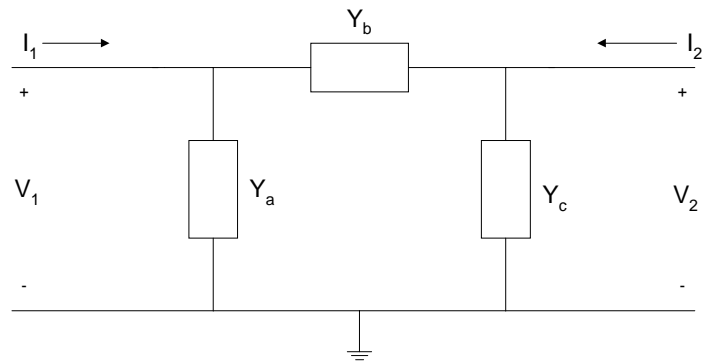Find the y-parameters of the pi (π) network shown in Figure 7.3.

Figure 7.3 Pi-Network

Solution

Using KCL, we have

$$I_1 = V_1 Y_a + (V_1 - V_2)Y_b = V_1(Y_a + Y_b) - V_2 Y_b \tag{7.19}$$

$$I_2 = V_2 Y_c + (V_2 - V_1)Y_b = -V_1 Y_b + V_2 (Y_b + Y_c) \qquad (7.20)$$

Comparing Equations (7.19) and (7.20) to Equations (7.12) and (7.13), the y-parameters are

$$[Y] = \begin{bmatrix} Y_a + Y_b & -Y_b \\ -Y_b & Y_b + Y_c \end{bmatrix} \qquad (7.21)$$

Example 7.3

Figure 7.4 shows the simplified model of a field effect transistor. Find its y-parameters.
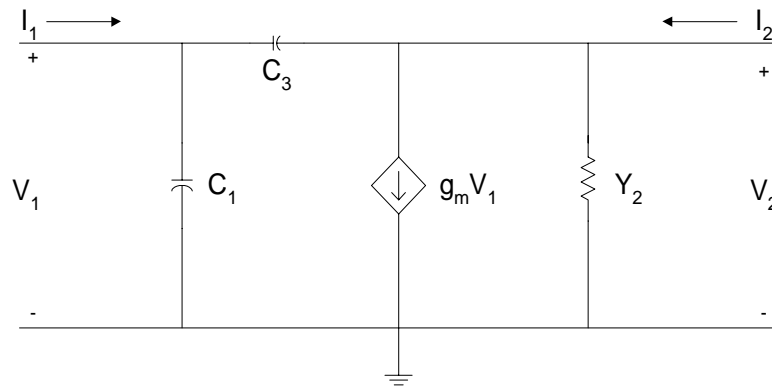

Figure 7.4 Simplified Model of a Field Effect Transistor

Using KCL,

$$I_1 = V_1 s C_1 + (V_1 - V_2)s C_3 = V_1 (s C_1 + s C_3) + V_2 (-s C_3) \qquad (7.22)$$

$$I_2 = V_2 Y_2 + g_m V_1 + (V_2 - V_1)s C_3 = V_1 (g_m - s C_3) + V_2 (Y_2 + s C_3) \qquad (7.23)$$

Comparing the above two equations to Equations (7.12) and (7.13), the y-parameters are

$$[Y] = \begin{bmatrix} sC_1 + sC_3 & -sC_3 \\ g_m - sC_3 & Y_2 + sC_3 \end{bmatrix}$$

(7.24)

7.1.3 h-parameters

A two-port network can be represented using the h-parameters. The describing equations for the h-parameters are

$$V_1 = h_{11}I_1 + h_{12}V_2$$

(7.25)

$$I_2 = h_{21}I_1 + h_{22}V_2$$

(7.26)

where

I_1 and V_2 are independent variables and

V_1 and I_2 are dependent variables.

In matrix form, the above two equations become

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

(7.27)

The h-parameters can be found as follows:

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0}$$

(7.28)

$$h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0}$$

(7.29)

$$h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0}$$

(7.30)

$$h_{22} = \left. \frac{I_2}{V_2} \right|_{I_1=0}$$

(7.31)

The h-parameters are also called hybrid parameters since they contain both open-circuit parameters ($I_1 = 0$) and short-circuit parameters ($V_2 = 0$). The h-parameters of a bipolar junction transistor are determined in the following example.

Example 7.4

A simplified equivalent circuit of a bipolar junction transistor is shown in Figure 7.5, find its h-parameters.
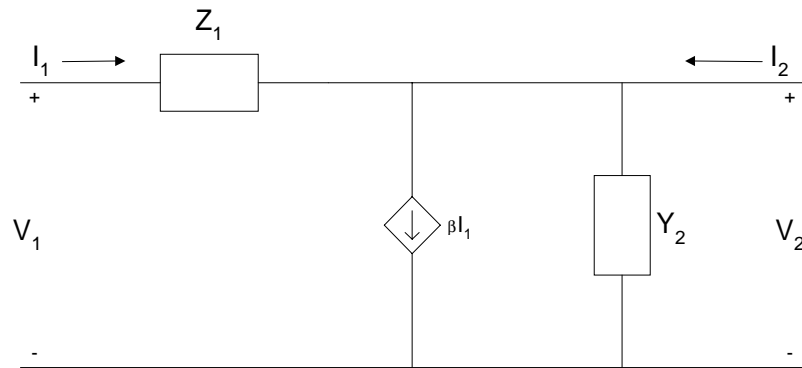

Figure 7.5 Simplified Equivalent Circuit of a Bipolar Junction Transistor

Solution

Using KCL for port 1,

$$V_1 = I_1 Z_1 \tag{7.32}$$

Using KCL at port 2, we get

$$I_2 = \beta I_1 + Y_2 V_2 \tag{7.33}$$

Comparing the above two equations to Equations (7.25) and (7.26) we get the h-parameters.

$$[h] = \begin{bmatrix} Z_1 & 0 \\ \beta & Y_2 \end{bmatrix} \tag{7.34}$$

7.1.4   Transmission parameters

A two-port network can be described by transmission parameters.   The describing equations are

$$V_1 = a_{11}V_2 - a_{12}I_2 \tag{7.35}$$

$$I_1 = a_{21}V_2 - a_{22}I_2 \tag{7.36}$$

where

V_2 and I_2 are independent variables and

V_1 and I_1 are dependent variables.

In matrix form, the above two equations can be rewritten as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{7.37}$$

The transmission parameters can be found as

$$a_{11} = \left. \frac{V_1}{V_2} \right|_{I_2=0} \tag{7.38}$$

$$a_{12} = -\left. \frac{V_1}{I_2} \right|_{V_2=0} \tag{7.39}$$

$$a_{21} = \left. \frac{I_1}{V_2} \right|_{I_2=0} \tag{7.40}$$

$$a_{22} = -\left. \frac{I_1}{I_2} \right|_{V_2=0} \tag{7.41}$$

The transmission parameters express the primary (sending end) variables V_1 and I_1 in terms of the secondary (receiving end) variables V_2 and $-I_2$. The negative of I_2 is used to allow the current to enter the load at the receiving end. Examples 7.5 and 7.6 show some techniques for obtaining the transmission parameters of impedance and admittance networks.

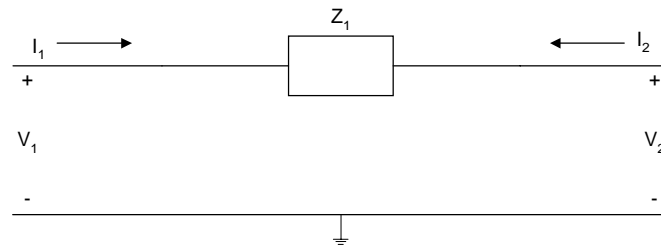**Example 7.5**

Find the transmission parameters of Figure 7.6.

Figure 7.6 Simple Impedance Network

Solution

By inspection,

$$I_1 = -I_2 \tag{7.42}$$

Using KVL,

$$V_1 = V_2 + Z_1 I_1 \tag{7.43}$$

Since $I_1 = -I_2$, Equation (7.43) becomes

$$V_1 = V_2 - Z_1 I_2 \tag{7.44}$$

Comparing Equations (7.42) and (7.44) to Equations (7.35) and (7.36), we have

$$
\begin{aligned}
a_{11} &= 1 & a_{12} &= Z_1 \\
a_{21} &= 0 & a_{22} &= 1
\end{aligned}
\tag{7.45}
$$

Example 7.6

Find the transmission parameters for the network shown in Figure 7.7.
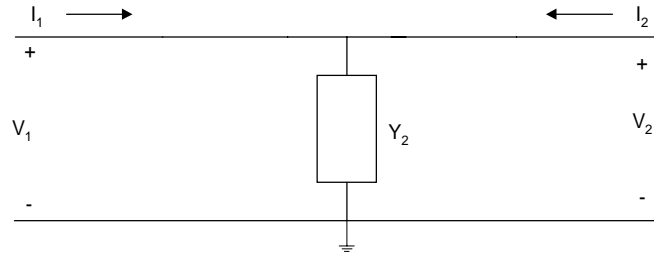

Figure 7.7 Simple Admittance Network

Solution
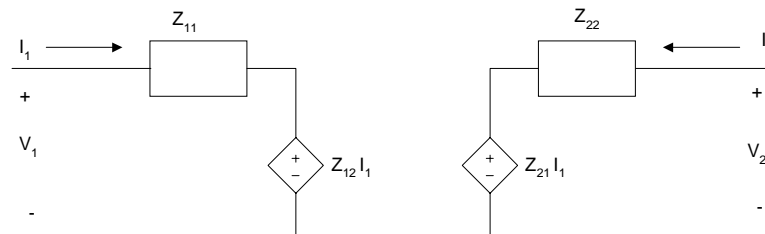
By inspection,

$$V_1 = V_2 \qquad (7.46)$$

Using KCL, we have
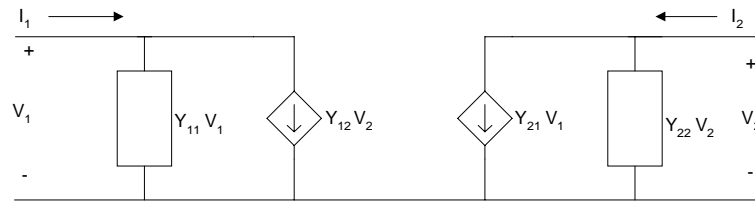
$$I_1 = V_2 Y_2 - I_2 \qquad (7.47)$$

Comparing Equations (7.46) and 7.47) to equations (7.35) and (7.36) we have

$$
\begin{aligned}
a_{11} &= 1 & a_{12} &= 0 \\
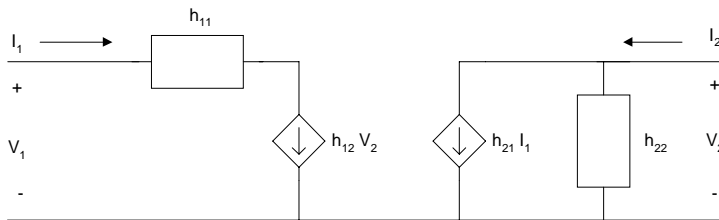a_{21} &= Y_2 & a_{22} &= 1
\end{aligned}
\qquad (7.48)
$$

Using the describing equations, the equivalent circuits of the various two-port network representations can be drawn. These are shown in Figure 7.8.
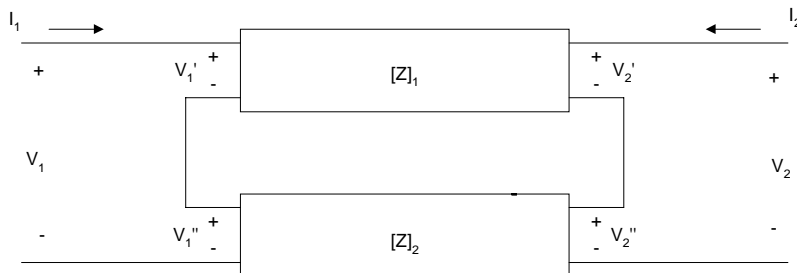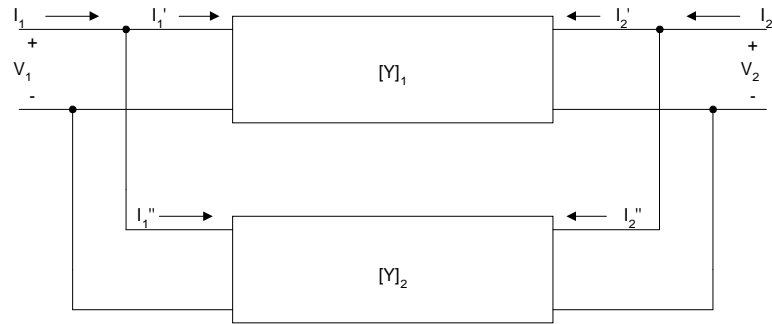

(a)

(b)

(c)

Figure 7.8 Equivalent Circuit of Two-port Networks (a) z-
parameters, (b) y-parameters and (c) h-parameters
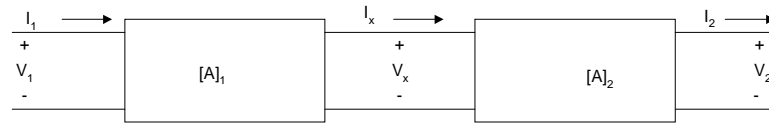
7.2   INTERCONNECTION OF TWO-PORT NETWORKS

Two-port networks can be connected in series, parallel or cascade. Figure 7.9
shows the various two-port interconnections.

(a) Series-connected Two-port Network

(b) Parallel-connected Two-port Network

(c) Cascade Connection of Two-port Network

Figure 7.9 Interconnection of Two-port Networks (a) Series
(b) Parallel (c) Cascade

It can be shown that if two-port networks with z-parameters $[Z]_1, [Z]_2, [Z]_3, ..., [Z]_n$ are connected in series, then the equivalent two-port z-parameters are given as

$$[Z]_{eq} = [Z]_1 + [Z]_2 + [Z]_3 + ...+[Z]_n \qquad (7.49)$$

If two-port networks with y-parameters $[Y]_1, [Y]_2, [Y]_3, ..., [Y]_n$ are connected in parallel, then the equivalent two-port y-parameters are given as

$$[Y]_{eq} = [Y]_1 + [Y]_2 + [Y]_3 + ...+[Y]_n \qquad (7.50)$$

When several two-port networks are connected in cascade, and the individual networks have transmission parameters $[A]_1, [A]_2, [A]_3, ..., [A]_n$, then the equivalent two-port parameter will have a transmission parameter given as

$$[A]_{eq} = [A]_1 * [A]_2 * [A]_3 * ...*[A]_n \qquad (7.51)$$

The following three examples illustrate the use of MATLAB for determining the equivalent parameters of interconnected two-port networks.

Example 7.7

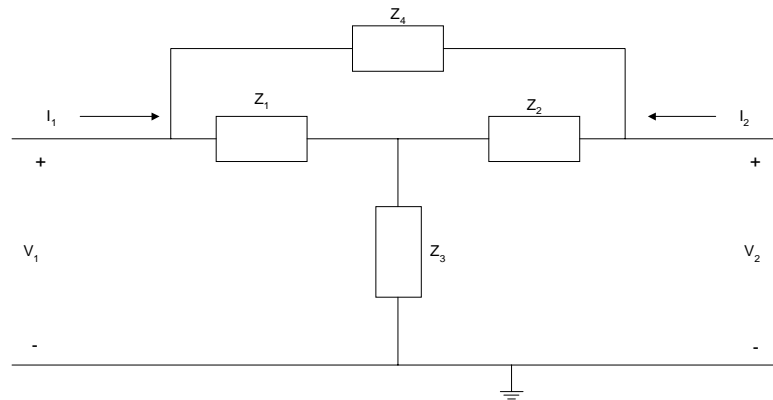Find the equivalent y-parameters for the bridge T-network shown in Figure 7.10.

Figure 7.10 Bridge-T Network

Solution

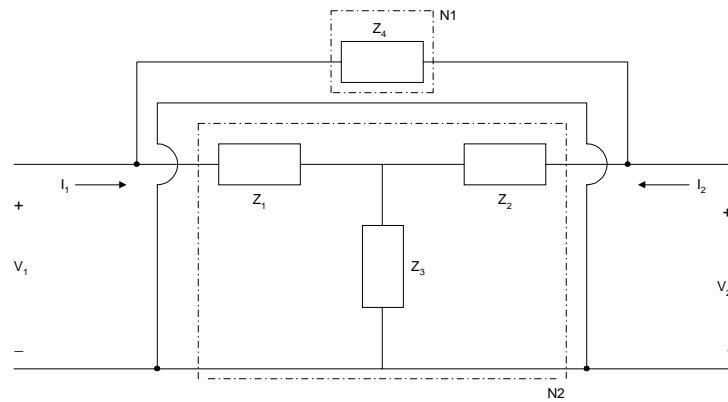The bridge-T network can be redrawn as

Figure 7.11 An Alternative Representation of Bridge-T Network

From Example 7.1, the z-parameters of network N2 are

$$[Z] = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}$$

We can convert the z-parameters to y-parameters [refs. 4 and 6] and we get

$$y_{11} = \frac{Z_2 + Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

$$y_{12} = \frac{-Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

$$y_{21} = \frac{-Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

$$(7.52)$$

$$y_{22} = -\frac{Z_1 + Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

From Example 7.5, the transmission parameters of network N1 are

$$a_{11} = 1 \qquad a_{12} = Z_4$$
$$a_{21} = 0 \qquad a_{22} = 1$$

We convert the transmission parameters to y-parameters[refs. 4 and 6] and we get

$$y_{11} = \frac{1}{Z_4}$$

$$y_{12} = -\frac{1}{Z_4}$$

$$(7.53)$$

$$y_{21} = -\frac{1}{Z_4}$$

$$y_{22} = \frac{1}{Z_4}$$

Using Equation (7.50), the equivalent y-parameters of the bridge-T network are

$$y_{11eq} = \frac{1}{Z_4} + \frac{Z_2 + Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

$$y_{12eq} = -\frac{1}{Z_4} - \frac{Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

$$y_{21eq} = -\frac{1}{Z_4} - \frac{Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \tag{7.54}$$

$$y_{22eq} = \frac{1}{Z_4} + \frac{Z_1 + Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}$$

Example 7.8

Find the transmission parameters of Figure 7.12.
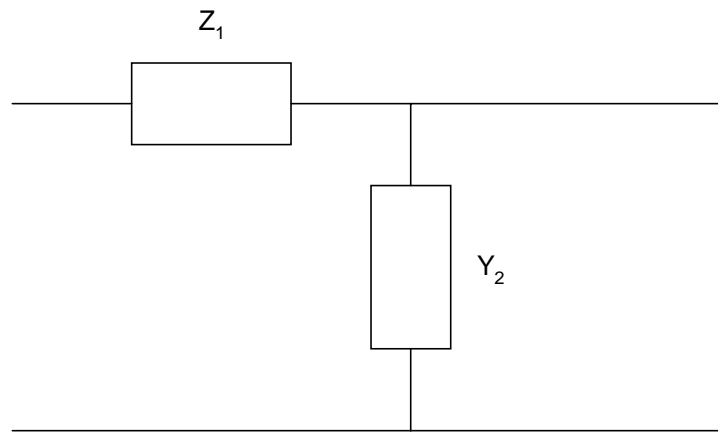
Figure 7.12 Simple Cascaded Network

Solution

can be redrawn as

Figure 7.13 Cascade of Two Networks N1 and N2

From Example 7.5, the transmission parameters of network N1 are

$$a_{11} = 1 \qquad a_{12} = Z_1$$
$$a_{21} = 0 \qquad a_{22} = 1$$

From Example 7.6, the transmission parameters of network N2 are

$$a_{11} = 1 \qquad a_{12} = 0$$
$$a_{21} = Y_2 \qquad a_{22} = 1$$

From Equation (7.51), the transmission parameters of are

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{eq} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ Y_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 + Z_1 Y_2 & Z_1 \\ Y_2 & 1 \end{bmatrix} \qquad (7.55)$$

Example 7.9

Find the transmission parameters for the cascaded system shown in Figure 7.14. The resistance values are in Ohms.
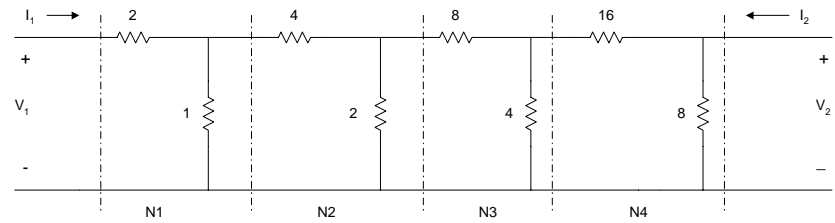
Figure 7.14 Cascaded Resistive Network

Solution

Figure 7.14 can be considered as four networks, N1, N2, N3, and N4 connected in cascade. From Example 7.8, the transmission parameters of Figure 7.12 are

$$[a]_{N1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$[a]_{N2} = \begin{bmatrix} 3 & 4 \\ 0.5 & 1 \end{bmatrix}$$

$$[a]_{N3} = \begin{bmatrix} 3 & 8 \\ 0.25 & 1 \end{bmatrix}$$

$$[a]_{N4} = \begin{bmatrix} 3 & 16 \\ 0.125 & 1 \end{bmatrix}$$

The transmission parameters of Figure 7.14 can be obtained using the following MATLAB program.

MATLAB Script

```
diary ex7_9.dat
% Transmission parameters of cascaded network

a1 = [3 2; 1 1];
a2 = [3 4; 0.5 1];
a3 = [3 8; 0.25 1];
a4 = [3 16; 0.125 1];

% equivalent transmission parameters
a = a1*(a2*(a3*a4))
diary
```

The value of matrix a is

```
a =
    112.2500   630.0000
     39.3750   221.0000
```

7.3   TERMINATED TWO-PORT NETWORKS

In normal applications, two-port networks are usually terminated. A terminated two-port network is shown in Figure 7.4.
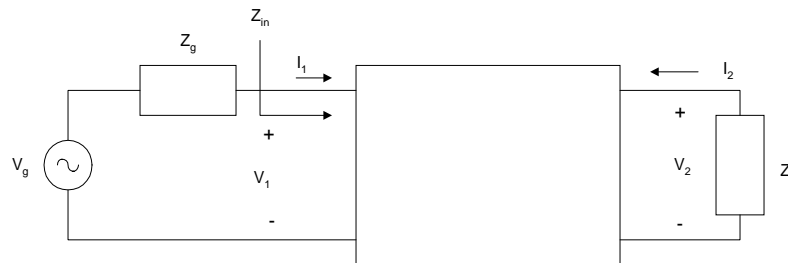
Figure 7.15 Terminated Two-Port Network

In the Figure 7.15, V_g and Z_g are the source generator voltage and impedance, respectively. Z_L is the load impedance. If we use z-parameter representation for the two-port network, the voltage transfer function can be shown to be

$$\frac{V_2}{V_g} = \frac{z_{21}Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}}$$

(7.56)

and the input impedance,

$$Z_{in} = z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L}$$

(7.57)

and the current transfer function,

$$\frac{I_2}{I_1} = -\frac{z_{21}}{z_{22} + Z_L}$$

(7.58)

A terminated two-port network, represented using the y-parameters, is shown in Figure 7.16.
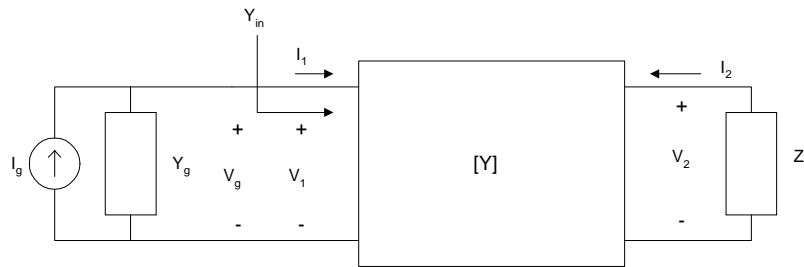
Figure 7.16 A Terminated Two-Port Network with y-parameters
Representation

It can be shown that the input admittance, Y_{in}, is

$$Y_{in} = y_{11} - \frac{y_{12}y_{21}}{y_{22} + Y_L}$$

(7.59)

and the current transfer function is given as

$$\frac{I_2}{I_g} = \frac{y_{21}Y_L}{(y_{11} + Y_g)(y_{22} + Y_L) - y_{12}y_{21}}$$

(7.60)

and the voltage transfer function

$$\frac{V_2}{V_g} = -\frac{y_{21}}{y_{22} + Y_L} \tag{7.61}$$

A doubly terminated two-port network, represented by transmission parameters, is shown in Figure 7.17.
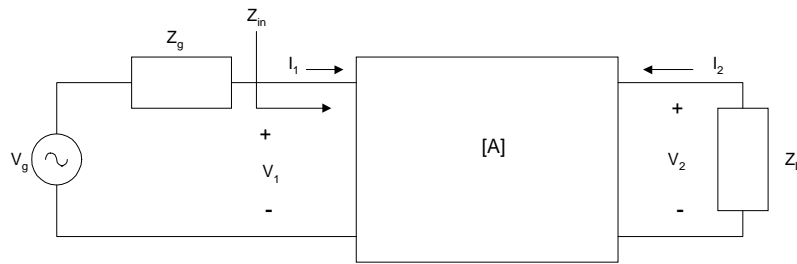
Figure 7.17 A Terminated Two-Port Network with Transmission Parameters Representation

The voltage transfer function and the input impedance of the transmission parameters can be obtained as follows. From the transmission parameters, we have

$$V_1 = a_{11}V_2 - a_{12}I_2 \tag{7.62}$$

$$I_1 = a_{21}V_2 - a_{22}I_2 \tag{7.63}$$

From Figure 7.6,

$$V_2 = -I_2 Z_L \tag{7.64}$$

Substituting Equation (7.64) into Equations (7.62) and (7.63), we get the input impedance,

$$Z_{in} = \frac{a_{11}Z_L + a_{12}}{a_{21}Z_L + a_{22}} \tag{7.65}$$

From Figure 7.17, we have

$$V_1 = V_g - I_1 Z_g \tag{7.66}$$

Substituting Equations (7.64) and (7.66) into Equations (7.62) and (7.63), we have

$$V_g - I_1 Z_g = V_2 [a_{11} + \frac{a_{12}}{Z_L}] \tag{7.67}$$

$$I_1 = V_2 [a_{21} + \frac{a_{22}}{Z_L}] \tag{7.68}$$

Substituting Equation (7.68) into Equation (7.67), we get

$$V_g - V_2 Z_g [a_{21} + \frac{a_{22}}{Z_L}] = V_2 [a_{11} + \frac{a_{12}}{Z_L}] \tag{7.69}$$

Simplifying Equation (7.69), we get the voltage transfer function

$$\frac{V_2}{V_g} = \frac{Z_L}{(a_{11} + a_{21} Z_g) Z_L + a_{12} + a_{22} Z_g} \tag{7.70}$$

The following examples illustrate the use of MATLAB for solving terminated two-port network problems.

Example 7.10

Assuming that the operational amplifier of Figure 7.18 is ideal,
(a)   Find the z-parameters of Figure 7.18.
(b)   If the network is connected by a voltage source with source resistance of 50Ω and a load resistance of 1 KΩ, find the voltage gain.
(c)   Use MATLAB to plot the magnitude response.
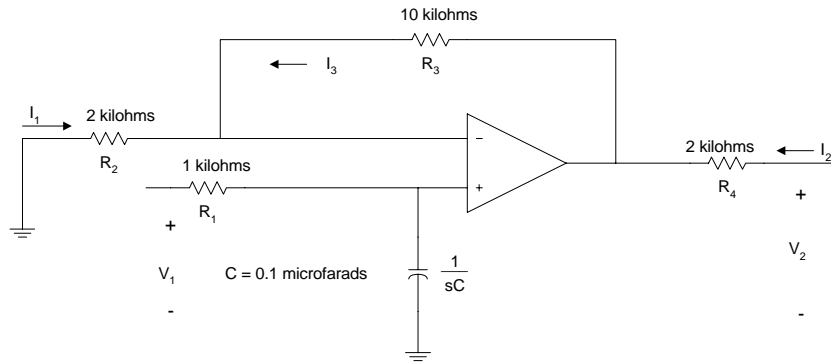
Figure 7.18   An Active Lowpass Filter

Solution

Using KVL,

$$V_1 = R_1 I_1 + \frac{I_1}{sC} \tag{7.71}$$

$$V_2 = R_4 I_2 + R_3 I_3 + R_2 I_3 \tag{7.72}$$

From the concept of virtual circuit discussed in Chapter 11,

$$R_2 I_3 = \frac{I_1}{sC} \tag{7.73}$$

Substituting Equation (7.73) into Equation (7.72), we get

$$V_2 = \frac{(R_2 + R_3) I_1}{sCR_2} + R_4 I_2 \tag{7.74}$$

Comparing Equations (7.71) and (7.74) to Equations (7.1) and (7.2), we have

$$z_{11} = R_1 + \frac{1}{sC}$$

$$z_{12} = 0$$

$$z_{21} = \left(1 + \frac{R_3}{R_2}\right)\left(\frac{1}{sC}\right)$$

(7.75)

$$z_{22} = R_4$$

From Equation (7.56), we get the voltage gain for a terminated two-port network. It is repeated here.

$$\frac{V_2}{V_g} = \frac{z_{21} Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12} z_{21}}$$

Substituting Equation (7.75) into Equation (7.56), we have

$$\frac{V_2}{V_g} = \frac{(1 + \frac{R_3}{R_2}) Z_L}{(R_4 + Z_L)[1 + sC(R_1 + Z_g)]}$$

(7.76)

For $Z_g = 50\ \Omega$, $Z_L = 1\ K\Omega$, $R_3 = 10\ K\Omega$, $R_2 = 1\ K\Omega$, $R_4 = 2\ K\Omega$ and $C = 0.1\ \mu F$, Equation (7.76) becomes

$$\frac{V_2}{V_g} = \frac{2}{[1 + 1.05 * 10^{-4} s]}$$

(7.77)

The MATLAB script is

```
%
num = [2];
den = [1.05e-4 1];
w = logspace(1,5);
h = freqs(num,den,w);
f = w/(2*pi);
mag = 20*log10(abs(h));  % magnitude in dB
semilogx(f,mag)
title('Lowpass Filter Response')
xlabel('Frequency, Hz')
```

ylabel('Gain in dB')

The frequency response is shown in Figure 7.19.
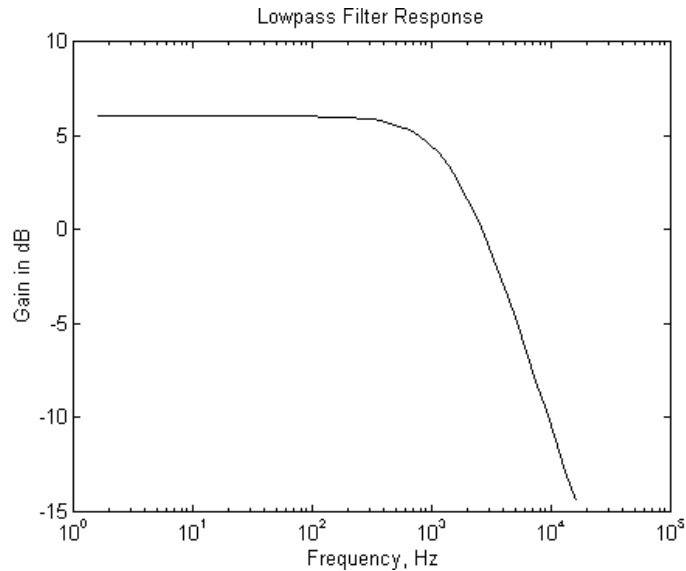
Figure 7.19 Magnitude Response of an Active Lowpass Filter

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1.   MathWorks, Inc., *MATLAB, High-Performance Numeric Computation Software*, 1995.

2.   Biran, A. and Breiner, M., *MATLAB for Engineers*, Addison-Wesley, 1995.

3.   Etter, D.M., *Engineering Problem Solving with MATLAB*, 2nd Edition, Prentice Hall, 1997.

4.   Nilsson, J.W., Electric Circuits, 3rd Edition, Addison-Wesley Publishing Company, 1990.

5.   Meader, D.A., *Laplace Circuit Analysis and Active Filters,* Prentice Hall, 1991.

6.   Johnson, D. E. Johnson, J.R., and Hilburn, J.L. *Electric Circuit Analysis*, 3rd Edition, Prentice Hall, 1997.

7.   Vlach, J.O., Network Theory and CAD, IEEE Trans. on Education, Vol. 36, No. 1, Feb. 1993, pp. 23 - 27.

EXERCISES

7.1   (a) Find the transmission parameters of the circuit shown in Figure P7.1a.   The resistance values are in ohms.
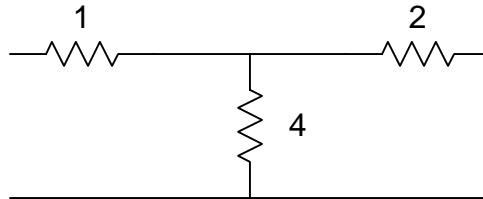
Figure P7.1a Resistive T-Network

(b) From the result of part (a), use MATLAB to find the transmission parameters of Figure P7.2b. The resistance values are in ohms.
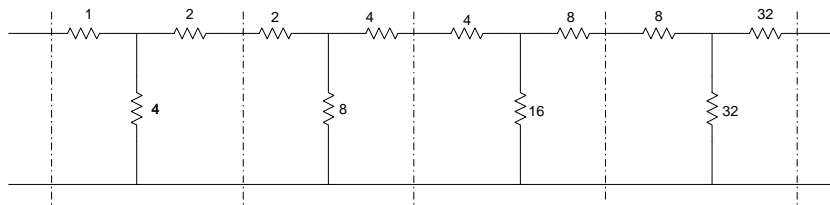
Figure P7.1b Cascaded Resistive Network

7.2   Find the y-parameters of the circuit shown in Figure P7.2 The resistance values are in ohms.
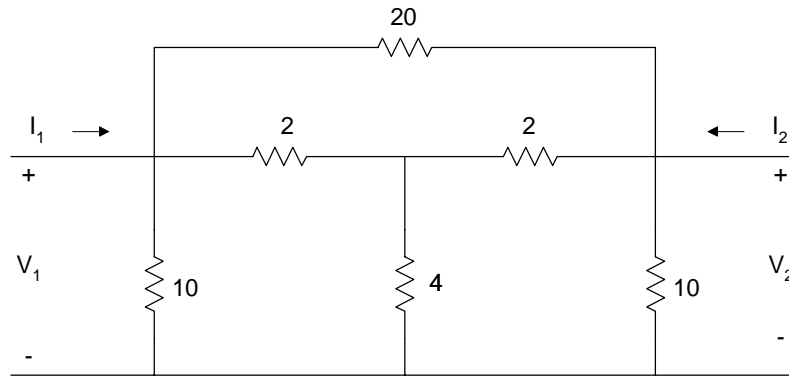
Figure P7.2 A Resistive Network

7.3 (a) Show that for the symmetrical lattice structure shown in Figure P7.3,

$$z_{11} = z_{22} = 0.5(Z_c + Z_d)$$
$$z_{12} = z_{21} = 0.5(Z_c - Z_d)$$

(b) If $Z_c = 10\ \Omega,\quad Z_d = 4\ \Omega,$ find the equivalent y-parameters.
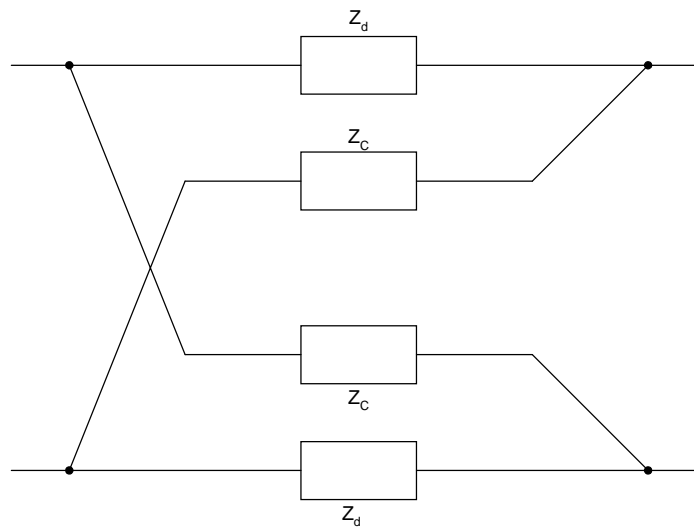
Figure P7.3 Symmetrical Lattice Structure

7.4 (a) Find the equivalent z-parameters of Figure P7.4.
(b) If the network is terminated by a load of 20 ohms and connected to a source of V_S with a source resistance of 4 ohms, use MATLAB to plot the frequency response of the circuit.
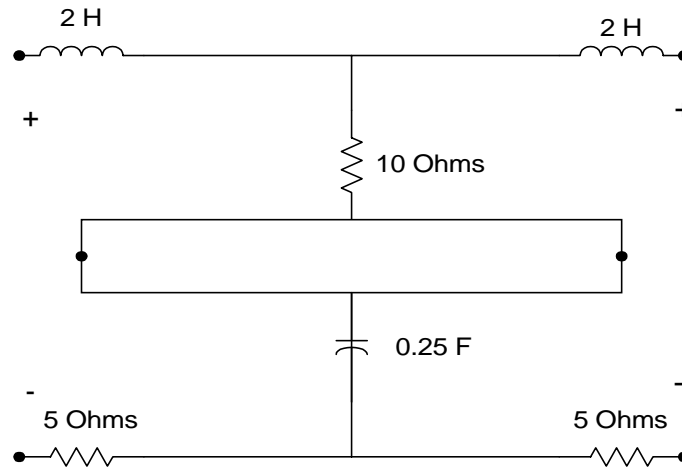
Figure P7.4 Circuit for Problem 7.4

7.5 For Figure P7.5
(a) Find the transmission parameters of the RC ladder network.
(b) Obtain the expression for $\dfrac{V_2}{V_1}$.
(c) Use MATLAB to plot the phase characteristics of $\dfrac{V_2}{V_1}$.
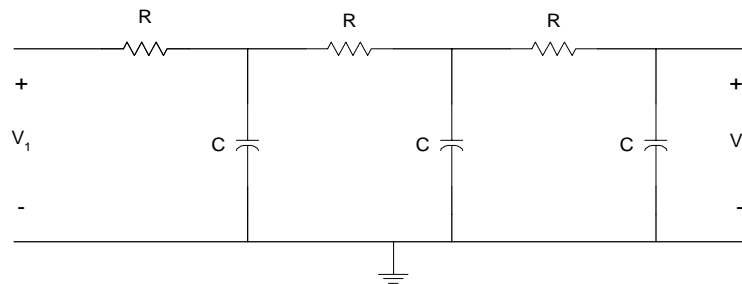
Figure P7.5 RC Ladder Network

7.6   For the circuit shown in Figure P7.6,
(a) Find the y-parameters.
(b) Find the expression for the input admittance.
(c) Use MATLAB to plot the input admittance as a function of
frequency.
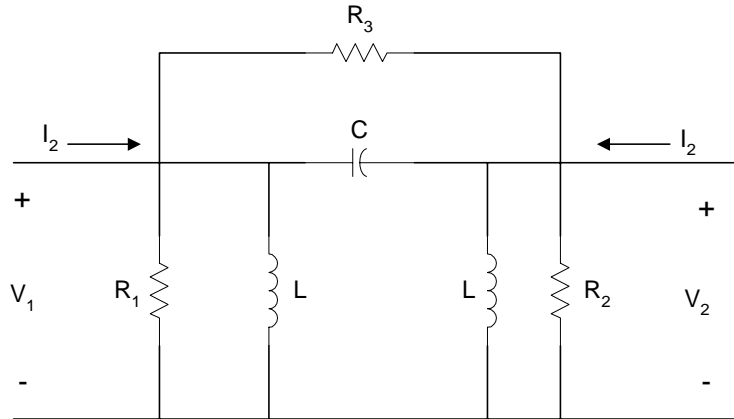
Figure P7.6 Circuit for Problem 7.6

7.7   For the op amp circuit shown in Figure P7.7, find the y-parameters.
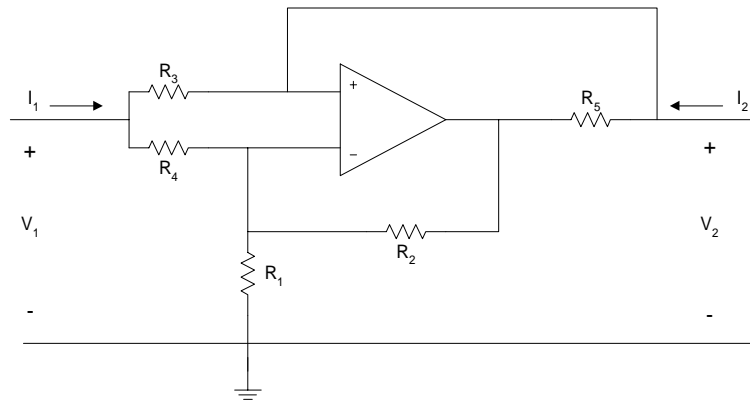
Figure P7.7   Op Amp Circuit

Attia, John Okyere. "Fourier Analysis."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

FOURIER ANALYSIS

In this chapter, Fourier analysis will be discussed. Topics covered are Fourier series expansion, Fourier transform, discrete Fourier transform, and fast Fourier transform. Some applications of Fourier analysis, using MATLAB, will also be discussed.

8.1   FOURIER SERIES

If a function $g(t)$ is periodic with period T_p, i.e.,

$$g(t) = g(t \pm T_p) \tag{8.1}$$

and in any finite interval $g(t)$ has at most a finite number of discontinuities and a finite number of maxima and minima (Dirichlets conditions), and in addition,

$$\int_0^{T_p} g(t)dt < \infty \tag{8.2}$$

then $g(t)$ can be expressed with series of sinusoids. That is,

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nw_0t) + b_n \sin(nw_0t) \tag{8.3}$$

where

$$w_0 = \frac{2\pi}{T_p} \tag{8.4}$$

and the Fourier coefficients a_n and b_n are determined by the following equations.

$$a_n = \frac{2}{T_p} \int_{t_o}^{t_o+T_p} g(t)\cos(nw_0t)dt \quad n = 0, 1, 2, \ldots \tag{8.5}$$

$$b_n = \frac{2}{T_p} \int_{t_o}^{t_o+T_p} g(t) \sin(nw_0 t) dt \quad n = 0, 1, 2 \dots \tag{8.6}$$

Equation (8.3) is called the trigonometric Fourier series. The term $\dfrac{a_0}{2}$ in Equation (8.3) is the dc component of the series and is the average value of $g(t)$ over a period. The term $a_n \cos(nw_0 t) + b_n \sin(nw_0 t)$ is called the n-th harmonic. The first harmonic is obtained when $n = 1$. The latter is also called the fundamental with the fundamental frequency of ω_o. When n = 2, we have the second harmonic and so on.

Equation (8.3) can be rewritten as

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nw_0 t + \Theta_n) \tag{8.7}$$

where

$$A_n = \sqrt{a_n^2 + b_n^2} \tag{8.8}$$

and

$$\Theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) \tag{8.9}$$

The total power in $g(t)$ is given by the Parseval's equation:

$$P = \frac{1}{T_p} \int_{t_o}^{t_o+T_p} g^2(t) dt = A_{dc}^2 + \sum_{n=1}^{\infty} \frac{A_n^2}{2} \tag{8.10}$$

where

$$A_{dc}^2 = \left(\frac{a_0}{2}\right)^2 \tag{8.11}$$

The following example shows the synthesis of a square wave using Fourier series expansion.

Example 8.1

Using Fourier series expansion, a square wave with a period of 2 ms, peak-to-peak value of 2 volts and average value of zero volt can be expressed as

$$g(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin[(2n-1)2\pi f_0 t \;] \qquad (8.12)$$

where

$$f_0 = 500 \text{ Hz}$$

if $a(t)$ is given as

$$a(t) = \frac{4}{\pi} \sum_{n=1}^{12} \frac{1}{(2n-1)} \sin[(2n-1)2\pi f_0 t \;] \qquad (8.13)$$

Write a MATLAB program to plot $a(t)$ from 0 to 4 ms at intervals of 0.05 ms and to show that $a(t)$ is a good approximation of $g(t)$.

Solution

MATLAB Script

```
% fourier series expansion
f = 500; c = 4/pi; dt = 5.0e-05;
tpts = (4.0e-3/5.0e-5) + 1;
for n = 1: 12
for m = 1: tpts
s1(n,m) = (4/pi)*(1/(2*n - 1))*sin((2*n - 1)*2*pi*f*dt*(m-1));
end
end
for m = 1:tpts
 a1 = s1(:,m);
 a2(m) = sum(a1);
end
f1 = a2';
t = 0.0:5.0e-5:4.0e-3;
clg
plot(t,f1)
xlabel('Time, s')
```

```
ylabel('Amplitude, V')
title('Fourier series expansion')
```

Figure 8.1 shows the plot of $a(t)$.
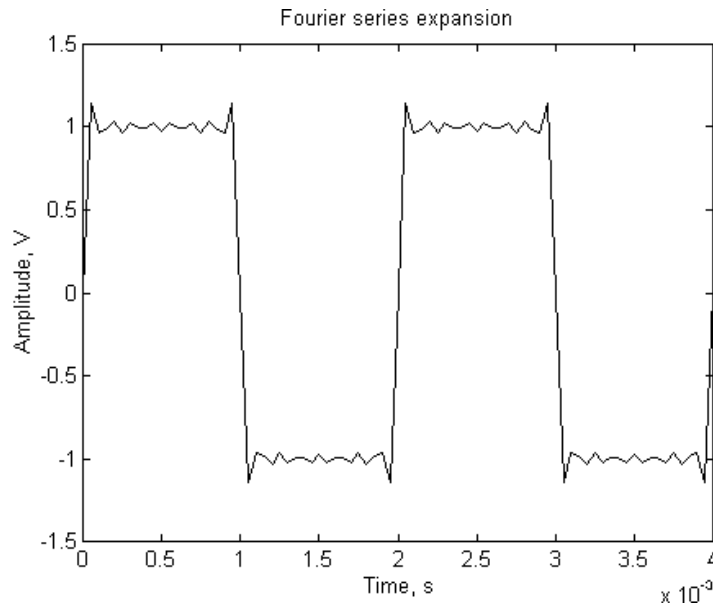
Figure 8.1 Approximation to Square Wave

By using the Euler's identity, the cosine and sine functions of Equation (8.3) can be replaced by exponential equivalents, yielding the expression

$$g(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jnw_0 t) \qquad (8.14)$$

where

$$c_n = \frac{1}{T_p} \int_{-t_p/2}^{T_p/2} g(t) \exp(-jnw_0 t) dt \qquad (8.15)$$

and

$$w_0 = \frac{2\pi}{T_p}$$

Equation (8.14) is termed the exponential Fourier series expansion. The coefficient c_n is related to the coefficients a_n and b_n of Equations (8.5) and (8.6) by the expression

$$c_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \angle - \tan^{-1}(\frac{b_n}{a_n}) \tag{8.16}$$

In addition, c_n relates to A_n and ϕ_n of Equations (8.8) and (8.9) by the relation

$$c_n = \frac{A_n}{2} \angle \Theta_n \tag{8.17}$$

The plot of $|c_n|$ versus frequency is termed the discrete amplitude spectrum or the line spectrum. It provides information on the amplitude spectral components of $g(t)$. A similar plot of $\angle c_n$ versus frequency is called the discrete phase spectrum and the latter gives information on the phase components with respect to the frequency of $g(t)$.

If an input signal $x_n(t)$

$$x_n(t) = c_n \exp(jnw_o t) \tag{8.18}$$

passes through a system with transfer function $H(w)$, then the output of the system $y_n(t)$ is

$$y_n(t) = H(jnw_o)c_n \exp(jnw_o t) \tag{8.19}$$

The block diagram of the input/output relation is shown in

$$x_n(t) \longrightarrow \boxed{\quad H(s) \quad} \longrightarrow y_n(t)$$
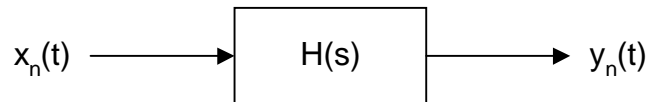
Figure 8.2 Input/Output Relationship

However, with an input $x(t)$ consisting of a linear combination of complex excitations,

$$x_n(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jnw_o t) \qquad (8.20)$$

the response at the output of the system is

$$y_n(t) = \sum_{n=-\infty}^{\infty} H(jnw_o)c_n \exp(jnw_o t) \qquad (8.21)$$

The following two examples show how to use MATLAB to obtain the coefficients of Fourier series expansion.

Example 8.2

For the full-wave rectifier waveform shown in Figure 8.3, the period is 0.0333s and the amplitude is 169.71 Volts.

(a)   Write a MATLAB program to obtain the exponential Fourier series coefficients c_n for $n = 0, 1, 2, .. , 19$

(b)   Find the dc value.

(c)   Plot the amplitude and phase spectrum.
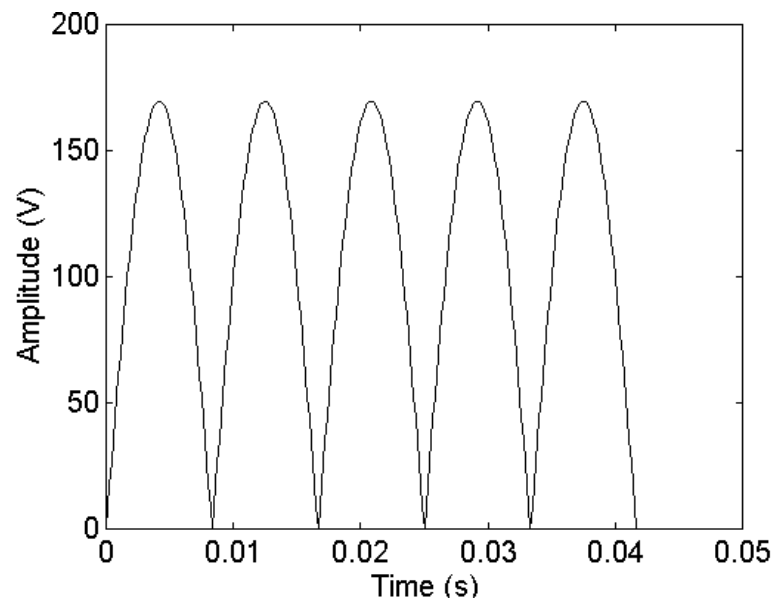
Figure 8.3 Full-wave Rectifier Waveform

Solution

```
diary ex8_2.dat
% generate the full-wave rectifier waveform
f1 = 60;
inv = 1/f1; inc = 1/(80*f1); tnum = 3*inv;
t = 0:inc:tnum;
g1 = 120*sqrt(2)*sin(2*pi*f1*t);
g = abs(g1);
N = length(g);
%
% obtain the exponential Fourier series coefficients

num = 20;
for i = 1:num
    for m = 1:N
      cint(m) = exp(-j*2*pi*(i-1)*m/N)*g(m);
      end
  c(i) = sum(cint)/N;
end
cmag = abs(c);
cphase = angle(c);

%print dc value
disp('dc value of g(t)'); cmag(1)
% plot the magnitude and phase spectrum

f = (0:num-1)*60;
subplot(121), stem(f(1:5),cmag(1:5))
title('Amplitude spectrum')
xlabel('Frequency, Hz')
subplot(122), stem(f(1:5),cphase(1:5))
title('Phase spectrum')
xlabel('Frequency, Hz')
diary
```

dc value of g(t)

```
ans =
      107.5344
```

Figure 8.4 shows the magnitude and phase spectra of Figure 8.3.
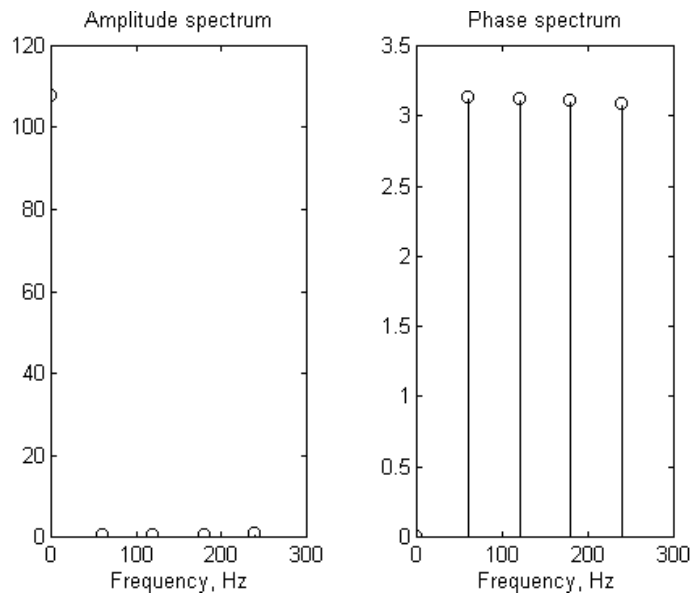
Figure 8.4 Magnitude and Phase Spectra of a Full-wave
Rectification Waveform

Example 8.3

The periodic signal shown in Figure 8.5 can be expressed as

$$g(t) = e^{-2t} \qquad -1 \le t < 1$$
$$g(t + 2) = g(t)$$

(i) Show that its exponential Fourier series expansion can be expressed as

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (e^2 - e^{-2})}{2(2 + jn\pi)} \exp(jn\pi t) \tag{8.22}$$

(ii) Using a MATLAB program, synthesize $g(t)$ using 20 terms, i.e.,

$$\hat{g}(t) = \sum_{n=-10}^{10} \frac{(-1)^n (e^2 - e^{-2})}{2(2 + jn\pi)} \exp(jn\pi t)$$
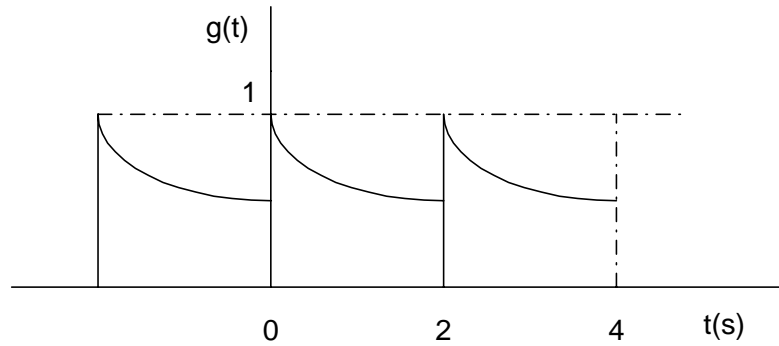
Figure 8.5 Periodic Exponential Signal

Solution

(i)

$$g(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jnw_o t)$$

where

$$c_n = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} g(t) \exp(-jnw_o t) dt$$

and

$$w_o = \frac{2\pi}{T_p} = \frac{2\pi}{2} = \pi$$

$$c_n = \frac{1}{2} \int_{-1}^{1} \exp(-2t) \exp(-jn\pi t) dt$$

$$c_n = \frac{(-1)^n (e^2 - e^{-2})}{2(2 + jn\pi)}$$

thus

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (e^2 - e^{-2})}{2(2 + jn\pi)} \exp(jn\pi t)$$

(ii) MATLAB Script

```
% synthesis of g(t) using exponential Fourier series expansion
dt = 0.05;
  tpts = 8.0/dt +1;
cst = exp(2) - exp(-2);

for n = -10:10
 for m = 1:tpts
   g1(n+11,m)  =  ((0.5*cst*((-1)^n))/(2+j*n*pi))*(exp(j*n*pi*dt*(m-
1)));
  end
end

for m = 1: tpts
 g2 = g1(:,m);
 g3(m) = sum(g2);
end

g = g3';
t = -4:0.05:4.0;
plot(t,g)
xlabel('Time, s')
ylabel('Amplitude')
title('Approximation of g(t)')
```

Figure 8.6 shows the approximation of $g(t)$.
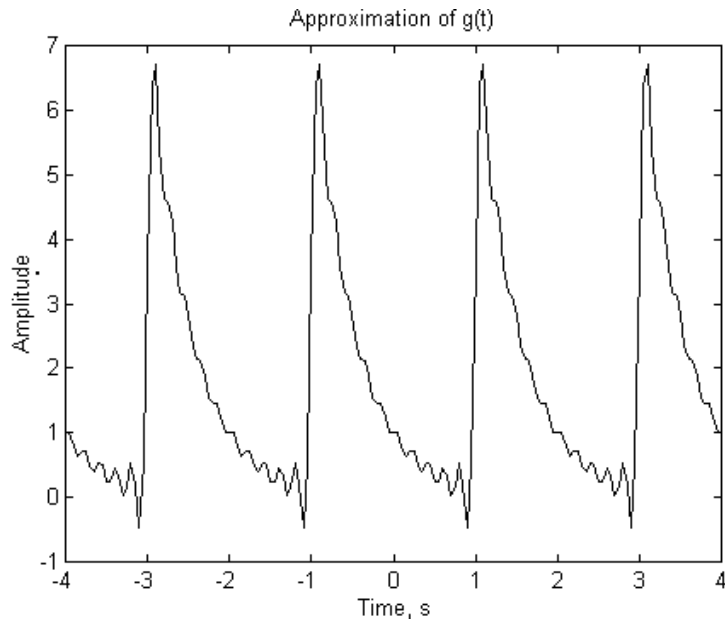
Figure 8.6 An Approximation of $g(t)$.

8.2 FOURIER TRANSFORMS

If $g(t)$ is a nonperiodic deterministic signal expressed as a function of time t, then the Fourier transform of $g(t)$ is given by the integral expression:

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt \qquad (8.23)$$

where

$$j = \sqrt{-1} \quad \text{and}$$

f denotes frequency

$g(t)$ can be obtained from the Fourier transform $G(f)$ by the Inverse Fourier Transform formula:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df \qquad (8.24)$$

For a signal $g(t)$ to be Fourier transformable, it should satisfy the Dirichlet's conditions that were discussed in Section 8.1. If $g(t)$ is continuous and non-periodic, then $G(f)$ will be continuous and periodic. However, if $g(t)$ is continuous and periodic, then $G(f)$ will discrete and nonperiodic; that is

$$g(t) = g(t \pm nT_p) \qquad (8.25)$$

where

T_p = period

then the Fourier transform of $g(t)$ is

$$G(f) = \frac{1}{T_p} \sum_{n=-\infty}^{\infty} c_n \delta(f - \frac{1}{T_p}) \qquad (8.26)$$

where

$$c_n = \frac{1}{T_p} \int_{-t_p/2}^{T_p/2} g(t)\exp(-j2\pi nf_o t)dt \qquad (8.27)$$

8.2.1   Properties of Fourier transform

If $g(t)$ and $G(f)$ are Fourier transform pairs, and they are expressed as

$$g(t) \Longleftrightarrow G(f) \qquad (8.28)$$

then the Fourier transform will have the following properties:

Linearity

$$ag_1(t) + bg_2(t) \Longleftrightarrow aG_1(f) + bG_2(f) \qquad (8.29)$$

where

a and b are constants

Time scaling

$$g(at) \Leftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right) \tag{8.30}$$

Duality

$$G(t) \Leftrightarrow g(-f) \tag{8.31}$$

Time shifting

$$g(t - t_0) \Leftrightarrow G(f)\exp(-j2\pi ft_0) \tag{8.32}$$

Frequency Shifting

$$\exp(j2f_c t)g(t) \Leftrightarrow G(f - f_c) \tag{8.33}$$

Definition in the time domain

$$\frac{dg(t)}{dt} \Leftrightarrow j2\pi fG(f) \tag{8.34}$$

Integration in the time domain

$$\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)\,\delta\,(f) \tag{8.35}$$

Multiplication in the time domain

$$g_1(t)g_2(t) \Leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)d\lambda \tag{8.36}$$

Convolution in the time domain

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau \Leftrightarrow G_1(f)G_2(f) \tag{8.37}$$

8.3   DISCRETE AND FAST FOURIER TRANSFORMS

Fourier series links a continuous time signal into the discrete-frequency domain. The periodicity of the time-domain signal forces the spectrum to be discrete. The discrete Fourier transform of a discrete-time signal $g[n]$ is given as

$$G[k] = \sum_{n=0}^{N-1} g[n]\exp(-j2\pi nk / N) \quad k = 0,1, ..., \text{N-1} \qquad (8.38)$$

The inverse discrete Fourier transform, $g[n]$ is

$$g[n] = \sum_{k=0}^{N-1} G[k]\exp(j2\pi nk / N) \quad n = 0,1,..., \text{N-1} \qquad (8.39)$$

where

N is the number of time sequence values of $g[n]$. It is also the total number frequency sequence values in $G[k]$.

T is the time interval between two consecutive samples of the input sequence $g[n]$.

F is the frequency interval between two consecutive samples of the output sequence $G[k]$.

N, T, and F are related by the expression

$$NT = \frac{1}{F} \qquad (8.40)$$

NT is also equal to the record length. The time interval, T, between samples should be chosen such that the Shannon's Sampling theorem is satisfied. This means that T should be less than the reciprocal of $2f_H$, where f_H is the highest significant frequency component in the continuous time signal $g(t)$ from which the sequence $g[n]$ was obtained.   Several fast DFT algorithms require N to be an integer power of 2.

A discrete-time function will have a periodic spectrum. In DFT, both the time function and frequency functions are periodic. Because of the periodicity of DFT, it is common to regard points from $n = 1$ through $n = N/2$ as positive,

and points from $n = N/2$ through $n = N - 1$ as negative frequencies. In addition, since both the time and frequency sequences are periodic, DFT values at points $n = N/2$ through $n = N - 1$ are equal to the DFT values at points $n = N/2$ through $n = 1$.

In general, if the time-sequence is real-valued, then the DFT will have real components which are even and imaginary components that are odd. Similarly, for an imaginary valued time sequence, the DFT values will have an odd real component and an even imaginary component.

If we define the weighting function W_N as

$$W_N = e^{\frac{-j2\pi}{N}} = e^{-j2\pi FT} \tag{8.41}$$

Equations (8.38) and (8.39) can be re-expressed as

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn} \tag{8.42}$$

and

$$g[n] = \sum_{k=0}^{N-1} G[k] W_N^{-kn} \tag{8.43}$$

The Fast Fourier Transform, FFT, is an efficient method for computing the discrete Fourier transform. FFT reduces the number of computations needed for computing DFT. For example, if a sequence has N points, and N is an integral power of 2, then DFT requires N^2 operations, whereas FFT requires $\frac{N}{2}\log_2(N)$ complex multiplication, $\frac{N}{2}\log_2(N)$ complex additions and $\frac{N}{2}\log_2(N)$ subtractions. For $N = 1024$, the computational reduction from DFT to FFT is more than 200 to 1.

The FFT can be used to (a) obtain the power spectrum of a signal, (b) do digital filtering, and (c) obtain the correlation between two signals.

8.3.1 MATLAB function fft

The MATLAB function for performing Fast Fourier Transforms is

$$fft(x)$$

where x is the vector to be transformed.

$$fft(x, N)$$

is also MATLAB command that can be used to obtain N-point fft. The vector x is truncated or zeros are added to N, if necessary.

The MATLAB functions for performing inverse fft is

$$ifft(x).$$

$$\left[z_m, z_p \right] = fftplot(x, ts)$$

is used to obtain fft and plot the magnitude z_m and z_p of DFT of x. The sampling interval is ts. Its default value is 1. The spectra are plotted versus the digital frequency F. The following three examples illustrate usage of MATLAB function fft.

Example 8.4

Given the sequence $x[n] = (1, 2, 1)$. (a) Calculate the DFT of $x[n]$. (b) Use the fft algorithm to find DFT of $x[n]$. (c) Compare the results of (a) and (b).

Solution

(a) From Equation (8.42)

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

From Equation (8.41)

$$W_3^0 = 1$$

$$W_3^1 = e^{-\frac{j2\pi}{3}} = -0.5 - j0.866$$

$$W_3^2 = e^{-\frac{j4\pi}{3}} = -0.5 + j0.866$$

$$W_3^3 = W_3^0 = 1$$

$$W_3^4 = W_3^1$$

Using Equation (8.41), we have

$$G[0] = \sum_{n=0}^{2} g[n]W_3^0 = 1 + 2 + 1 = 4$$

$$G[1] = \sum_{n=0}^{2} g[n]W_3^n = g[0]W_3^0 + g[1]W_3^1 + g[2]W_3^2$$

$$= 1 + 2(-0.5 - j0.866) + (-0.5 + j0.866) = -0.5 - j0.866$$

$$G[2] = \sum_{n=0}^{2} g[n]W_3^{2n} = g[0]W_3^0 + g[1]W_3^2 + g[2]W_3^4$$

$$= 1 + 2(-0.5 + j0.866) + (-0.5 - j0.866) = -0.5 + j0.866$$

(b) The MATLAB program for performing the DFT of $x[n]$ is

MATLAB Script

```
diary ex8_4.dat
%
x = [1 2 1];
xfft = fft(x)
diary
```

The results are

```
xfft =
     4.0000   -0.5000 - 0.8660i     -0.5000 + 0.8660i
```

(c) It can be seen that the answers obtained from parts (a) and (b) are identical.

Example 8.5

Signal $g(t)$ is given as

$$g(t) = 4e^{-2t} \cos\left[2\pi(10)t\right]u(t)$$

(a)   Find the Fourier transform of $g(t)$, i.e., $G(f)$.
(b)   Find the DFT of $g(t)$ when the sampling interval is 0.05 s with N = 1000.
(c)   Find the DFT of $g(t)$ when the sampling interval is 0.2 s with N = 250.
(d)   Compare the results obtained from parts a, b, and c.

Solution

(a) $g(t)$ can be expressed as

$$g(t) = 4e^{-2t}\left[\frac{1}{2}e^{j20\pi t} + \frac{1}{2}e^{-j20\pi t}\right]u(t)$$

Using the frequency shifting property of the Fourier Transform, we get

$$G(f) = \frac{2}{2 + j2\pi(f - 10)} + \frac{2}{2 + j2\pi(f + 10)}$$

(b, c) The MATLAB program for computing the DFT of $g(t)$ is

MATLAB Script

```
% DFT of g(t)
%  Sample 1, Sampling interval of 0.05 s
ts1 = 0.05;  % sampling interval
fs1 = 1/ts1;  %  Sampling frequency
n1 = 1000;    %  Total Samples
m1 = 1:n1;    %  Number of bins
sint1 = ts1*(m1 - 1);  %  Sampling instants
freq1 = (m1 - 1)*fs1/n1;  % frequencies
gb = (4*exp(-2*sint1)).*cos(2*pi*10*sint1);
gb_abs = abs(fft(gb));
subplot(121)
```

```
plot(freq1, gb_abs)
title('DFT of g(t), 0.05s Sampling interval')
xlabel('Frequency (Hz)')

%  Sample 2,  Sampling interval of 0.2 s
ts2 = 0.2;   % sampling interval
fs2 = 1/ts2;  %  Sampling frequency
n2 = 250;      %  Total Samples
m2 = 1:n2;    %  Number of bins
sint2 = ts2*(m2 - 1); %  Sampling instants
freq2 = (m2 - 1)*fs2/n2;  % frequencies
gc = (4*exp(-2*sint2)).*cos(2*pi*10*sint2);
gc_abs = abs(fft(gc));
subplot(122)
plot(freq2, gc_abs)
title('DFT of g(t), 0.2s Sampling interval')
xlabel('Frequency (Hz)')
```

The two plots are shown in Figure 8.7.
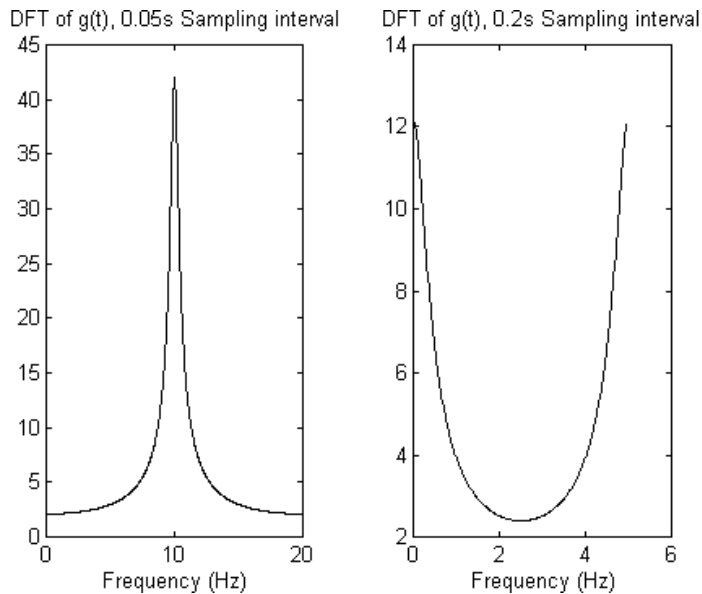
Figure 8.7 DFT of $g(t)$

(d) From Figure 8.7, it can be seen that with the sample interval of 0.05 s, there was no aliasing and spectrum of $G[k]$ in part (b) is almost the same

as that of $G(f)$ of part (a). With the sampling interval being 0.2 s (less than the Nyquist rate), there is aliasing and the spectrum of $G[k]$ is different from that of $G(f)$.

Example 8.6

Given a noisy signal

$$g(t) = \sin(2\pi f_1 t) + 0.5n(t)$$

where

$$f_1 \;=\; 100 \text{ Hz}$$

n(t) is a normally distributed white noise. The duration of $g(t)$ is 0.5 seconds. Use MATLAB function rand to generate the noise signal.   Use MATLAB to obtain the power spectral density of $g(t)$.

Solution

A representative program that can be used to plot the noisy signal and obtain the power spectral density is

MATLAB Script

```
% power spectral estimation of noisy signal
t = 0.0:0.002:0.5;
f1 =100;

% generate the sine portion of signal
x = sin(2*pi*f1*t);

% generate a normally distributed white noise
n = 0.5*randn(size(t));

% generate the noisy signal
y = x+n;
subplot(211), plot(t(1:50),y(1:50)),
title('Nosiy time domain signal')

% power spectral estimation is done
yfft = fft(y,256);
```

```
len = length(yfft);
pyy = yfft.*conj(yfft)/len;
f = (500./256)*(0:127);

subplot(212), plot(f,pyy(1:128)),
title('power spectral density'),
xlabel('frequency in Hz')
```

The plot of the noisy signal and its spectrum is shown in Figure 8.8. The amplitude of the noise and the sinusoidal signal can be changed to observe their effects on the spectrum.
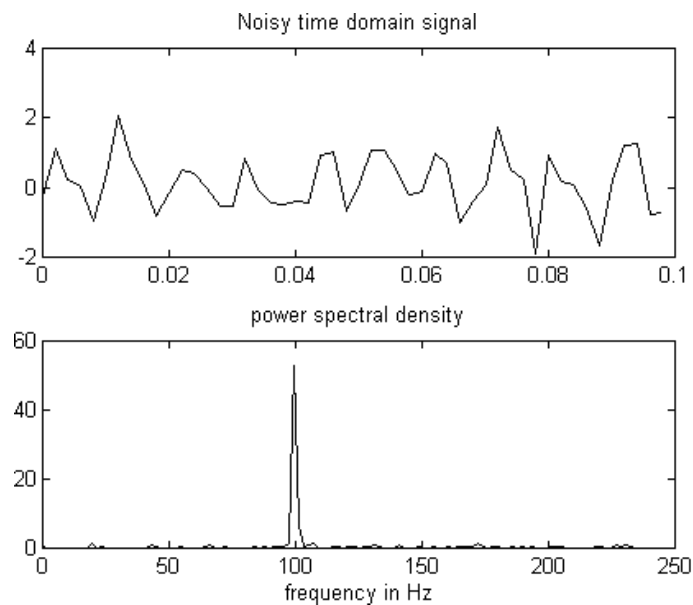
Figure 8.8 Noisy Signal and Its Spectrum

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2.   Etter, D. M., *Engineering Problem Solving with MATLAB,* 2nd Edition, Prentice Hall, 1997.

3.   Nilsson, J. W., *Electric Circuits*, 3rd Edition, Addison-Wesley
          Publishing Company, 1990.

4.   Johnson, D. E., Johnson, J.R., and Hilburn, J.L., *Electric Circuit
          Analysis*, 3rd Edition, Prentice Hall, 1997.

EXERCISES

8.1   The triangular waveform, shown in Figure P8.1 can be expressed as

$$g(t) = \frac{8A}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cos\big((2n-1)w_0 t\big)$$

where

$$w_0 = \frac{1}{T_p}$$
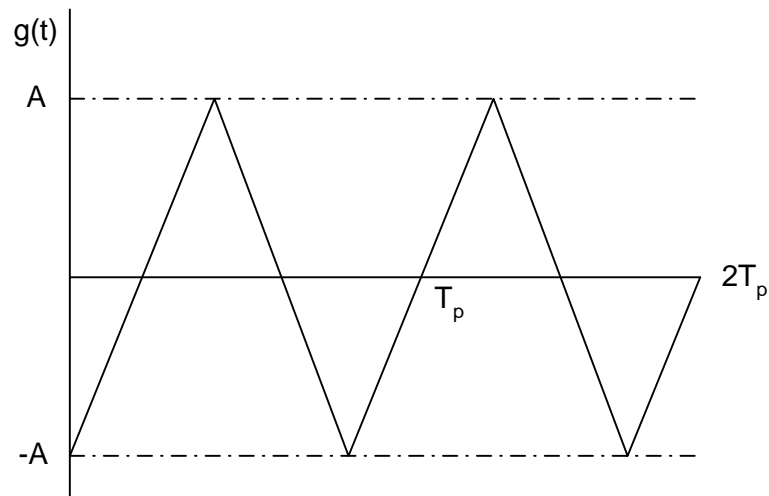
Figure P8.1 Triangular Waveform

If $A = 1$, $T = 8$ ms, and sampling interval is 0.1 ms.

(a)   Write MATLAB program to resynthesize $g(t)$ if 20

terms are used.

(b)   What is the root-mean-squared value of the function that is the difference between $g(t)$ and the approximation to $g(t)$ when 20 terms are used for the calculation of $g(t)$?

8.2   A periodic pulse train $g(t)$ is shown in Figure P8.2.
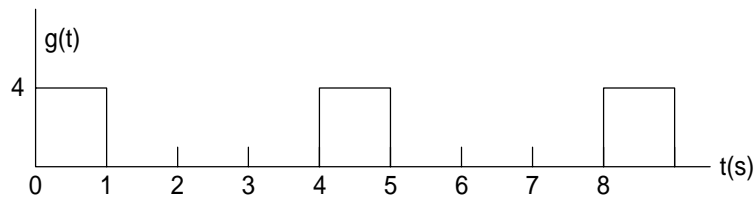
Figure P8.2 Periodic Pulse Train

If $g(t)$ can be expressed by Equation (8.3),

(a)   Derive expressions for determining the Fourier Series coefficients a_n and b_n.

(b)   Write a MATLAB program to obtain a_n and b_n for $n = 0$, 1, ..., 10 by using Equations (8.5) and (8.6).

(c)   Resynthesis $g(t)$ using 10 terms of the values a_n, b_n obtained from part (b).

8.3   For the half-wave rectifier waveform, shown in Figure P8.3, with a period of 0.01 s and a peak voltage of 17 volts.

(a)   Write a MATLAB program to obtain the exponential Fourier series coefficients c_n for $n = $ 0, 1, ..., 20.

(b)   Plot the amplitude spectrum.

(c)   Using the values obtained in (a), use MATLAB to regenerate the approximation to $g(t)$ when 20 terms of the exponential Fourier series are used.
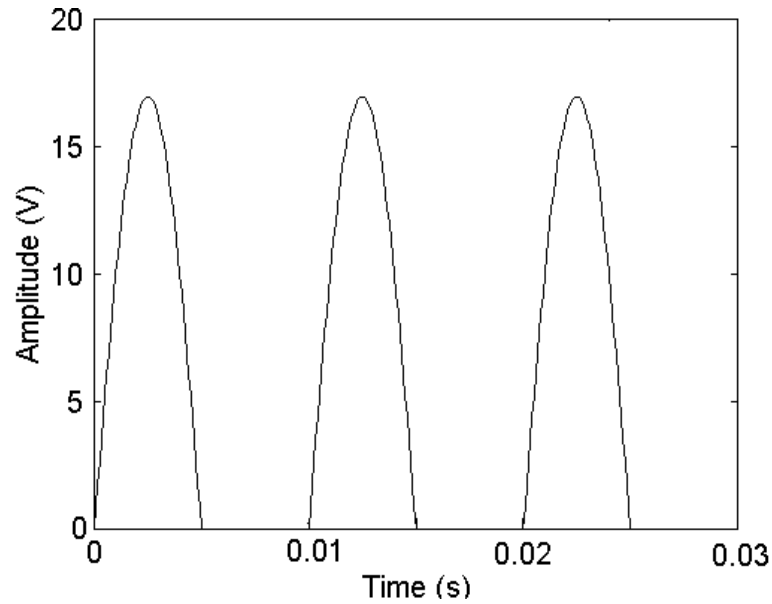
Figure P8.3 Half-Wave Rectifier Waveform

8.4   Figure P8.4(a) is a periodic triangular waveform.
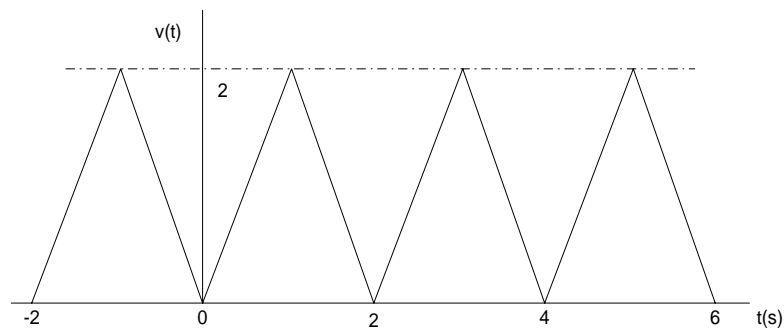
Figure P8.4(a) Periodic Triangular Waveform

(a)   Derive the Fourier series coefficients a_n and b_n .

(b)   With the signal $v(t)$ of the circuit shown in P8.4(b),
        derive the expression for the current $i(t)$.
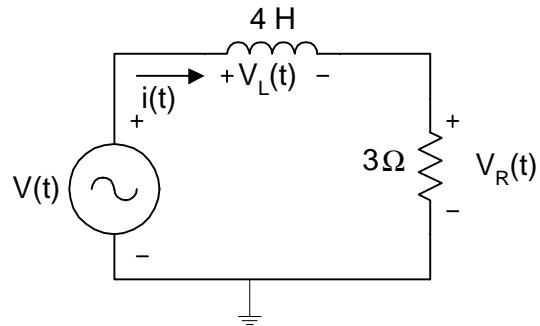
Figure P8.4(b) Simple RL Circuit

(c)   Plot the voltages $v_R(t)$, $v_L(t)$ and also the sum of $v_R(t)$ and $v_L(t)$.

(d)   Compare the voltages of $v_R(t) + v_L(t)$ to $V(t)$.

8.5   If the periodic waveform shown in Figure 8.5 is the input of the circuit shown in Figure P8.5.

(a)   Derive the mathematical expression for $v_C(t)$.

(b)   Use MATLAB to plot the signals $g(t)$ and $v_C(t)$.
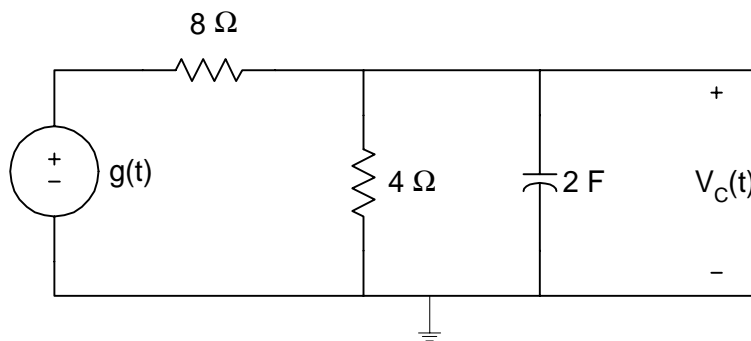
Figure P8.5 RC Circuit

8.6 The unit sample response of a filter is given as

$$h[n] = \begin{pmatrix} 0 & -1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) Find the discrete Fourier transform of $h[n]$; assume that the values of $h[n]$ not shown are zero.

(b) If the input to the filter is $x[n] = \sin\left[\dfrac{n}{8}\right]u[n]$, find the output of the filter.

8.7 $g(t) = \sin(200\pi t) + \sin(400\pi t)$

(a) Generate 512 points of $g(t)$. Using the FFT algorithm, generate and plot the frequency content of $g(t)$. Assume a sampling rate of 1200 Hz. Find the power spectrum.

(b) Verify that the frequencies in $g(t)$ are observable in the FFT plot.

8.8 Find the DFT of

$$g(t) = e^{-5t}u(t)$$

(a) Find the Fourier transform of $g(t)$.

(b) Find the DFT of $g(t)$ using the sampling interval of 0.01 s and time duration of 5 seconds.

(c) Compare the results obtained from parts (a) and (b).

Attia, John Okyere. "Diodes."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER NINE

DIODES

In this chapter, the characteristics of diodes are presented. Diode circuit analysis techniques will be discussed. Problems involving diode circuits are solved using MATLAB.

9.1   DIODE CHARACTERISTICS

Diode is a two-terminal device. The electronic symbol of a diode is shown in Figure 9.1(a). Ideally, the diode conducts current in one direction. The current versus voltage characteristics of an ideal diode are shown in Figure 9.1(b).

(a)

(b)
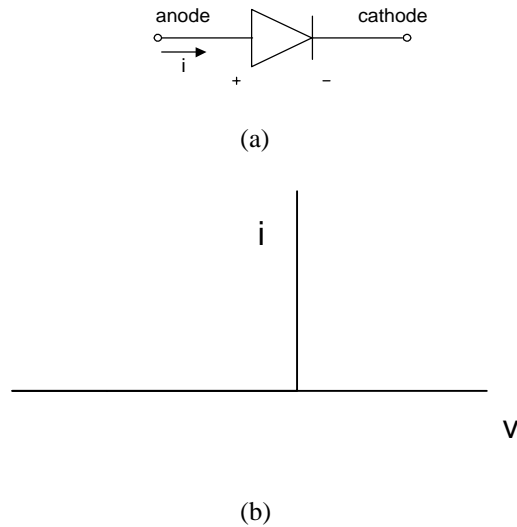
Figure 9.1   Ideal Diode   (a) Electronic Symbol
(b) I-V Characteristics

The I-V characteristic of a semiconductor junction diode is shown in Figure 9.2. The characteristic is divided into three regions: forward-biased, reversed-biased, and the breakdown.
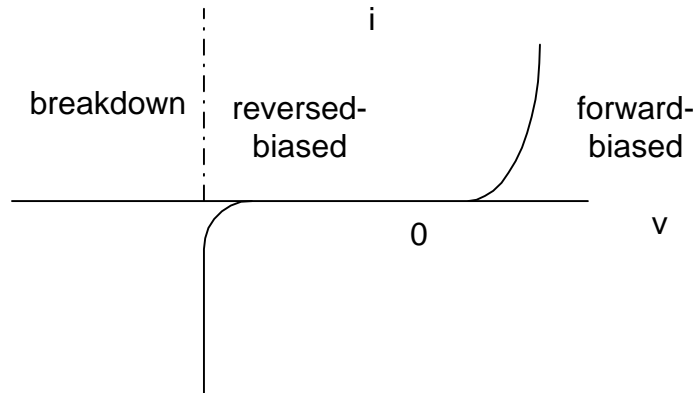
Figure 9.2 I-V Characteristics of a Semiconductor Junction Diode

In the forward-biased and reversed-biased regions, the current, i, and the voltage, v, of a semiconductor diode are related by the diode equation

$$i = I_S[e^{(v/nV_T)} - 1] \tag{9.1}$$

where

I_S is reverse saturation current or leakage current,

n is an empirical constant between 1 and 2,

V_T is thermal voltage, given by

$$V_T = \frac{kT}{q} \tag{9.2}$$

and

k is Boltzmann's constant $= 1.38x10^{-23}$ J / °K,

q is the electronic charge $= 1.6x10^{-19}$ Coulombs,

T is the absolute temperature in °K

At room temperature (25 °C), the thermal voltage is about 25.7 mV.

9.1.1   Forward-biased region

In the forward-biased region, the voltage across the diode is positive. If we assume that the voltage across the diode is greater than 0.1 V at room temperature, then Equation (9.1) simplifies to

$$i = I_S e^{(v/nV_T)} \tag{9.3}$$

For a particular operating point of the diode ($i = I_D$ and $v = V_D$), we have

$$i_D = I_S e^{(v_D/nV_T)} \tag{9.4}$$

To obtain the dynamic resistance of the diode at a specified operating point, we differentiate Equation (9.3) with respect to v, and we have

$$\frac{di}{dv} = \frac{I_s e^{(v/nV_T)}}{nV_T}$$

$$\left.\frac{di}{dv}\right|_{v=V_D} = \frac{I_s e^{(v_D/nV_T)}}{nV_T} = \frac{I_D}{nV_T}$$

and the dynamic resistance of the diode, r_d , is

$$r_d = \left.\frac{dv}{di}\right|_{v=V_D} = \frac{nV_T}{I_D} \tag{9.5}$$

From Equation (9.3), we have

$$\frac{i}{I_S} = e^{(v/nV_T)}$$

thus

$$\ln(i) = \frac{v}{nV_T} + \ln(I_S) \tag{9.6}$$

Equation (9.6) can be used to obtain the diode constants n and I_S , given the data that consists of the corresponding values of voltage and current. From

Equation (9.6), a curve of v versus $\ln(i)$ will have a slope given by $\dfrac{1}{nV_T}$ and y-intercept of $\ln(I_S)$. The following example illustrates how to find n and I_S from an experimental data. Since the example requires curve fitting, the MATLAB function **polyfit** will be covered before doing the example.

9.1.2   MATLAB function polyfit

The **polyfit** function is used to compute the best fit of a set of data points to a polynomial with a specified degree. The general form of the function is

$$coeff\_xy = polyfit(x,\ y,\ n) \qquad\qquad (9.7)$$

where

 x and y are the data points.

 n is the nth degree polynomial that will fit the vectors x and y.

 $coeff\_xy$ is a polynomial that fits the data in vector y to x in the least square sense. $coeff\_xy$ returns n+1 coefficients in descending powers of x.

Thus, if the polynomial fit to data in vectors x and y is given as

$$coeff\_xy(x) = c_1 x^n + c_2 x^{n-1} + ... + c_m$$

The degree of the polynomial is n and the number of coefficients $m = n+1$ and the coefficients $(c_1, c_2,\ ...,\ c_m)$ are returned by the MATLAB polyfit function.

Example 9.1

A forward-biased diode has the following corresponding voltage and current. Use MATLAB to determine the reverse saturation current, I_S and diode parameter n.

Forward Voltage, V	Forward Current, A
0.1	0.133e-12
0.2	1.79e-12
0.3	24.02e-12
0.4	0.321e-9
0.5	4.31e-9
0.6	57.69e-9
0.7	7.726e-7

Solution

```
diary ex9_1.dat
% Diode  parameters

vt = 25.67e-3;
v = [0.1 0.2 0.3 0.4 0.5 0.6 0.7];
i  =  [0.133e-12  1.79e-12  24.02e-12  321.66e-12  4.31e-9  57.69e-9
772.58e-9];

%
lni = log(i); % Natural log of current

% Coefficients of Best fit linear model is obtained
p_fit = polyfit(v,lni,1);

% linear equation is   y = m*x + b
b = p_fit(2);
m = p_fit(1);
ifit = m*v + b;

% Calculate Is and n
Is = exp(b)
n = 1/(m*vt)

% Plot v versus ln(i), and best fit linear model
plot(v,ifit,'w', v, lni,'ow')
axis([0,0.8,-35,-10])
```

```
xlabel('Voltage (V)')
ylabel('ln(i)')
title('Best fit linear model')
diary
```

The results obtained from MATLAB are

Is =   9.9525e-015

n =   1.5009

Figure 9.3 shows the best fit linear model used to determine the reverse satura-
tion current, I_S, and diode parameter, n.
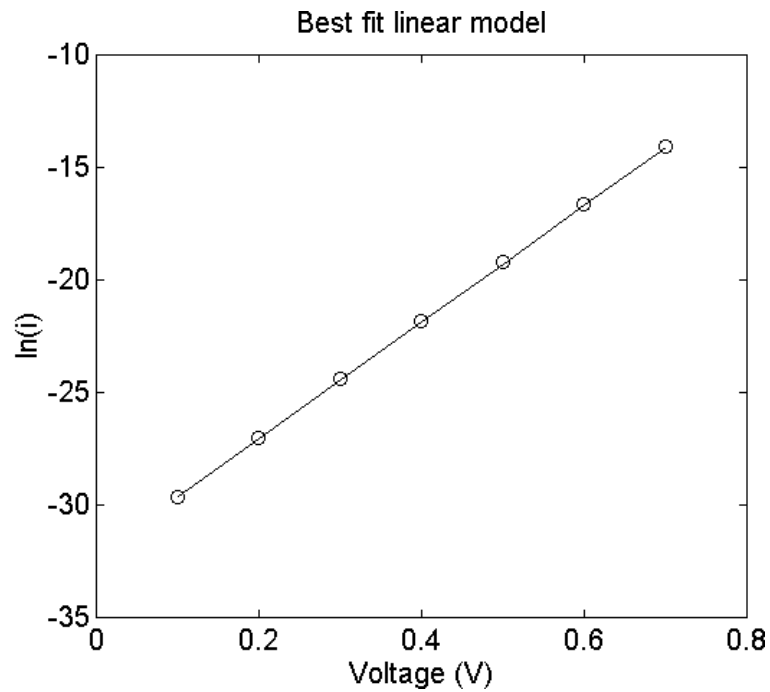
Figure 9.3 Best Fit Linear Model of Voltage versus Natural
Logarithm of Current

9.1.3 Temperature effects

From the diode equation (9.1), the thermal voltage and the reverse saturation current are temperature dependent. The thermal voltage is directly proportional to temperature. This is expressed in Equation (9.2). The reverse saturation current I_S increases approximately 7.2% /°C for both silicon and germanium diodes. The expression for the reverse saturation current as a function of temperature is

$$I_S(T_2) = I_S(T_1)e^{[k_S(T_2-T_1)]} \tag{9.8}$$

where

$$k_S = 0.072 \; /\,°C.$$
T_1 and T_2 are two different temperatures.

Since $e^{0.72}$ is approximately equal to 2, Equation (9.8) can be simplified and rewritten as

$$I_S(T_2) = I_S(T_1)2^{(T_2-T_1)/10} \tag{9.9}$$

Example 9.2

The saturation current of a diode at 25 °C is 10^{-12} A. Assuming that the emission constant of the diode is 1.9, (a) Plot the i-v characteristic of the diode at the following temperatures: $T_1 = 0$ °C, $T_2 = 100$ °C.

Solution

MATLAB Script

```
% Temperature effects on diode characteristics
%
k = 1.38e-23; q = 1.6e-19;
t1 = 273 + 0;
t2 = 273 + 100;

ls1 = 1.0e-12;
ks = 0.072;
ls2 = ls1*exp(ks*(t2-t1));

v = 0.45:0.01:0.7;
```

```
l1 = ls1*exp(q*v/(k*t1));
l2 = ls2*exp(q*v/(k*t2));

plot(v,l1,'wo',v,l2,'w+')
axis([0.45,0.75,0,10])
title('Diode I-V Curve at two Temperatures')
xlabel('Voltage (V)')
ylabel('Current (A)')
text(0.5,8,'o is for 100 degrees C')
text(0.5,7, '+ is for 0 degree C')
```

Figure 9.4 shows the temperature effects of the diode forward characteristics.
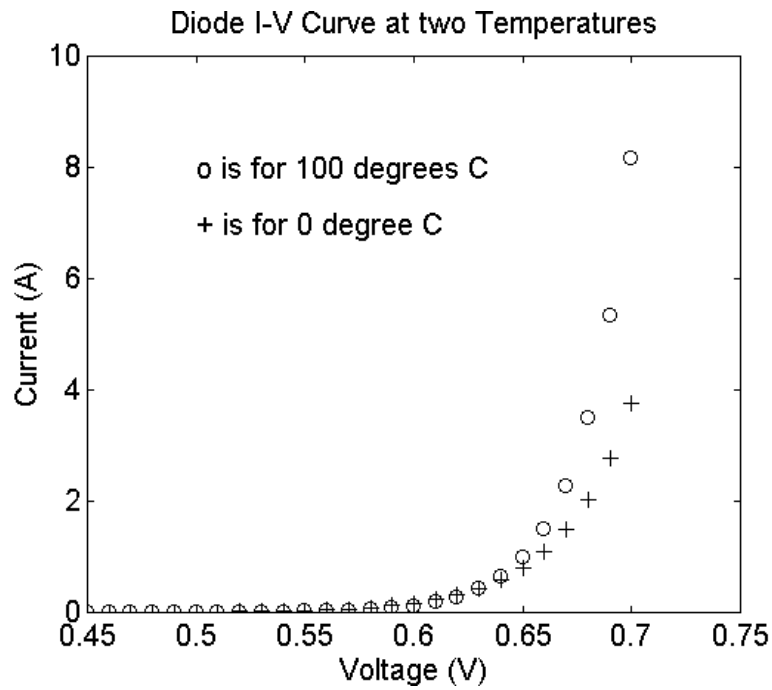
Figure 9.4 Temperature Effects on the Diode Forward
Characteristics

9.2   ANALYSIS OF DIODE CIRCUITS

Figure 9.5 shows a diode circuit consisting of a dc source V_{DC}, resistance R, and a diode. We want to determine the diode current I_D and the diode voltage V_D.
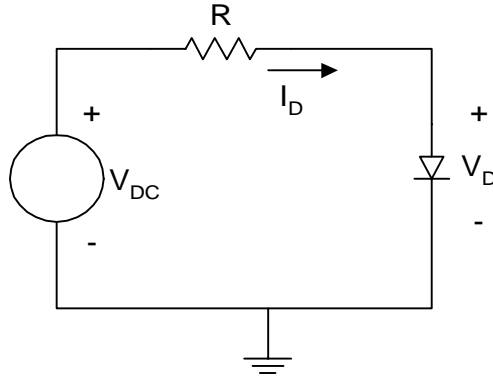
Figure 9.5 Basic Diode Circuit

Using Kirchoff Voltage Law, we can write the loadline equation

$$V_{DC} = RI_D + V_D \qquad\qquad (9.10)$$

The diode current and voltage will be related by the diode equation

$$i_D = I_S e^{(v_D/nV_T)} \qquad\qquad (9.11)$$

Equations (9.10) and (9.11) can be used to solve for the current I_D and voltage V_D.

There are several approaches for solving I_D and V_D. In one approach, Equations (9.10) and (9.11) are plotted and the intersection of the linear curve of Equation (9.10) and the nonlinear curve of Equation (9.11) will be the operating point of the diode. This is illustrated by the following example.

Example 9.3

For the circuit shown in Figure 9.5, if $R = 10 \text{ k}\Omega$, $V_{DC} = 10\text{V}$, and the reverse saturation current of the diode is 10^{-12} A and $n = 2.0$. (Assume a temperature of $25 \, ^{\circ}\text{C}$.)

(a)   Use MATLAB to plot the diode forward characteristic curve and the loadline.

(b)   From the plot estimate the operating point of the diode.

Solution

MATLAB Script

```
% Determination of operating point using
% graphical technique
%
% diode equation
k = 1.38e-23;q = 1.6e-19;
t1 = 273 + 25; vt = k*t1/q;
v1 = 0.25:0.05:1.1;
i1 = 1.0e-12*exp(v1/(2.0*vt));

% load line 10=(1.0e4)i2 + v2
vdc = 10;
r = 1.0e4;
v2 = 0:2:10;
i2 = (vdc - v2)/r;

% plot
plot(v1,i1,'w', v2,i2,'w')
axis([0,2, 0, 0.0015])
title('Graphical method - operating point')
xlabel('Voltage (V)')
ylabel('Current (A)')
text(0.4,1.05e-3,'Loadline')
text(1.08,0.3e-3,'Diode curve')
```

Figure 9.6 shows the intersection of the diode forward characteristics and the loadline.
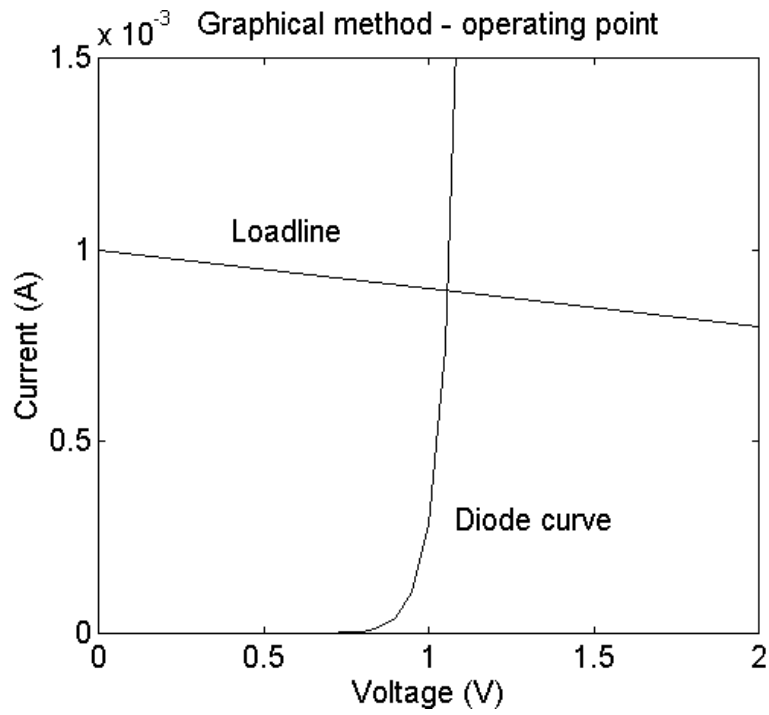
Figure 9.6 Loadline and Diode Forward Characteristics

From Figure 9.6, the operating point of the diode is the intersection of the loadline and the diode forward characteristic curve. The operating point is approximately

$$I_D = 0.9 \ \text{mA}$$
$$V_D = 0.7 \ \text{V}$$

The *second approach* for obtaining the diode current I_D and diode voltage V_D of Figure 9.5 is to use iteration. Assume that $\left(I_{D1}, V_{D1}\right)$ and $\left(I_{D2}, V_{D2}\right)$ are two corresponding points on the diode forward characteristics. Then, from Equation (9.3), we have

$$i_{D1} = I_S e^{(v_{D1}/nV_T)} \tag{9.12}$$

$$i_{D2} = I_S e^{(v_{D2}/nV_T)} \tag{9.13}$$

Dividing Equation (9.13) by (9.12), we have

$$\frac{I_{D2}}{I_{D1}} = e^{(V_{D2} - V_{D1}/nV_T)}$$

(9.14)

Simplifying Equation (9.14), we have

$$v_{D2} = v_{D1} + nV_T \ln\left(\frac{I_{D2}}{I_{D1}}\right)$$

(9.15)

Using iteration, Equation (9.15) and the loadline Equation (9.10) can be used to obtain the operating point of the diode.

To show how the iterative technique is used, we assume that $I_{D1} = 1\text{mA}$ and $V_{D1} = 0.7$ V. Using Equation (9.10), I_{D2} is calculated by

$$I_{D2} = \frac{V_{DC} - V_{D1}}{R}$$

(9.16)

Using Equation (9.15), V_{D2} is calculated by

$$V_{D2} = V_{D1} + nV_T \ln\left(\frac{I_{D2}}{I_{D1}}\right)$$

(9.17)

Using Equation (9.10), I_{D3} is calculated by

$$I_{D3} = \frac{V_{DC} - V_{D2}}{R}$$

(9.18)

Using Equation (9.15), V_{D3} is calculated by

$$V_{D3} = V_{D1} + nV_T \ln\left(\frac{I_{D3}}{I_{D1}}\right)$$

(9.19)

Similarly, I_{D4} and V_{D4} are calculated by

$$I_{D4} = \frac{V_{DC} - V_{D3}}{R} \qquad (9.20)$$

$$V_{D4} = V_{D1} + nV_T \ln(\frac{I_{D4}}{I_{D1}}) \qquad (9.21)$$

The iteration is stopped when V_{Dn} is approximately equal to V_{Dn-1} or I_{Dn} is approximately equal to I_{Dn-1} to the desired decimal points. The iteration technique is particularly facilitated by using computers. Example 9.4 illustrates the use of MATLAB for doing the iteration technique.

Example 9.4

Redo Example 9.3 using the iterative technique. The iteration can be stopped when the current and previous value of the diode voltage are different by 10^{-7} volts.

Solution

MATLAB Script

```
% Determination of diode operating point using
% iterative method
k = 1.38e-23;q = 1.6e-19;
t1 = 273 + 25; vt = k*t1/q;
vdc = 10;
r = 1.0e4;
n = 2;
id(1) = 1.0e-3; vd(1) = 0.7;
reltol = 1.0e-7;
i = 1;
vdiff = 1;
while vdiff > reltol
  id(i+1) = (vdc - vd(i))/r;
  vd(i+1) = vd(i) + n*vt*log(id(i+1)/id(i));
  vdiff = abs(vd(i+1) - vd(i));
  i = i+1;
end
k = 0:i-1;
% operating point of diode is (vdiode, idiode)
idiode = id(i)
```

```
vdiode = vd(i)
% Plot the voltages during iteration process
plot(k,vd,'wo')
axis([-1,5,0.6958,0.701])
title('Diode Voltage during Iteration')
xlabel('Iteration Number')
ylabel('Voltage, V')
```

From the MATLAB program, we have

```
idiode =
        9.3037e-004

vdiode =
        0.6963
```

Thus $I_D = 0.9304$ mA and $V_D = 0.6963$ V. shows the diode voltage during the iteration process.
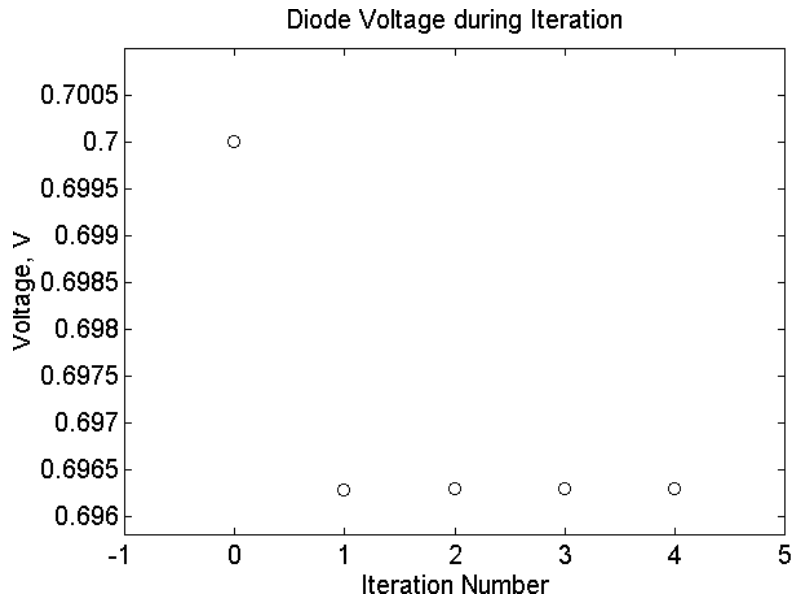
Figure 9.7 Diode Voltage during Iteration Process

9.3   HALF-WAVE RECTIFIER

A half-wave rectifier circuit is shown in Figure 9.8. It consists of an alternating current (ac) source, a diode and a resistor.
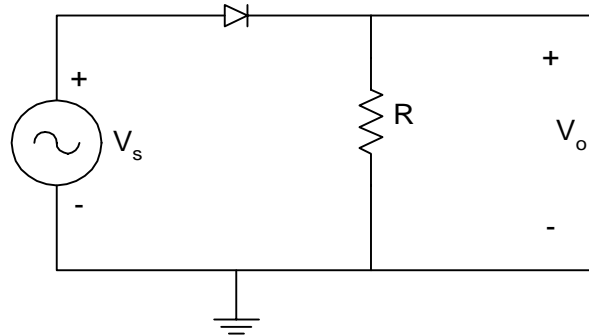
Figure 9.8 Half-wave Rectifier Circuit

Assuming that the diode is ideal, the diode conducts when source voltage is positive, making

$$v_0 = v_S \qquad\qquad \text{when}\ \ v_S\ \geq 0 \qquad\qquad (9.22)$$

When the source voltage is negative, the diode is cut-off, and the output voltage is

$$v_0 = 0 \qquad\qquad \text{when}\ v_S\ < 0 \qquad\qquad (9.23)$$

Figure 9.9 shows the input and output waveforms when the input signal is a sinusoidal signal.

The battery charging circuit, explored in the following example, consists of a source connected to a battery through a resistor and a diode.
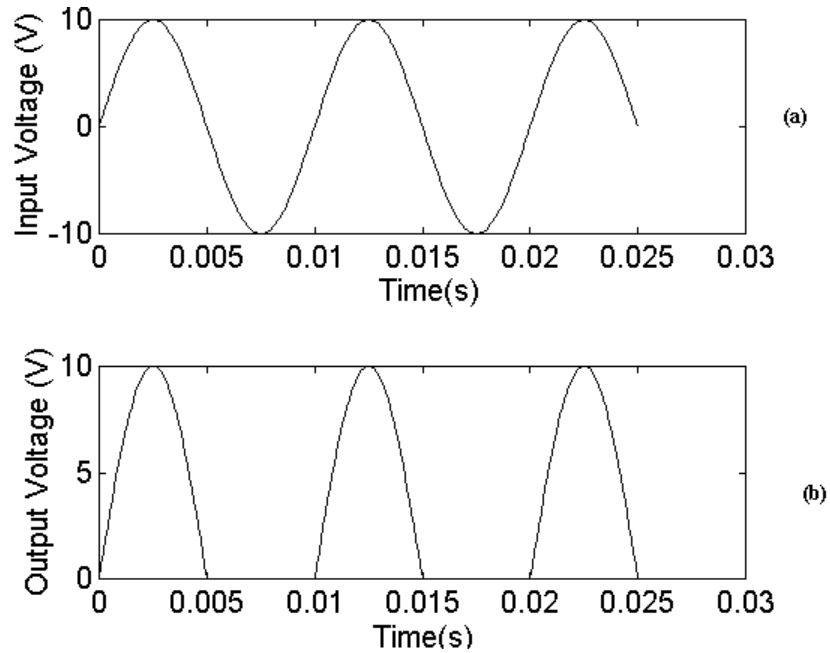
Figure 9.9 (a) Input and (b) Output Waveforms of a Half-wave Rectifier Circuit

Example 9.5

A battery charging circuit is shown in Figure 9.10. The battery voltage is $V_B = 11.8$ V. The source voltage is $v_S(t) = 18\sin(120\pi t)$ V and the resistance is $R = 100\ \Omega$. Use MATLAB (a) to sketch the input voltage, (b) to plot the current flowing through the diode, (c) to calculate the conduction angle of the diode, and (d) calculate the peak current. (Assume that the diode is ideal.)
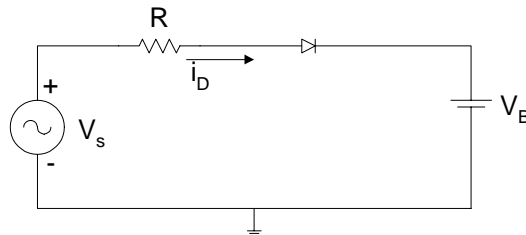
Figure 9.10 A Battery Charging Circuit

Solution:

When the input voltage v_S is greater than V_B, the diode conducts and the diode current, i_d, is given as

$$i_d = \frac{V_S - V_B}{R} \tag{9.24}$$

The diode starts conducting at an angle θ, given by $v_S \geq V_B$, i.e.,

$$18 \sin\theta_1 = 18\sin(120\pi t_1) = V_B = 11.8$$

The diode stops conducting current when $v_s \leq V_B$

$$18 \sin\theta_2 = 18\sin(120\pi t_2) = V_B$$

due to the symmetry

$$\theta_2 = \pi - \theta_1$$

MATLAB Program:

```
diary ex9_5.dat
% Baltery charging circuit
period = 1/60;
 period2 = period*2;
inc =period/100;
npts = period2/inc;
vb = 11.8;

t = [];
for i = 1:npts
 t(i) = (i-1)*inc;
 vs(i) = 18*sin(120*pi*t(i));
   if vs(i) > vb
    idiode(i) = (vs(i) -vb)/r;
   else
    idiode(i) = 0;
   end
end
```

```
subplot(211), plot(t,vs)
%title('Input Voltage')
xlabel('Time (s)')
ylabel('Voltage (V)')
text(0.027,10, 'Input Voltage')
subplot(212), plot(t,idiode)
%title('Diode Current')
xlabel('Time (s)')
ylabel('Current(A)')
text(0.027, 0.7e-3, 'Diode Current')

% conduction angle
theta1 = asin(vb/18); theta2 = pi - theta1;
acond = (theta2 -theta1)/(2*pi)
% peak current
pcurrent = (18*sin(pi/2) - vb)/r
% pcurrent = max(idiode)
diary
```

The conduction angle, acond, and the peak current, pcurrent, are

```
acond =
        0.2724

pcurrent =
         0.0620
```

Figure 9.11 shows the input voltage and diode current.

The output of the half-wave rectifier circuit of Figure 9.8 can be smoothed by connecting a capacitor across the load resistor. The smoothing circuit is shown in Figure 9.12.

When the amplitude of the source voltage V_S is greater than the output voltage, the diode conducts and the capacitor is charged. When the source voltage becomes less than the output voltage, the diode is cut-off and the capacitor discharges with the time constant CR. The output voltage and the diode current waveforms are shown in Figure 9.13.
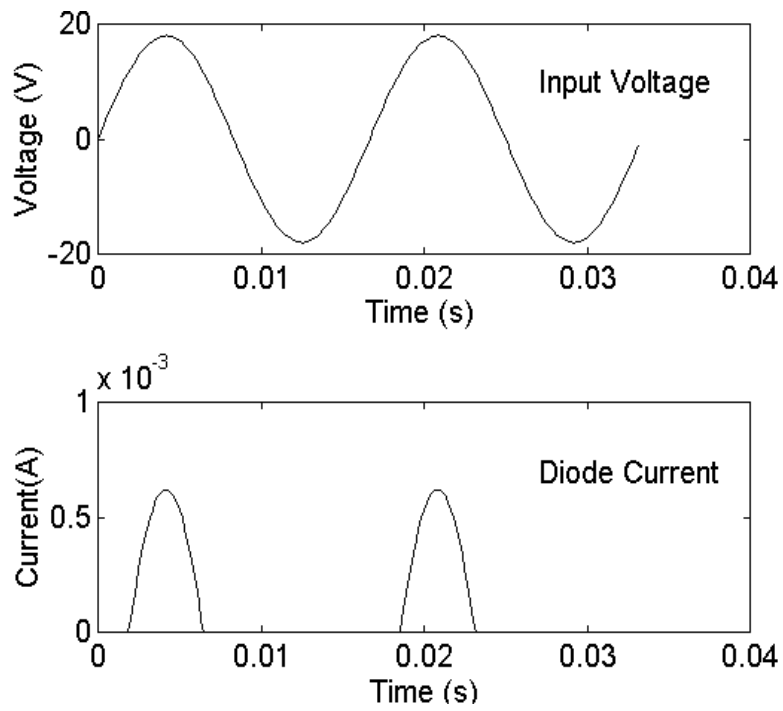
Figure 9.11 Input Voltage and Diode Current
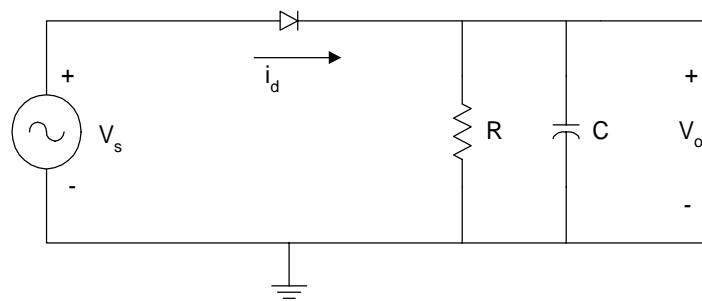
Figure 9.12 Capacitor Smoothing Circuit
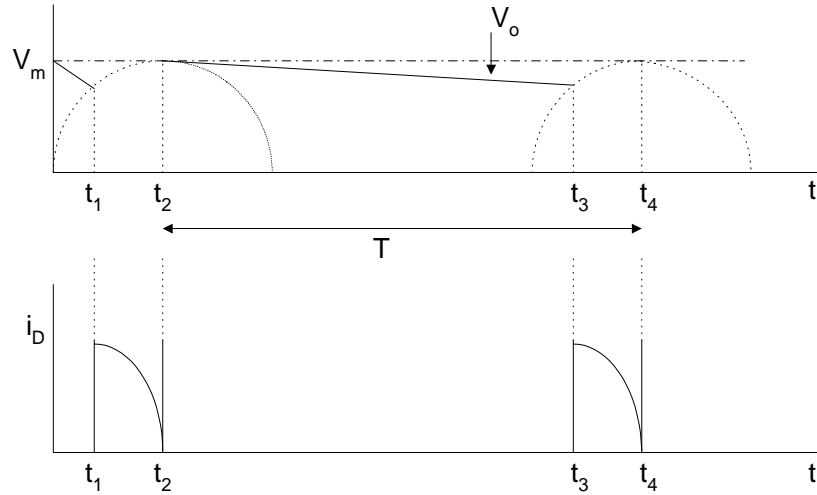
Figure 9.13   (a) Output Voltage and (b) Diode Current for Half-wave Rectifier with Smoothing Capacitor Filter

In Figure 9.12(a), the output voltage reaches the maximum voltage V_m, at time $t = t_2$ to $t = t_3$, the diode conduction ceases, and capacitor discharges through R. The output voltage between times t_2 and t_3 is given as

$$v_0(t) = V_m e^{-\left(\frac{t-t_2}{RC}\right)} \qquad\qquad t_2 < \text{t} < t_3 \qquad\qquad (9.25)$$

The peak to peak ripple voltage is defined as

$$
\begin{aligned}
V_r &= v_0(t_2) - v_0(t_3) = V_m - V_m e^{-\left(\frac{t_3-t_2}{RC}\right)} \\
&= V_m\left[1 - e^{-\left(\frac{t_3-t_2}{RC}\right)}\right]
\end{aligned}
\qquad\qquad (9.26)
$$

For large values C such that $CR >> \left(t_3 - t_2\right)$, we can use the well-known exponential series approximation

$$e^{-x} \cong 1 - x \qquad \text{for } |\text{x}| << 1$$

Thus, Equation (9.26) approximates to

$$V_r = \frac{V_m(t_3 - t_2)}{RC} \qquad (9.27)$$

The discharging time for the capacitor, $(t_3 - t_2)$, is approximately equal to the period of the input ac signal, provided the time constant is large. That is,

$$t_3 - t_2 \cong T = \frac{1}{f_0} \qquad (9.28)$$

where

f_0 is the frequency of the input ac source voltage.

Using Equation (9.28), Equation (9.27) becomes

$$V_{r(peak-to-peak)} = \frac{V_m}{f_0 CR} \qquad (9.29)$$

For rectifier circuits, because $RC \gg T$, the output voltage decays for a small fraction of its fully charged voltage, and the output voltage may be regarded as linear. Therefore, the output waveform of Figure 9.12 is approximately triangular. The *rms* value of the triangular wave is given by

$$V_{rms} = \frac{V_{peak-to-peak}}{2\sqrt{3}} = \frac{V_m}{2\sqrt{3}f_o CR} \qquad (9.30)$$

The approximately dc voltage of the output waveform is

$$V_{dc} = V_m - \frac{V_r}{2} = V_m - \frac{V_m}{2f_o CR} \qquad (9.31)$$

9.3.1   MATLAB function fzero

The MATLAB **fzero** is used to obtain the zero of a function of one variable. The general form of the **fzero** function is

$$fzero(' function', x1)$$

$$fzero(' function', x1, tol)$$

where

$fzero(' funct', x1)$ finds the zero of the function $funct(x)$ that is near the point $x1$.

$fzero(' funct', x1, tol)$ returns zero of the function $funct(x)$ accurate to within a relative error of tol.

The MATLAB function fzero is used in the following example.

Example 9.6

For a capacitor smoothing circuit of Figure 9.12, if $R = 10\text{K}\Omega$, $C = 100\mu\text{F}$, and $v_S(t) = 120\sqrt{2}\ \sin(120\pi t)$,

(a) use MATLAB to calculate the times t_2, t_3, of Figure 9.12;
(b) compare the capacitor discharge time with period of the input signal.

Solution

The maximum value of $v_S(t)$ is $120\sqrt{2}$, and it occurs at $120\pi t_2 = \dfrac{\pi}{2}$, thus

$$t_2 = \frac{1}{240} = 0.00417 \text{ s}$$

The capacitor discharge waveform is given by

$$v_C(t) = 120\sqrt{2}\ \exp\left(-\frac{(t - t_2)}{RC}\right) \qquad t_2 < t < t_3$$

At $t = t_3 \qquad v_C(t) = v_S(t)$,

Defining $v(t)$ as

$$v(t) = 120\sqrt{2}\,\sin\left(120\pi(t - t_p)\right) - 120\sqrt{2}\,\exp\left(-\frac{(t - t_2)}{RC}\right)$$

Then,

$$v(t_3) = 0 = 120\sqrt{2}\,\sin\left(120\pi(t_3 - t_p)\right) - 120\sqrt{2}\,\exp\left(-\frac{(t_3 - t_2)}{RC}\right)$$

Thus,

$$v(t_3) = 0 = \sin\left(120\pi(t_3 - t_p)\right) - \exp\left(-\frac{(t_3 - t_2)}{RC}\right) \qquad (9.32)$$

MATLAB is used to solve Equation (9.32)

MATLAB Script

```
diary ex9_6.dat
% Capacitance discharge time for smoothing capacitor
% filter circuit
vm = 120*sqrt(2);
f0 = 60; r =10e3; c = 100e-6;
t2 = 1/(4*f0);
tp = 1/f0;

% use MATLAB function fzero to find the zero of a
% function of one variable
rc = r*c;
t3 = fzero('sinexpf1',4.5*t2);
tdis_cap = t3- t2;
fprintf('The value of t2 is %9.5f  s\n', t2)
fprintf('The value of t3 is %9.5f s\n', t3)
fprintf('The capacitor discharge time is %9.5f  s\n', tdis_cap)
fprintf('The period of input signal is %9.5f  s\n', tp)
diary

%
function y = sinexpf1(t)
t2 = 1/240; tp = 1/60;
rc = 10e3*100e-6;
y = sin(120*pi*(t-tp)) - exp(-(t-t2)/rc);
end
```

The results are

The value of t2 is   0.00417 s
The value of t3 is   0.02036 s
The capacitor discharge time is   0.01619 s
The period of input signal is   0.01667 s

9.4   FULL-WAVE RECTIFICATION

A full-wave rectifier that uses a center-tapped transformer is shown in Figure 9.14.
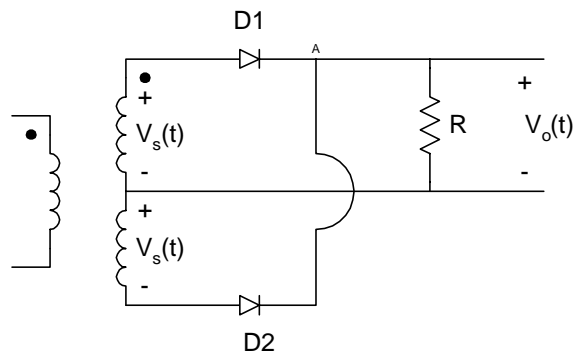
Figure 9.14 Full-wave Rectifier Circuit with Center-tapped
                    Transformer

When $v_S(t)$ is positive, the diode D1 conducts but diode D2 is off, and the output voltage $v_0(t)$ is given as

$$v_0(t) = v_S(t) - V_D \qquad (9.33)$$

where

V_D is a voltage drop across a diode.

When $v_S(t)$ is negative, diode D1 is cut-off but diode D2 conducts. The current flowing through the load R enters it through node A. The output voltage is

$$v(t) = \left| v_S(t) \right| - V_D \qquad (9.34)$$

A full-wave rectifier that does not require a center-tapped transformer is the bridge rectifier of Figure 9.15.
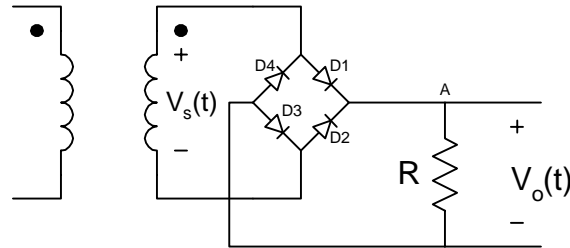
Figure 9.15 Bridge Rectifier

When $v_S(t)$ is negative, the diodes D2 and D4 conduct, but diodes D1 and D3 do not conduct. The current entering the load resistance R enters it through node A. The output voltage is

$$v(t) = \left| v_S(t) \right| - 2V_D \qquad (9.35)$$

Figure 9.16 shows the input and output waveforms of a full-wave rectifier circuit assuming ideal diodes.

The output voltage of a full-wave rectifier circuit can be smoothed by connecting a capacitor across the load. The resulting circuit is shown in Figure 9.17.

The output voltage and the current waveforms for the full-wave rectifier with RC filter are shown in Figure 9.18.
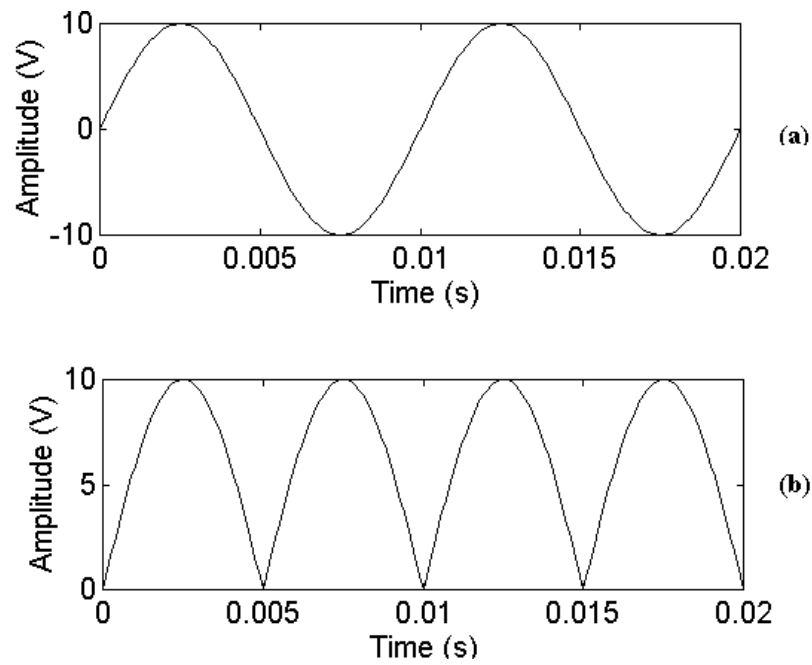
Figure 9.16   (a) Input and (b) Output Voltage Waveforms for Full-
wave Rectifier Circuit
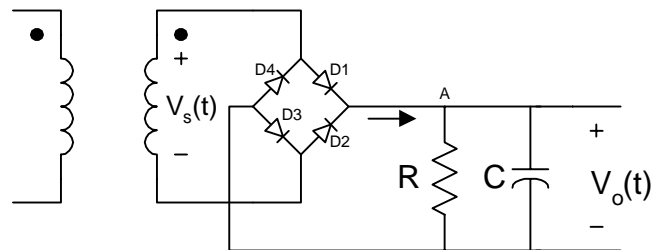
Figure 9.17 Full-wave Rectifier with Capacitor Smoothing Filter
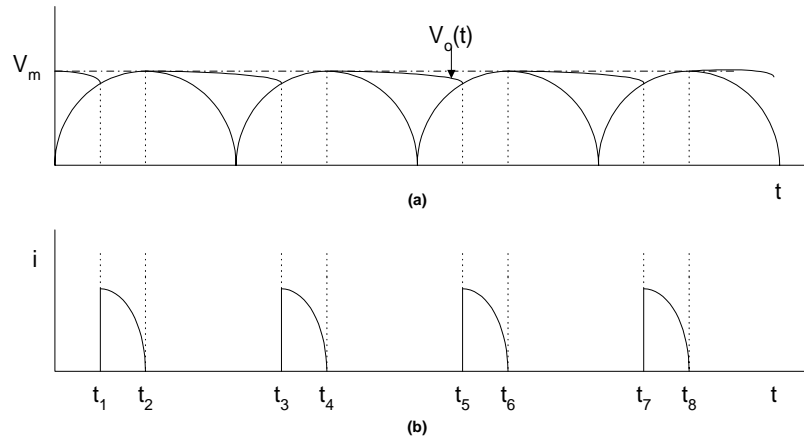
Figure 9.18 (a) Voltage and (b) Current Waveform of a Full-wave
Rectifier with RC Filter

From Figures 9.13 and 9.18, it can be seen that the frequency of the ripple voltage is twice that of the input voltage. The capacitor in Figure 9.17 has only half the time to discharge. Therefore, for a given time constant, *CR*, the ripple voltage will be reduced, and it is given by

$$V_{r(peak-to-peak)} = \frac{V_m}{2f_o CR} \qquad (9.36)$$

where

V_m is peak value of the input sinusoidal waveform

f_0 frequency of the input sinusoidal waveform

The rms value of the ripple voltage is

$$V_{rms} = \frac{V_m}{4\sqrt{3}f_o CR} \qquad (9.37)$$

and the output dc voltage is approximately

$$V_{dc} = V_m - \frac{V_r}{2} = V_m - \frac{V_m}{4f_o CR} \tag{9.38}$$

Example 9.7

For the full-wave rectifier with RC filter shown in Figure 9.17, if $v_S(t) = 20\sin(120\pi t)$ and $R = 10\text{K}\Omega$, $C = 100\mu\text{F}$, use MATLAB to find the

(a) peak-to-peak value of ripple voltage,
(b) dc output voltage,
(c) discharge time of the capacitor,
(d) period of the ripple voltage.

Solution

Peak-to-peak ripple voltage and dc output voltage can be calculated using Equations (9.36) and (9.37), respectively. The discharge time of the capacitor is the time $\left(t_3 - t_1\right)$ of Figure 9.19.
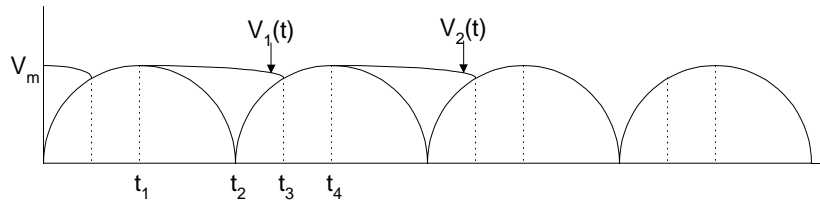
Figure 9.19 Diagram for Calculating Capacitor Discharge Time

$$v_1(t) = V_m \exp\left[-\frac{(t - t_1)}{RC}\right] \tag{9.39}$$

$$v_2(t) = \left|V_m \sin[2\pi(t - t_2)]\right| \tag{9.40}$$

$v_1(t)$ and $v_2(t)$ intersect at time t_3.

The period of input waveform, $v_S(t)$ is $T = \dfrac{1}{240}$ s

Thus,

$$t_1 = \frac{T}{4} = \frac{1}{240} \text{ s,} \quad \text{and} \quad t_2 = \frac{T}{2} = \frac{1}{120} \text{ s} \qquad (9.41)$$

MATLAB Script

```
diary ex9_7.dat
% Full-wave rectifier
%
period = 1/60;
t1 = period/4;
vripple = 20/(2*60*10e3*100e-6);
vdc = 20 - vripple/2;
t3 = fzero('sinexpf2',0.7*period);
tdis_cap = t3 - t1;
fprintf('Ripple value (peak-peak) is %9.5f V\n', vripple)
fprintf('DC output voltage is %9.5f V\n', vdc)
fprintf('Capacitor discharge time is %9.5f s\n', tdis_cap)
fprintf('Period of ripple voltage is %9.5f s\n', 0.5*period)
diary

%
%
function y = sinexpf2(t)
t1 = 1/240; t2 = 2*t1; rc = 10e3*100e-6;
y = 20(sin(120*pi*(t - t2))) - exp(-(t-t1)/rc);
end
```

The results are

```
Ripple value (peak-peak) is   0.16667 V
DC output voltage is  19.91667 V
Capacitor discharge time is   0.00800 s
Period of ripple voltage is   0.00833 s
```

9.5   ZENER DIODE VOLTAGE REGULATOR CIRCUITS

The zener diode is a pn junction diode with controlled reverse-biased break-down voltage. Figure 9.20 shows the electronic symbol and the current-voltage characteristics of the zener diode.

(a)

(b)
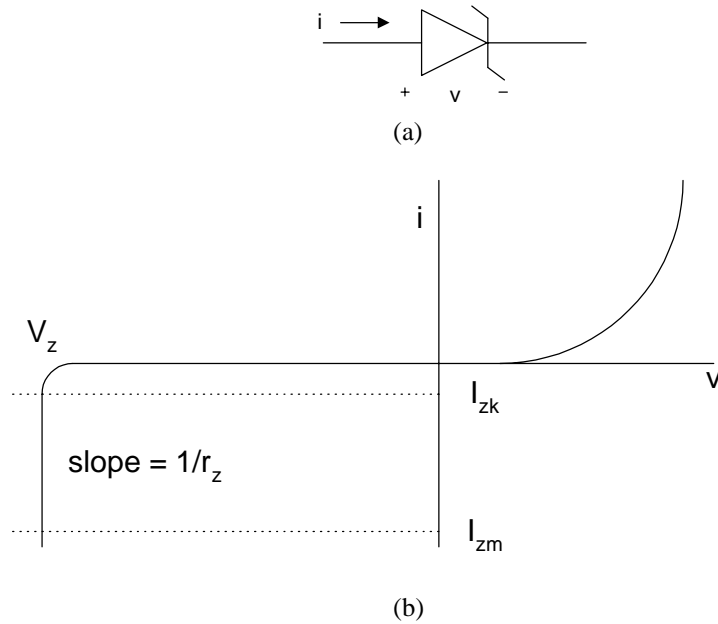
Figure 9.20   Zener Diode (a) Electronic Symbol (b) I-V
Characteristics

I_{ZK} is the minimum current needed for the zener to breakdown. I_{ZM} is the maximum current that can flow through the zener without being destroyed. It is obtained by

$$I_{ZM} = \frac{P_Z}{V_Z}$$ (9.42)

where P_Z is the zener power dissipation.

The incremental resistance of the zener diode at the operating point is specified by

$$r_z = \frac{\Delta V_z}{\Delta I_z} \qquad (9.43)$$

One of the applications of a zener diode is its use in the design of voltage reference circuits. A zener diode shunt voltage regulator circuit is shown in Figure 9.21
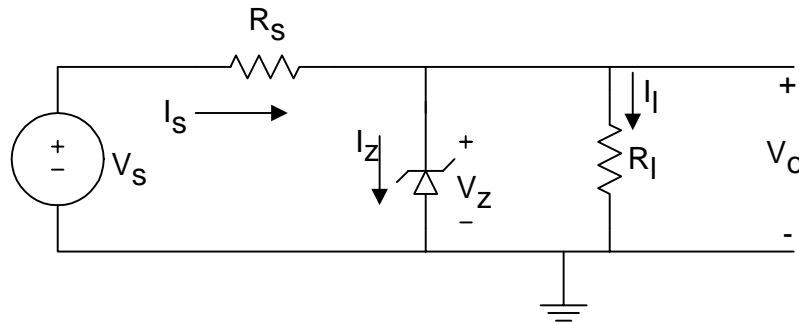
Figure 9.21 Zener Diode Shunt Voltage Regulator Circuit

The circuit is used to provide an output voltage, V_0, which is nearly constant. When the source voltage is greater than the zener breakdown voltage, the zener will break down ` and the output voltage will be equal to the zener breakdown voltage. Thus,

$$V_0 = V_Z \qquad (9.44)$$

From Kirchoff current law, we have

$$I_S = I_Z + I_L \qquad (9.45)$$

and from Ohm's Law, we have

$$I_S = \frac{V_S - V_Z}{R_S} \qquad (9.46)$$

and

$$I_L = \frac{V_O}{R_L} \qquad (9.47)$$

Assuming the load resistance R_L is held constant and V_S (which was originally greater than V_Z) is increased, the source current I_S will increase; and since I_L is constant, the current flowing through the zener will increase. Conversely, if R is constant and V_S decreases, the current flowing through the zener will decrease since the breakdown voltage is nearly constant; the output voltage will remain almost constant with changes in the source voltage V_S.

Now assuming the source voltage is held constant and the load resistance is decreased, then the current I_L will increase and I_Z will decrease. Conversely, if V_S is held constant and the load resistance increases, the current through the load resistance I_L will decrease and the zener current I_Z will increase.

In the design of zener voltage regulator circuits, it is important that the zener diode remains in the breakdown region irrespective of the changes in the load or the source voltage. There are two extreme input/output conditions that will be considered:

(1)   The diode current I_Z is minimum when the load current I_L is maximum and the source voltage V_S is minimum.

(2)   The diode current I_Z is maximum when the load current I_L is minimum and the source voltage V_S is maximum.

From condition (1) and Equation (9.46), we have

$$R_S = \frac{V_{S,\min} - V_Z}{I_{L,\max} + I_{Z,\min}} \tag{9.48}$$

Similarly, from condition (2), we get

$$R_S = \frac{V_{S,\max} - V_Z}{I_{L,\min} + I_{Z,\max}} \tag{9.49}$$

Equating Equations (9.48) and (9.49), we get

$$(V_{S,\min} - V_Z)(I_{L,\min} + I_{Z,\max}) = (V_{S,\max} - V_Z)(I_{L,\max} + I_{Z,\min}) \tag{9.50}$$

We use the rule of thumb that the maximum zener current is about ten times the minimum value, that is

$$I_{Z,min} = 0.1 I_{Z,max} \qquad (9.51)$$

Substituting Equation (9.49) into Equation (9.51), and solving for $I_{Z,max}$, we obtain

$$I_{Z,max} = \frac{I_{L,min}(V_Z - V_{S,min}) + I_{L,max}(V_{S,max} - V_Z)}{V_{S,min} - 0.9 V_Z - 0.1 V_{S,max}} \qquad (9.52)$$

Knowing $I_{Z,max}$, we can use Equation (9.49) to calculate R_S. The following example uses MATLAB to solve a zener voltage regulator problem.

Example 9.8

A zener diode voltage regulator circuit of Figure 9.21 has the following data:

$$30 \le V_S \le 35\text{V}; \quad R_L = 10\text{K}, \quad R_S = 2\text{K}$$

$$V_Z = -20 + 0.05 I \qquad \text{for } -100 \text{ mA} \le I < 0 \qquad (9.53)$$

Use MATLAB to

(a) plot the zener breakdown characteristics, (b) plot the loadline for $V_S = 30$V and $V_S = 35$ V, (c) determine the output voltage when $V_S = 30$V and $V_S = 35$V.

Solution

Using Thevenin Theorem, Figure 9.21 can be simplified into the form shown in Figure 9.22.
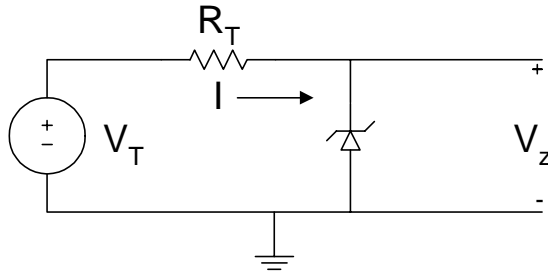
Figure 9.22 Equivalent Circuit of Voltage Regulator Circuit

$$V_T = \frac{V_S R_L}{R_L + R_S} \tag{9.54}$$

and

$$R_T = R_L \| R_S \tag{9.55}$$

Since $R_L = 10\text{K}$, $R_S = 2\text{K}$, $R_T = (10)(2\text{K}) / 12 \text{ K} = 1.67 \text{ K}\Omega$

when $V_S = 30\text{V}$, $V_T = (30)(10) / 12 = 25 \text{ V}$

when $V_S = 35\text{V}$, $V_T = (35)(10) / 12 = 29.17 \text{ V}$

The loadline equation is

$$V_T = R_T I + V_Z \tag{9.56}$$

Equations (9.53) and (9.56) are two linear equations solving for I, so we get

$$V_Z = V_T - R_T I = -20 + 0.05I$$

$$\Rightarrow \quad I = \frac{(V_T + 20)}{R_T + 0.05} \tag{9.57}$$

From Equations (9.56) and (9.57), the output voltage (which is also zener voltage) is

$$V_Z = V_T - R_T I = V_T - \frac{R_T(V_T + 20)}{R_T + 0.05} \qquad (9.58)$$

MATLAB program

```
diary ex9_8.dat
% Zener diode voltage regulator
vs1 = -30; vs2 = -35; rl =10e3; rs = 2e3;
i = -50e-3: 5e-3 :0;
vz = -20 + 0.05*i;
m = length(i);
i(m+1) = 0; vz(m+1) = -10;
i(m+2) = 0; vz(m+2) = 0;
% loadlines
vt1 = vs1*rl/(rl+rs);
vt2 = vs2*rl/(rl+rs);
rt = rl*rs/(rl+rs);
l1 = vt1/20;
l2 = vt2/20;
v1 = vt1:abs(l1):0;
i1 = (vt1 - v1)/rt;
v2 = vt2:abs(l2):0;
i2 = (vt2 - v2)/rt;
% plots of Zener characteristics, loadlines
plot(vz,i,'w',v1,i1,'w',v2,i2,'w')
axis([-30,0,-0.03,0.005])
title('Zener Voltage Regulator Circuit')
xlabel('Voltage (V)')
ylabel('Current (A)')
text(-19.5,-0.025,'Zener Diode Curve')
text(-18.6,-0.016, 'Loadline (35 V Source)')
text(-14.7,-0.005,'Loadline (30 V Source)')
% output voltage when vs = -30v
ip1 = (vt1 + 20)/(rt + 0.05)
vp1 = vt1 - rt*(vt1+20)/(rt + 0.05)
% output voltage when vs = -35v
ip2 = (vt2 + 20)/(rt + 0.05)
vp2 = vt2 - rt*(vt2+20)/(rt + 0.05)
diary
```

The results obtained are

ip1 =

       -0.0030

vp1 =

       -20.0001

ip2 =

       -0.0055

vp2 =

       -20.0003

When the source voltage is 30 V, the output voltage is 20.0001 V.
In addition, when the source voltage is 35 V, the output voltage is 20.0003 V.

The zener breakdown characteristics and the loadlines are shown in Figure 9.23.
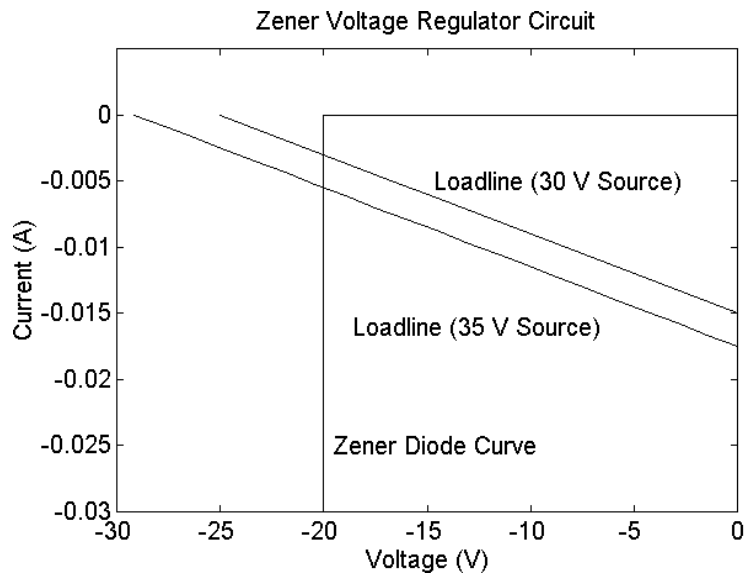
Figure 9.23 Zener Characteristics and Loadlines

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EXERCISES

9.1   Use the iteration technique to find the voltage V_D and the I_D of Figure P9.1. Assume that $T = 25$ °C, $n = 1.5$, $I_S = 10^{-16}$ A. Stop current the iteration when $\left| V_n - V_{n-1} \right| < 10^{-9}$ V.
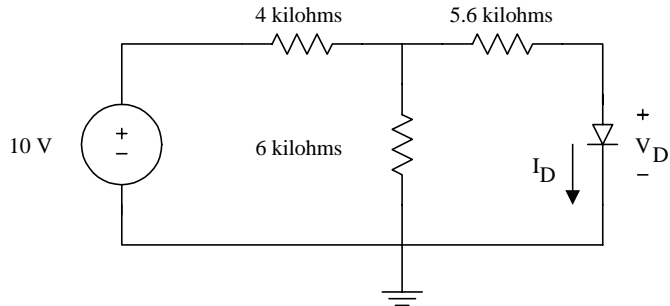
Figure P9.1 A Diode Circuit

9.2 A zener diode has the following I-V characteristics

Reverse Voltage (V)	Reverse Current (A)
-2	-1.0e-10
-4	-1.0e-10
-6	-1.0e-8
-8	-1.0e-5
-8.5	-2.0e-5
-8.7	-15.0e-3
-8.9	-43.5 e-3

(a) Plot the reverse characteristics of the diode. (b) What is the breakdown voltage of the diode? (c) Determine the dynamic resistance of the diode in its breakdown region.

9.3 A forward-biased diode has the following corresponding voltage and current.
(a) Plot the static I-V characteristics.

(b) Determine the diode parameters I_S and n.

(c) Calculate the dynamic resistance of the diode at $V_S = 0.5$ V.

Forward Voltage, V	Forward Current, A
0.2	7.54e-7
0.3	6.55e-6
0.4	5.69e-5
0.5	4.94e-4
0.6	4.29e-3
0.7	3.73e-2

.

9.4   For Figure P9.4,
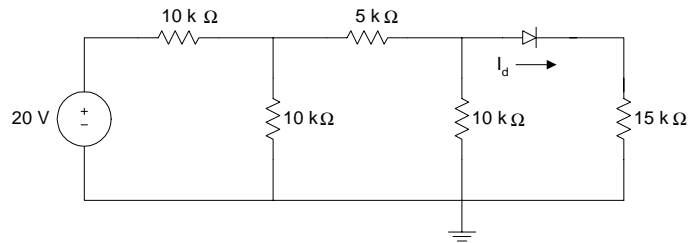
Figure P9.4 Diode Circuit

(a) Use iteration to find the current through the diode. The iteration can be stopped when $\left| I_{dn} - I_{dn-1} \right| < 10^{-12}$ A.

(b) How many iterations were performed before the required result was obtained?   Assume a temperature of 25 °C, emission coefficient, n , of 1.5, and the reverse saturation current, I_S , is 10^{-16} A.

9.5 For a full-wave rectifier circuit with smoothing capacitor shown in   Figure 9.17, if $v_S(t) = 100\sin(120\pi t)$ V, $R = 50\text{K}\Omega$, $C = 250\mu\text{F}$,   using MATLAB

    (a) Plot the input and output voltages when the capacitor is disconnected from the load resistance R.

    (b) When the capacitor is connected across load resistance R, determine the conduction time of the diode.

    (c) What is the diode conduction time?

9.6 For the voltage regulator circuit shown in Figure 9.21, assume that $50 <$ $V_S < 60$ V, $R_L = 50\text{K}$, $R_S = 5\text{K}$, $V_S = $ -40 + 0.01 I. Use MATLAB to

    (a) Plot the zener diode breakdown characteristics.

    (b) Plot the loadline for $V_S = 50$ V and $V_S = 60$V.

    (c) Determine the output voltage and the current flowing through the source resistance R_S when $V_S = 50$V and $V_S = 60$V.

9.7 For the zener voltage regulator shown in Figure 9.21, If $V_S = 35$V, R_S = 1KΩ, $V_Z = -25 + 0.02I$ and $5\text{K} < R_L < 50\text{K}$, use MATLAB to

    (a) Plot the zener breakdown characteristics

    (b) Plot the loadline when $R_L = 5\text{K}$ and $R_L = 50\text{K}$.

    (c) Determine the output voltage when $R_L = 5\text{K}\Omega$ and $R_L = 50\text{K}\Omega$.

    (d) What is the power dissipation of the diode when $R_L = 50\text{K}\Omega$?

Attia, John Okyere. "Semiconductor Physics."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER TEN

SEMICONDUCTOR PHYSICS

In this chapter, a brief description of the basic concepts governing the flow of current in a pn junction are discussed. Both intrinsic and extrinsic semiconductors are discussed. The characteristics of depletion and diffusion capacitance are explored through the use of example problems solved with MATLAB. The effect of doping concentration on the breakdown voltage of pn junctions is examined.

10.1 INTRINSIC SEMICONDUCTORS

10.1.1 Energy bands

According to the planetary model of an isolated atom, the nucleus that contains protons and neutrons constitutes most of the mass of the atom. Electrons surround the nucleus in specific orbits. The electrons are negatively charged and the nucleus is positively charged. If an electron absorbs energy (in the form of a photon), it moves to orbits further from the nucleus. An electron transition from a higher energy orbit to a lower energy orbit emits a photon for a direct band gap semiconductor.

The energy levels of the outer electrons form energy bands. In insulators, the lower energy band (valence band) is completely filled and the next energy band (conduction band) is completely empty. The valence and conduction bands are separated by a forbidden energy gap.
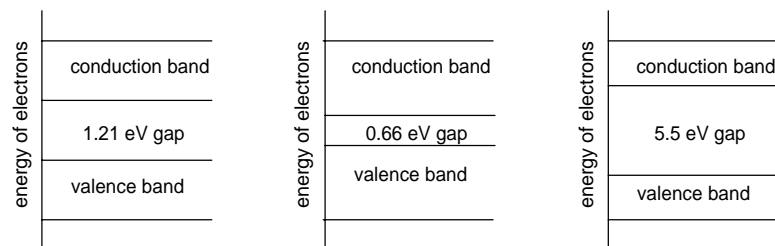

Figure 10.1 Energy Level Diagram of (a) Silicon, (b) Germanium, and (c) Insulator (Carbon)

In conductors, the valence band partially overlaps the conduction band with no forbidden energy gap between the valence and conduction bands. In semiconductors the forbidden gap is less than 1.5eV. Some semiconductor materials are silicon (Si), germanium (Ge), and gallium arsenide (GaAs). Figure 10.1 shows the energy level diagram of silicon, germanium and insulator (carbon).

10.1.2 Mobile carriers

Silicon is the most commonly used semiconductor material in the integrated circuit industry. Silicon has four valence electrons and its atoms are bound together by covalent bonds. At absolute zero temperature the valence band is completely filled with electrons and no current flow can take place. As the temperature of a silicon crystal is raised, there is increased probability of breaking covalent bonds and freeing electrons. The vacancies left by the freed electrons are holes. The process of creating free electron-hole pairs is called ionization. The free electrons move in the conduction band. The average number of carriers (mobile electrons or holes) that exist in an intrinsic semiconductor material may be found from the mass-action law:

$$n_i = AT^{1.5}e^{[-E_g/(kT)]} \tag{10.1}$$

where

T is the absolute temperature in °K

k is Boltzmann's constant (k = 1.38 x 10^{-23} J/K or 8.62x10^{-5} eV/K)

E_g is the width of the forbidden gap in eV. E_g is 1.21 and 1.1eV for Si at 0°K and 300°K, respectively. It is given as

$$E_g = E_c - E_v \tag{10.2}$$

A is a constant dependent on a given material and it is given as

$$A = \frac{2}{h^3}(2\pi m_0 k)^{3/2}(\frac{m_n^*}{m_0}\frac{m_p^*}{m_o})^{3/4} \tag{10.3}$$

where

h is Planck's constant ($h = 6.62$ x 10^{-34} J s   or 4.14 x 10^{-15} eV s).

m_o is the rest mass of an electron

m_n* is the effective mass of an electron in a material

m_p* is effective mass of a hole in a material

The mobile carrier concentrations are dependent on the width of the energy gap, E_g, measured with respect to the thermal energy kT.   For small values of T ($kT << E_g$), n_i is small implying, there are less mobile carriers. For silicon, the equilibrium intrinsic concentration at room temperature is

$$n_i = 1.52 \text{ x } 10^{10} \text{ electrons/cm}^3 \tag{10.4}$$

Of the two carriers that we find in semiconductors, the electrons have a higher mobility than holes.   For example, intrinsic silicon at 300°K has electron mobility of 1350 cm^2 / volt-sec and hole mobility of 480 cm^2 / volt-sec.   The conductivity of an intrinsic semiconductor is given by

$$\sigma_i = q(n_i \mu_n + p_i \mu_p) \tag{10.5}$$

where

q	is the electronic charge (1.6 x 10^{-19} C)
n_i	is the electron concentration
p_i	is the hole concentration. $p_i = n_i$ for the intrinsic semiconductor
μ_n	electron mobility in the semiconductor material
μ_p	hole mobility in the semiconductor material.

Since electron mobility is about three times that of hole mobility in silicon, the electron current is considerably more than the hole current.   The following example illustrates the dependence of electron concentration on temperature.

Example 10.1

Given that at $T = 300$ °K, the electron concentration in silicon is 1.52 x 10^{10} electrons /cm^3 and $E_g = 1.1$ eV at 300 °K.

(a) Find the constant A of Equation (10.1).
(b) Use MATLAB to plot the electron concentration versus temperature.

Solution

From Equation (10.1), we have

$$1.52 x 10^{10} = A(300)^{1.5} e^{[-1.1/300*8.62*10^{-5})]}$$

We use MATLAB to solve for A.   The width of energy gap with temperature is given as [1].

$$E_g(T) = 1.17 - 4.37 x 10^{-4} \left(\frac{T^2}{T + 636} \right) \qquad (10.6)$$

Using Equations (10.1) and (10.6), we can calculate the electron concentration at various temperatures.

MATLAB Script

```
%
% Calculation of the constant A
diary ex10_1.dat
k = 8.62e-5;
na = 1.52e10;   ta = 300;
ega = 1.1;
ka  = -ega/(k*ta);
t32a = ta.^1.5;
A = na/(t32a*exp(ka));
fprintf('constant A is %10.5e \n', A)

% Electron Concentration vs. temperature

for i = 1:10
  t(i) = 273 + 10*(i-1);
```

```
    eg(i) = 1.17 - 4.37e-4*(t(i)*t(i))/(t(i) + 636);
    t32(i) = t(i).^1.5;
    ni(i) = A*t32(i)*exp(-eg(i)/(k*t(i)));
end
semilogy(t,ni)
title('Electron Concentration vs. Temperature')
xlabel('Temperature, K')
ylabel('Electron Concentration, cm-3')
```

Result for part (a)

constant A is 8.70225e+024

Figure 10.2 shows the plot of the electron concentration versus temperature.
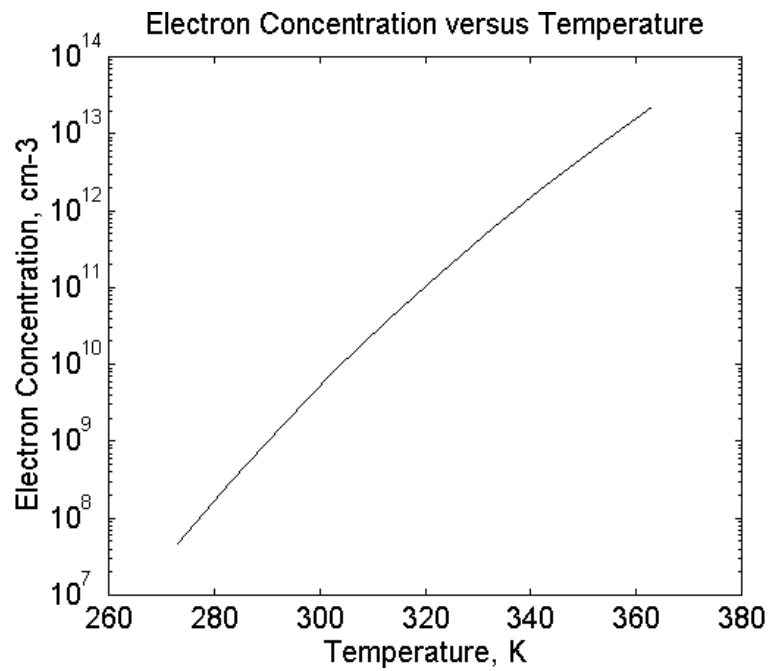

Figure 10.2 Electron Concentration versus Temperature

10.2   EXTRINSIC SEMICONDUCTOR

10.2.1   Electron and hole concentrations

Extrinsic semiconductors are formed by adding specific amounts of impurity atoms to the silicon crystal. An n-type semiconductor is formed by doping the silicon crystal with elements of group V of the periodic table (antimony, arsenic, and phosphorus). The impurity atom is called a donor. The majority carriers are electrons and the minority carriers are holes. A p-type semiconductor is formed by doping the silicon crystal with elements of group III of the periodic table (aluminum, boron, gallium, and indium). The impurity atoms are called acceptor atoms. The majority carriers are holes and minority carriers are electrons.

In a semiconductor material (intrinsic or extrinsic), the law of mass action states that

$$pn = \text{constant} \tag{10.7}$$

where

p is the hole concentration

n is the electron concentration.

For intrinsic semiconductors,

$$p = n = n_i \tag{10.8}$$

and Equation (10.5) becomes

$$pn = n_i^2 \tag{10.9}$$

and n_i is given by Equation (10.1).

The law of mass action enables us to calculate the majority and minority carrier density in an extrinsic semiconductor material. The charge neutrality condition of a semiconductor implies that

$$p + N_D = n + N_A \tag{10.10}$$

where

N_D is the donor concentration

N_A is the acceptor concentration

p is the hole concentration

n is the electron concentration.

In an n-type semiconductor, the donor concentration is greater than the intrinsic electron concentration, i.e., N_D is typically 10^{17} cm^{-3} and $n_i = 1.5$ x 10^{10} cm^{-3} in Si at room temperature. Thus, the majority and minority concentrations are given by

$$n_n \cong N_D \tag{10.11}$$

$$p \cong \frac{n_i^2}{N_D} \tag{10.12}$$

In a p-type semiconductor, the acceptor concentration N_A is greater than the intrinsic hole concentration $p_i = n_i$. Thus, the majority and minority concentrations are given by

$$p_p \cong N_A \tag{10.13}$$

$$n \cong \frac{n_i^2}{N_A} \tag{10.14}$$

The following example gives the minority carrier as a function of doping concentration.

Example 10.2

For an n-type semiconductor at 300°K, if the doping concentration is varied from 10^{13} to 10^{18} atoms/cm^3, determine the minority carriers in the doped semiconductors.

Solution

From Equation (10.11) and (10.12),

Electron concentration = N_D and

Hole concentration = $\dfrac{n_i^2}{N_D}$

where

$n_i = 1.5\,2 \times 10^{10}$ electrons/cm^3

The MATLAB program is as follows:

```
% hole concentration in a n-type semiconductor
nd = logspace(13,18);
n = nd;
ni = 1.52e10;
ni_sq = ni*ni;
p = ni_sq./nd;
semilogx(nd,p,'b')
title('Hole concentration')
xlabel('Doping concentration, cm-3')
ylabel('Hole concentration, cm-3')
```
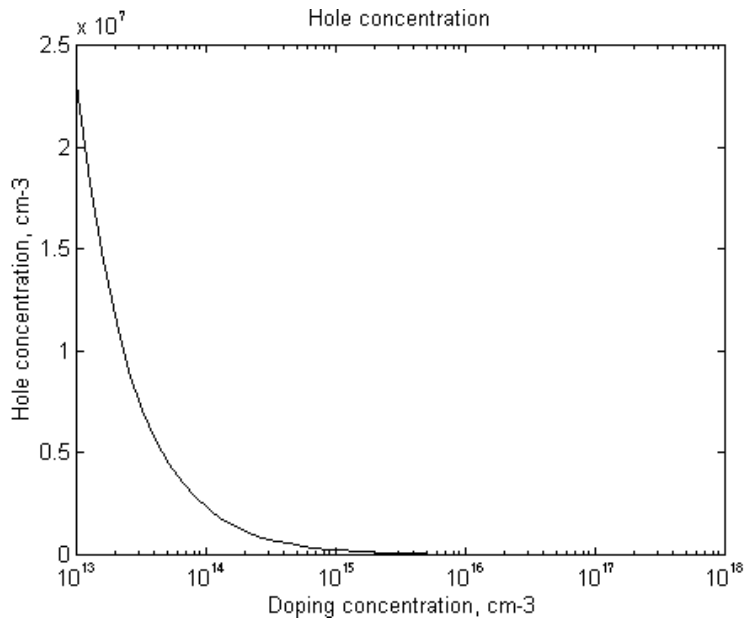
Figure 10.3 shows the hole concentration versus doping.

Figure 10.3   Hole Concentration in N-type Semiconductor (Si)

10.2.2 Fermi level

The Fermi level, E_F, is a chemical energy of a material. It is used to describe the energy level of the electronic state at which an electron has the probability of 0.5 occupying that state. It is given as

$$E_F = \frac{1}{2}(E_C + E_V) - \frac{4}{3}KT\ln(\frac{m_n^*}{m_p^*})$$
(10.15)

where

E_C = energy in the conduction band

E_V = energy in the valence band

and k, T, m_n^* and m_p^* were defined in Section 10.1.

In an intrinsic semiconductor (Si and Ge) m_n^* and m_p^* are of the same order of magnitude and typically, $E_F \gg kT$. Equation (10.15) simplifies to

$$E_F = E_i \cong \frac{1}{2}(E_C + E_V)$$
(10.16)

Equation (10.16) shows that the Fermi energy occurs near the center of the energy gap in an intrinsic semiconductor. In addition, the Fermi energy can be thought of as the average energy of mobile carriers in a semiconductor material.

In an n-type semiconductor, there is a shift of the Fermi level towards the edge of the conduction band. The upward shift is dependent on how much the doped electron density has exceeded the intrinsic value. The relevant equation is

$$n = n_i e^{\left[(E_F - E_i)/kT\right]}$$
(10.17)

where

n is the total electron carrier density

n_i is the intrinsic electron carrier density

E_F is the doped Fermi level

E_i is the intrinsic Fermi level.

In the case of a p-type semiconductor, there is a downward shift in the Fermi level. The total hole density will be given by

$$p = n_i e^{\left[(E_i - E_F)/kT\right]} \tag{10.18}$$

Figure 10.4 shows the energy band diagram of intrinsic and extrinsic semiconductors.

| (a) | (b) | (c) |
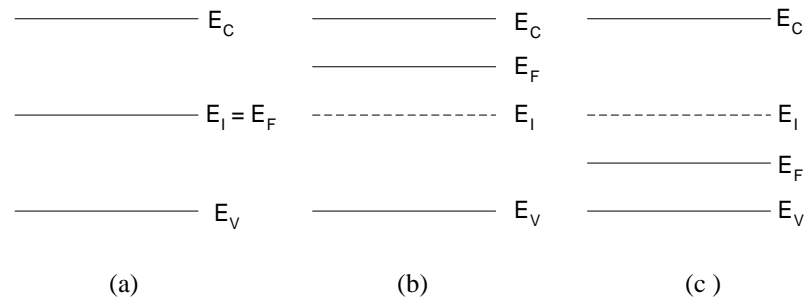
Figure 10.4 Energy-band Diagram of (a) Intrinsic, (b) N-type, and (c) P-type Semiconductors.

10.2.3 Current density and mobility

Two mechanisms account for the movement of carriers in a semiconductor material: drift and diffusion. Drift current is caused by the application of an electric field, whereas diffusion current is obtained when there is a net flow of carriers from a region of high concentration to a region of low concentration. The total drift current density in an extrinsic semiconductor material is

$$J = q(n\mu_n + p\mu_p)\mathrm{E} \tag{10.19}$$

where

J	is current density
n	is mobile electron density
p	is hole density,
μ_n	is mobility of an electron
μ_p	is mobility of a hole
q	is the electron charge

$$E \quad \text{is the electric field.}$$

The total conductivity is

$$\sigma = q(n\mu_n + p\mu_p)E \qquad (10.20)$$

Assuming that there is a diffusion of holes from an area of high concentration to that of low concentration, then the current density of holes in the x-direction is

$$J_p = -qD_p \frac{dp}{dx} \qquad \text{A/cm}^2 \qquad (10.21)$$

where

q is the electronic charge

D_p is the hole diffusion constant

p is the hole concentration.

Equation (10.21) also assumes that, although the hole concentration varies along the x-direction, it is constant in the y and z-directions. Similarly, the electron current density, J_n, for diffusion of electrons is

$$J_n = qD_n \frac{dn}{dx} \quad \text{A / cm}^2 \qquad (10.22)$$

where

D_n is the electron diffusion constant

n is the electron concentration.

For silicon, $D_p = 13 \text{ cm}^2/\text{s}$, and $D_n = 200 \text{ cm}^2/\text{s}$. The diffusion and mobility constants are related, under steady-state conditions, by the Einstein relation

$$\frac{D_n}{\mu_n} = \frac{D_p}{\mu_p} = \frac{kT}{q} \qquad (10.23)$$

The following two examples show the effects of doping concentration on mobility and resistivity.

Example 10.3

From measured data, an empirical relationship between electron (μ_n) and hole (μ_p) mobilities versus doping concentration at 300°K is given as [2]

$$\mu_n(N_D) = \frac{5.1x10^{18} + 92N_D^{0.91}}{3.75x10^{15} + N_D^{0.91}} \tag{10.24}$$

$$\mu_{pn}(N_A) = \frac{2.9x10^{15} + 47.7N_A^{0.76}}{5.86x10^{12} + N_A^{0.76}} \tag{10.25}$$

where

N_D and N_A are donor and acceptor concentration per cm^3, respectively.

Plot the μ_n (N_D) and μ_p (N_A) for the doping concentrations from 10^{14} to 10^{20} cm^{-3}.

Solution

MATLAB Script

```
% nc - is doping concentration
%
nc = logspace(14,20);
un = (5.1e18 + 92*nc.^0.91)./(3.75e15 + nc.^0.91);
up = (2.90e15 + 47.7*nc.^0.76)./(5.86e12 + nc.^0.76);
semilogx(nc,un,'w',nc,up,'w')
text(8.0e16,1000,'Electron Mobility')
text(5.0e14,560,'Hole Mobility')
title('Mobility versus Doping')
xlabel('Doping Concentration in cm-3')
ylabel('Bulk Mobility (cm2/v.s)')
```

Figure 10.5 shows the plot of mobility versus doping concentration.
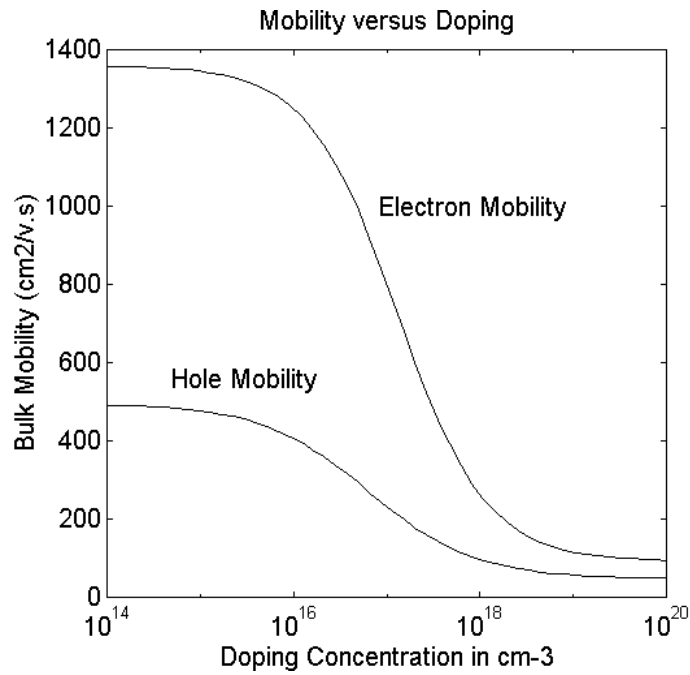
Figure 10.5 Mobility versus Doping Concentration

Example 10.4

At the temperature of 300°K, the resistivity of silicon doped by phosphorus is given as [3]

$$\rho_n = \frac{3.75x10^{15} + N_D^{0.91}}{1.47x10^{-17}\,N_D^{1.91} + 8.15x10^{-1}\,N_D} \qquad (10.26)$$

A similar relation for silicon doped with boron is given as [4]

$$\rho_p = \frac{5.86x10^{12} + N_A^{0.76}}{7.63x10^{-18}\,N_A^{1.76} + 4.64*10^{-4}\,N_A} \qquad (10.27)$$

where

N_D and N_A are donor and acceptor concentrations, respectively.

Use MATLAB to plot the resistivity versus doping concentration (cm^{-3}).

Solution

MATLAB Script

```
% nc is doping concentration
% rn - resistivity of n-type
% rp - resistivity of p-type

nc = logspace(14,20);
rn = (3.75e15 + nc.^0.91)./(1.47e-17*nc.^1.91 + 8.15e-1*nc);
rp = (5.86e12 + nc.^0.76)./(7.63e-18*nc.^1.76 + 4.64e-4*nc);

semilogx(nc,rn,'w',nc,rp,'w')
axis([1.0e14, 1.0e17,0,140])
title('Resistivity versus Doping')
ylabel('Resistivity (ohm-cm)')
xlabel('Doping Concentration cm-3')
text(1.1e14,12,'N-type')
text(3.0e14,50,'P-type')
```

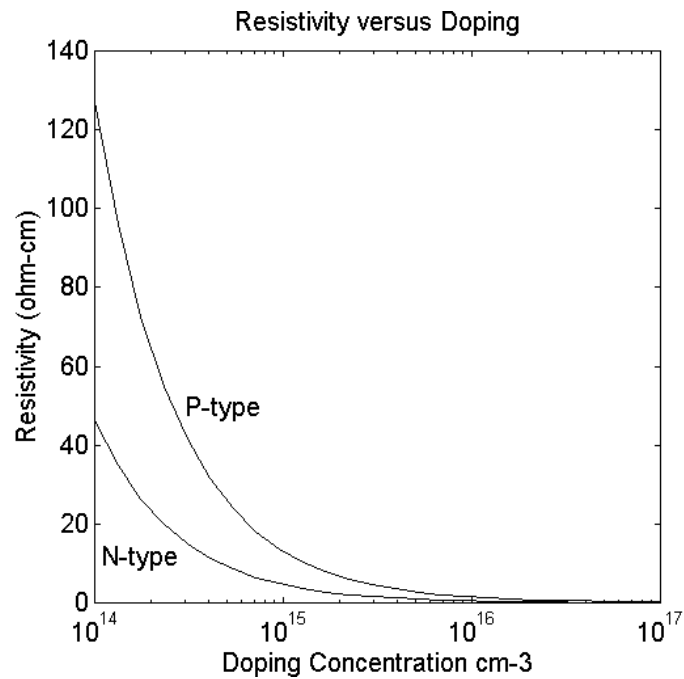Figure 10.6 shows the resistivity of N- and P-type silicon.

Figure 10.6 Resistivity versus Doping Concentration

10.3   PN JUNCTION: CONTACT POTENTIAL, JUNCTION CURRENT

10.3.1   Contact potential

An ideal pn junction is obtained when a uniformly doped p-type material abruptly changes to n-type material. This is shown in Figure 10.7.
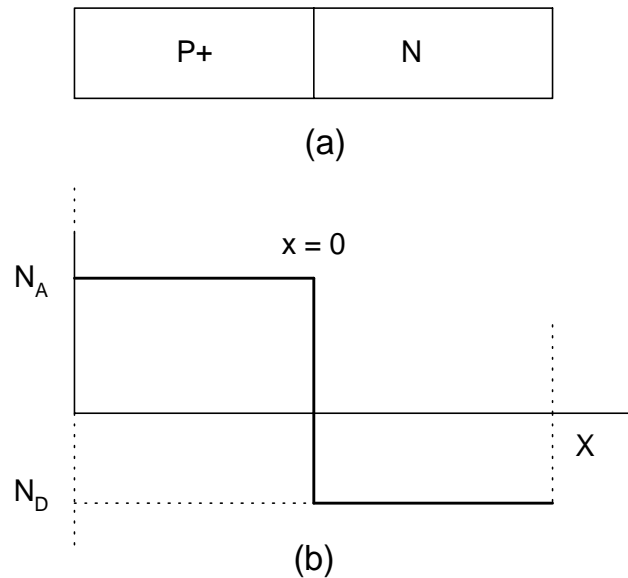
Figure 10.7 Ideal pn Junction (a) Structure, (b) Concentration of Donors (N_D), and acceptor (N_A) impurities.

Practical pn junctions are formed by diffusing into an n-type semiconductor a p-type impurity atom, or vice versa. Because the p-type semiconductor has many free holes and the n-type semiconductor has many free electrons, there is a strong tendency for the holes to diffuse from the p-type to the n-type semiconductors. Similarly, electrons diffuse from the n-type to the p-type material. When holes cross the junction into the n-type material, they recombine with the free electrons in the n-type. Similarly, when electrons cross the junction into the p-type region, they recombine with free holes. In the junction a transition region or depletion region is created.

In the depletion region, the free holes and electrons are many magnitudes lower than holes in p-type material and electrons in the n-type material. As electrons and holes recombine in the transition region, the region near the junction within the n-type semiconductor is left with a net positive charge. The region near the junction within the p-type material will be left with a net negative charge. This is illustrated in Figure 10.8.

Because of the positive and negative fixed ions at the transition region, an electric field is established across the junction. The electric field creates a potential difference across the junction, the potential barrier. The latter is also

called diffusion potential or contact potential, V_C. The potential barrier prevents the flow of majority carriers across the junction under equilibrium conditions.

$$N_A = N_D$$
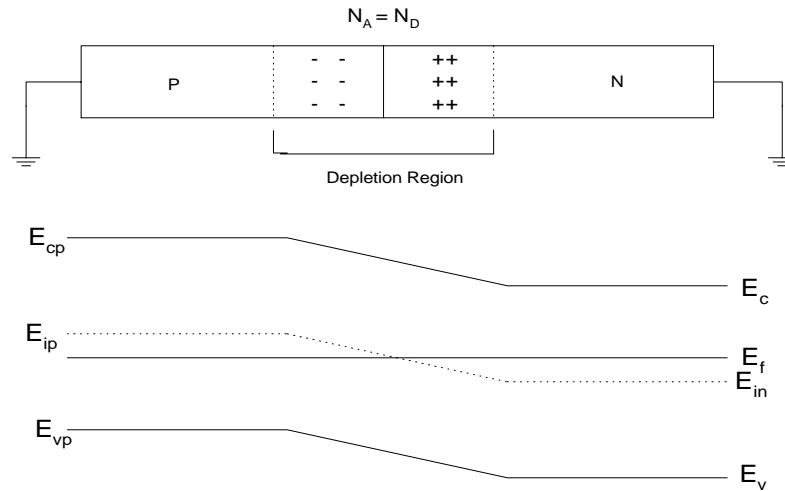
Figure 10.8 pn Junction (a) Depletion region with Positive and Negative Ions (b) Energy Band Diagram near a pn Junction.

The contact potential, V_C, may be obtained from the relations

$$\frac{n_n}{n_p} = e^{\left(\frac{qV_C}{kT}\right)} = \frac{p_p}{p_n} \tag{10.28}$$

or

$$V_C = \frac{kT}{q} \ln(\frac{n_n}{n_p}) = \frac{kT}{q} \ln(\frac{p_p}{p_n}) \tag{10.29}$$

But, noting that $p_p \cong N_A$, $n_p \cong \dfrac{n_i^2}{N_A}$, $n_n \cong N_D$, $p_n \cong \dfrac{n_i^2}{N_D}$,

Equation (10.29) becomes

$$V_C = \frac{kT}{q} \ln(\frac{N_A N_D}{n_i^2}) \tag{10.30}$$

The contact potential can also be obtained from the band-bending diagram of the pn junction shown in Figure 10.8. That is, from Figure 10.8

$$V_C = \frac{E_{in} - E_{ip}}{q} \tag{10.31}$$

or

$$V_C = -(\phi_{fn} + |\phi_{fp}|) \tag{10.32}$$

where

ϕ_{FN} and ϕ_{FP} are the electron and hole Fermi potentials, respectively. They are given as

$$\phi_{FN} = \frac{E_F - E_{IN}}{q} = \frac{kT}{q} \ln\left(\frac{N_D}{n_i}\right) \tag{10.33}$$

and

$$\phi_{FP} = \frac{E_F - E_{IP}}{q} = \frac{kT}{q} \ln\left(\frac{N_A}{n_i}\right) \tag{10.34}$$

Using Equations (10.31) to (10.34), we have

$$V_C = \frac{kT}{q} \ln\left(\frac{N_A N_D}{n_i^2}\right) \tag{10.35}$$

It should be noted that Equations (10.30) and (10.35) are identical. Typically, V_C is from 0.5 to 0.8 V for the silicon pn junction. For germanium, V_C is approximately 0.1 to 0.2, and that for gallium arsenide is 1.5V.

When a positive voltage V_S is applied to the p-side of the junction and n-side is grounded, holes are pushed from the p-type material into the transition region. In addition, electrons are attracted to transition region. The depletion region decreases, and the effective contact potential is reduced. This allows majority carriers to flow through the depletion region. Equation (10.28) modifies to

$$\frac{n_n}{n_p} = e^{\left[\frac{q(V_G - V_S)}{kT}\right]} = \frac{p_p}{p_n} \tag{10.36}$$

When a negative voltage V_S is applied to the p-side of a junction and the n-side is grounded, the applied voltage adds directly to the contact potential. The depletion region increases and it become more difficult for the majority carriers to flow across the junction. The current flow is mainly due to the flow of minority carriers. Equation (10.28) modifies to

$$\frac{n_n}{n_p} = e^{\left[\frac{q(V_C+V_S)}{kT}\right]} = \frac{p_p}{p_n} \qquad (10.37)$$

Figure 10.9 shows the potential across the diode when a pn junction is forward-biased and reversed-biased.
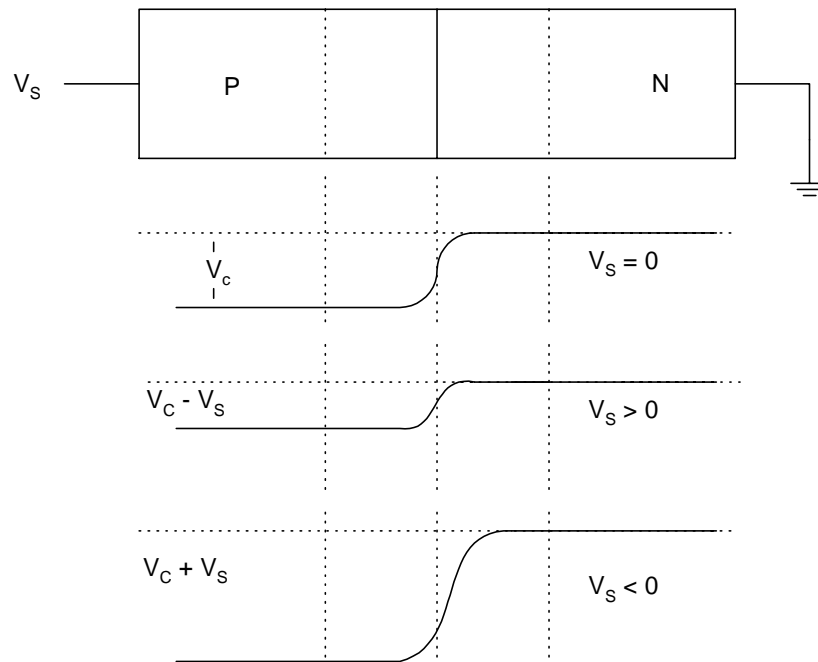

Figure 10.9 PN Junction (a) with Depletion Layer and Source Connection (b) Contact Potential with No Source Voltage (V_S = 0) (c) Junction Potential for Forward-biased pn Junction (V_S > 0) and (d) Junction Potential for Reversed-biased pn Junction (V_S < 0)

The following example illustrates the effect of source voltage on the junction potential.

Example 10.5

For a Silicon pn junction with $N_D = 10^{14}$ cm^{-3} and $N_A = 10^{17}$ cm^{-3} and with $n_i^2 = 1.04 \times 10^{26}$ cm^{-6} at $T = 300$ °K,

(a)   Calculate the contact potential.

(b)   Plot the junction potential when the source voltage V_S of Figure 10.9 increases from -1.0 to 0.7 V.

Solution

MATLAB Script

```
diary ex10_5.dat
% Junction potential versus source voltage
% using equation(10.36) contact potential is

t = 300;
na = 1.0e17;
nd = 1.0e14;
nisq = 1.04e20;
q = 1.602e-19;
k = 1.38e-23;

% calculate contact potential
vc = (k*t/q)*(log(na*nd/nisq))
vs = -1.0:0.1:0.7;
jct_pot = vc - vs;

% plot curve
plot(vs,jct_pot)
title('Junction potential vs. source voltage')
xlabel('Source voltage, V')
ylabel('Junction potential, V')
diary
```

(a) The contact potential is

$$vc =$$
$$0.6535$$

(b) Figure 10.10 shows the graph of the junction potential versus the source voltage.
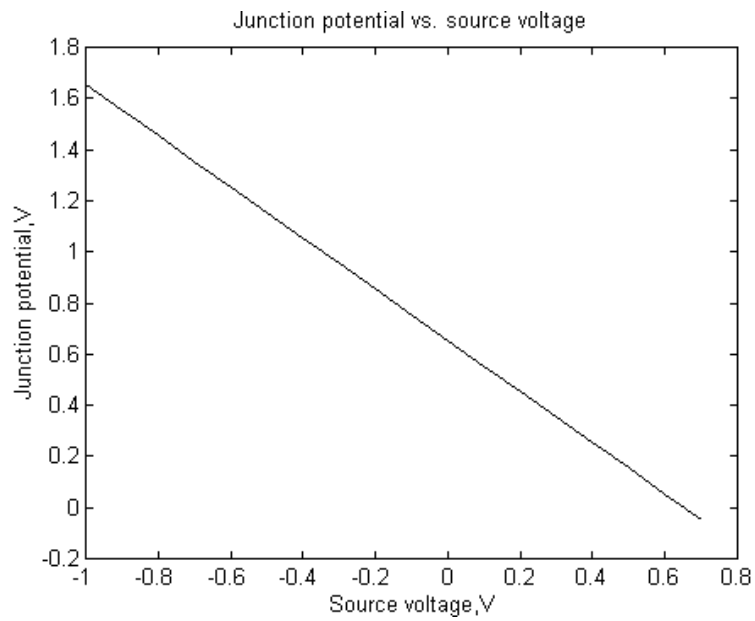

Figure 10.10 Junction Potential versus Source Voltage.

10.3.2 Junction current

The pn junction current is given as

$$I = I_s \left[e^{\left(\frac{qV_S}{kT} \right)} - 1 \right] \tag{10.38}$$

where

V_S is the voltage across the pn junction [see Figure 10.9 (a)]

q is the electronic charge

T is the absolute temperature

k is Boltzmann's constant

I_S is reverse saturation current. It is given as

$$I_S = qA(\frac{D_p p_n}{L_p} + \frac{D_n n_p}{L_n})$$ (10.39)

where

A	is the diode cross-sectional area
L_p, L_n	are the hole and electron diffusion lengths
p_n, n_p	are the equilibrium minority carrier concentrations
D_p, D_n	are the hole and electron diffusion coefficients, respectively.

Since $p_n \cong \dfrac{n_i^2}{N_D}$ and $n_p \cong \dfrac{n_i^2}{N_A}$, Equation (10.39) becomes

$$I_S = qA\left(\frac{D_p}{L_p N_D} + \frac{D_n}{L_n N_A}\right)n_i^2$$ (10.40)

The diffusion coefficient and diffusion length are related by the expression

$$L_p = \sqrt{D_p \tau_p}$$ (10.41)

and

$$L_n = \sqrt{D_n \tau_n}$$ (10.42)

where

τ_p, τ_n are the hole minority and electron minority carrier lifetime, respectively.

Equation (10.38) is the diode equation. It is applicable for forward-biased ($V_S > 0$) and reversed-biased ($V_S < 0$) pn junctions.

Using Equations (10.1) and (10.39), the reverse saturation current can be rewritten as

$$I_S = k_1 T^3 e^{\left[-E_g/(kT)\right]} \tag{10.43}$$

where k_1 is a proportionality constant

$$\frac{dI_S}{dT} = 3k_1 T^2 e^{-\frac{E_g}{kT}} + k_1 T^3 \frac{-E_g}{kT^2} e^{-\frac{E_g}{kT}}$$

Thus

$$\frac{1}{I_S}\frac{dI_S}{dT} = \frac{3}{T} + \frac{1}{T}\frac{E_g}{kT} = \frac{3}{T} + \frac{1}{T}\frac{V_g}{V_T} \tag{10.44}$$

where

$$V_T = \frac{kT}{q} \qquad \text{and} \qquad V_g = \frac{E_g}{q}$$

For silicon at room temperature,

$$\frac{V_g}{V_T} = 44.4.$$

Thus

$$\frac{dI_S}{dT} = (3 + \frac{V_g}{V_T})\frac{dT}{T} = 47.4\frac{dT}{T} \tag{10.45}$$

At room temperature (300° K), the saturation current approximately doubles every 5° C [5]. The following example shows how I_S is affected by temperature.

Example 10.6

A silicon diode has $I_S = 10^{-15}$ A at 25° C and assuming I_S increases by 15% per °C rise in temperature, find and plot the value of I_S from 25 °C to 125 °C.

Solution

From the information given above, the reverse saturation current can be expressed as

$$I_S = 10^{-15}(1.15)^{(T-25)}$$

MATLAB is used to find I_S at various temperatures.

MATLAB Script

```
% Saturation current
%

t = 25:5:125;
is = 1.0e-15*(1.15).^(t-25);

plot(t,is)
title('Reverse Saturation Current vs. Temperature')
xlabel('Temperature, C')
ylabel('Current, A')
```

Figure 10.11 shows the effect of temperature on the reverse saturation current.
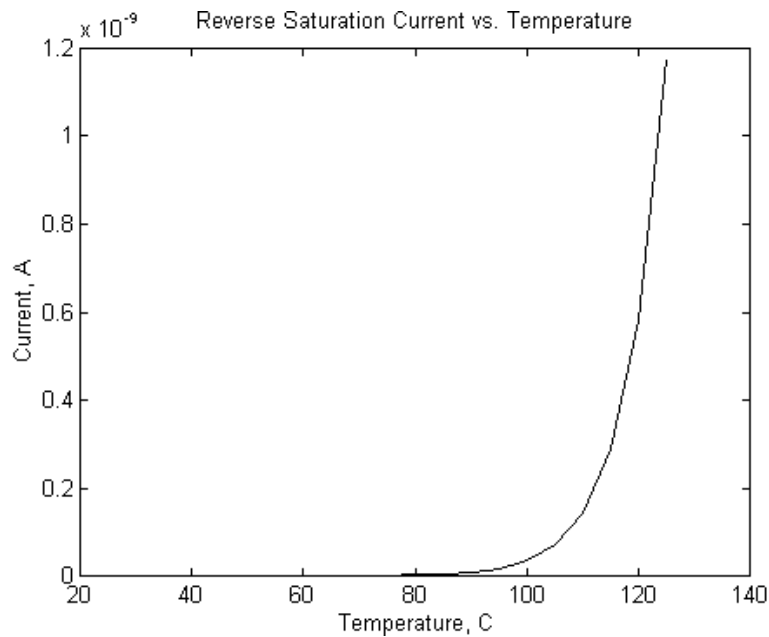
Figure 10.11 Reverse Saturation Current versus Temperature

10.4 DEPLETION AND DIFFUSION CAPACITANCES

10.4.1 Depletion capacitance

As mentioned previously, a pn junction is formed when a p-type material is joined to an n-type region. During device fabrication, a p-n junction can be formed using process such as ion-implantation diffusion or epitaxy. The doping profile at the junction can take several shapes. Two popular doping profiles are abrupt (step) junction and linearly graded junction.

In the abrupt junction, the doping of the depletion region on either side of the metallurgical junction is a constant. This gives rise to constant charge densities on either side of the junction. This is shown in Figure 10.12.

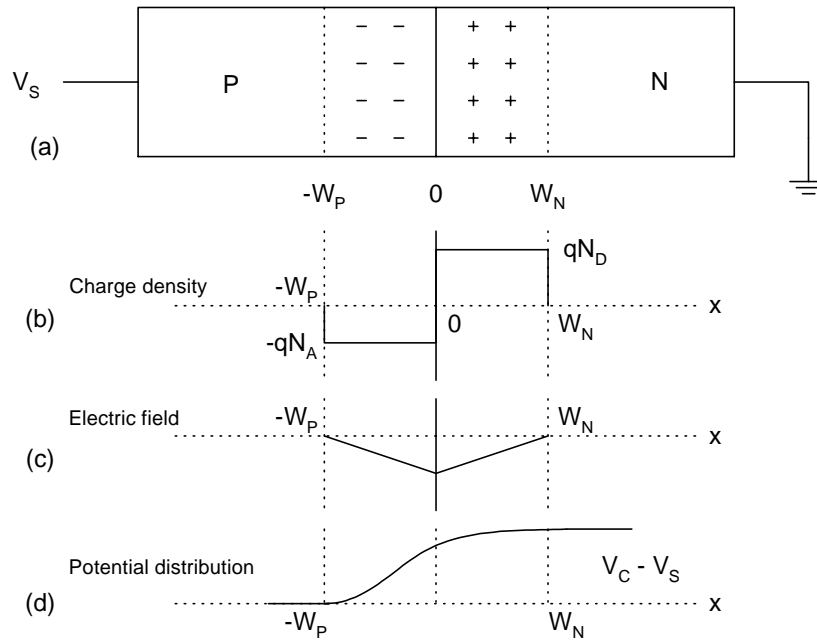**Figure 10.12**   PN Junction with Abrupt Junction (a) Depletion Region (b) Charge Density (c) Electric Field and (d) Potential Distribution

For charge equality,

$$qN_A W_P = qN_D W_N \qquad (10.46)$$

it can be shown [6] that the depletion width in the p-type (W_P) and that of the n-type material (W_N) can be given as

$$W_P = \sqrt{\frac{2\varepsilon N_D (V_C - V_s)}{qN_A (N_D + N_A)}} \qquad (10.47)$$

$$W_N = \sqrt{\frac{2\varepsilon N_A (V_C - V_s)}{q N_D (N_D + N_A)}}$$

(10.48)

where

ε is the relative dielectric constant
       ($\varepsilon = 12\varepsilon_0$ for Si, and ε_0 = 8.85 x 10^{-12} F/m)

N_D is donor concentration

N_A is acceptor concentration

q is electronic charge

V_C is contact potential obtained from Equation (10.30)

V_S is source voltage.

If the doping density on one side of the metallurgical junction is greater than that on the other side (i.e., $N_A \gg N_D$ or $N_D \gg N_A$), then the junction properties are controlled entirely by the lightly doped side. This condition is termed the one-sided step junction approximation. This is the practical model for shallow junctions formed by a heavily doped diffusion into a lightly doped region of opposite polarity [7].

In a linearly graded junction, the ionized doping charge density varies linearly across the depletion region. The charge density passes through zero at the metallurgical junction. Figure 10.13 shows the profile of the linearly graded junction.
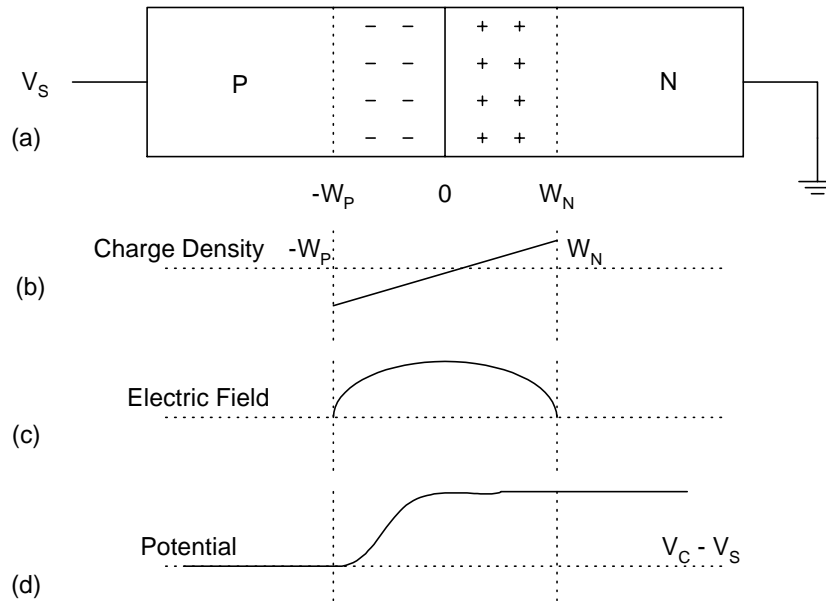
Figure 10.13   PN Junction with Linearly Graded Junction
(a)   Depletion Region   (b) Charge Density
(c) Electric Field   (d) Potential Distribution

For a linearly graded junction, the depletion width in the p-type and n-type material, on either side of the metallurgical junction, can be shown to be

$$W_N = |W_P| = \left[\frac{12\varepsilon(V_C - V_S)}{qa} \right]^{\frac{1}{3}}$$ (10.49)

where

a is the slope of the graded junction impurity profile.

The contact potential is given as [6]

$$V_C = \frac{kT}{q} \ln(\frac{aW_N}{2n_i})$$ (10.50)

The depletion capacitance, C_j, is due to the charge stored in the depletion region. It is generally given as

$$C_j = \frac{\varepsilon A}{W_T} \tag{10.51}$$

where

$$W_T = W_N + \left| W_P \right| \tag{10.52}$$

A is cross-sectional area of the pn junction.

For abrupt junction, the depletion capacitance is given as

$$C_j = A\sqrt{\frac{\varepsilon q N_A N_D}{2(N_D + N_A)(V_C - V_S)}} \tag{10.53}$$

For linearly graded junction, the depletion capacitance is given as

$$C_j = 0.436(aq)^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} A(V_C - V_S)^{-\frac{1}{3}}$$

$$C_j = 0.436A[\frac{aq\varepsilon^2}{(V_C - V_S)}]^{\frac{1}{3}} \tag{10.54}$$

In general, we may express the depletion capacitance of a pn junction by

$$C_j = \frac{C_{j0}}{\left[1 - \frac{V_S}{V_C}\right]^m} \qquad \frac{1}{3} \le m \le \frac{1}{2} \tag{10.55}$$

where

$$m = \frac{1}{3} \quad \text{for linearly graded junction and}$$

$$m = \frac{1}{2} \quad \text{for step junction}$$

C_{j0} = zero-biased junction capacitance. It can be obtained from
Equations (10.53) and (10.54) by setting V_S equal to zero.

Equations (10.53 to 10.55) are, strictly speaking, valid under the conditions of reversed-biased $V_S < 0$. The equations can, however, be used when $V_S < 0.2$V. The positive voltage, V_C , is the contact potential of the pn junction. As the pn junction becomes more reversed biased ($V_S < 0$), the depletion capacitance decreases. However, when the pn junction becomes slightly forward biased, the capacitance increases rapidly. This is illustrated by the following example.

Example 10.7

For a certain pn junction, with contact potential 0.065V, the junction capacitance is 4.5 pF for V_S = -10 and C_j is 6.5 pF for V_S = -2 V.

(a) Find m and C_{j0} of Equation (10.55).

(b) Use MATLAB to plot the depletion capacitance from -30V to 0.4V.

Solution

From Equation (10.55)

$$C_{j1} = \frac{C_{j0}}{[1 - \dfrac{V_{S1}}{V_C}]^m}$$

$$C_{j2} = \frac{C_{j0}}{[1 - \dfrac{V_{S2}}{V_C}]^m}$$

therefore

$$\frac{C_{j1}}{C_{j2}} = \left[\frac{V_C - V_{S2}}{V_C - V_{S1}}\right]^m$$

$$m = \frac{\log_{10}\left[\dfrac{C_{j1}}{C_{j2}}\right]}{\log_{10}\left[\dfrac{V_C - V_{S2}}{V_C - V_{S1}}\right]} \qquad (10.56)$$

and

$$C_{j0} = C_{j1}\left[1 - \frac{V_{S1}}{V_C}\right]^m \qquad (10.57)$$

MATLAB is used to find m and C_{j0}. It is also used to plot the depletion capacitance.

MATLAB Script

```
% depletion capacitance
%
cj1 = 4.5e-12; vs1 = -10;
cj2 = 6.5e-12; vs2 = -2;
vc = 0.65;

num = cj1/cj2;
den = (vc-vs2)/(vc-vs1);
m = log10(num)/log10(den);
cj0 = cj1*(1 - (vs1/vc))^m;
vs = -30:0.2:0.4;
k = length(vs);
for i = 1:k
 cj(i) = cj0/(1-(vs(i)/vc))^m;
end
plot(vs,cj,'w')
xlabel('Voltage,V')
ylabel('Capacitance,F')
title('Depletion Capacitance vs. Voltage')
axis([-30,2,1e-12,14e-12])
```

(a) The values of m, C_{j0} are

```
m =
    0.02644
```

cj0 =
        9.4246e-012

(b) Figure 10.14 shows the depletion capacitance versus the voltage across the
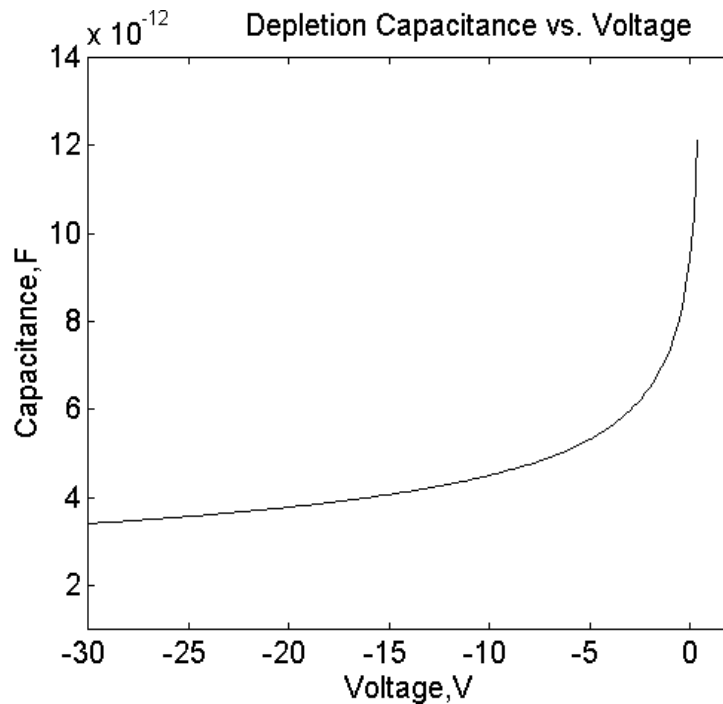junction.

Figure 10.14 Depletion Capacitance of a pn Junction

10.4.2 Diffusion capacitance

When a pn junction is forward biased, holes are injected from the p-side of the
metallurgical junction into the n-type material. The holes are momentarily
stored in the n-type material before they recombine with the majority carriers
(electrons) in the n-type material. Similarly, electrons are injected into and
temporarily stored in the p-type material. The electrons then recombine with
the majority carriers (holes) in the p-type material. The diffusion capacitance,
C_d, is due to the buildup of minority carriers charge around the metallurgical

junction as the result of forward biasing the pn junction. Changing the forward current or forward voltage, ΔV, will result in the change in the value of the stored charge ΔQ, the diffusion capacitance, C_d, can be found from the general expression

$$C_d = \frac{\Delta Q}{\Delta V} \qquad (10.58)$$

It turns out that the diffusion capacitance is proportional to the forward-biased current. That is

$$C_d = K_d I_{DF} \qquad (10.59)$$

where

$\quad K_d \qquad$ is constant at a given temperature

$\quad I_{DF} \qquad$ is forward-biased diode current.

The diffusion capacitance is usually larger than the depletion capacitance [1, 6]. Typical values of C_d ranges from 80 to 1000 pF.

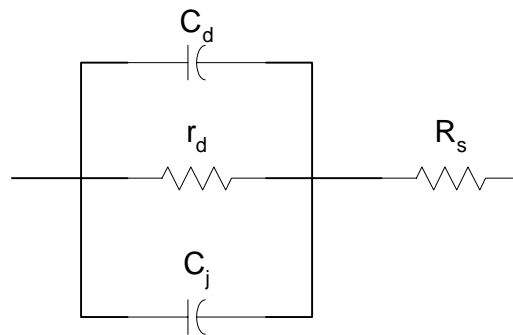A small signal model of the diode is shown in Figure 10.15.

Figure 10.15 Small-signal Model of a Forward-biased pn Junction

In Figure 10.15, C_d and C_j are the diffusion and depletion capacitance, respectively. R_S is the semiconductor bulk and contact resistance. The dynamic resistance, r_d, of the diode is given as

$$r_d = \frac{nkT}{qI_{DF}} \qquad (10.60)$$

where

n	is constant
k	is Boltzmann's constant
T	is temperature in degree Kelvin
q	is electronic charge.

When a pn junction is reversed biased, $C_d = 0$. The model of the diode is shown in Figure 10.16.
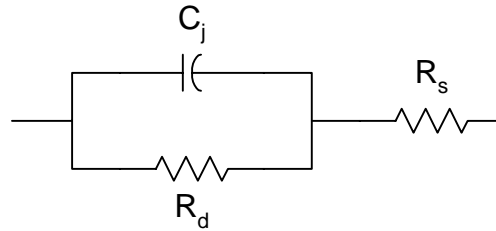

Figure 10.16 Model of a Reverse-biased pn Junction

In Figure 10.16, C_j is the depletion capacitance. The diffusion capacitance is zero. The resistance R_d is reverse resistance of the pn junction (normally in the mega-ohms range).

Example 10.8

A certain diode has contact potential; $V_C = 0.55$V, $C_{j0} =$ diffusion capacitance at zero biased is   8 pF;   the diffusion capacitance at 1mA is 100 pF. Use MATLAB to plot the diffusion and depletion capacitance for forward- biased voltages from 0.0 to 0.7 V. Assume that $I_S = 10^{-14}$ A, $n = 2.0$ and step-junction profile.

Solution

Using Equations (10.38) and (10.59), we write the MATLAB program to obtain the diffusion and depletion capacitance.

MATLAB Script

```
%
% Diffusion and depletion Capacitance
%
cd1 = 100e-12; id1 = 1.0e-3; cj0 = 8e-12; vc =0.55;
m = 0.5;
is = 1.0e-14; nd = 2.0;
k = 1.38e-23; q = 1.6e-19; T = 300;
kd = cd1/id1;
vt = k*T/q;
v = 0.0:0.05:0.55;
nv = length(v);

for i = 1:nv
 id(i) = is*exp(v(i)/(nd*vt));
 cd(i) = kd*id(i);
 ra(i) = v(i)/vc;
 cj(i) = cj0/((1 - ra(i)).^m);
end

subplot(121)
plot(v,cd)
title('Diffusion Cap.')
xlabel('Voltage, V'), ylabel('Capacitance, F')

subplot(122)
plot(v,cj)
title('Depletion Cap.')
xlabel('Voltage, V'), ylabel('Capacitance, F')
```

Figure 10.17 shows the depletion and diffusion capacitance of a forward-biased pn junction.
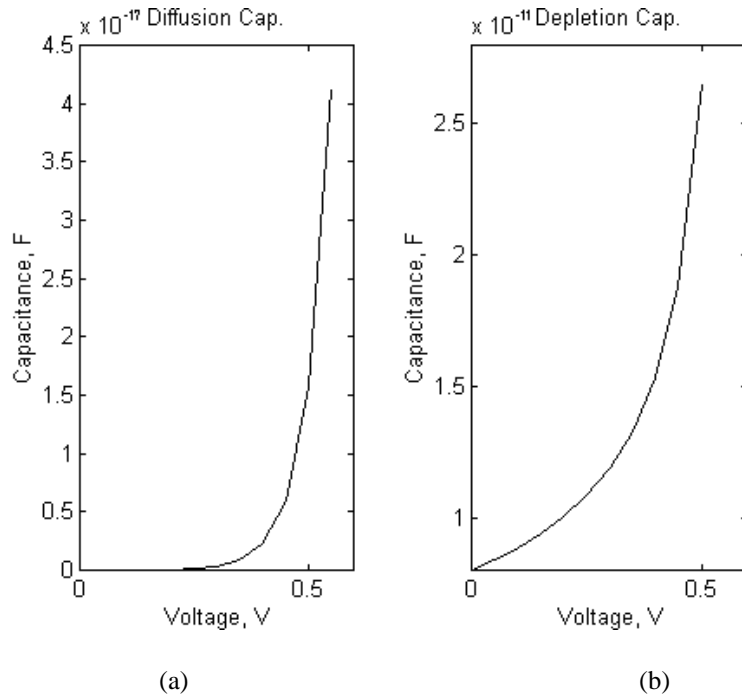
Figure 10.17 (a) Depletion and (b) Diffusion Capacitance

10.5   BREAKDOWN VOLTAGES OF PN JUNCTIONS

The electric field E is related to the charge density through the Poisson's equation

$$\frac{dE(x)}{dx} = \frac{\rho(x)}{\varepsilon_S \varepsilon_0} \tag{10.61}$$

where

ε_S is the semiconductor dielectric constant

ε_0 is the permittivity of free space, $8.86 * 10^{-14}$ F/cm

$\rho(x)$ is the charge density.

For an abrupt junction with charge density shown in Figure 10.12, the charge density

$$\rho(x) = -qN_A \qquad -W_P < x < 0$$
$$= qN_D \qquad 0 < x < W_N \tag{10.62}$$

The maximum electric field

$$\left|E_{max}\right| = \frac{qN_A W_P}{\varepsilon_s \varepsilon_0} = \frac{qN_D W_N}{\varepsilon_s \varepsilon_0} \tag{10.63}$$

Using Equation (10.47) or (10.48, Equation (10.63) becomes

$$\left|E_{max}\right| = \sqrt{\frac{2qN_D N_A (V_C - V_S)}{\varepsilon_S \varepsilon_0 (N_A + N_D)}} \tag{10.64}$$

For a linearly graded junction, the charge density, $\rho(x)$ is given as (see Figure 10.13)

$$\rho(x) = ax \qquad -\frac{W}{2} < x < \frac{W}{2} \tag{10.65}$$

and the maximum electric field can be shown to be

$$\left|E_{max}\right| = \frac{aq}{8\varepsilon_S \varepsilon_0} W^2 \tag{10.66}$$

where

      a is slope of charge density
      W is width of depletion layer and

$$\frac{W}{2} = W_N = W_P$$

The width of the depletion region, W, can be obtained from Equation (10.49).

Equation (10.64) indicates that as the reverse voltage increases, the magnitude of the electric field increases. The large electric field accelerates the carriers

crossing the junction. At a critical field, E_{crit}, the accelerated carriers in the depletion region have sufficient energy to create new electron-hole pairs as they collide with other atoms. The secondary electrons can in turn create more carriers in the depletion region. This is termed the avalanche breakdown process. For silicon with an impurity concentration of 10^{16} cm^{-3}, the critical electric field is about 2.0×10^5 V/cm.

In a highly doped pn junctions, where the impurity concentration is about 10^{18} cm^{-3}, the critical electric field is about 10^6 V/cm. This high electric field is able to strip electrons away from the outer orbit of the silicon atoms, thus creating hole-electron pairs in the depletion region. This mechanism of breakdown is called zener breakdown. This breakdown mechanism does not involve any multiplication effect. Normally, when the breakdown voltage is less than 6V, the mechanism is zener breakdown process. For breakdown voltages beyond 6V, the mechanism is generally an avalanche breakdown process.

For an abrupt junction, where one side is heavily doped, the electrical properties of the junction are determined by the lightly doped side. Experimentally, the breakdown voltage of semiconductor step junction (n$^+$p or p$^+$n) as the function of doping concentration in the lightly doped side is given as [7]

$$V_{BR} = k \left[\frac{N_B}{10^{16}} \right]^{-0.75}$$

(10.67)

where

k = 25V for Ge
= 60V for Si

and

N_B is the doping concentration of lightly doped side.

The following example shows the effect of doping concentration on breakdown voltage.

Example 10.9

Use MATLAB to plot the breakdown voltage versus doping concentration for a one-sided step junction for silicon and germanium, and using doping concentration from 10^{14} to 10^{19} cm^{-3}.

Solution

Using Equation (10.67), we calculate the breakdown voltage for various doping concentration.

MATLAB Script

```
%
% Breakdown voltage
%
k1 = 25;
 k2 = 60;
nb = logspace(14,19);
n = length(nb);

for i = 1:n
 vbr1(i) = k1*(nb(i)/1.0e16)^(-0.75); % Ge breakdown voltage
 vbr2(i) = k2*(nb(i)/1.0e16)^(-0.75); % Si breakdown voltage
end

semilogx(nb,vbr1,'w', nb,vbr2,'w')
xlabel('Impurity Concentration, cm-3')
ylabel('Breakdown Voltage,V')
title('Breakdown Voltage vs. Impurity Concentration')
axis([1.0e14,1.0e17,0,2000])
text(2.0e14,270,'Ge')
text(3.0e14,1000,'Si')
```

Figure 10.18 shows the plot of breakdown voltage of one-sided abrupt junction.
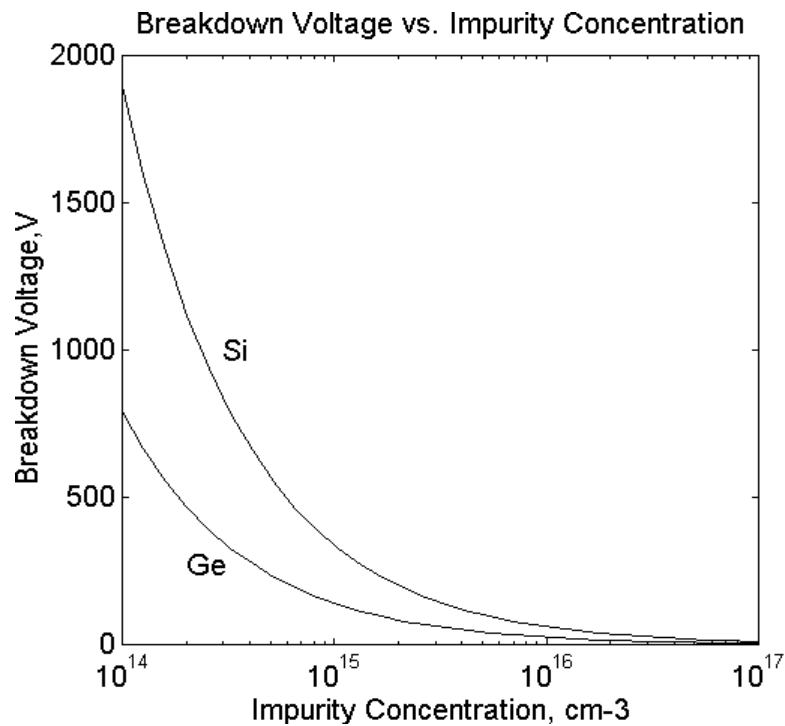
Figure 10.18 Breakdown Voltage versus Impurity Concentration

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EXERCISES

10.1   In the case of silicon for temperature below 700 $^\circ$K, the density of intrinsic created carriers, n_i, can be approximated as [8]

$$n_i = 3.87 * 10^{16} T^{3/2} e^{-\left(\frac{7.02*10^3}{T}\right)}$$

(10.68)

(a)   Use MATLAB to plot the intrinsic carrier concentration versus (1000/T) where T is temperature in degrees Kelvin.

(b)   Compare the above relation for intrinsic concentration with that of Example 10.1. Plot the difference between of n_i for Equations (10.1) and (10.68).

10.2   Assuming that at 300 $^\circ$K the mobile carrier concentrations of intrinsic germanium and silicon semiconductor materials are $2.390*10^{13}$ and $1.52*10^{10}$, respectively, use MATLAB to plot the $E_F - E_i$ versus donor concentration for Ge and Si. Assume donor concentrations from 10^{10} to 10^{18}.

10.3   For power devices with breakdown voltages above 100V and resistivities greater than 1 ohm-cm (n-type silicon) and 3 ohm-cm (p-type silicon), the resistivity versus doping concentrations can be simplified to

$$\rho_n = 4.596 * 10^{15} N_D^{-1}$$

$$\rho_{pn} = 1.263 * 10^{16} N_A^{-1}$$

(a) Use MATLAB to plot resistivity versus doping concentration (from 10^{12} to 10^{18} cm^{-3}).
(b) Compare your results with those obtained in Example 10.4.

10.4   For Ge pn junction with $N_A = 10^{18}$ cm^{-3} , $N_D = 10^{15}$ cm^{-3} and n_i at 300 °K is 2.39*10^{13} ,

(a) Calculate the contact potential.
(b) Plot the junction potential for source voltages of -1.0V to 0.3V.

10.5   For the small signal model of the forward-biased pn junction, shown in Figure 10.15, $R_S = 5\Omega$, $r_d = 10\ \Omega$, $C_d = 110$ pF at I_{DF} of 1 mA. Use MATLAB to plot the equivalent input impedance (magnitude and phase) for frequencies from 10^4 to 10^{10} Hz.

10.6   Empirically, the breakdown voltage of a linearly graded junction can be approximated as [9]

$$V_{BR} = k \left[\frac{a^{-4}}{10^{21}} \right]^{-0.75}$$

where $k = 18$ V for Ge or 40 V for Si.

Use MATLAB to plot the breakdown voltage vs. impurity gradient of Ge and Si. Use impurity gradient values from 10^{19} to 10^{24}.

Attia, John Okyere. "Operational Amplifiers."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER ELEVEN

OPERATIONAL AMPLIFIERS

The operational amplifier (Op Amp) is one of the versatile electronic circuits. It can be used to perform the basic mathematical operations: addition, subtraction, multiplication, and division. They can also be used to do integration and differentiation. There are several electronic circuits that use an op amp as an integral element. Some of these circuits are amplifiers, filters, oscillators, and flip-flops. In this chapter, the basic properties of op amps will be discussed. The non-ideal characteristics of the op amp will be illustrated, whenever possible, with example problems solved using MATLAB.

11.1   PROPERTIES OF THE OP AMP

The op amp, from a signal point of view, is a three-terminal device: two inputs and one output. Its symbol is shown in Figure 11.1. The inverting input is designated by the '-' sign and non-inverting input by the '+' sign.

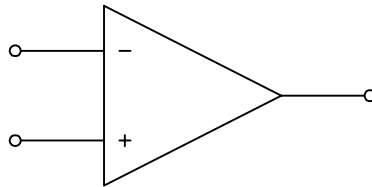**Figure 11.1** Op Amp Circuit Symbol

An ideal op amp has an equivalent circuit shown in Figure 11.2. It is a difference amplifier, with output equal to the amplified difference of the two inputs.

An ideal op amp has the following properties:

- infinite input resistance,
- zero output resistance,
- zero offset voltage,
- infinite frequency response and
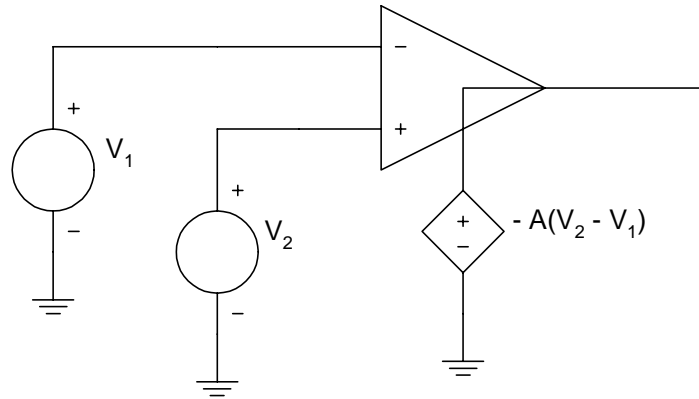- infinite common-mode rejection ratio,
- infinite open-loop gain, A.

Figure 11.2   Equivalent Circuit of an Ideal Op Amp

A practical op amp will have large but finite open-loop gain in the range from 10^5 to 10^9. It also has a very large input resistance 10^6 to 10^{10} ohms. The output resistance might be in the range of 50 to 125 ohms. The offset voltage is small but finite and the frequency response will deviate considerably from the infinite frequency response. The common-mode rejection ratio is not infinite but finite. Table 11.1 shows the properties of the general purpose 741 op amp.

Table 11.1
Properties of 741 Op Amp

Property	Value (Typical)
Open Loop Gain	2×10^5
Input resistance	2.0 M
Output resistance	75 Ω
Offset voltage	1 mV
Input bias current	30 nA
Unity-gain bandwidth	1 MHz
Common-mode rejection ratio	95 dB
Slew rate	0.7 V/μV

Whenever there is a connection from the output of the op amp to the inverting input as shown in Figure 11.3, we have a negative feedback connection
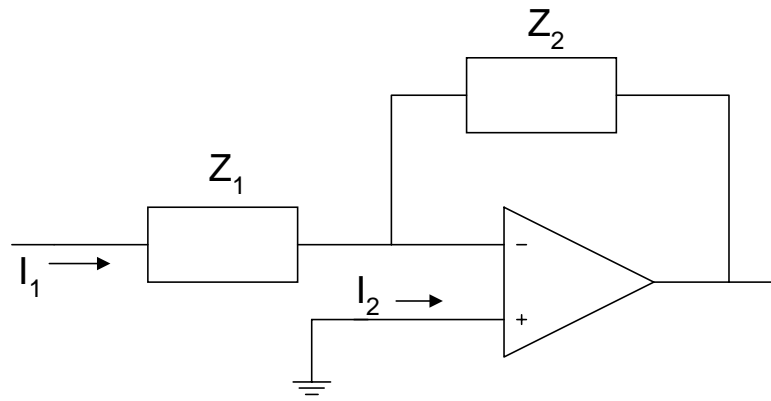
(a)
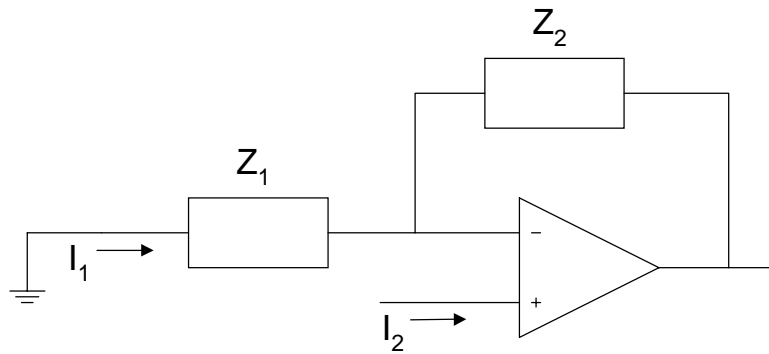
(b)

Figure 11.3   Negative Feedback Connections for Op Amp
(a) Inverting (b) Non-inverting configurations

With negative feedback and finite output voltage, Figure 11.2 shows that

$$V_O = A(V_2 - V_1) \qquad (11.1)$$

Since the open-loop gain is very large,

$$(V_2 - V_1) = \frac{V_O}{A} \cong 0 \qquad (11.2)$$

Equation (11.2) implies that the two input voltages are also equal. This condition is termed the concept of the virtual short circuit. In addition, because of the large input resistance of the op amp, the latter is assumed to take no current for most calculations.

11.2   INVERTING CONFIGURATION

An op amp circuit connected in an inverted closed loop configuration is shown in Figure 11.4.
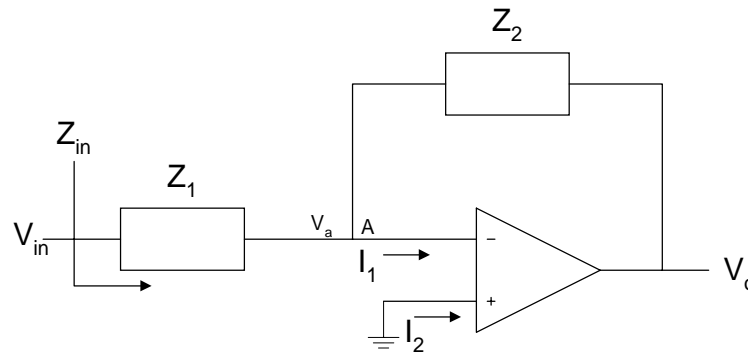

Figure 11.4 Inverting Configuration of an Op Amp

Using nodal analysis at node A, we have

$$\frac{V_a - V_{in}}{Z_1} + \frac{V_a - V_O}{Z_2} + I_1 = 0 \tag{11.3}$$

From the concept of a virtual short circuit,

$$V_a = V_b = 0 \tag{11.4}$$

and because of the large input resistance, $I_1 = 0$. Thus, Equation (11.3) simplifies to

$$\frac{V_O}{V_{IN}} = -\frac{Z_2}{Z_1} \tag{11.5}$$

The minus sign implies that V_{IN} and V_0 are out of phase by 180°. The input impedance, Z_{IN}, is given as

$$Z_{IN} = \frac{V_{IN}}{I_1} = Z_1 \qquad (11.6)$$

If $Z_1 = R_1$ and $Z_2 = R_2$, we have an inverting amplifier shown in Figure 11.5.
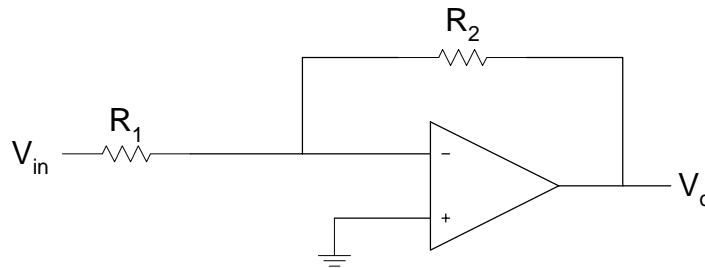

Figure 11.5 Inverting Amplifier

The closed-loop gain of the amplifier is

$$\frac{V_O}{V_{IN}} = -\frac{R_2}{R_1} \qquad (11.7)$$

and the input resistance is R_1. Normally, $R_2 > R_1$ such that $|V_0| > |V_{IN}|$. With the assumptions of very large open-loop gain and high input resistance, the closed-loop gain of the inverting amplifier depends on the external components R_1, R_2, and is independent of the open-loop gain.

For Figure 11.4, if $Z_1 = R_1$ and $Z_2 = \dfrac{1}{jwC}$, we obtain an integrator circuit shown in Figure 11.6. The closed-loop gain of the integrator is

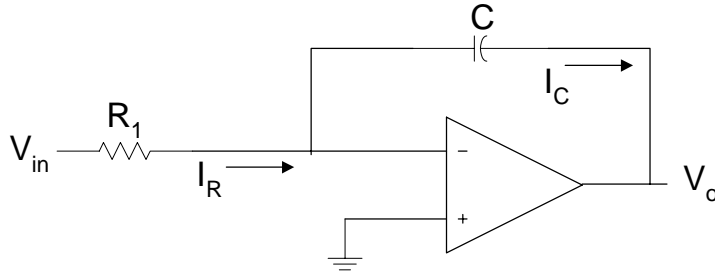$$\frac{V_O}{V_{IN}} = -\frac{1}{jwCR_1} \qquad (11.8)$$

Figure 11.6 Op Amp Inverting Integrator

In the time domain

$$\frac{V_{IN}}{R_1} = I_R \text{ and } I_C = -C\frac{dV_O}{dt} \tag{11.9}$$

Since $I_R = I_C$

$$V_O(t) = -\frac{1}{R_1 C}\int_0^t V_{IN}(t)d\tau + V_O(0) \tag{11.10}$$

The above circuit is termed the Miller integrator. The integrating time constant is CR_1. It behaves as a lowpass filter, passing low frequencies and attenuating high frequencies. However, at dc the capacitor becomes open circuited and there is no longer a negative feedback from the output to the input. The output voltage then saturates. To provide finite closed-loop gain at dc, a resistance R_2 is connected in parallel with the capacitor. The circuit is shown in . The resistance R_2 is chosen such that R_2 is greater than R.
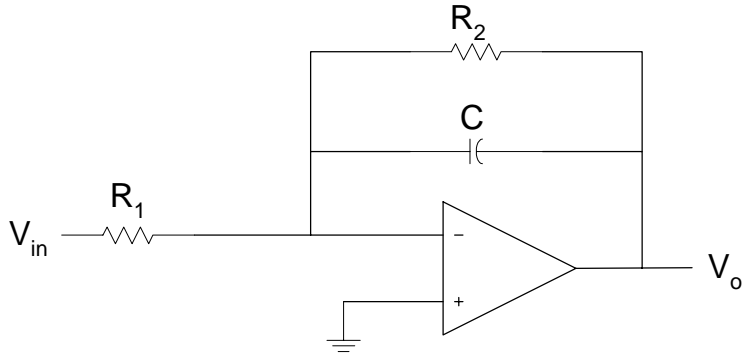
Figure 11.7 Miller Integrator with Finite Closed Loop Gain at DC

For Figure 11.4, if $Z_1 = \dfrac{1}{jwC}$ and $Z_2 = R,$ we obtain a differentiator circuit shown in Figure 11.8. From Equation (11.5), the closed-loop gain of the differentiator is
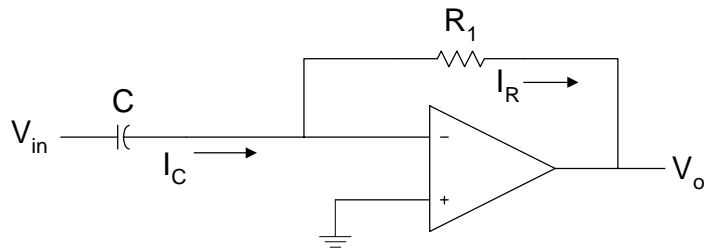
$$\frac{V_O}{V_{IN}} = -jwCR \qquad (11.11)$$

Figure 11.8 Op Amp Differentiator Circuit

In the time domain

$$I_C = C\frac{dV_{IN}}{dt} \ , \text{ and } V_O(t) = -I_R R \qquad (11.12)$$

Since

$$I_C(t) = I_R(t)$$

we have

$$V_O(t) = -CR\frac{dV_{IN}(t)}{dt} \tag{11.13}$$

Differentiator circuits will differentiate input signals. This implies that if an input signal is rapidly changing, the output of the differentiator circuit will appear " spike-like."

The inverting configuration can be modified to produce a weighted summer. This circuit is shown in Figure 11.9.
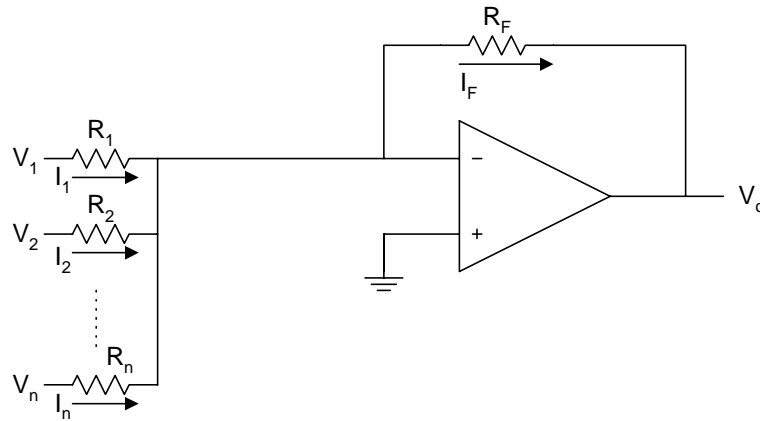

Figure 11.9 Weighted Summer Circuit

From Figure 11.9

$$I_1 = \frac{V_1}{R_1}, \quad I_2 = \frac{V_2}{R_2}, \quad ...., \quad I_n = \frac{V_n}{R_n} \tag{11.14}$$

also

$$I_F = I_1 + I_2 + ... I_N \tag{11.15}$$

$$V_O = -I_F R_F \tag{11.16}$$

Substituting Equations (11.14) and (11.15) into Equation (11.16) we have

$$V_O = -\left(\frac{R_F}{R_1}V_1 + \frac{R_F}{R_2}V_2 + \ldots \frac{R_F}{R_N}V_N\right) \qquad (11.17)$$

The frequency response of Miller integrator, with finite closed-loop gain at dc, is obtained in the following example.

Example 11.1

For Figure 11.7, (a) Derive the expression for the transfer function $\dfrac{V_o}{V_{in}}(jw)$.

(b) If $C = 1$ nF and $R_1 = 2K\Omega$, plot the magnitude response for R_2 equal to (i) 100 KΩ, (ii) 300KΩ, and (iii) 500KΩ.

Solution

$$Z_2 = R_2 \left\| \frac{1}{sC_2} = \frac{R_2}{1 + sC_2R_2} \qquad (11.18)$$

$$Z_1 = R_1 \qquad (11.19)$$

$$\frac{V_o}{V_{in}}(s) = \frac{-\dfrac{R_2}{R_1}}{1 + sC_2R_2} \qquad (11.20)$$

$$\frac{V_o}{V_{in}}(s) = \frac{-\dfrac{1}{C_2R_1}}{s + \dfrac{1}{C_2R_2}} \qquad (11.21)$$

MATLAB Script

```
% Frequency response of lowpass circuit
c = 1e-9; r1 = 2e3;
r2 = [100e3, 300e3, 500e3];
n1 = -1/(c*r1); d1 = 1/(c*r2(1));
num1 = [n1]; den1 = [1 d1];
w = logspace(-2,6);
h1 = freqs(num1,den1,w);
f = w/(2*pi);
```

```
d2 = 1/(c*r2(2)); den2 = [1 d2];
h2 = freqs(num1, den2, w);
d3 = 1/(c*r2(3)); den3 = [1 d3];
h3 = freqs(num1,den3,w);
semilogx(f,abs(h1),'w',f,abs(h2),'w',f,abs(h3),'w')
xlabel('Frequency, Hz')
ylabel('Gain')
axis([1.0e-2,1.0e6,0,260])
text(5.0e-2,35,'R2 = 100 Kilohms')
text(5.0e-2,135,'R2 = 300 Kilohms')
text(5.0e-2,235,'R2 = 500 Kilohms')
title('Integrator Response')
```

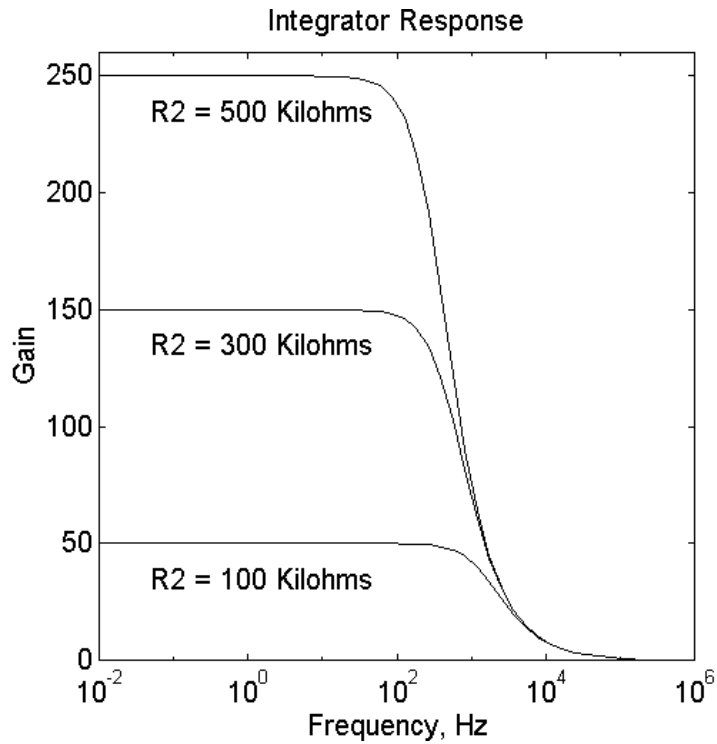Figure 11.10 shows the frequency response of Figure 11.7.

Figure 11.10 Frequency Response of Miller Integrator with Finite
Closed-Loop Gain at DC

11.3   NON-INVERTING CONFIGURATION

An op amp connected in a non-inverting configuration is shown in Figure 11.11.
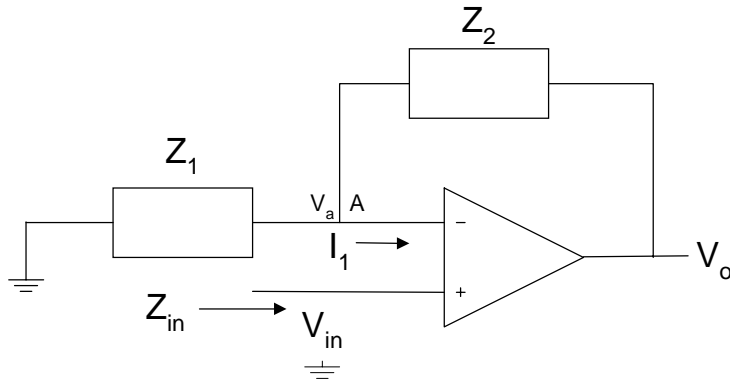
Figure 11.11 Non-Inverting Configuration

Using nodal analysis at node A

$$\frac{V_a}{Z_1} + \frac{V_a - V_O}{Z_2} + I_1 = 0 \tag{11.22}$$

From the concept of a virtual short circuit,

$$V_{IN} = V_a \tag{11.23}$$

and because of the large input resistance ($\dot{i}_1 = 0$), Equation (11.22) simplifies to

$$\frac{V_O}{V_{IN}} = 1 + \frac{Z_2}{Z_1} \tag{11.24}$$

The gain of the inverting amplifier is positive. The input impedance of the amplifier Z_{IN} approaches infinity, since the current that flows into the positive input of the op-amp is almost zero.

If $Z_1 = R_1$ and $Z_2 = R_2$, Figure 11.10 becomes a voltage follower with gain. This is shown in Figure 11.11.
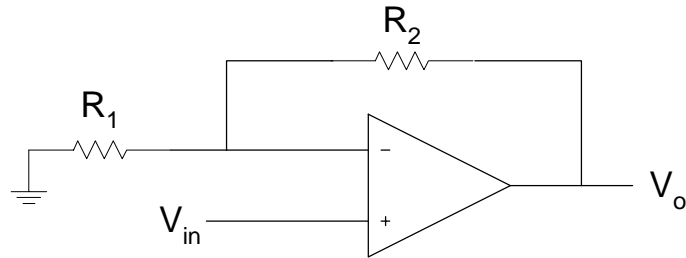
Figure 11.12 Voltage Follower with Gain

The voltage gain is

$$\frac{V_O}{V_{IN}} = \left(1 + \frac{R_2}{R_1}\right) \qquad (11.25)$$

The zero, poles and the frequency response of a non-inverting configuration are obtained in Example 11.2.

Example 11.2

For the Figure 11.13 (a) Derive the transfer function. (b) Use MATLAB to find the poles and zeros. (c) Plot the magnitude and phase response, assume that $C_1 = 0.1\text{uF}$, $C_2 = 1000\ 0.1\text{uF}$, $R_1 = 10\text{K}\Omega$, and $R_2 = 10\ \Omega$.

Figure 11.13 Non-inverting Configuration

Solution

Using voltage division

$$\frac{V_1}{V_{IN}}(s) = \frac{1/sC_1}{R_1 + 1/sC_1} \qquad (11.26)$$

From Equation (11.24)

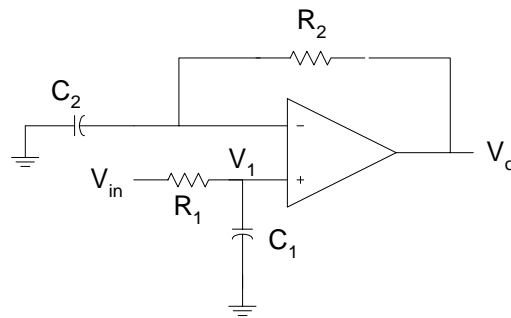$$\frac{V_O}{V_1}(s) = 1 + \frac{R_2}{1/sC_2} \qquad (11.27)$$

Using Equations (11.26) and (11.27), we have

$$\frac{V_O}{V_{IN}}(s) = \left(\frac{1 + sC_2R_2}{1 + sC_1R_1}\right) \qquad (11.28)$$

The above equation can be rewritten as

$$\frac{V_O}{V_{IN}}(s) = \frac{C_2R_2\left(s + \dfrac{1}{C_2R_2}\right)}{C_1R_1\left(s + \dfrac{1}{C_1R_1}\right)} \qquad (11.29)$$

The MATLAB program that can be used to find the poles, zero and plot the frequency response is as follows:

```
diary ex11_2.dat
% Poles and zeros, frequency response of Figure 11.13
%
%
c1 = 1e-7; c2 = 1e-3; r1 = 10e3; r2 = 10;

% poles and zeros
b1 = c2*r2;
a1 = c1*r1;
num = [b1 1];
den = [a1 1];
disp('the zero is')
z = roots(num)
```

```
disp('the poles are')
p = roots(den)

% the frequency response
w = logspace(-2,6);
h = freqs(num,den,w);
gain = 20*log10(abs(h));
f = w/(2*pi);
phase = angle(h)*180/pi;
subplot(211),semilogx(f,gain,'w');
xlabel('Frequency, Hz')
ylabel('Gain, dB')
axis([1.0e-2,1.0e6,0,22])
text(2.0e-2,15,'Magnitude Response')
subplot(212),semilogx(f,phase,'w')
xlabel('Frequency, Hz')
ylabel('Phase')
axis([1.0e-2,1.0e6,0,75])
text(2.0e-2,60,'Phase Response')

diary
```

The results are:

the zero is

z =

-100

the pole is

p =

-1000

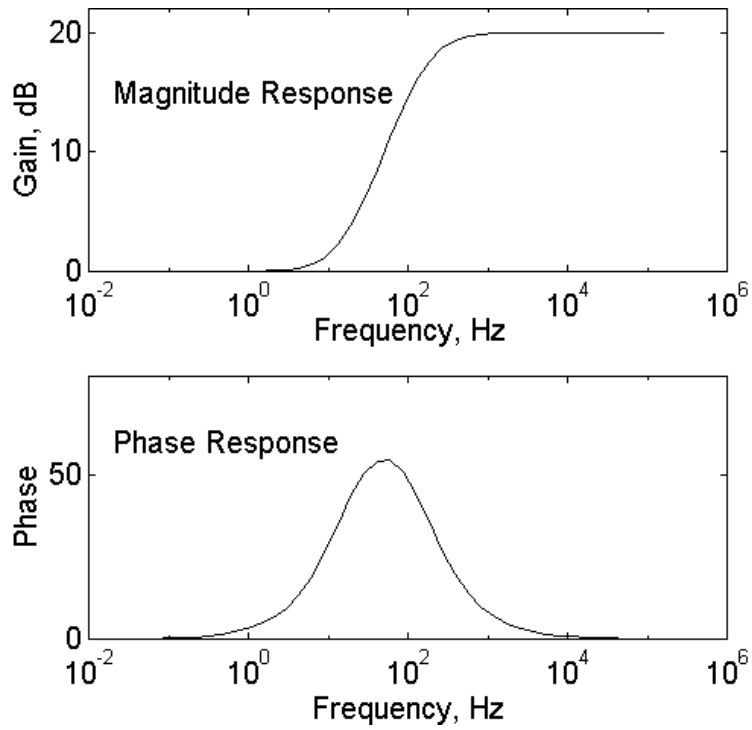The magnitude and phase plots are shown in Figure 11.14

Figure 11.14 Frequency Response of Figure 11.13

11.4   EFFECT OF FINITE OPEN-LOOP GAIN

For the inverting amplifier shown in Figure 11.15, if we assume a finite open-loop gain A, the output voltage V_0 can be expressed as

$$V_O = A(V_2 - V_1) \tag{11.30}$$

Since $V_2 = 0$,

$$V_1 = -\frac{V_O}{A}$$
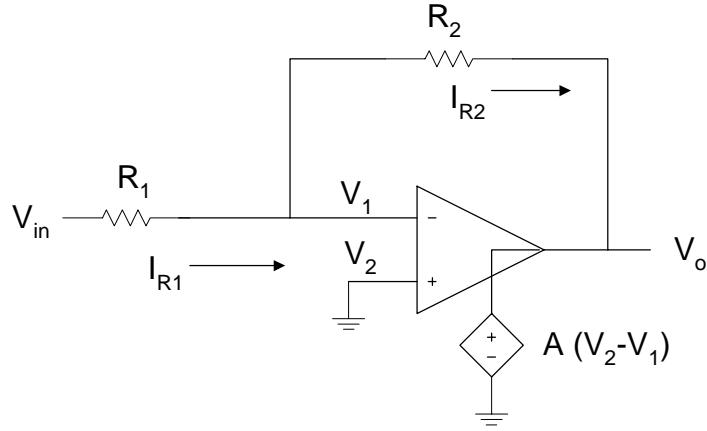
Figure 11.15 Inverter with Finite Open-loop Gain

Because the op amp has a very high input resistance, $i_1 = 0$, we have

$$I_{R1} = I_{R2} \tag{11.31}$$

But

$$I_{R1} = \frac{V_{IN} - V_1}{R_1} = \frac{V_{IN} - V_0/A}{R_1} \tag{11.32}$$

Also

$$V_O = V_1 - I_{R2}R_2 \tag{11.33}$$

Using Equations (11.30), (11.31) and (11.32), Equation (11.33) becomes

$$V_O = -\frac{V_O}{A} - \frac{R_2}{R_1}\left(V_{IN} + V_O/A\right) \tag{11.34}$$

Simplifying Equation (11.34), we get

$$\frac{V_O}{V_{IN}} = -\frac{R_2/R_1}{1 + \left(1 + R_2/R_1\right)/A} \tag{11.35}$$

It should be noted that as the open-loop gain approaches infinity, the closed-loop gain becomes

$$\frac{V_O}{V_{IN}} \cong -\frac{R_2}{R_1}$$

The above expression is identical to Equation (11.7). In addition, from Equation (11.30), the voltage V_1 goes to zero as the open-loop gain goes to infinity. Furthermore, to minimize the dependence of the closed-loop gain on the value of the open-loop gain, A, we should make

$$\left(1 + \frac{R_2}{R_1}\right) << A \qquad (11.36)$$

This is illustrated by the following example.

Example 11.3

In , R_1 = 500 Ω, and R_2 = 50 KΩ. Plot the closed-loop gain as the open-loop gain increases from 10^2 to 10^8.

Solution

```
% Effect of finite open-loop gain
%
a = logspace(2,8);
r1 = 500; r2 = 50e3; r21 = r2/r1;
g = [];
n = length(a);
for i = 1:n
 g(i) = r21/(1+(1+r21)/a(i));
end
semilogx(a,g,'w')
xlabel('Open loop gain')
ylabel('Closed loop gain')
title('Effect of Finite Open Loop Gain')
axis([1.0e2,1.0e8,40,110])
```

Figure 11.16 shows the characteristics of the closed-loop gain as a function of the open-loop gain.

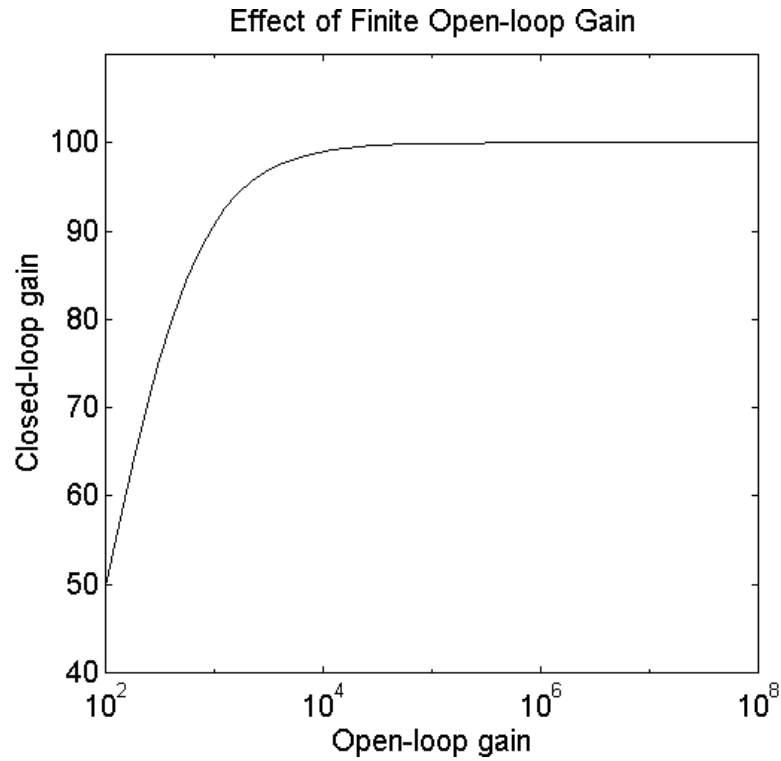## Effect of Finite Open-loop Gain

Figure 11.16   Closed-Loop Gain versus Open-Loop Gain

For the voltage follower with gain shown in Figure 11.12, it can be shown that the closed-loop gain of the amplifier with finite open-loop gain is

$$\frac{V_O}{V_{IN}} = -\frac{\left(1 + R_2/R_1\right)}{1 + \left(1 + R_2/R_1\right)\big/A} \tag{11.37}$$

11.5   FREQUENCY RESPONSE OF OP AMPS

The simplified block diagram of the internal structure of the operational amplifier is shown in Figure 11.17.
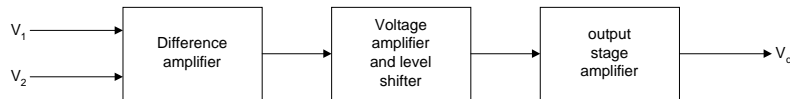

Figure 11.17 Internal Structure of Operational Amplifier

Each of the individual sections of the operational amplifier contains a lowpass RC section, with its corner (pole) frequency. Thus, an op amp will have an open-loop gain with frequency that can be expressed as

$$A(s) = \frac{A_O}{\left(1 + s/w_1\right)\left(1 + s/w_2\right)\left(1 + s/w_3\right)} \tag{11.38}$$

where

$$w_1 < w_2 < w_3$$

$$A_O = \text{gain at dc}$$

For most operational amplifiers, w_1 is very small (approx. 20π radians /s) and w_2 might be in the range of 2 to 6 mega-radians/s.

Example 11.4

The constituent parts of an operational amplifier have the following internal characteristics: the pole of the difference amplifier is at 200 Hz and the gain is - 500. The pole of the voltage amplifier and level shifter is 400 KHz and has a gain of 360. The pole of the output stage is 800KHz and the gain is 0.92. Sketch the magnitude response of the operational amplifier open-loop gain.

Solution

The lowpass filter response can be expressed as

$$\frac{V_O}{V_{IN}}(jw) = -\frac{C_{rstage}}{1 + jf/f_p} \qquad (11.39)$$

or

$$\frac{V_O}{V_{IN}}(s) = \frac{C_{rstage}}{1 + s/w_p} \qquad (11.40)$$

The transfer function of the amplifier is given as

$$A(s) = \frac{-500}{(1 + s/400\pi)} \frac{360}{(1 + s/8\pi 10^5)} \frac{0.92}{(1 + s/1.6\pi 10^6)} \qquad (11.41)$$

The above expression simplifies to

$$A(s) = \frac{2.62 x 10^{21}}{(s + 400\pi)(s + 8\pi 10^5)(s + 1.6\pi 10^6)} \qquad (11.42)$$

MATLAB script

```
% Frequency response of op amp
% poles are
p1 = 400*pi; p2 = 8e5*pi; p3 = 1.6e6*pi;
p = [p1 p2 p3];
% zeros
z = [0];
const = 2.62e21;

% convert to poles and zeros  and
% find the frequency response
a3 = 1;
a2 = p1 + p2 + p3;
a1 = p1*p2 + p1*p3 + p2*p3;
a0 = p1*p2*p3;
den = [a3 a2 a1 a0];
num = [const];
w = logspace(1,8);
```

```
h = freqs(num,den,w);
f = w/(2*pi);
g_db = 20*log10(abs(h));

% plot the magnitude response
semilogx(f,g_db)
title('Magnitude response')
xlabel('Frequency, Hz')
ylabel('Gain, dB')
```

The frequency response of the operational amplifier is shown in Figure 11.18.
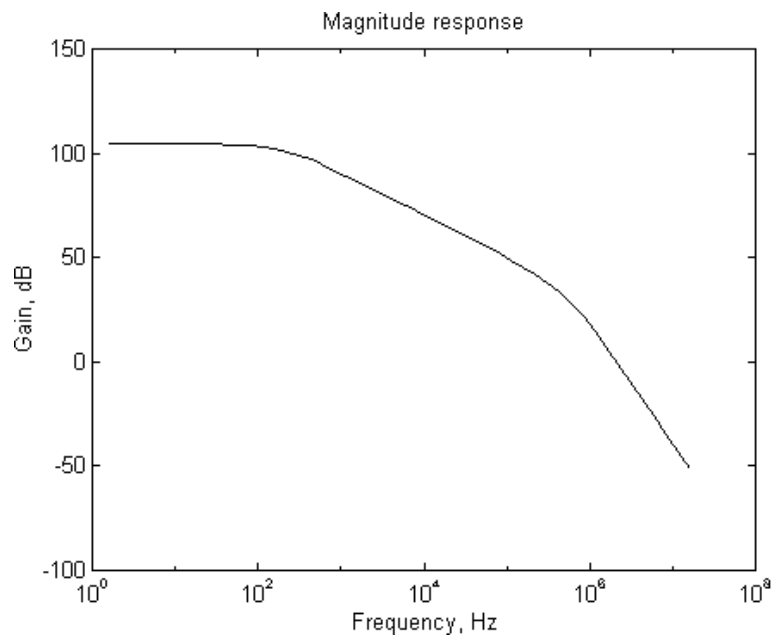
Figure 11.18   Open-Loop Gain Characteristics of an Op Amp

For an internally compensated op amp, there is a capacitor included on the IC chip. This causes the op amp to have a single pole lowpass response. The process of making one pole dominant in the open-loop gain characteristics is called frequency compensation, and the latter is done to ensure the stability of the op amp. For an internally compensated op amp, the open-loop gain $A(s)$ can be written as

$$A(s) = \frac{A_O}{\left(1 + s/w_b\right)} \tag{11.43}$$

where

A_0 is   dc open-loop gain

w_b is   break frequency.

For the 741 op amp, $A_0 = 10^5$ and $w_b = 20\,\pi$ radians/s. At physical frequencies $s = jw$, Equation (11.43) becomes

$$A(jw) = \frac{A_O}{\left(1 + jw/w_b\right)} \tag{11.44}$$

For frequencies $w > w_b$, Equation (11.44) can be approximated by

$$A(jw) = \frac{A_O w_b}{jw} \tag{11.45}$$

The unity gain bandwidth, w_t (the frequency at which the gain goes to unity), is given as

$$w_t = A_O w_b \tag{11.46}$$

For the inverting amplifier shown in , if we substitute Equation (11.43) into Equation (11.35), we get a closed-loop gain

$$\frac{V_O}{V_{IN}}(s) = -\frac{R_2/R_1}{1 + \left(1 + R_2/R_1\right)/A_o + \dfrac{s}{w_t/\left(1 + R_2/R_1\right)}} \tag{11.47}$$

In the case of non-inverting amplifier shown in , if we substitute Equation (11.43) into Equation (11.37), we get the closed-loop gain expression

$$\frac{V_O}{V_{IN}}(s) = \frac{1 + R_2/R_1}{1 + \left(1 + R_2/R_1\right)/A_o + \dfrac{s}{w_t/\left(1 + R_2/R_1\right)}} \qquad (11.48)$$

From Equations (11.47) and (11.48), it can be seen that the break frequency for the inverting and non-inverting amplifiers is given by the expression

$$w_{3dB} = \frac{w_t}{1 + R_2/R_1} \qquad (11.49)$$

The following example illustrates the effect of the ratio $\dfrac{R_2}{R_1}$ on the frequency response of op amp circuits.

Example 11.5

An op amp has an open-loop dc gain of 10^7, the unity gain bandwidth of 10^8 Hz. For an op amp connected in an inverting configuration (Figure 11.5), plot the magnitude response of the closed-loop gain.

if $\dfrac{R_2}{R_1} = 100$, 600, 1100

Solution

Equation (11.47) can be written as

$$\frac{V_o}{V_{IN}}(s) = \frac{\dfrac{w_t R_2}{R_1(1 + R_2/R_1)}}{s + \dfrac{w_t}{A_0} + \dfrac{w_t}{(1 + R_2/R_1)}} \qquad (11.50)$$

MATLAB script

```
% Inverter closed-loop gain versus frequency
w = logspace(-2,10);   f = w/(2*pi);
r12 = [100 600 1100];
```

```
a =[];    b = [];   num = [];   den = [];  h = [];
for i = 1:3
 a(i) = 2*pi*1.0e8*r12(i)/(1+r12(i));
 b(i) = 2*pi*1.0e8*((1/(1+r12(i))) + 1.0e-7);
 num = [a(i)];
 den = [1 b(i)];
 h(i,:) = freqs(num,den,w);
end
semilogx(f,abs(h(1,:)),'w',f,abs(h(2,:)),'w',f,abs(h(3,:)),'w')
title('Op Amp Frequency Characteristics')
xlabel('Frequency, Hz')
ylabel('Gain')
axis([1.0e-2,1.0e10,0,1200])
text(1.5e-2, 150, 'Resistance ratio of 100')
text(1.5e-2, 650, 'Resistance ratio of 600')
text(1.50e-2, 1050, 'Resistance ratio of 1100')
```

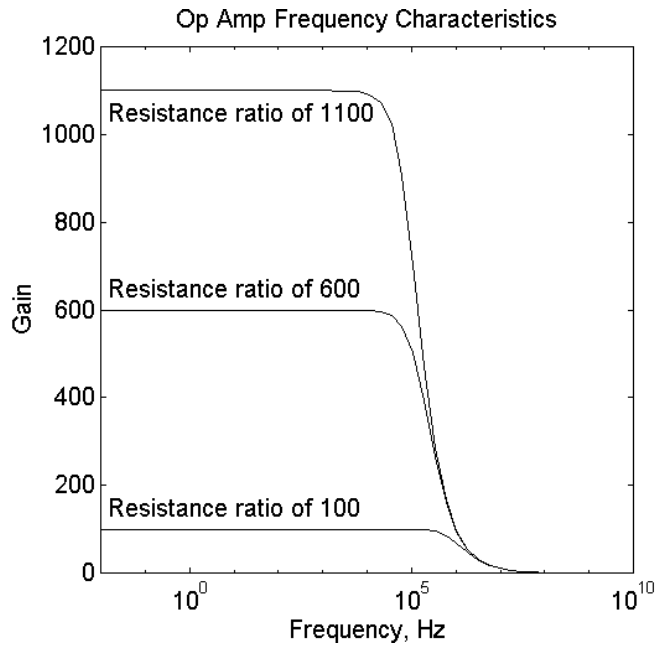Figure 11.19 shows the plots obtained from the MATLAB program.

Figure 11.19   Frequency Response of an Op Amp Inverter with
Different Closed Loop Gain

11.6   SLEW RATE AND FULL-POWER BANDWIDTH

Slew rate *(SR)* is a measure of the maximum possible rate of change of the output voltage of an op amp. Mathematically, it is defined as

$$SR = \frac{dV_O}{dt}\bigg|_{\text{max}} \qquad (11.51)$$

The slew rate is often specified on the op amp data sheets in V/μs. Poor op amps might have slew rates around 1V/μs and good ones might have slew rates up to 1000 V/μs are available, but the good ones are relatively expensive.

Slew rate is important when an output signal must follow a large input signal that is rapidly changing. If the slew rate is lower than the rate of change of the input signal, then the output voltage will be distorted. The output voltage will become triangular, and attenuated. However, if the slew rate is higher than the rate of change of the input signal, no distortion occurs and input and output of the op amp circuit will have similar wave shapes.

As mentioned in the Section (11.5), frequency compensated op amp has an internal capacitance that is used to produce a dominant pole.   In addition, the op amp has a limited output current capability, due to the saturation of the input stage. If we designate I_{max} as the maximum possible current that is available to charge the internal capacitance of an op amp, the charge on the frequency-compensation capacitor is

$$CdV = Idt$$

Thus, the highest possible rate of change of the output voltage is

$$SR = \frac{dV_O}{dt}\bigg|_{\text{max}} = \frac{I_{\text{max}}}{C} \qquad (11.52)$$

For a sinusoidal   input signal given by

$$v_i(t) = V_m \sin wt \qquad (11.53)$$

The rate of change of the input signal is

$$\frac{dv_i(t)}{dt} = wV_m \cos wt \tag{11.54}$$

Assuming that the input signal is applied to a unity gain follower, then the output rate of change

$$\frac{dV_O}{dt} = \frac{dv_i(t)}{dt} = wV_m \cos wt \tag{11.55}$$

The maximum value of the rate of change of the output voltage occurs when $\cos(wt) = 1$, i.e., $wt = 0, 2\pi, 4\pi. \ldots,$ the slew rate

$$SR = \frac{dV_O}{dt}\bigg|_{max} = wV_m \tag{11.56}$$

Equation (11.56) can be used to define full-power bandwidth. The latter is the frequency at which a sinusoidal rated output signal begins to show distortion due to slew rate limiting. Thus

$$w_m V_{o,rated} = SR \tag{11.57}$$

Thus

$$f_m = \frac{SR}{2\pi, V_{o,rated}} \tag{11.58}$$

The full-power bandwidth can be traded for output rated voltage, thus, if the output rated voltage is reduced, the full-power bandwidth increases. The following example illustrates the relationship between the rated output voltage and the full-power bandwidth.

Example 11.6

The LM 741 op amp has a slew rate of 0.5 V/µs. Plot the full-power bandwidth versus the rated output voltage if the latter varies from ± 1 to ± 10 V.

Solution

% Slew rate and full-power bandwidth

sr = 0.5e6;

```
v0 = 1.0:10;
fm = sr./(2*pi*v0);

plot(v0,fm)
title('Full-power Bandwidth vs. Rated Output Voltage')
xlabel('Rated output voltage, V')
ylabel('Bandwidth, Hz')
```

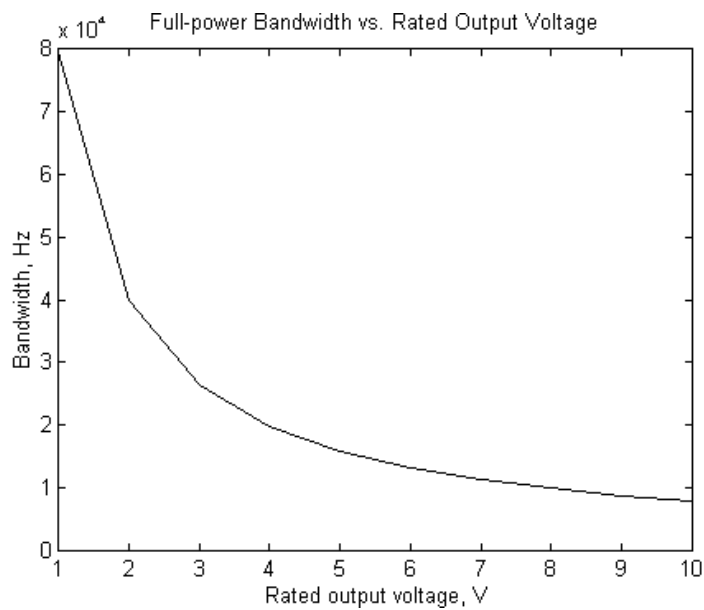Figure 11.20 shows the plot for Example 11.6.

Figure 11.20 Rated Output Voltage versus Full-power Bandwidth

11.7   COMMON-MODE REJECTION

For practical op amps, when two inputs are tied together and a signal applied to the two inputs, the output will be nonzero. This is illustrated in Figure 11.21a, where the
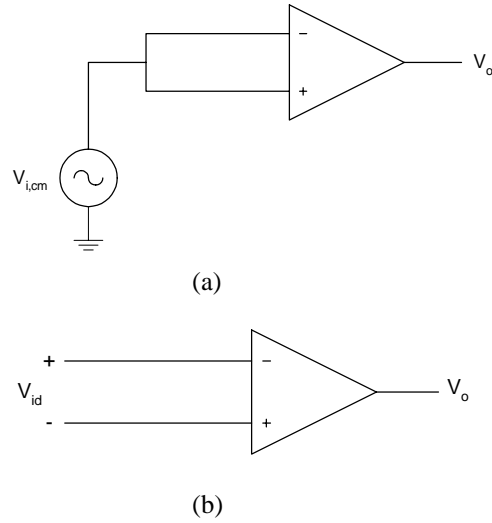
(a)

(b)

Figure 11.21 Circuits Showing the Definitions of (a) Common-mode Gain and (b) Differential-mode Gain

common-mode gain, A_{cm}, is defined as

$$A_{cm} = \frac{v_o}{v_{i,cm}} \tag{11.59}$$

The differential-mode gain, A_d, is defined as

$$A_d = \frac{v_o}{v_{id}} \tag{11.60}$$

For an op amp with arbitrary input voltages, V_1 and V_2 (see Figure 11.21b), the differential input signal, v_{id}, is

$$v_{id} = V_2 - V_1 \tag{11.61}$$

and the common mode input voltage is the average of the two input signals,

$$V_{i,cm} = \frac{V_2 + V_1}{2} \tag{11.62}$$

The output of the op amp can be expressed as

$$V_O = A_d v_{id} + A_{cm} v_{i,cm} \tag{11.63}$$

The common-mode rejection ratio (CMRR) is defined as

$$CMRR = \left| \frac{A_d}{A_{cm}} \right| \tag{11.64}$$

The CMRR represents the op amp's ability to reject signals that are common to the two inputs of an op amp. Typical values of CMRR range from 80 to 120 dB. CMRR decreases as frequency increases.

For an inverting amplifier as shown in Figure 11.5, because the non-inverting input is grounded, the inverting input will also be approximately 0 V due to the virtual short circuit that exists in the amplifier. Thus, the common-mode input voltage is approximately zero and Equation (11.63) becomes

$$V_O \cong A_d V_{id} \tag{11.65}$$

The finite CMRR does not affect the operation of the inverting amplifier.

A method normally used to take into account the effect of finite CMRR in calculating the closed-loop gain is as follows: The contribution of the output voltage due to the common-mode input is $A_{cm} V_{i,cm}$. This output voltage contribution can be obtained if a differential input signal, V_{error}, is applied to the input of an op amp with zero common-mode gain.

Thus

$$V_{error} A_d = A_{cm} V_{i,cm} \tag{11.66}$$

$$V_{error} = \frac{A_{cm} V_{i,cm}}{A_d} = \frac{V_{i,cm}}{CMRR} \tag{11.67}$$

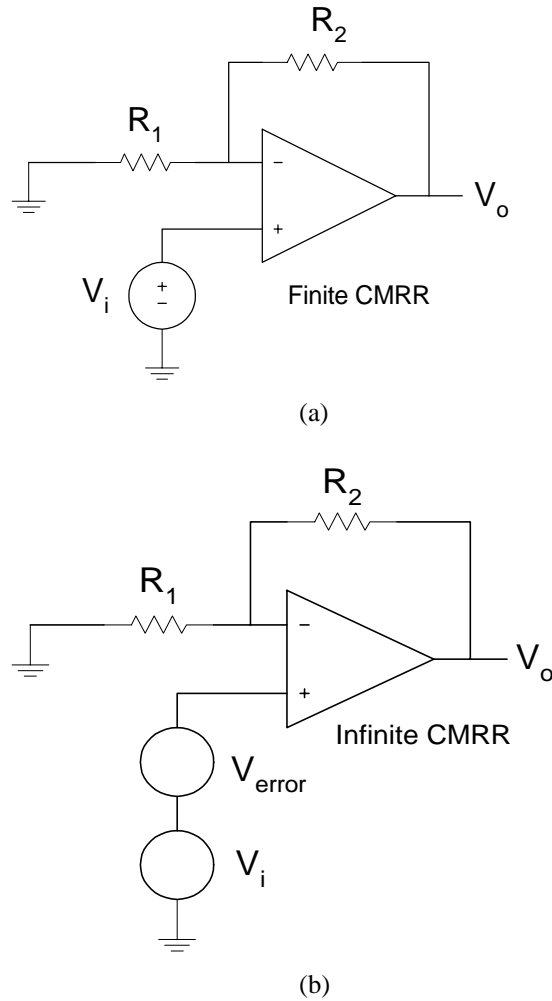Figure 11.22 shows how to use the above technique to analyze a non-inverting amplifier with a finite CMRR.

Figure 11.22 Non-inverting Amplifier (a) Finite CMRR
(b) Infinite CMRR

From Figure 11.22b, the output voltage is given as

$$V_O = V_i\left(1 + R_2/R_1\right) + \frac{V_i}{CMRR}\left(1 + R_2/R_1\right)$$
(11.68)

The following example illustrates the effect of a finite CMRR on the closed-loop gain of a non-inverting amplifier.

Example 11.7

For the amplifier shown in Figure 11.22, if $R_2 = 50K\Omega$ and $R_1 = 1K\Omega$, plot the closed-loop gain versus CMRR for the following values of the latter: $10^4, 10^5, 10^6, 10^7, 10^8$ and 10^9.

Solution

MATLAB Script

```
% Non-inverting amplifier with finite CMRR
r2 = 50e3; r1 = 1.0e3; rr = r2/r1;
cmrr = logspace(4,9,6);  gain = (1+rr)*(1+1./cmrr);
semilogx(cmrr,gain,'wo')
xlabel('Common-mode Rejection Ratio')
ylabel('Closed Loop Gain')
title('Gain versus CMRR')
axis([1.0e3,1.0e10,50.998, 51.008])
```

Figure 11.23 shows the effect of CMRR on the closed loop of a non-inverting amplifier.
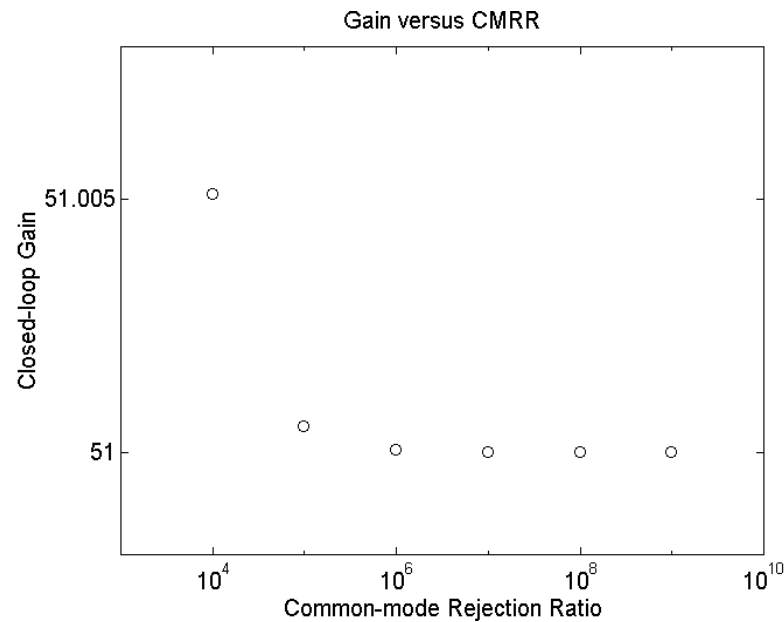

Figure 11.23   Effect of finite CMRR on the Gain of a Non-inverting Amplifier

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3. Sedra, A.S. and Smith, K.C., *Microelectronics Circuits*, 4th Edition, Oxford University Press, 1997.

4. Ferris, C.D., *Elements of Electronic Design*, West Publishing, 1995.

5. Irvine, R.G., *Operational Amplifiers - Characteristics and Applications*, Prentice Hall, 1981.

6. Ghausi, M.S., *Electronic Devices and Circuits: Discrete and Integrated*, HRW, 1985.

EXERCISES

11.1   For the circuit shown in Figure P11.1, (a) derive the transfer function $\dfrac{V_O}{V_{IN}}(s)$. (b) If $R_1 = 1K\Omega$, obtain the magnitude response.
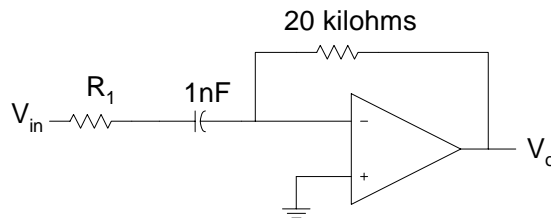

Figure P11.1 An Op Amp Filter

11.2   For Figure 11.12, if the open-loop gain is finite, (a) show that the closed-loop gain is given by the expression shown in Equation (11.37). (b) If $R_2 = 100K$ and $R_1 = 0.5K$, plot the percentage error

in the magnitude of the closed-loop gain for open-loop gains of $10^2, 10^4, 10^6$ and 10^8.

11.3 Find the poles and zeros of the circuit shown in Figure P11.3. Use MATLAB to plot the magnitude response. The resistance values are in kilohms.
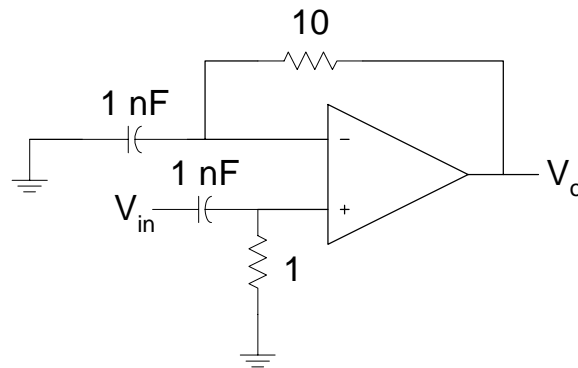
Figure P11.3 An Op Amp Circuit

11.4 For the amplifier shown in Figure 11.12, if the open-loop gain is 106, $R_2 = 24K$, and $R_1 = 1K$, plot the frequency response for a unity gain bandwidth of $10^6, 10^7$, and 10^8 Hz.

11.5 For the inverting amplifier, shown in Figure 11.5, plot the 3-dB frequency versus resistance ratio $\dfrac{R_2}{R_1}$ for the following values of the resistance ratio: 10, 100, 1000, 10,000 and 100,000. Assume that $A_O = 10^6$ and $f_t = 10^7$ Hz.

11.6 For the inverting amplifier, shown in Figure 11.5, plot the closed loop gain versus resistance ratio $\dfrac{R_2}{R_1}$ for the following open-loop gain, A_O: 103, 105 and 107. Assume a unity gain bandwidth of

$$f_t = 10^7 \text{ Hz and resistance ratio, } \frac{R_2}{R_1} \text{ has the following values: 10,}$$

100, 1000, 10,000 and 100,000.

11.7 An op amp with a slew rate of 1 V/μs is connected in the unity gain follower configuration. A square wave of zero dc voltage and a peak voltage of 1 V and a frequency of 100 KHz is connected to the input of the unity gain follower. Write a MATLAB program to plot the output voltage of the amplifier.

11.8 For the non-inverting amplifier, if R_{icm} = 400 MΩ, R_{id} = 50 MΩ, R_1 = 2KΩ and R_2 = 30KΩ, plot the input resistance versus the dc open-loop gain A_0. Assume the following values of the open-loop gain: $10^3, 10^5, 10^7$ and 10^9.

Attia, John Okyere. "Transistor Circuits."
Electronics and Circuit Analysis using MATLAB.
Ed. John Okyere Attia
Boca Raton: CRC Press LLC, 1999

CHAPTER TWELVE

TRANSISTOR CIRCUITS

In this chapter, MATLAB will be used to solve problems involving metal-oxide semiconductor field effect and bipolar junction transistors. The general topics to be discussed in this chapter are dc model of BJT and MOSFET, biasing of discrete and integrated circuits,   and frequency response of amplifiers.

12.1   BIPOLAR JUNCTION TRANSISTORS

Bipolar junction transistor (BJT) consists of two pn junctions connected back-to-back. The operation of the BJT depends on the flow of both majority and minority carriers. There are two types of BJT: npn and pnp transistors. The electronic symbols of the two types of transistors are shown in Figure 12.1.
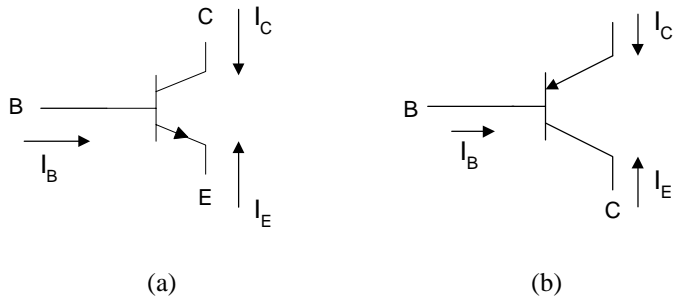

(a)   (b)

Figure 12.1   (a) NPN transistor   (b) PNP Transistor

The dc behavior of the BJT can be described by the Ebers-Moll Model. The equations for the model are

$$I_F = I_{ES}\left[\exp\left(\frac{V_{BE}}{V_T}\right) - 1\right] \tag{12.1}$$

$$I_R = I_{CS}\left[\exp\left(\frac{V_{BC}}{V_T}\right) - 1\right] \tag{12.2}$$

and

$$I_C = \alpha_F I_F - I_R \tag{12.3}$$

$$I_E = -I_F + \alpha_R I_R \tag{12.4}$$

and

$$I_B = (1 - \alpha_F)I_F + (1 - \alpha_R)I_R \tag{12.5}$$

where

I_{ES} and I_{CS} are the base-emitter and base-collector saturation currents, respectively

α_R is large signal reverse current gain of a common-base configuration

α_F is large signal forward current gain of the common-base configuration.

and

$$V_T = \frac{kT}{q} \tag{12.6}$$

where

k is the Boltzmann's constant ($k = 1.381 \times 10^{-23}$ V.C/$^\circ$ K),
T is the absolute temperature in degrees Kelvin, and
q is the charge of an electron (q = 1.602×10^{-19} C).

The forward and reverse current gains are related by the expression

$$\alpha_R I_{CS} = \alpha_F I_{ES} = I_S \tag{12.7}$$

where

I_S is the BJT transport saturation current.

The parameters α_R and α_F are influenced by impurity concentrations and junction depths. The saturation current, I_S, can be expressed as

$$I_S = J_S A \tag{12.8}$$

where

A is the area of the emitter and

J_S is the transport saturation current density, and it can be further expressed as

$$J_S = \frac{qD_n n_i^2}{Q_B} \tag{12.9}$$

where

D_n is the average effective electron diffusion constant

n_i is the intrinsic carrier concentration in silicon ($n_i = 1.45$ x 10^{10} atoms / cm^3 at 300° K)

Q_B is the number of doping atoms in the base per unit area.

The dc equivalent circuit of the BJT is based upon the Ebers-Moll model. The model is shown in Figure 12.2. The current sources $\alpha_R I_R$ indicate the interaction between the base-emitter and base-collector junctions due to the narrow base region.

In the case of a pnp transistor, the directions of the diodes in Figure 12.2 are reversed. In addition, the voltage polarities of Equations (12.1) and (12.2) are reversed. The resulting Ebers-Moll equations for pnp transistors are

$$I_E = I_{ES}\left[\exp\left(\frac{V_{EB}}{V_T}\right) - 1\right] - \alpha_R I_{CS}\left[\exp\left(\frac{V_{CB}}{V_T}\right) - 1\right] \tag{12.10}$$

$$I_C = -\alpha_F I_{ES}\left[\exp\left(\frac{V_{EB}}{V_T}\right) - 1\right] + I_{CS}\left[\exp\left(\frac{V_{CB}}{V_T}\right) - 1\right] \tag{12.11}$$
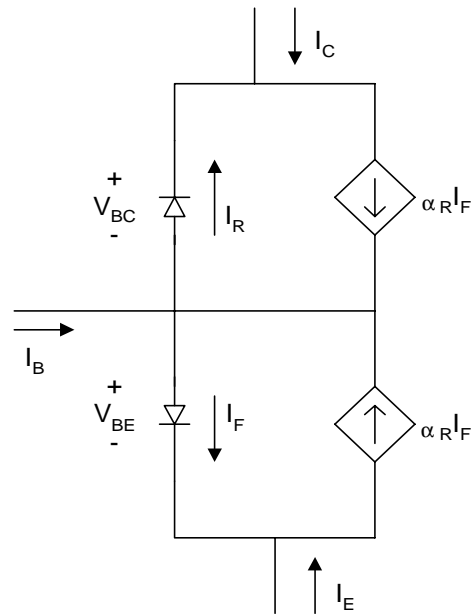
Figure 12.2 Ebers-Moll Static Model for an NPN transistor
(Injection Version)

The voltages at the base-emitter and base-collector junctions will define the regions of operation. The four regions of operations are forward-active, reverse-active, saturation and cut-off. Figure 12.3 shows the regions of operation based on the polarities of the base-emitter and base collector junctions.

Forward-Active Region

The forward-active region corresponds to forward biasing the emitter-base junction and reverse biasing the base-collector junction. It is the normal operational region of transistors employed for amplifications. If $V_{BE} > 0.5$ V and $V_{BC} < 0.3$V, then equations (12.1) to (12.4) and (12.6) can be rewritten as

$$I_C = I_S \exp\left(\frac{V_{BE}}{V_T}\right) \tag{12.12}$$

$$I_E = -\frac{I_S}{\alpha_F} \exp\left(\frac{V_{BE}}{V_T}\right) \tag{12.13}$$

From Figure 12.1,

$$I_B = -\left(I_C + I_E\right) \tag{12.14}$$

Substituting Equations (12.12) and (12.13) into (12.14), we have

$$I_B = I_S \frac{\left(1 - \alpha_F\right)}{\alpha_F} \exp\left(\frac{V_{BE}}{V_T}\right) \tag{12.15}$$

$$= \frac{I_S}{\beta_F} \exp\left(\frac{V_{BE}}{V_T}\right) \tag{12.16}$$

where

$\beta_F = $ large signal forward current gain of common-emitter configuration

$$\beta_F = \frac{\alpha_F}{1 - \alpha_F} \tag{12.17}$$

From Equations (12.12) and (12.16), we have

$$I_C = \beta_F I_B \tag{12.18}$$

We can also define, β_R, the large signal reverse current gain of the common-emitter configuration as

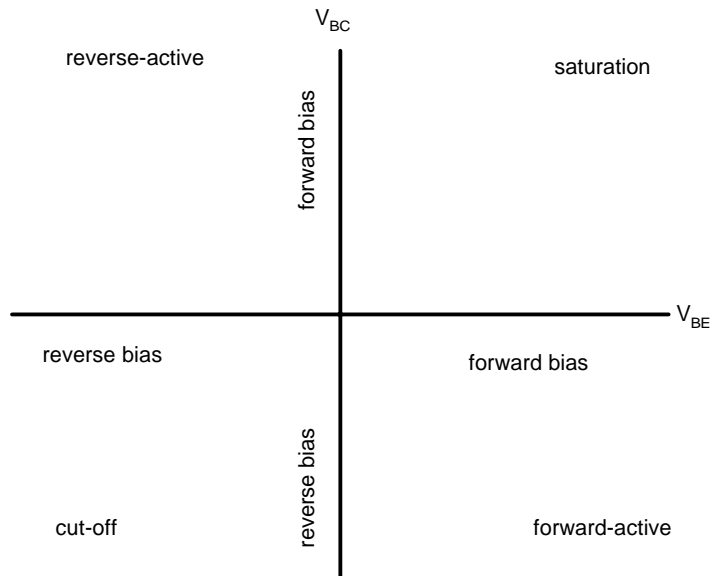$$\beta_R = \frac{\alpha_R}{1 - \alpha_R} \tag{12.19}$$

Figure 12.3 Regions of Operation for a BJT as Defined by the Bias of V_{BE} and V_{BC}

Reverse-Active Region

The reverse-active region corresponds to reverse biasing the emitter-base junction and forward biasing the base-collector junction. The Ebers-Moll model in the reverse-active region ($V_{BC} > 0.5$V and $V_{BE} < 0.3$V) simplifies to

$$I_E = I_S \left[\frac{V_{BC}}{V_T} \right] \tag{12.20}$$

$$I_B = \frac{I_S}{\beta_R} \exp \left[\frac{V_{BC}}{V_T} \right] \tag{12.21}$$

Thus,

$$I_E = \beta_R I_B \tag{12.22}$$

The reverse-active region is seldom used.

Saturation and Cut-off Regions

The saturation region corresponds to forward biasing both base-emitter and base-collector junctions. A switching transistor will be in the saturation region when the device is in the conducting or "ON" state.

The cut-off region corresponds to reverse biasing the base-emitter and base-collector junctions. The collector and base currents are very small compared to those that flow when transistors are in the active-forward and saturation regions. In most applications, it is adequate to assume that $I_C = I_B = I_E = 0$ when a BJT is in the cut-off region. A switching transistor will be in the cut-off region when the device is not conducting or in the "OFF" state.

Example 12.1

Assume that a BJT has an emitter area of 5.0 mil², $\beta_F = 120, \ \beta_R = 0.3$ transport current density, $J_S = 2 * 10^{-10} \ \mu A / mil^2$ and $T = 300°$K. Plot I_E versus V_{BE} for $V_{BC} = -1$V. Assume $0 < V_{BE} < 0.7$ V.

Solution

From Equations (12.1), (12.2) and (12.4), we can write the following MATLAB program.

MATLAB Script

```
%Input characteristics of a BJT
diary ex12_1.dat
diary on
k=1.381e-23; temp=300; q=1.602e-19;
cur_den=2e-10; area=5.0; beta_f=120; beta_r=0.3;
vt=k*temp/q; is=cur_den*area;
alpha_f=beta_f/(1+beta_f);
alpha_r = beta_r/(1+beta_r);
ies=is/alpha_f;
vbe=0.3:0.01:0.65;
ics=is/alpha_r;
m=length(vbe)
for i = 1:m
 ifr(i) = ies*exp((vbe(i)/vt)-1);
```

```
 ir1(i) = ics*exp((-1.0/vt)-1);
 ie1(i) = abs(-ifr(i) + alpha_r*ir1(i));
end
plot(vbe,ie1)
title('Input characteristics')
xlabel('Base-emitter voltage, V')
ylabel('Emitter current, A')
```

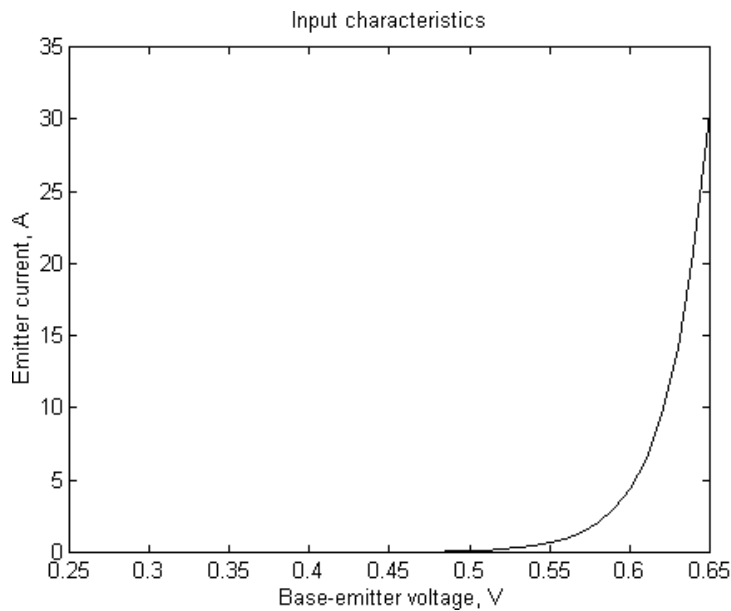Figure 12.4 shows the input characteristics.

Figure 12.4 Input Characteristics of a Bipolar Junction Transistor

Experimental studies indicate that the collector current of the BJT in the forward-active region increases linearly with the voltage between the collector-emitter V_{CE}. Equation 12.12 can be modified as

$$I_C \cong I_S \exp\left[\frac{V_{BE}}{V_T}\right]\left(1 + \frac{V_{CE}}{V_{AF}}\right) \qquad (12.23)$$

where

V_{AF} is a constant dependent on the fabrication process.

Example 12.2

For an npn transistor with emitter area of 5.5 mil^2, $\alpha_F = 0.98$, $\alpha_R = 0.35$, $V_{AF} = 250\,V$ and transport current density is $2.0\,x\,10^{-9}\ \mu A/mil^2$. Use MATLAB to plot the output characteristic for $V_{BE} = 0.65$ V. Neglect the effect of V_{AF} on the output current I_C. Assume a temperature of 300 °K.

Solution

MATLAB Script

```
%output characteristic of an npn transistor
%
diary ex12_2.dat
k=1.381e-23; temp=300; q=1.602e-19;
cur_den=2.0e-15; area=5.5; alpha_f=0.98;
alpha_r=0.35; vt=k*temp/q; is=cur_den*area;
ies=is/alpha_f; ics=is/alpha_r;
vbe= [0.65];
vce=[0 0.07 0.1 0.2 0.3 0.4 0.5 0.6 0.7 1 2 4 6];
n=length(vbe);
m=length(vce);
for i=1:n
  for j=1:m
    ifr(i,j)= ies*exp((vbe(i)/vt) - 1);
    vbc(j) = vbe(i) - vce(j);
    ir(i,j) = ics*exp((vbc(j)/vt) - 1);
    ic(i,j) = alpha_f*ifr(i,j) - ir(i,j);
  end
end
ic1 = ic(1,:);
plot(vce, ic1,'w')
title('Output Characteristic')
xlabel('Collector-emitter Voltage, V')
ylabel('Collector current, A')
text(3,3.1e-4, 'Vbe = 0.65 V')
axis([0,6,0,4e-4])
```

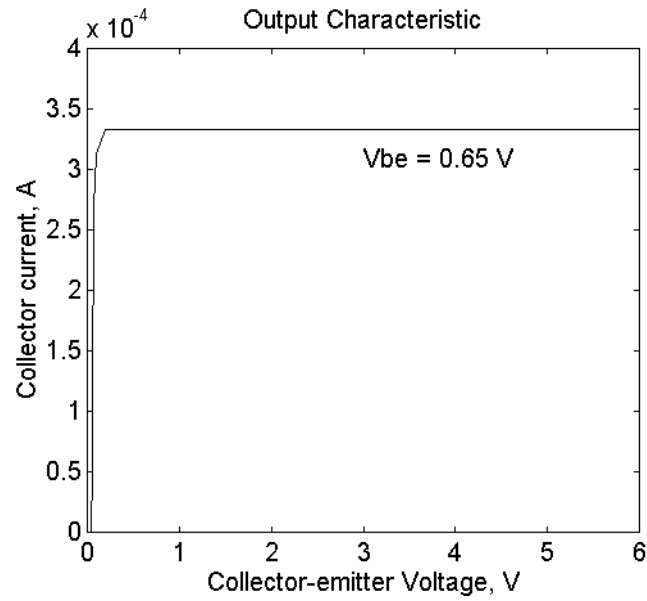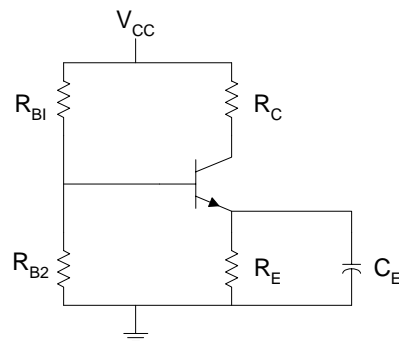Figure 12.5 shows the output characteristic.

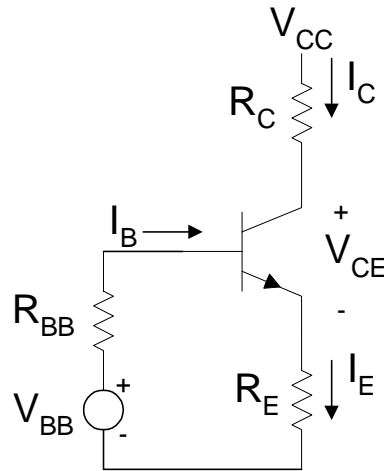**Figure 12.5** Output Characteristic on an NPN Transistor

12.2 BIASING BJT DISCRETE CIRCUITS

12.2.1 Self-bias circuit

One of the most frequently used biasing circuits for discrete transistor circuits is the self-bias of the emitter-bias circuit shown in .

(a)

(b)

Figure 12.6 (a) Self-Bias Circuit (b) DC Equivalent Circuit of (a)

The emitter resistance, R_E, provides stabilization of the bias point. If V_{BB} and R_B are the Thevenin equivalent parameters for the base bias circuit, then

$$V_{BB} = \frac{V_{CC}R_{B2}}{R_{B1} + R_{B2}}$$ (12.24)

$$R_B = R_{B1}\|R_{B2}$$ (12.25)

Using Kirchoff's Voltage Law for the base circuit, we have

$$V_{BB} = I_B R_B + V_{BE} + I_E R_E$$ (12.26)

Using Equation (12.18) and Figure 12.6b, we have

$$I_E = I_B + I_C = I_B + \beta_F I_B = (\beta_F + 1)I_B$$ (12.27)

Substituting Equations (12.18) and (12.27) into (12.26), we have

$$I_B = \frac{V_{BB} - V_{BE}}{R_B + (\beta_F + 1)R_E} \tag{12.28}$$

or

$$I_C = \frac{V_{BB} - V_{BE}}{\dfrac{R_B}{\beta_F} + \dfrac{(\beta_F + 1)}{\beta_F}R_E} \tag{12.29}$$

Applying KVL at the output loop of Figure 12.6b gives

$$V_{CE} = V_{CC} - I_C R_C - I_E R_E \tag{12.30}$$

$$= V_{CC} - I_C\left(R_C + {R_E}\big/{\alpha_F}\right) \tag{12.31}$$

12.2.2   Bias stability

Equation (12.30) gives the parameters that influence the bias current I_C. The voltage V_{BB} depends on the supply voltage V_{CC}. In some cases, V_{CC} would vary with I_C, but by using a stabilized voltage supply we can ignore the changes in V_{CC}, and hence V_{BB}. The changes in the resistances R_{BB} and R_E are negligible. There is a variation of β_F with respect to changes in I_C. A typical plot of β_F versus I_C is shown in Figure 12.7.
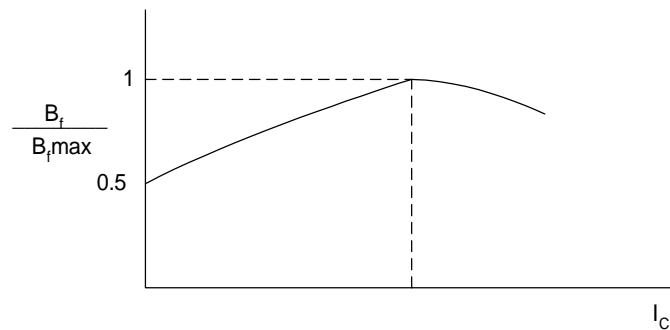

Figure 12.7 Normalized plot of β_F as a Function of Collector Current

Temperature changes cause two transistor parameters to change. These are (1) base-emitter voltage (V_{BE}) and (2) collector leakage current between the base and collector (I_{CBO}). The variation on V_{BE} with temperature is similar to the changes of the pn junction diode voltage with temperature. For silicon transistors, the voltage V_{BE} varies almost linearly with temperature as

$$\Delta V_{BE} \cong -2(T_2 - T_1) \qquad mV \qquad (12.32)$$

where

T_1 and T_2 are in degrees Celsius.

The collector-to-base leakage current, I_{CBO}, approximately doubles every 10° temperature rise. As discussed in Section 9.1, if I_{CBO1} is the reverse leakage current at room temperature (25 °C), then

$$I_{CBO2} = 2^{\left(\frac{T_2 - 25^O C/10}{}\right)} I_{CBO1}$$

and

$$\Delta I_{CBO} = I_{CBO2} - I_{CBO1} = I$$

$$= I_{CBO}\left[2^{\left(\frac{T_2 - 25^O C/10}{}\right)} - 1\right] \qquad (12.33)$$

Since the variations in I_{CBO} and V_{BE} are temperature dependent, but changes in V_{CC} and β_F are due to factors other than temperature, the information about the changes in V_{CC} and β_F must be specified.

From the above discussion, the collector current is a function of four variables: $V_{BE}, I_{CBO}, \beta_F, V_{CC}$. The change in collector current can be obtained using partial derivatives. For small parameter changes, a change in collector current is given as

$$\Delta I_C = \frac{\partial I_C}{\partial V_{BE}} \Delta V_{BE} + \frac{\partial I_C}{\partial V_{CBO}} \Delta I_{CBO} + \frac{\partial I_C}{\partial \beta_F} \Delta \beta_F + \frac{\partial I_C}{\partial V_{CC}} \Delta V_{CC} \qquad (12.34)$$

The stability factors can be defined for the four variables as

$$S_\beta = \frac{\partial I_C}{\partial \beta_F} \cong \frac{\Delta I_C}{\Delta \beta_F}$$

$$S_v = \frac{\partial I_C}{\partial V_{BE}} \cong \frac{\Delta I_C}{\Delta V_{BE}}$$

$$S_I = \frac{\partial I_C}{\partial I_{CBO}} \cong \frac{\Delta I_C}{\Delta I_{CBO}}$$

and

$$S_{VCC} = \frac{\partial I_C}{\partial V_{CC}} \cong \frac{\Delta I_C}{\Delta V_{CC}} \tag{12.35}$$

Using the stability factors, Equation (12.34) becomes

$$\Delta I_C = S_V \Delta V_{BE} + S_\beta \Delta \beta_F + S_I \Delta I_{CBO} + S_{VCC} \Delta V_{CC} \tag{12.36}$$

From Equation (12.30),

$$S_V = \frac{dI_C}{dV_{BE}} = -\frac{1}{R_B/\beta_F + R_E \left(\beta_F + 1/\beta_F \right)} \tag{12.37}$$

From Equation (12.31),

$$I_C = \frac{V_{CC} - V_{CE}}{R_C + R_E/\alpha_F} \tag{12.38}$$

Thus, the stability factor S_{VCC} is given as

$$S_{VCC} = \frac{dI_C}{dV_{CC}} = \frac{1}{R_C + R_E/\alpha_F} \tag{12.39}$$

To obtain the stability factor S_I, an expression for I_C involving I_{CBO} needs to be derived. The derivation is assisted by referring to Figure 12.8.
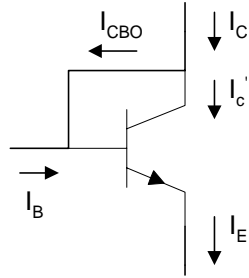
Figure 12.8 Current in Transistor including I_{CBO}

The current

$$I_C = I_C' + I_{CBO} \qquad (12.40)$$

and

$$I_C' = \beta_F \left(I_B + I_{CBO} \right) \qquad (12.41)$$

From Equations (12.40) and (12.41), we have

$$I_C = \beta_F I_B + \left(\beta_F + 1 \right) I_{CBO} \qquad (12.42)$$

Assuming that $\beta_F + 1 \cong \beta_F$, then

$$I_C = \beta_F I_B + \beta_F I_{CBO} \qquad (12.43)$$

so

$$I_B = \frac{I_C}{\beta_F} - I_{CBO} \qquad (12.44)$$

The loop equation of the base-emitter circuit of Figure 12.6(b) gives

$$V_{BB} - V_{BE} = I_B R_{BB} + R_E \left(I_B + I_C \right)$$

$$= I_B \left(R_{BB} + R_E \right) + R_E I_C \qquad (12.45)$$

Assuming that $\beta_F + 1 \cong \beta_F$ and substituting Equation (12.44) into (12.45), we get

$$V_{BB} - V_{BE} = \left(R_{BB} + R_E \right)\left(\frac{I_C}{\beta_F} - I_{CBO} \right) + I_C R_E \qquad (12.46)$$

Solving for I_C, we have

$$I_C = \frac{V_{BB} - V_{BE} + \left(R_{BB} + R_E \right)I_{CBO}}{\left(R_{BB} + R_E \right)\Big/ \beta_F + R_E} \qquad (12.47)$$

Taking the partial derivative,

$$S_I = \frac{\partial I_C}{\partial I_{CBO}} = \frac{R_{BB} + R_E}{\left(R_{BB} + R_E \right)\Big/ \beta_F + R_E} \qquad (12.48)$$

The stability factor involving β_F and S_β can also be found by taking the partial derivative of Equation (12.47). Thus,

$$S_\beta = \frac{\partial I_C}{\partial \beta} = \frac{\left(R_B + R_E \right)\left[V_{BB} - V_{BE} + \left(R_B + R_E \right)I_{CBO} \right]}{\left(R_B + R_E + \beta R_E \right)^2} \qquad (12.49)$$

The following example shows the use of MATLAB for finding the changes in the quiescent point of a transistor due variations in temperature, base-to-emitter voltage and common emitter current gain.

Example 12.3

The self-bias circuit of Figure 12.6 has the following element values: $R_{B1} = 50\ K$, $R_{B2} = 10\ K$, $R_E = 1.2\ K$, $R_C = 6.8\ K$, β_F varies from 150 to 200 and V_{CC} is 10 ± 0.05 V. I_{CBO} is 1 μA at 25 ^{0}C. Calculate the collector current at 25 ^{o}C and plot the change in collector current for temperatures between 25 and 100 ^{o}C. Assume V_{BE} and β_F at 25 ^{o}C are 0.7 V and 150, respectively.

Solution

Equations (12.25), (12.26), and (12.30) can be used to calculate the collector current. At each temperature, the stability factors are calculated using Equations (12.37), (12.39), (12,48) and (12.49). The changes in V_{BE} and I_{CBO} with temperature are obtained using Equations (12.32) and (12.33), respectively. The change in I_C for each temperature is calculated using Equation (12.36).

MATLAB Script:

```
% Bias stability
%
rb1=50e3; rb2=10e3; re=1.2e3; rc=6.8e3;
vcc=10; vbe=0.7; icbo25=1e-6; beta=(150+200)/2;
vbb=vcc*rb2/(rb1+rb2);
rb=rb1*rb2/(rb1+rb2);
ic=beta*(vbb-vbe)/(rb+(beta+1)*re);

%stability factors are calculated
svbe=-beta/(rb+(beta+1)*re);
alpha=beta/(beta+1);
svcc=1/(rc + (re/alpha));
svicbo=(rb+re)/(re+(rb+re)/alpha);
sbeta=((rb+re)*(vbb-vbe+icbo25*(rb+re))/(rb+re+beta*re)^2);
% Calculate changes in Ic for various temperatures

t=25:1:100;
len_t = length(t);
dbeta = 50; dvcc=0.1;
for i=1:len_t
    dvbe(i)= -2e-3*(t(i)-25);
    dicbo(i)=icbo25*(2^((t(i)-25)/10)-1);
    dic(i)=svbe*dvbe(i)+svcc*dvcc...
                +svicbo+dicbo(i)+sbeta*dbeta;
end
plot(t,dicbo)
title('Change in collector current vs. temperature')
xlabel('Temperature, degree C')
ylabel('Change in collector current, A')
```

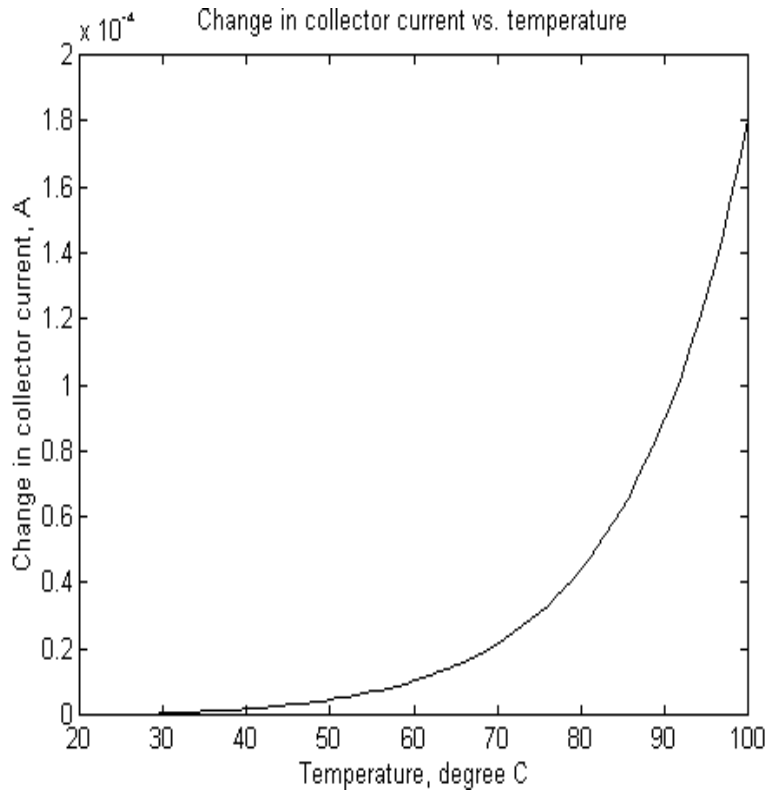Figure 12.9 shows I_C versus temperature.

Figure 12.9 I_C versus Temperature

12.3   INTEGRATED CIRCUIT BIASING

Biasing schemes for discrete electronic circuits are not suitable for integrated circuits (IC) because of the large number of resistors and the large coupling and bypass capacitor required for biasing discrete electronic circuits. It is uneconomical to fabricate IC resistors since they take a disproportionately large area on an IC chip.   In addition, it is almost impossible to fabricate IC inductors. Biasing of ICs is done using mostly transistors that are connected to create constant current sources.   Examples of integrated circuit biasing schemes are discussed in this section.

12.3.1 Simple current mirror

A simple current mirror is shown in Figure 12.10. The current mirror consists of two matched transistors Q_1 and Q_2 with their bases and emitters connected. The transistor Q_1 is connected as a diode by shorting the base to its collector.
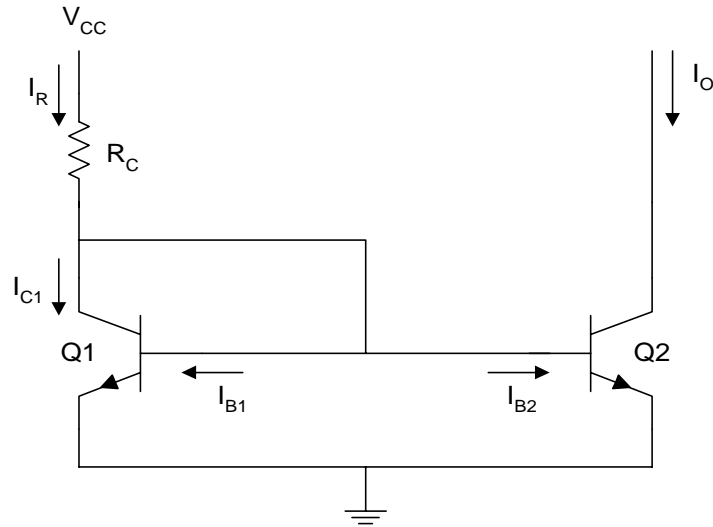

Figure 12.10 Simple Current Mirror

From Figure 12.10, we observe that

$$I_R = \frac{V_{CC} - V_{BE}}{R_C} \tag{12.50}$$

Using KCL, we get

$$I_R = I_{C1} + I_{B1} + I_{B2}$$

$$= I_{E1} + I_{B2} \tag{12.51}$$

But

$$I_{B2} = \frac{I_{E2}}{\beta + 1}$$

Assuming matched transistors

$$I_{B1} \cong I_{B2}$$

$$I_{E1} \cong I_{E2} \qquad (12.52)$$

From Equations (12.51) and (12.52), we get

$$I_R = I_{E1} + \frac{I_{E2}}{\beta+1} \cong I_{E2}\left[1 + \frac{1}{\beta+1}\right] = \left[\frac{\beta+2}{\beta+1}\right]I_{E2} \qquad (12.53)$$

and

$$I_O = I_{C2} = \beta I_{B2} = \frac{\beta I_{E2}}{\beta+1}$$

Therefore

$$I_O = \left[\frac{\beta}{\beta+1}\right]\left[\frac{\beta+1}{\beta+2}\right]I_R = \frac{\beta}{\beta+2}I_R \qquad (12.54)$$

$$I_O \cong I_R \text{ if } \beta \gg 1 \qquad (12.55)$$

Equation (15.55) is true provided Q$_2$ is in the active mode. In the latter mode of transistor operation, the device Q$_2$ behaves as a current source. For Q$_2$ to be in the active mode, the following relation should be satisfied

$$V_{CE2} > V_{CEsat}$$

12.3.2 Wilson current source

The Wilson current source, shown in Figure 12.11, achieves high output resistance and an output current that is less dependent on transistor β_F. To obtain an expression for the output current, we assume that all three transistors are identical. Thus

$$I_{C1} = I_{C2}$$

$$V_{BE1} = V_{BE2}$$

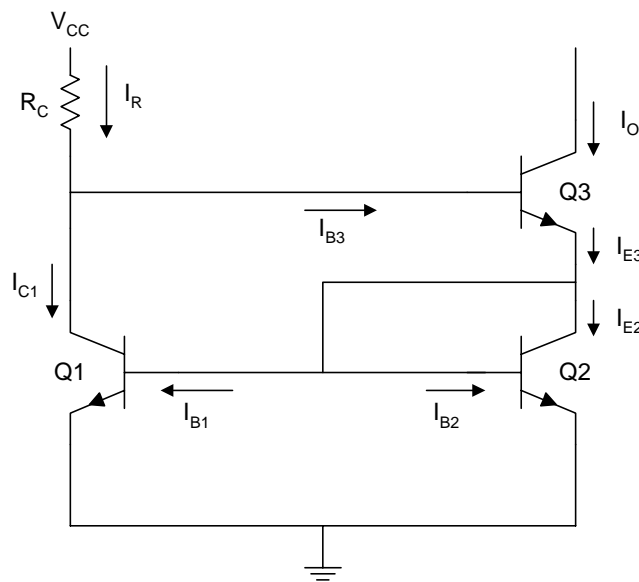$$\beta_{F1} = \beta_{F2} = \beta_{F3} = \beta_F \qquad (12.56)$$

Figure 12.11 Wilson Current Source

Using KCL at the collector of transistor Q_3, we get

$$I_{C1} = I_R - I_{B3} = I_R - \frac{I_O}{\beta_F}$$

therefore,

$$I_O = \beta_F\left(I_R - I_{C1}\right) \qquad (12.57)$$

Using KCL at the emitter of Q_3, we obtain

$$I_{E3} = I_{C2} + I_{B1} + I_{B2} = I_{C1} + 2I_{B1}$$

$$= I_{C1}\left(1 + \frac{2}{\beta_F}\right) \tag{12.58}$$

But

$$I_0 = \alpha_F I_{E3} = \frac{\beta_F}{\beta_F + 1} I_{E3} \tag{12.59}$$

Substituting Equation (12.58) into (12.59), we have

$$I_0 = \left(\frac{\beta_F}{\beta_F + 1}\right)\left(1 + \frac{2}{\beta_F}\right) I_{C1} \tag{12.60}$$

Simplifying Equation (12.60), we get

$$I_{C1} = \left(\frac{\beta_F + 1}{\beta_F + 2}\right) I_0 \tag{12.61}$$

Combining Equations (12.57) and (12.61), we obtain

$$I_0 = \beta_F\left[I_R - \left(\frac{\beta_F + 1}{\beta_F + 2}\right) I_0\right] \tag{12.62}$$

Simplifying Equation (12.62), we get

$$I_0 = \left(\frac{\beta_F^2 + 2\beta_F}{\beta_F^2 + 2\beta_F + 2}\right) I_R$$

$$= \left(1 - \frac{2}{\beta_F^2 + 2\beta_F + 2}\right) I_R \tag{12.63}$$

For reasonable values of β_F

$$\left(\frac{2}{\beta_F^2 + 2\beta_F + 2}\right) \ll 1$$

and Equation (12.63) becomes

$$I_0 \cong I_R$$

Thus, β has little effect on the output current, and

$$I_R = \frac{V_{CC} - V_{BE3} - V_{BE1}}{R_C} \qquad (12.64)$$

Example 12.4

For Figures 12.10 and 12.11, what are the percentage difference between the reference and output currents for the β_F from 40 to 200. Assume that for both figures, $V_{CC} = 10\,V$, $R_C = 50\,K\Omega$ and $V_{BE} = 0.7\,V$.

Solution

We use Equation (12.50) to calculate I_R and Equation (12.53) to find I_0 of the simple current mirror. Similarly, we use Equation (12.64) to find I_R and Equation (12.63) to calculate I_0 of the Wilson current source.

MATLAB Script

```
% Integrated circuit Biasing
vcc=10; rc=50e3; vbe=0.7;
beta =40:5:200; ir1=(vcc-vbe)/rc;
ir2=(vcc-2*vbe)/rc; m=length(beta);
for i=1:m
  io1(i) = beta(i)*ir1/(beta(i) + 2);
  pd1(i)=abs((io1(i)-ir1)*100/ir1);
  io2(i)=(beta(i)^2+2*beta(i))/(beta(i)^2+2*beta(i)+2);
  pd2(i)=abs((io2(i)*ir2-ir2)*100/ir2);
end
subplot(211), plot(beta,pd1)
%title('error for simple current mirror')
xlabel('Transistor beta')
ylabel('Percentage error')
text(90,5,'Error for simple current mirror')
subplot(212),plot(beta,pd2)
```

```
%title('Error for Wilson current mirror')
xlabel('Transistor beta')
ylabel('Percentage error')
text(90, 0.13, 'Error for Wilson current source')
```

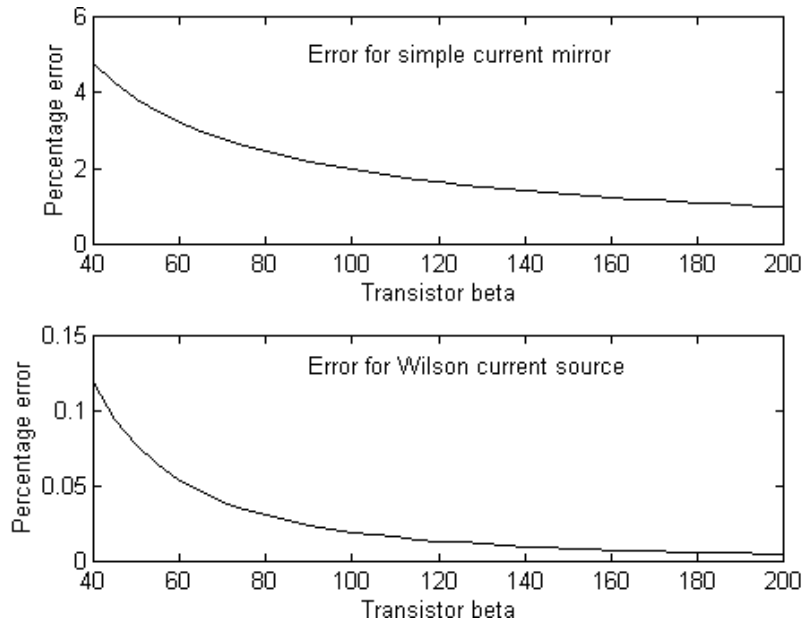Figure 12.12 shows the percentage errors obtained for the simple current mirror and Wilson current source.

Figure 12.12   Percentage Error between Reference and Output
Currents for Simple Current Mirror and Wilson
Current Source

12.4   FREQUENCY RESPONSE OF COMMON EMITTER AMPLIFIER

The common-emitter amplifier, shown in Figure 12.13, is capable of generating a relatively high current and voltage gains. The input resistance is medium and is essentially independent of the load resistance R_L. The output resistance is relatively high and is essentially independent of the source resistance.

The coupling capacitor, C_{C1}, couples the source voltage v_S to the biasing network. Coupling capacitor C_{C2} connects the collector resistance R_C to the load R_L. The bypass capacitance C_E is used to increase the midband gain, since it effectively short circuits the emitter resistance R_E at midband frequencies. The resistance R_E is needed for bias stability. The external capacitors C_{C1}, C_{C2}, C_E will influence the low frequency response of the common emitter amplifier. The internal capacitances of the transistor will influence the high frequency cut-off. The overall gain of the common-emitter amplifier can be written as

$$A(s) = \frac{A_m s^2 \left(s + w_z\right)}{\left(s + w_{L1}\right)\left(s + w_{L2}\right)\left(s + w_{L3}\right)\left(1 + s/w_H\right)} \tag{12.65}$$

where

A_M is the midband gain.

w_H is the frequency of the dominant high frequency pole

w_{L1}, w_{L2}, w_{L3} are low frequency poles introduced by the
                 coupling and bypass capacitors

w_Z is the zero introduced by the bypass capacitor.
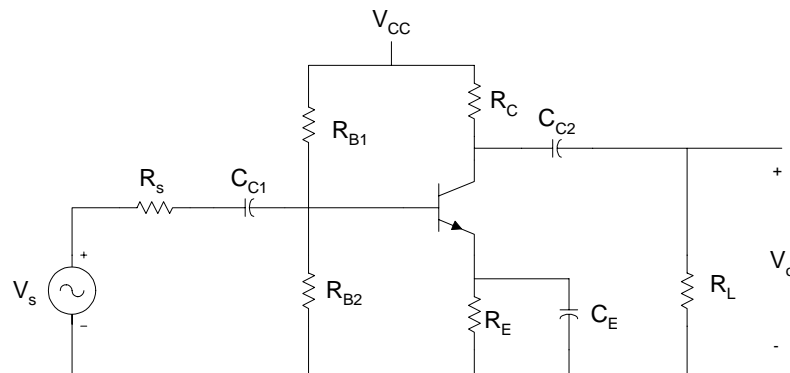

Figure 12.13 Common Emitter Amplifier

The midband gain is obtained by short circuiting all the external capacitors and open circuiting the internal capacitors. Figure 12.14 shows the equivalent for calculating the midband gain.
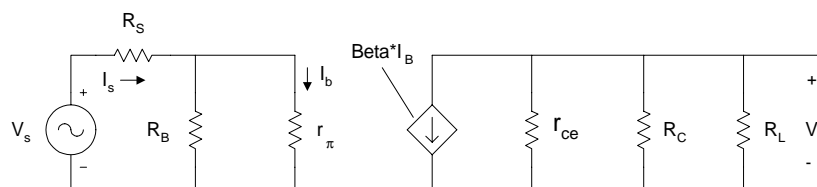

Figure 12.14 Equivalent Circuit for Calculating Midband Gain

From Figure 12.14, the midband gain, A_m, is

$$A_m = \frac{V_O}{V_S} = -\beta \Big[r_{CE} \big\| R_C \big\| R_L \Big] \left[\frac{R_B}{R_B + r_\pi} \right] \left[\frac{1}{R_S + \big[R_B \big\| r_\pi \big]} \right] \qquad (12.66)$$

It can be shown that the low frequency poles, w_{L1}, w_{L2}, w_{L3}, can be obtained by the following equations

$$\tau_1 = \frac{1}{w_{L1}} = C_{C1} R_{IN} \qquad (12.67)$$

where

$$R_{IN} = R_S + \Big[R_B \big\| r_\pi \Big] \qquad (12.68)$$

$$\tau_2 = \frac{1}{w_{L2}} = C_{C2} \Big[R_L + \big(R_C \big\| r_{ce} \big) \Big] \qquad (12.69)$$

and

$$\tau_3 = \frac{1}{w_{L3}} = C_E R_E^{'} \qquad (12.70)$$

where

$$R'_E = R_E \left\| \left[\frac{r_\pi}{\beta_F + 1} + \left(\frac{R_B \| R_S}{\beta_F + 1} \right) \right] \right. \tag{12.71}$$

and the zero

$$w_Z = \frac{1}{R_E C_E} \tag{12.72}$$

Normally, $w_Z < w_{L3}$ and the low frequency cut-off w_L is larger than the largest pole frequency. The low frequency cut-off can be approximated as

$$w_L \cong \sqrt{\left(w_{L1} \right)^2 + \left(w_{L2} \right)^2 + \left(w_{L3} \right)^2} \tag{12.73}$$

The high frequency equivalent circuit of the common-emitter amplifier is shown in Figure 12.15.
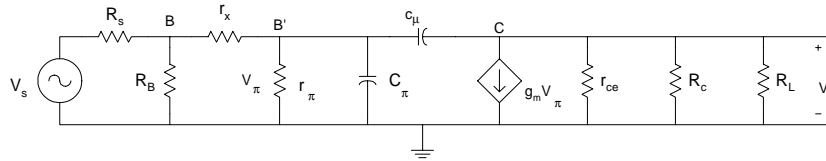

Figure 12.15   Equivalent Circuit of CE Amplifier at High
Frequencies

In Figure 12.15, C_μ is the collector-base capacitance, C_π is the emitter to base capacitance, r_X is the resistance of silicon material of the base region between the base terminal B and an internal or intrinsic base terminal B'. Using the Miller Theorem, it can be shown that the 3-dB frequency at high frequencies is approximately given as

$$w_H^{-1} = \left(r_\pi \left\| \left[r_x + \left(R_B \| R_S \right) \right] \right) C_T \right. \tag{12.74}$$

where

$$C_T = C_\pi + C_\mu \left[1 + g_m \left(R_L \| R_C \right) \right] \tag{12.75}$$

and

$$g_m = \frac{I_C}{V_T} \qquad (12.76)$$

In the following example, MATLAB is used to obtain the frequency response of a common-emitter amplifier.

Example 12.5

For a CE amplifier shown in Figure 12.13,
$\beta = 150, \ R_L = 2 \ K\Omega, \ R_C = 4 \ K\Omega, C_\pi = 100 \ pF, \ C_\mu = 5 \ pF, \ V_{CC} = 10 \ V,$

$r_{ce} = r_o = 60 \ K\Omega, \ R_E = 1.5 \ K\Omega, \ C_{C1} = 2 \ \mu F, C_{C2} = 4 \ \mu F, \ C_E = 150 \ \mu F,$

$R_{B1} = 60 \ K\Omega, \ R_{B2} = 40 \ K\Omega, R_S = 100\Omega, \ r_x = 10 \ \Omega.$
Use MATLAB to plot the magnitude response of the amplifier.

Solution

Using Equations (12.67), (12.69), (12.70) and (12.74) are used to calculate the poles of Equation (12.65). The zero of the overall amplifier gain is calculated using Equation (12.66). The MATLAB program is as follows:

MATLAB Script

```
%Frequency response of CE Amplifier
rc=4e3; rb1=60e3; rb2=40e3; rs=100; rce=60e3;
re=1.5e3; rl=2e3; beta=150; vcc=10; vt=26e-3; vbe =0.7;
cc1=2e-6; cc2=4e-6; ce=150e-6;, rx=10; cpi=100e-12;
cmu=5e-12;
%  Ic is calculated
rb = (rb1 * rb2)/(rb1 + rb2);
vbb = vcc * rb2/(rb1 + rb2);
icq = beta * (vbb - vbe)/(rb + (beta + 1)*re);

% Calculation of low frequency poles
% using equations (12.67), (12.69) and (12.70)
rpi=beta * vt/icq;
rb_rpi=rpi * rb/(rpi + rb);
rin=rs + rb_rpi;
wl1=1/(rin * cc1);
rc_rce=rc * rce/(rc + rce);
```

```
wl2=1/(cc2 * (rl + rc_rce));
rb_rs=rb * rs/(rb + rs);
rx1=(rpi + rb_rs)/(beta + 1);
re_prime=re * rx1/(re + rx1);
wl3=1/(re_prime * ce);

% Calculate the low frequency zero using equation (12.72)
wz = 1/(re*ce);
% Calculate the high frequency pole using equation (12.74)
gm = icq/vt;
rbrs_prx = ( rb * rs/(rb + rs)) + rx;
rt = (rpi * rbrs_prx)/(rpi + rbrs_prx);
rl_rc = rl * rc/(rl + rc);
ct = cpi + cmu * (1 + gm * rl_rc);
wh = 1/(ct * rt);
% Midband gain is calculated
rcercrl = rce * rl_rc/(rce + rl_rc);
am = -beta * rcercrl * (rb/(rb + rpi)) * (1/(rin));

% Frequency response calculation using equation (12.65)
a4 = 1; a3 = wl1 + wl2 + wl3 + wh;
a2 = wl1*wl2 + wl1*wl3 + wl2*wl3 + wl1*wh + wl2*wh + wl3*wh;
a1 = wl1*wl2*wl3 +wl1*wl2*wh + wl1*wl3*wh + wl2*wl3*wh;
a0 = wl1*wl2*wl3*wh;
den=[a4 a3 a2 a1 a0];
b3 = am*wh;
b2 = b3*wz; b1 =0; b0 = 0;
num = [b3 b2 b1 b0];
w = logspace(1,10);
h = freqs(num,den,w);
mag = 20*log10(abs(h));
f = w/(2*pi);
% Plot the frequency response
semilogx(f,mag,'w')
title('Magnitude Response')
xlabel('Frequency, Hz')
ylabel('Gain, dB')
axis([1, 1.0e10, 0, 45])
```
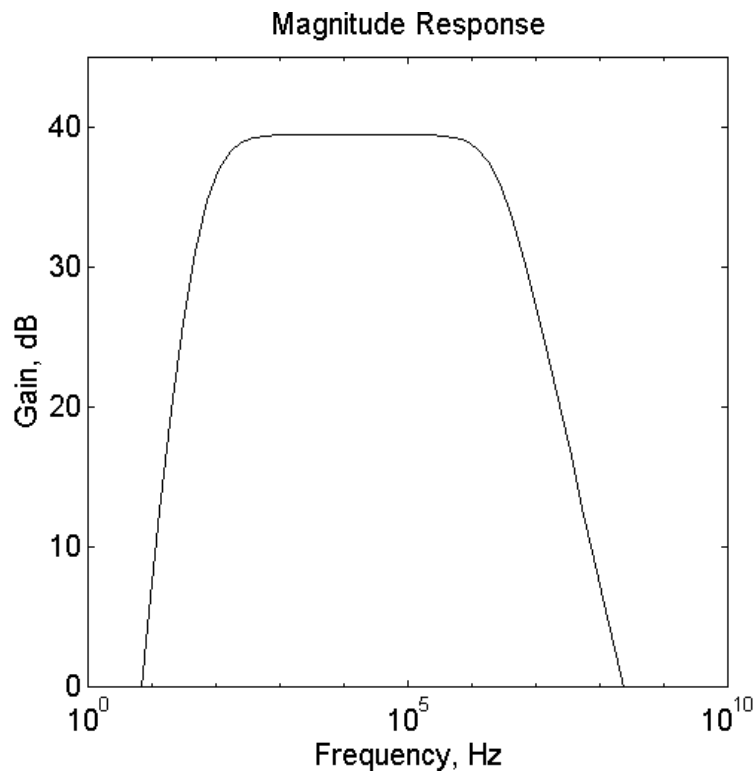
The frequency response is shown in Figure 12.16.

Figure 12.16 Frequency Response of a CE Amplifier

12.5   MOSFET CHARACTERISTICS

Metal-oxide-semiconductor field effect transistor (MOSFET) is a four-terminal device. The terminals of the device are the gate, source, drain, and substrate. There are two types of mosfets: the enhancement type and the depletion type. In the enhancement type MOSFET, the channel between the source and drain has to be induced by applying a voltage on the gate. In the depletion type mosfet, the structure of the device is such that there exists a channel between the source and drain. Because of the oxide insulation between the gate and the channel, mosfets   have high input resistance. The electronic symbol of a mosfet is shown in Figure 12.19.
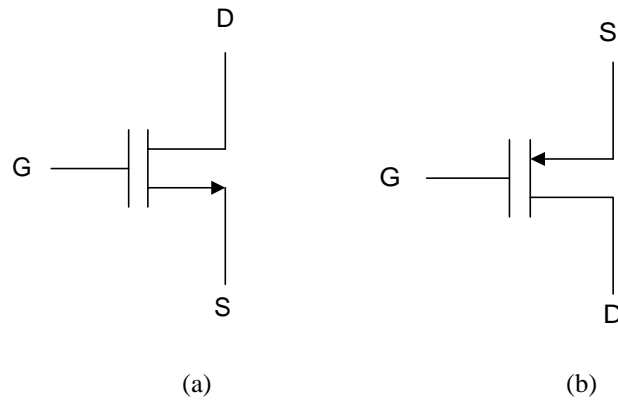
Figure 12.17 Circuit Symbol of (a) N-channel and (b) P-channel MOSFETs

Mosfets can be operated in three modes: cut-off, triode, and saturation regions. Because the enhancement mode mosfet is widely used, the presentation in this section will be done using an enhancement-type mosfet. In the latter device, the channel between the drain and source has to be induced by applying a voltage between the gate and source. The voltage needed to create the channel is called the threshold voltage, V_T. For an n-channel enhancement-type mosfet, V_T is positive and for a p-channel device it is negative.

Cut-off Region

For an n-channel mosfet, if the gate-source voltage V_{GS} satisfies the condition

$$V_{GS} < V_T \tag{12.77}$$

then the device is cut-off. This implies that the drain current is zero for all values of the drain-to-source voltage.

Triode Region

When $V_{GS} > V_T$ and V_{DS} is small, the mosfet will be in the triode region. In the latter region, the device behaves as a non-linear voltage-controlled resistance. The I-V characteristics are given by

$$I_D = k_n \left[2 \left(V_{GS} - V_T \right) V_{DS} - V_{DS}{}^2 \right] \qquad (12.78)$$

provided

$$V_{DS} \le V_{GS} - V_T \qquad (12.79)$$

where

$$k_n = \frac{\mu_n \varepsilon \varepsilon_{ox}}{2 t_{ox}} \frac{W}{L} = \frac{\mu_n C_{ox}}{2} \left(\frac{W}{L} \right) \qquad (12.80)$$

and

μ_n is surface mobility of electrons
ε is permittivity of free space (8.85 E-14 F/cm)
ε_{ox} is dielectric constant of SiO_2
t_{ox} is thickness of the oxide
L is length of the channel
W is width of the channel

Saturation Region

Mosfets can operate in the saturation region. A mosfet will be in saturation provided

$$V_{DS} \ge V_{GS} - V_T \qquad (12.81)$$

and I-V characteristics are given as

$$I_D = k_n \left(V_{GS} - V_T \right)^2 \qquad (12.82)$$

The dividing locus between the triode and saturation regions is obtained by substituting

$$V_{DS} = V_{GS} - V_T \qquad (12.83)$$

into either Equation (12.78) or (12.82), so we get

$$I_D = k_n V_{DS}^{\,2} \tag{12.84}$$

In the following example, I-V characteristics and the locus that separates triode and saturation regions are obtained using MATLAB.

Example 12.6

For an n-channel enhancement-type MOSFET with $k_n = 1 \ mA/V^2$ and $V = 1.5\,V$, use MATLAB to sketch the I-V characteristics for $V_{GS} = 4, 6, 8\,V$ and for V_{DS} between 0 and $12\,V$.

Solution

MATLAB Script

```
% I-V characteristics of mosfet
%
kn=1e-3; vt=1.5;
vds=0:0.5:12;
vgs=4:2:8;
m=length(vds);
n=length(vgs);

for i=1:n
        for j=1:m
          if vgs(i) < vt
                cur(i,j)=0;
          elseif vds(j) >= (vgs(i) - vt)
                cur(i,j)=kn * (vgs(i) - vt)^2;
          elseif vds(j) < (vgs(i) - vt)
                cur(i,j)= kn*(2*(vgs(i)-vt)*vds(j) - vds(j)^2);
          end
        end
end
plot(vds,cur(1,:),'w',vds,cur(2,:),'w',vds,cur(3,:),'w')
xlabel('Vds, V')
ylabel('Drain Current,A')
title('I-V Characteristics of a MOSFET')
text(6, 0.009, 'Vgs = 4 V')
```

text(6, 0.023, 'Vgs = 6 V')
text(6, 0.045, 'Vgs = 8 V')
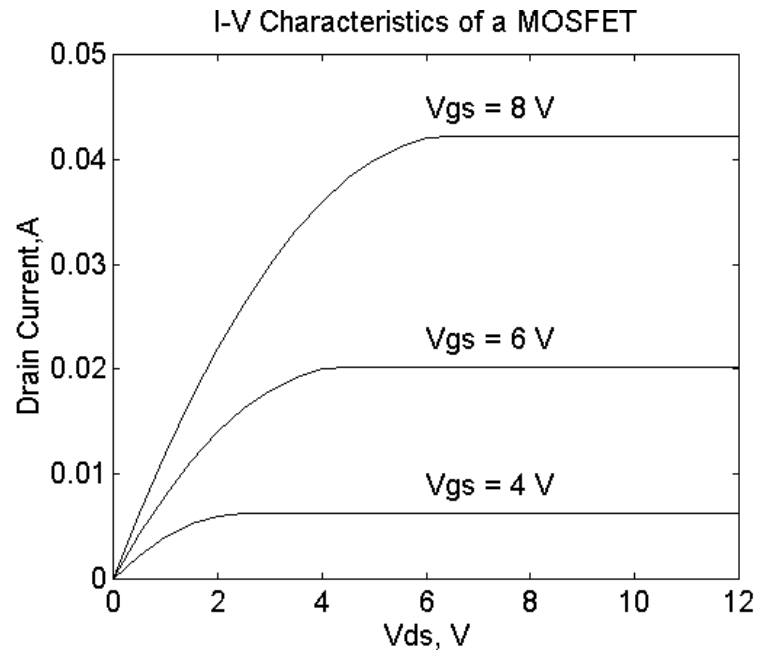
Figure 12.18 shows the I-V characteristics.

Figure 12.18   I-V Characteristics of N-channel Enhancement-type
Mosfet

12.6   BIASING OF MOSFET CIRCUITS

A popular circuit for biasing discrete mosfet amplifiers is shown in Figure 12.19. The resistances R_{G1} and R_{G2} will define the gate voltage. The resistance R_S improves operating point stability.
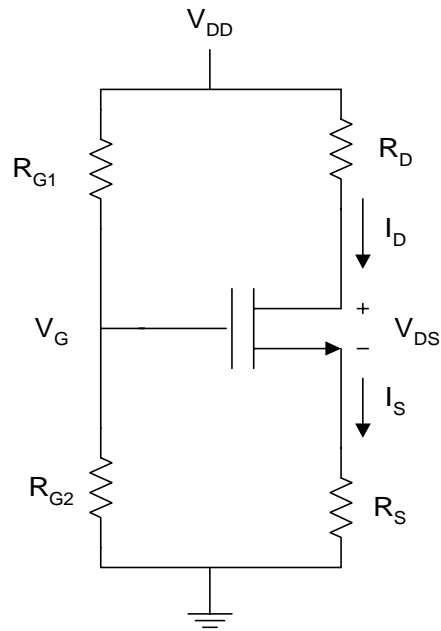
Figure 12.19 Simple Biasing Circuit for Enhancement-type NMOS

Because of the insulated gate, the current that passes through the gate of the MOSFET is negligible. The gate voltage is given as

$$V_G = \frac{R_{G1}}{R_{G1} + R_{G2}} V_{DD} \qquad (12.85)$$

The gate-source voltage V_{GS} is

$$V_{GS} = V_G - I_S R_S \qquad (12.86)$$

For conduction of the MOSFET, the gate-source voltage V_{GS} should be greater than the threshold voltage of the mosfet, V_T. Since $I_D = I_S$, Equation (12.86) becomes

$$V_{GS} = V_G - I_D R_S \qquad (12.87)$$

The drain-source voltage is obtained by using KVL for the drain-source circuit

$$V_{DS} = V_{DD} - I_D R_D - I_S R_S$$

$$= V_{DD} - I_D (R_D - R_S) \tag{12.88}$$

For proper operation of the bias circuit,

$$V_{GS} > V_T \tag{12.89}$$

When Equation (12.89) is satisfied, the MOSFET can either operate in the triode or saturation region. To obtain the drain current, it is initially assumed that the device is in saturation and Equation (12.82) is used to calculate I_D. Equation (12.81) is then used to confirm the assumed region of operation. If Equation (12.82) is not satisfied, then Equation (12.78) is used to calculate I_D. The method is illustrated by the following example.

Example 12.7

For Figure 12.19, $V_T = 2$ V, $k_n = 0.5$ mA/V^2, $V_{DD} = 9$V, $R_{G1} = R_{G2} = 10\ M\Omega$, $R_S = R_D = 10\ K\Omega$. Find I_D and V_{DS}.

Solution

Substituting Equation (12.86) into Equation (12.82), we have

$$I = k_n \left(V_g - I_D R_D - V_T \right)^2 \tag{12.90}$$

Simplifying Equation (12.90), we have

$$0 = k_n R_D^2 I_D^2 - \left[1 + 2(V_g - V_T)R_D \right] I_D + k\left(V_g - V_T \right)^2 \tag{12.91}$$

The above quadratic equation is solved to obtain I_D. Two solutions of I_D are obtained. However, only one is sensible and possible. The possible one is

the one that will make $V_{GS} > V_T$. With the possible value of I_D obtained, V_{DS} is calculated using Equation (12.88). It is then verified whether

$$V_{DS} > V_{GS} - V_T$$

The above condition ensures saturation of the device. If the device is not in saturation, then substituting Equation (12.86) into Equation (12.78), we get

$$I_D = k_n \left[2(V_g - I_D R_D - V_T)(V_{DD} - (R_D + R_S)I_D) - \left(V_{DD} - (R_D + R_S)I_D\right)^2 \right] \quad (12.92)$$

Simplifying Equation (12.92), we get the quadratic equation

$$0 = I_D^2 \left[(R_S + R_D)^2 + 2R_D(R_D + R_S) \right]$$
$$+ I_D \left[2V_{DD}(R_D + R_S) - 2V_{DD}R_D - 2(V_g - V_T)(R_D + R_S) - \frac{1}{k_n} \right] \quad (12.93)$$
$$+ 2(V_g - V_T)V_{DD} - V_{DD}^2$$

Two roots are obtained by solving Equation (12.93). The sensible and possible root is the one that will make

$$V_{GS} > V_T$$

The MATLAB program for finding I_D is shown below.

MATLAB Script

```
%
% Analysis of MOSFET bias circuit
%
diary ex12_7.dat
diary on
vt=2; kn=0.5e-3; vdd=9;
rg1=10e6; rg2=10e6; rs=10e3; rd=10e3;
vg=vdd * rg2/(rg1 + rg2);

% Id is calculated assuming device is in saturation

a1=kn*(rd^2);
a2=-(1 + 2*(vg - vt)*rd * kn);
```

```
a3=kn * (vg - vt)^2;
p1=[a1,a2,a3];
r1=roots(p1);

% check for the sensible value of the drain current

vgs = vg - rs * r1(1);
 if vgs > vt
         id = r1(1);
   else
         id = r1(2);
% check for sensible value of the drain current
   vgs = vg - rs*r2(1);
         if vgs > vt
           id = r2(1);
          else
           id=r2(2);
         end
vds=vdd - (rs + rd)*id;
end

% print out results
fprintf('Drain current is %7.3e Amperes\n',id)
fprintf('Drain-source voltage is %7.3e Volts\n', vds)
```

The results are

> Drain current is 1.886e-004 Amperes
> Drain-source voltage is 5.228e+000 Volts

The circuit shown in Figure 12.20 is a mosfet transistor with the drain connected to the gate. The circuit is normally referred to as diode-connected enhancement transistor.

From Equation (12.88), the MOSFET is in saturation provided

$$V_{DS} > V_{GS} - V_T$$

i.e.,

$$V_{DS} - V_{GS} > -V_T \quad \text{or} \quad V_{DS} + V_{SG} > -V_T$$

or

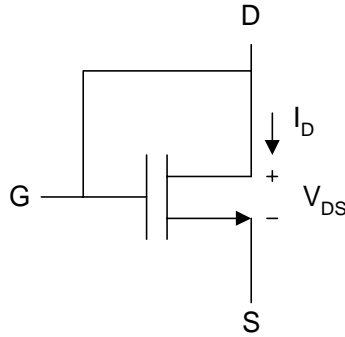$$V_{DG} > -V_T \qquad\qquad\qquad (12.94)$$

Figure 12.20 Diode-connected Enhancement Type MOSFET

Since $V_{DG} = 0$ and V_T is positive for n-channel MOSFET, the device is in saturation and

$$i_D = k_n \left(V_{GS} - V_T\right)^2 \qquad\qquad\qquad (12.95)$$

But if $V_{GS} = V_{DS}$, Equation (12.101) becomes

$$i_D = k_n \left(V_{DS} - V_T\right)^2$$

The diode-connected enhancement mosfet can also be used to generate dc currents for nMOS and CMOS analog integrated circuits. A circuit for generating dc currents that are constant multiples of a reference current is shown in Figure 12.21. It is a MOSFET version of current mirror circuits discussed in Section 12.3.

Assuming the threshold voltages of the transistors of Figure 12.21 are the same, then since transistor T1 is in saturation,

$$I_{REF} = k_1 \left(V_{GS1} - V_T\right)^2 \qquad\qquad\qquad (12.96)$$

Since transistors T1 and T2 are connected in parallel, we get

$$V_{GS1} = V_{GS2} = V_{GS} \tag{12.97}$$

and

$$I_0 = k_2 \left(V_{GS2} - V_T\right)^2$$

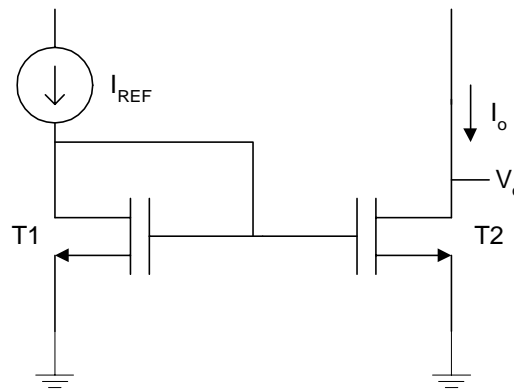$$I_0 = k_2 \left(V_{GS} - V_T\right)^2 \tag{12.98}$$

Figure 12.21 Basic MOSFET Current Mirror

Combining Equations (12.96) and (12.98), the current

$$I_0 = I_{REF} \left(\frac{k_2}{k_1}\right) \tag{12.99}$$

and using Equation (12.74), Equation (12.99) becomes

$$I_0 = I_{REF} \left(\frac{\left(W/L\right)_2}{\left(W/L\right)_1}\right) \tag{12.100}$$

Thus, I_0 will be a multiple of I_{REF}, and the scaling constant is determined by the device geometry. In practice, because of the finite output resistance of transistor T2, I_0 will be a function of the output voltage v_0.

Example 12.8

For the circuit shown in Figure 12.22, $R_1 = 1.5 \ M\Omega$, $L_1 = L_2 = 6 \ \mu m$, $W_1 = 12 \ \mu m$, $W_2 = 18 \ \mu m$, $V_T = 2.0$ V and $V_{DD} = 5$ V. Find the output current I_{D1}, V_{GS1}, I_0 and R_2. Assume that $V_0 = 2.5$ V, $\mu C_{OX} = 30 \ \mu A / V^2$. Neglect channel length modulation.
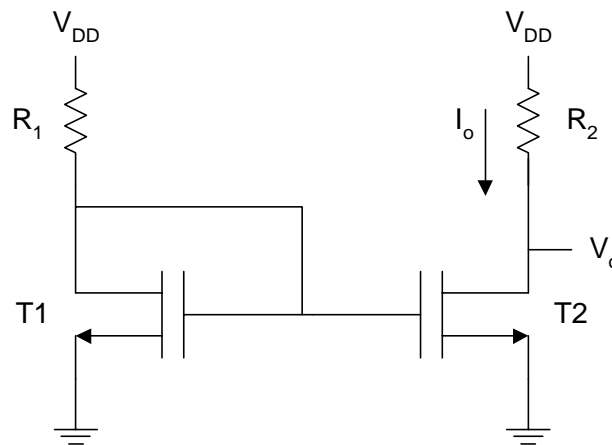

Figure 12.22 Circuit for Example 12.8

Solution

Since T1 is in saturation,

$$I_{D1} = k_{n1}\left(V_{GS} - V_T\right)^2 = k_{n1}(V_{DS} - V_T)^2 \qquad (12.101)$$

$$V_{DS} = V_{DD} - I_{D1}R_1 \qquad (12.102)$$

Substituting Equation (12.100) into (12.99), we get

$$I_{D1} = k_{n1}\left(V_{DD} - V_T - R_1 I_{D1}\right)^2$$

$$\frac{I_{D1}}{k_{n1}} = \left(V_{DD} - V_T\right)^2 - 2(V_{DD} - V_T)R_1 I_{D1} + R_1^2 I_{D1}^2$$

$$0 = R_1^2 I_{D1}^2 - \left(2(V_{DD} - V_T)R_1 + \frac{1}{k_{n1}}\right)I_{D1} + \left(V_{DD} - V_T\right)^2 \qquad (12.103)$$

The above quadratic equation will have two solutions, but only one of the solution of I_{D1} will be valid. The valid solution will result in $V_{GS} > V_T$.

Using equation (12.100), we obtain

$$I_0 = I_{D1}\left(\frac{\left(W/L\right)_2}{\left(W/L\right)_1}\right) \qquad (12.104)$$

and

$$R = \frac{5 - V_0}{I_0} \qquad (12.105)$$

The MATLAB program is as follows:

MATLAB Script

```
%
% Current mirror
%
diary ex12_8.dat
diary on
ucox = 30e-6; l1 = 6e-6; l2 = 6e-6;
w1 = 12e-6; w2=18e-6;
r1=1.5e6; vt=2.0; vdd=5; vout=2.5;

% roots of quadratic equation(12.103) is obtained
kn = ucox * w1/(2 * l1);
a1 = r1^2;
a2 = -2*(vdd - vt)*r1 - (1/kn);
```

```
a3 = (vdd - vt)^2;
p = [a1,a2,a3];
i = roots(p);

% check for realistic value of drain current
vgs=vdd - r1*i(1);
if vgs > vt
        id1 = i(1);
  else
        id1 = i(2);
end

% output current is calculated from equation(12.100)
% r2 is obtained using equation (12.105)
iout = id1*w2*l1/(w1 * l2);
r2=(vdd - vout)/iout;

% print results
fprintf('Gate-source Voltage of T1 is %8.3e Volts\n',vgs)
fprintf('Drain Current of T1 is %8.3e Ampers\n', id1)
fprintf('Drain Current Io is %8.3e Ampers\n', iout)
fprintf('Resistance R2 is %8.3e Ohms\n', r2)
```

The results are

> Gate-source Voltage of T1 is 1.730e+000 Volts
> Drain Current of T1 is 1.835e-006 Ampers
> Drain Current Io is 2.753e-006 Ampers
> Resistance R2 is 9.082e+005 Ohms

12.7   FREQUENCY RESPONSE OF COMMON-SOURCE AMPLIFIER

The common-source amplifier has characteristics similar to those of the common-emitter amplifier discussed in Section 12.4. However, the common-source amplifier has higher input resistance than that of the common-emitter amplifier. The circuit for the common source amplifier is shown in Figure 12.23.
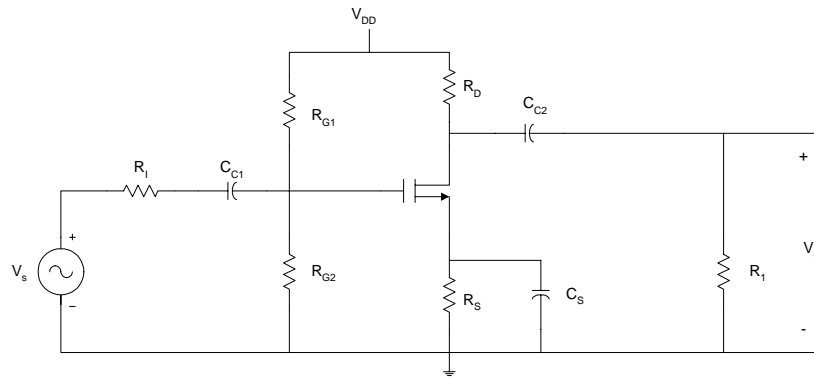
Figure 12.23 Common-Source Amplifier

The external capacitors C_{C1}, C_{C2} and C_S will influence the low frequency response. The internal capacitances of the FET will affect the high frequency response of the amplifier. The overall gain of the common-source amplifier can be written in a form similar to Equation (12.65).

The midband gain, A_m, is obtained from the midband equivalent circuit of the common-source amplifier. This is shown in Figure 12.24. The equivalent circuit is obtained by short-circuiting all the external capacitors and open-circuiting all the internal capacitances of the FET.
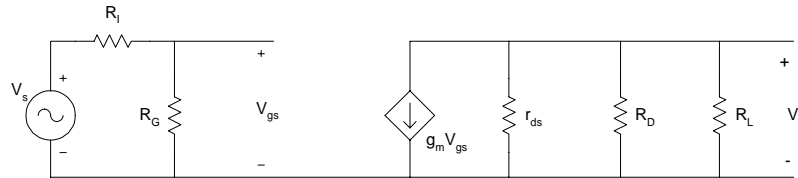

Figure 12.24 Midband Equivalent Circuit of Common-Source Amplifier

Using voltage division,

$$v_{gs} = \frac{R_G}{R_I + R_G} v_S \qquad (12.106)$$

From Ohm's Law,

$$v_0 = -g_m v_{gs} \left(r_{ds} \| R_D \| R_L \right) \tag{12.107}$$

Substituting Equation (12.106) into (12.107), we obtain the midband gain as

$$A_m = \frac{v_0}{v_s} = -g_m \left(\frac{R_G}{R_G + R_I} \right) \left(r_{ds} \| R_D \| R_L \right) \tag{12.108}$$

At low frequencies, the small signal equivalent circuit of the common-source amplifier is shown in Figure 12.25.
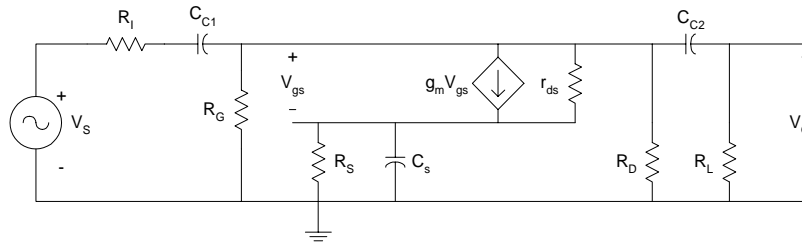

Figure 12.25   Equivalent Circuit for Obtaining the Poles at Low
Frequencies of Common-source Amplifier

It can be shown that the low frequency poles due to C_{C1} and C_{C2} can be written as

$$\tau_1 = \frac{1}{w_{L1}} \cong C_{C1}(R_g + R_I) \tag{12.109}$$

$$\tau_2 = \frac{1}{w_{L2}} \cong C_{C2}(R_L + R_D \| r_{ds}) \tag{12.110}$$

Assuming r_d is very large, the pole due to the bypass capacitance C_S can be shown to be

$$\tau_3 = \frac{1}{w_{L3}} \cong C_S \left(\frac{R_S}{1 + g_m R_S} \right) \tag{12.111}$$

and the zero of C_S is

$$w_Z = \frac{1}{R_S C_S} \tag{12.112}$$

The 3-dB frequency at the low frequency can be approximated as

$$w_L \cong \sqrt{\left(w_{L1}\right)^2 + \left(w_{L2}\right)^2 + \left(w_{L3}\right)^2} \tag{12.113}$$

For a single stage common-source amplifier, the source bypass capacitor is usually the determining factor in establishing the low 3-dB frequency.

The high frequency equivalent circuit of a common-source amplifier is shown in Figure 12.26. In the figure, the internal capacitances of the FET, C_{gs}, C_{gd} and C_{ds} are shown. The external capacitors of the common of common-source amplifier are short-circuited at high frequencies.
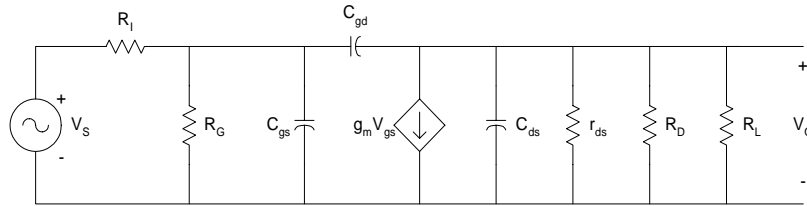

Figure 12.26   High Frequency Equivalent Circuit of Common-source Amplifier

Using the Miller theorem, Figure 12.26 can be simplified. This is shown in Figure 12.27.

The voltage gain at high frequencies is

$$A_V = \frac{v_0}{v_s} \cong -\left(\frac{R_G}{R_G + R_I}\right)\left(\frac{g_m R_L^{'}}{\left(1 + s(R_G \| R_I)C_1\right)\left(1 + sR_L^{'}C_2\right)}\right) \tag{12.114}$$

where
$$C_1 = C_{gs} + C_{gd}\left(1 + g_m R_L^{'}\right) \tag{12.115}$$
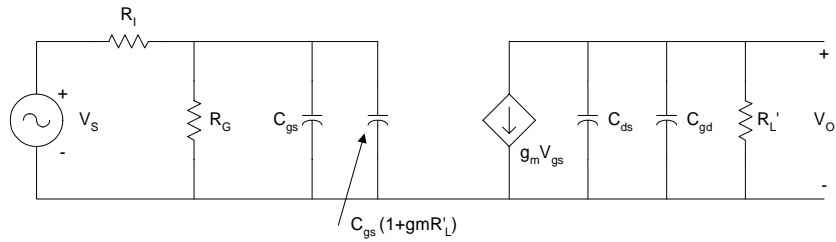and
$$C_2 = C_{ds} + C_{gd} \tag{12.116}$$

Figure 12.27 Simplified High Frequency Equivalent Circuit for Common-source Amplifier

The high frequency poles are

$$w_{H1} = \frac{1}{C_1 \left(R_G \| R_I \right)} \tag{12.117}$$

$$w_{H2} = \frac{1}{C_2 \left(R_L \| R_D \| r_{ds} \right)} \tag{12.118}$$

The approximate high frequency cut-off is

$$w_H = \frac{1}{\sqrt{\left(\dfrac{1}{w_{H1}} \right)^2 + \left(\dfrac{1}{w_{H2}} \right)^2}} \tag{12.119}$$

In the following example, MATLAB is used to obtain the midband gain, cut-off frequencies and bandwidth of a common-source amplifier.

Example 12.9

For the common-source amplifier, shown in Figure 12.23,
$C_{C1} = C_{C2} = 1\mu F, \; C_S = 50\mu F$. The FET parameters are
$C_{gd} = C_{ds} = 1 \; pF, \; C_{gs} = 10 \; pF, \; g_m = 10 \; mA/V, \; r_{ds} = 50 \; K\Omega$.
$R_D = 8 \; K\Omega, \; R_L = 10 \; K\Omega, \; R_S = 2 \; K\Omega, \; R_I = 50 \; \Omega$,
$R_{G1} = 5 \; M\Omega, \; R_{G2} = 5 \; M\Omega$.
Determine (a) midband gain, (b) the low frequency cut-off, (c) high frequency cut-off, and (d) bandwidth of the amplifier.

Solution

MATLAB Script

```
%
% common-source amplifier
%
diary ex12_9.dat
diary on
rg1=5e6; rg2=5e6; rd=8e3; rl=10e3;
ri=50; rs=2e3; rds=50e3;
cc1=1e-6; cc2=1e-6; cs=50e-6;
gm=10e-3; cgs=10e-12; cgd=1e-12; cds=1e-12;

% Calculate midband gain using equation (12.108)
a = (1/rds) + (1/rd) + (1/rl);
rlprime = 1/a;
rg = rg1*rg2/(rg1 + rg2);
gain_mb = -gm*rg*rlprime/(ri + rg);

% Calculate Low cut-off frequency using equation (12.113)
t1 = cc1*(rg + ri);
wl1 = 1/t1;
rd_rds = (rd*rds)/(rd + rds);
t2 = cc2 * (rl + rd_rds);
wl2=1/t2;
t3=cs * rs/(1 + gm * rs);
wl3=1/t3;
wl=sqrt(wl1^2 + wl2^2 + wl3^2);
```

% Calculate high frequency cut-off using equations (12.115 to 12.119)
c1=cgs + cgd * (1 + gm * rlprime);
c2=cds + cgd;
rg_ri=rg * ri/(rg + ri);
wh1=1/(rg_ri * c1);
wh2=1/(rlprime * c2);
int_term = sqrt((1/wh1)^2 + (1/wh2)^2);
wh = 1/int_term;
bw = wh-wl;

% Print results
fprintf('Midband Gain is %8.3f\n', gain_mb)
fprintf('Low frequency cut-off is %8.3e\n', wl)
fprintf('High frequency cut-off is %8.3e\n', wh)
fprintf('Bandwidth is %8.3e Hz\n', bw)

The results are

Midband Gain is -40.816
Low frequency cut-off is 2.182e+002
High frequency cut-off is 1.168e+008
Bandwidth is 1.168e+008 Hz

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EXERCISES

12.1 For the data provided in Example 12.2, Use MATLAB to sketch the output characteristics for $V_{BE} = 0.3, 0.5, 0.7$ V. Do not neglect the effect of V_{AF} on the collector current.

12.2 For the self-bias circuit, shown in Figure 12.6, the collector current involving I_{CBO} is given by Equation (12.47). Assuming that $R_{B1} = 75\ K\Omega$, $R_{B2} = 25\ K\Omega$, $R_E = 1\ K\Omega$, $R_C = 7.5\ K\Omega$, $\beta_F = 100$, and at 25° C, $V_{BE} = 0.6$ V and $I_{CBO} = 0.01\ \mu A$, determine the collector currents for temperatures between 25 °C and 85 °C. If R_E is changed to 3 $K\Omega$, what will be the value of I_C?

12.3 For Figure 12.13, if $R_{B1} = 50\ K\Omega$, $R_{B2} = 40\ K\Omega$, $r_S = 50\ \Omega$, $r_X = 10\ \Omega$, $R_L = 5\ K\Omega$, $R_C = 5\ K\Omega$, $r_{ce} = 100\ K\Omega$, $C_{C1} = C_{C2} = 2\mu F$, $C_\pi = 50 pF$, $C_\mu = 2 pF$, $\beta_F = 100$, V_{CC} = 10 V, explore the low frequency response for the following values of R_E: 0.1 KΩ, 1 KΩ, 5 KΩ. Calculate the high frequency cut-off for R_E = 0.1 KΩ.

12.4 For the Widlar current source, shown in Figure P12.4, determine the output current if $R_C = 40\ K\Omega$, V_{CC} = 10 V, $V_{BE1} = 0.7$ V and $R_2 = 25\ K\Omega$.
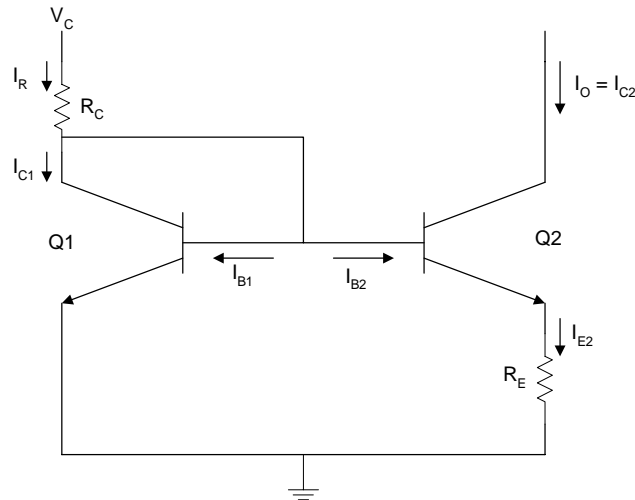
Figure P12.4 Widlar Current Source

12.5 For the n-channel enhancement-type MOSFET with $k_n = 2 \, mA/V^2$ and $V_T = 1\,V$, write a MATLAB program to plot the triode characteristics for $V_{GS} = 2, 3, 4, 5$ V when $V_{DS} < 1\,V$.

12.6 For Figure 12.19, V_T = 1.5 V, k_n = 0.5 mA/V², V_{DD} = 10V, $R_{G1} = 10 \, M\Omega$, $R_{G2} = 12 \, M\Omega$, and $R_D = 10 \, K\Omega$. Find I_D for the following values of R_S: 2, 4, 6, 8 $K\Omega$. Indicate the region of operation for each value of R_S.

12.7 For the common-source amplifier shown in Figure 12.23,
$R_D = 10 \, K\Omega, R_L = 1 \, M\Omega, R_{SB} = 1.5 \, K\Omega, R_S = 100 \, \Omega,$
$R_{G1} = 10 \, M\Omega, R_{G2} = 10 \, M\Omega, C_{C1} = C_{C2} = 2 \, \mu F,$
$C_S = 40 \, \mu F.$ The FET parameters are $C_{gs} = 10 \, pF,$
$C_{gd} = C_{ds} = 1.5 \, pF, \; g_m = 5 \, mA/V,$ and
$r_{ds} = 100 \, K\Omega.$ Use MATLAB to plot the frequency response of the amplifier.