Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

Homer Reid

August 22, 2000

Chapter 1

Problem 1.1

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV (million electron volt) is a unit of energy, used in modern physics, equal to 1.60 x 10^{-6} erg. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34 x 10^{-17} gm-cm/sec.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90 x 10^{-22} gm, what is its kinetic energy, in electron volts?

Place the nucleus at the origin, and suppose the electron is emitted in the positive y direction, and the neutrino in the positive x direction. Then the resultant of the electron and neutrino momenta has magnitude

$$|\mathbf{p_{e+\nu}}| = \sqrt{(1.73)^2 + 1^2} = 2 \text{ MeV/c},$$

and its direction makes an angle

$$\theta = \tan^{-1}\frac{1.73}{1} = 60^\circ$$

with the x axis. The nucleus must acquire a momentum of equal magnitude and directed in the opposite direction. The kinetic energy of the nucleus is

$$T = \frac{p^2}{2m} = \frac{4 \text{ MeV}^2 \text{ c}^{-2}}{2 \cdot 3.9 \cdot 10^{-22} \text{ gm}} \cdot \frac{1.78 \cdot 10^{-27} \text{ gm}}{1 \text{ MeV c}^{-2}} = 9.1 \text{ ev}$$

This is much smaller than the nucleus rest energy of several hundred GeV, so the non-relativistic approximation is justified.

Problem 1.2

The *escape velocity* of a particle on the earth is the minimum velocity required at the surface of the earth in order that the particle can escape from the earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for the earth, ignoring the presence of the moon, is 6.95 mi/sec.

If the particle starts at the earth's surface with the escape velocity, it will just manage to break free of the earth's field and have nothing left. Thus after it has escaped the earth's field it will have no kinetic energy left, and also no potential energy since it's out of the earth's field, so its total energy will be zero. Since the particle's total energy must be constant, it must also have zero total energy at the surface of the earth. This means that the kinetic energy it has at the surface of the earth must exactly cancel the gravitational potential energy it has there:

$$\frac{1}{2}mv_e^2 - G\frac{mM_R}{R_R} = 0$$

so

$$\begin{aligned} v &= \sqrt{\left(\frac{2GM_R}{R_R}\right)} = \left(\frac{2 \cdot (6.67 \cdot 10^{11} \text{ m}^3 \text{ kg}^{-3} \text{ s}^{-2}) \cdot (5.98 \cdot 10^{24} \text{ kg})}{6.38 \cdot 10^6 \text{ m}}\right)^{1/2} \\ &= 11.2 \text{ km/s} \cdot \frac{1 \text{ m}}{1.61 \text{ km}} = 6.95 \text{ mi/s}. \end{aligned}$$

Problem 1.3

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric resistance, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 6800 ft/sec and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Suppose that, at time t, the rocket has mass $m(t)$ and velocity $v(t)$. The total external force on the rocket is then $F = gm(t)$, with $g = 32.1$ ft/s^2, pointed downwards, so that the total change in momentum between t and $t + dt$ is

$$F dt = -gm(t)dt. \tag{1}$$

At time t, the rocket has momentum

$$p(t) = m(t)v(t). \tag{2}$$

On the other hand, during the time interval dt the rocket releases a mass Δm of gas at a velocity v' with respect to the rocket. In so doing, the rocket's velocity increases by an amount dv. The total momentum at time $t + dt$ is the sum of the momenta of the rocket and gas:

$$p(t + dt) = p_r + p_g = [m(t) - \Delta m][v(t) + dv] + \Delta m[v(t) + v'] \tag{3}$$

Subtracting (2) from (3) and equating the difference with (1), we have (to first order in differential quantities)

$$-gm(t)dt = m(t)dv + v'\Delta m$$

or

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\frac{\Delta m}{dt}$$

which we may write as

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\gamma \tag{4}$$

where

$$\gamma = \frac{\Delta m}{dt} = \frac{1}{60} m_0 s^{-1}.$$

This is a differential equation for the function $v(t)$ giving the velocity of the rocket as a function of time. We would now like to recast this as a differential equation for the function $v(m)$ giving the rocket's velocity as a function of its mass. To do this, we first observe that since the rocket is *releasing* the mass Δm every dt seconds, the time derivative of the rocket's mass is

$$\frac{dm}{dt} = -\frac{\Delta m}{dt} = -\gamma.$$

We then have

$$\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt} = -\gamma\frac{dv}{dm}.$$

Substituting into (4), we obtain

$$-\gamma\frac{dv}{dm} = -g - \frac{v'}{m}\gamma$$

or

$$dv = \frac{g}{\gamma}dm + v'\frac{dm}{m}.$$

Integrating, with the condition that $v(m_0) = 0$,

$$v(m) = \frac{g}{\gamma}(m - m_0) + v' \ln\left(\frac{m}{m_0}\right).$$

Now, γ=(1/60)m_0 s^{-1}, while v' =-6800 ft/s. Then

$$v(m) = 1930 \text{ ft/s} \cdot \left(\frac{m}{m_0} - 1\right) + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right)$$

For $m_0 \gg m$ we can neglect the first term in the parentheses of the first term, giving

$$v(m) = -1930 \text{ ft/s} + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right).$$

The escape velocity is $v = 6.95$ mi/s $= 36.7 \cdot 10^3$ ft/s. Plugging this into the equation above and working backwards, we find that escape velocity is achieved when m_0/m=293.

Thanks to Brian Hart for pointing out an inconsistency in my original choice of notation for this problem.

Problem 1.4

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F}\cdot\mathbf{p}.$$

We have

$$\mathbf{F} = \dot{\mathbf{p}} \tag{5}$$

If m is constant,

$$\mathbf{F} = m\dot{\mathbf{v}}$$

Dotting $\mathbf{v}$ into both sides,

$$\begin{aligned}\mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\dot{\mathbf{v}} &= \frac{1}{2}m\frac{d}{dt}|\mathbf{v}|^2\\ &= \frac{dT}{dt}\end{aligned} \tag{6}$$

On the other hand, if m is not constant, instead of $\mathbf{v}$ we dot $\mathbf{p}$ into (5):

$$\begin{aligned}\mathbf{F}\cdot\mathbf{p} &= \mathbf{p}\cdot\dot{\mathbf{p}}\\ &= m\mathbf{v}\cdot\frac{d(m\mathbf{v})}{dt}\\ &= m\mathbf{v}\cdot\left(\mathbf{v}\frac{dm}{dt} + m\frac{d\mathbf{v}}{dt}\right)\\ &= \frac{1}{2}v^2\frac{d}{dt}m^2 + \frac{1}{2}m^2\frac{d}{dt}(v^2)\\ &= \frac{1}{2}\frac{d}{dt}(m^2v^2) = \frac{d(mT)}{dt}.\end{aligned}$$

Problem 1.5

Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2.$$

We have

$$R_x = \frac{1}{M}\sum_i m_i x_i$$

so

$$R_x^2 = \frac{1}{M^2}\left[\sum_i m_i^2 x_i^2 + \sum_{i\neq j} m_i m_j x_i x_j\right]$$

and similarly

$$R_y^2 = \frac{1}{M^2}\left[\sum_i m_i^2 y_i^2 + \sum_{i\neq j} m_i m_j y_i y_j\right]$$

$$R_z^2 = \frac{1}{M^2}\left[\sum_i m_i^2 z_i^2 + \sum_{i\neq j} m_i m_j z_i z_j\right].$$

Adding,

$$R^2 = \frac{1}{M^2}\left[\sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j (\mathbf{r_i}\cdot\mathbf{r_j})\right]. \qquad (7)$$

On the other hand,

$$r_{ij}^2 = r_i^2 + r_j^2 - 2\mathbf{r_i}\cdot\mathbf{r_j}$$

and, in particular, $r_{ii}^2 = 0$, so

$$\begin{aligned}\sum_{i,j} m_i m_j r_{ij}^2 &= \sum_{i\neq j}[m_i m_j r_i^2 + m_i m_j r_j^2 - 2m_i m_j (r_i\cdot r_j)]\\ &= 2\sum_{i\neq j} m_i m_j r_i^2 - 2\sum_{i\neq j} m_i m_j (r_i\cdot r_j). \qquad (8)\end{aligned}$$

Next,

$$M\sum_i m_i r_i^2 = \sum_j m_j\left(\sum_i m_i r_i^2\right) = \sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j r_i^2. \qquad (9)$$

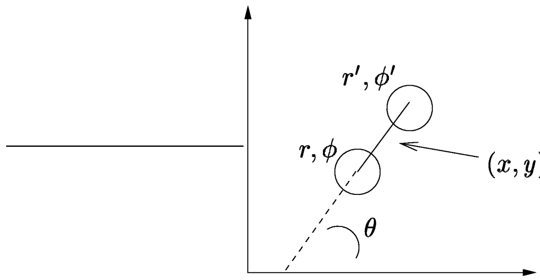

Figure 1: My conception of the situation of Problem 1.8

Subtracting half of (8) from (9), we have

$$M \sum m_i r_i^2 - \frac{1}{2} \sum ij m_i m_j r_{ij}^2 = \sum_i m_i^2 r_i^2 + \sum_{i \neq j} m_i m_j (\mathbf{r_i} \cdot \mathbf{r_j})$$

and comparing this with (7) we see that we are done.

Problem 1.8

Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\cos\theta\, dx + \sin\theta\, dy = 0$$

$$\sin\theta\, dx - \cos\theta\, dy = a(d\phi + d\phi')$$

(where θ, ϕ, and ϕ' have meanings similar to the problem of a single vertical disc, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

My conception of the situation is illustrated in Figure 1. θ is the angle between the x axis and the axis of the two wheels. ϕ and ϕ' are the rotation angles of the two wheels, and $\mathbf{r}$ and $\mathbf{r}'$ are the locations of their centers. The center of the wheel axis is the point just between $\mathbf{r}$ and $\mathbf{r}'$:

$$(x, y) = \frac{1}{2}(r_x + r'_x, r_y + r'_y).$$

If the ϕ wheel rotates through an angle $d\phi$, the vector displacement of its center will have magnitude $ad\phi$ and direction determined by θ. For example, if $\theta = 0$ then the wheel axis is parallel to the x axis, in which case rolling the ϕ wheel clockwise will cause it to move in the negative y direction. In general, referring to the Figure, we have

$$\mathbf{dr} = a\,d\phi[\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}] \tag{10}$$

$$\mathbf{dr}' = a\,d\phi'[\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}] \tag{11}$$

Adding these componentwise we have[1]

$$\begin{aligned} dx &= \frac{a}{2}[d\phi + d\phi']\sin\theta \\ dy &= -\frac{a}{2}[d\phi + d\phi']\cos\theta \end{aligned}$$

Multiplying these by $\sin\theta$ or $-\cos\theta$ and adding or subtracting, we obtain

$$\begin{aligned} &\sin\theta\,dx - \cos\theta\,dy = a[d\phi + d\phi'] \\ &\cos\theta\,dx + \sin\theta\,dy = 0. \end{aligned}$$

Next, consider the vector $\mathbf{r}_{12} = \mathbf{r} - \mathbf{r}'$ connecting the centers of the two wheels. The definition of θ is such that its tangent must just be the ratio of the y and x components of this vector:

$$\begin{aligned} \tan\theta &= \frac{y_{12}}{x_{12}} \\ \to \sec^2\theta\,d\theta &= -\frac{y_{12}}{x_{12}^2}\,dx_{12} + \frac{1}{x_{12}}\,dy_{12}. \end{aligned}$$

Subtracting (11) from (10),

$$\sec^2\theta d\theta = a[d\phi - d\phi']\left(-\frac{y_{12}}{x_{12}^2}\sin\theta - \frac{1}{x_{12}}\cos\theta\right)$$

Again substituting for y_{12}/x_{12} in the first term in parentheses,

$$\sec^2\theta d\theta = -a[d\phi - d\phi']\frac{1}{x_{12}}(\tan\theta\sin\theta + \cos\theta)$$

or

$$\begin{aligned} d\theta &= -a[d\phi - d\phi']\frac{1}{x_{12}}(\sin^2\theta\cos\theta + \cos^3\theta) \\ &= -a[d\phi - d\phi']\frac{1}{x_{12}}\cos\theta. \end{aligned} \tag{12}$$

[1]Thanks to Javier Garcia for pointing out a factor-of-two error in the original version of these equations.

However, considering the definition of θ, we clearly have

$$\cos\theta = \frac{x_{12}}{(x_{12}^2 + y_{12}^2)^{1/2}} = \frac{x_{12}}{b}$$

because the magnitude of the distance between r_1 and r_2 is constrained to be b by the rigid axis. Then (12) becomes

$$d\theta = -\frac{a}{b}[d\phi - d\phi']$$

with immediate solution

$$\theta = C - \frac{a}{b}[\phi - \phi'].$$

with C a constant of integration.

Problem 1.9

A particle moves in the $x - y$ plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

The particle's position is $(x(t), y(t))$, while the position of the moving point is $(f(t), 0)$. Then the vector $\mathbf{d}$ from the particle to the point has components

$$d_x = x(t) - f(t) \qquad d_y = y(t). \tag{13}$$

The particle's velocity $\mathbf{v}$ has components

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \tag{14}$$

and for the vectors in (13) and (**??** to be in the same direction, we require

$$\frac{v_y}{v_x} = \frac{d_y}{d_x}$$

or

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

so

$$\frac{dy}{y} = \frac{dx}{x - f(t)} \tag{15}$$

For example, if $f(t) = \alpha t$, then we may integrate to find

$$\ln y(t) = \ln[x(t) - \alpha(t)] + C$$

or

$$y(t) = C \cdot [x(t) - \alpha t]$$

which is a holonomic constraint. But for general $f(t)$ the right side of (15) is not integrable, so the constraint is nonholonomic.

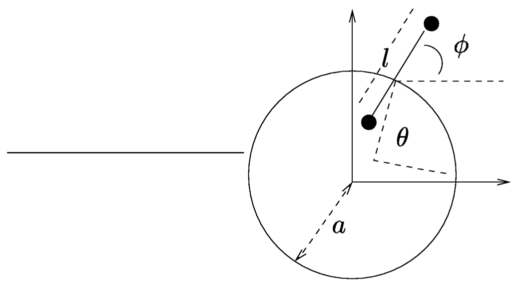

Figure 2: My conception of the situation of Problem 1.10

Problem 1.10

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Set up the kinetic energy in generalized coordinates.

My conception of this one is shown in Figure 2. θ is the angle representing how far around the circle the center of the rod has moved. ϕ is the angle the rod makes with the x axis.

The position of the center of the rod is $(x, y) = (a\cos\theta, a\sin\theta)$. The positions of the masses relative to the center of the rod are $(x_{rel}, y_{rel}) = \pm(1/2)(l\cos\phi, l\sin\phi)$. Then the absolute positions of the masses are

$$(x, y) = (a\cos\theta \pm \frac{l}{2}\cos\phi, a\sin\theta \pm \frac{l}{2}\sin\phi)$$

and their velocities are

$$(v_x, v_y) = (-a\sin\theta\,\dot{\theta} \mp \frac{l}{2}\sin\phi\,\dot{\phi}, a\cos\theta\,\dot{\theta} \pm \frac{l}{2}\cos\phi\,\dot{\phi}).$$

The magnitudes of these are

$$\begin{aligned}|v| &= a^2\dot{\theta}^2 + \frac{l^2}{4}\dot{\phi}^2 \pm al\,\dot{\theta}\,\dot{\phi}(\sin\theta\sin\phi + \cos\theta\cos\phi)\\ &= a^2\dot{\theta}^2 + \frac{l^2}{4}\dot{\phi}^2 \pm al\,\dot{\theta}\,\dot{\phi}\cos(\theta - \phi)\end{aligned}$$

When we add the kinetic energies of the two masses, the third term cancels, and we have

$$T = \frac{1}{2}\sum mv^2 = m(a^2\dot{\theta}^2 + \frac{l^2}{4}\dot{\phi}^2).$$

Problem 1.11

Show that Lagrange's equations in the form of Eq. 1-53 can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes known as the *Nielsen* form of the Lagrange equations.

Problem 1.12

A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \sigma \cdot \mathbf{L}$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and σ is a fixed vector in space.

(a) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq. (1-58).

(b) Show that the components in the two coordinate systems are related to each other as in Eq. (1-49).

(c) Obtain the equations of motion in spherical polar coordinates.

Problem 1.13

A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right),$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics).

If we take

$$U(r) = \frac{1}{r}\left(1 + \frac{v^2}{c^2}\right) = \frac{1}{r} + \frac{(\dot{r})^2}{c^2 r}$$

then

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2}$$

and

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{d}{dt}\left(\frac{2\dot{r}}{c^2 r}\right) = \frac{2\ddot{r}}{c^2 r} - \frac{2(\dot{r})^2}{c^2 r^2}$$

so

$$Q_r = -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2}\left(1 + \frac{2r\ddot{r} - (\dot{r})^2}{c^2}\right)$$

The Lagrangian for motion in a plane is

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{r}^2\dot{\theta}^2 - \frac{1}{r^2}\left(1 + \frac{2r\ddot{r} - (\dot{r})^2}{c^2}\right).$$

Problem 1.14

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, \dots, q_n, t)}{dt}$$

also satisfies Lagrange's equations, where F is any arbitrary, but differentiable, function of its arguments.

We have

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} \tag{16}$$

and

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}. \tag{17}$$

For the function F we may write

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$$

and from this we may read off

$$\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt} = \frac{\partial F}{\partial q_i}.$$

Then taking the time derivative of (17) gives

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i}$$

so we have

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} - \frac{d}{dt}\frac{\partial F}{\partial q_i}.$$

The first two terms on the RHS cancel because L satisfies the Euler-Lagrange equations, while the second two terms cancel because F is differentiable. Hence L' satisfies the Euler-Lagrange equations.

Problem 1.16

A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0, c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1-56) is related to L' by a point transformation (cf. Exercise 15 above). What is the significance of the condition on the value of $b^2 - ac$?

Clearly we have

$$\frac{\partial L}{\partial x} = -Kax - Kby \qquad \frac{\partial L}{\partial \dot{x}} = ma\dot{x} + mb\dot{y}$$

so the Euler-Lagrange equation for x is

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \qquad \rightarrow \qquad m(a\ddot{x} + b\ddot{y}) = -K(ax + by).$$

Similarly, for y we obtain

$$m(b\ddot{y} + c\ddot{y}) = -K(bx + cy).$$

These are the equations of motion for a particle of mass m undergoing simple harmonic motion in two dimensions, as if bound by two springs of spring constant K. Normally we would express the Lagrangian in unravelled form, by transforming to new coordinates u_1 and u_2 with

$$u_1 = ax + by \qquad u_2 = bx + cy.$$

The condition $b^2 - ac \neq 0$ is the condition that the coordinate transformation not be degenerate, i.e. that there are actually two distinct dimensions in which the particle experiences a restoring force. If $b^2 = ac$ then we have just a one-dimensional problem.

Problem 1.17

Obtain the Lagrangian equations of motion for a spherical pendulum, i.e. a mass point suspended by a rigid weightless rod.

Denoting the mass of the particle by m, the length of the rod by L, and the angle between the rod and the vertical by θ, we have the particle's linear velocity given in magnitude by $v = L\dot{\theta}$, while its height is $h = -L\cos\theta$ (where the fulcrum of the pendulum is taken as the origin of coordinates). Then

$$L = T - V = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

so the equation of motion is

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \qquad \to \qquad -g\sin\theta = L\ddot{\theta}.$$

Problem 1.18

A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x),$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

We have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= m\dot{x}^2\frac{dV}{dx} - 2V(x)\frac{dV}{dx} \\
\frac{\partial L}{\partial \dot{x}} &= \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m^2(\dot{x})^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}\frac{d}{dt}V(x)
\end{aligned}$$

In the last equation we can use

$$\frac{d}{dt}V(x) = \dot{x}\frac{dV}{dx}.$$

Then the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \to \quad m^2(\dot{x})^2\ddot{x} + 2m\ddot{x}V(x) + m\dot{x}^2\frac{dV}{dx} + 2V(x)\frac{dV}{dx}$$

or

$$\left(m\ddot{x} + \frac{dV}{dx}\right)\left(m\dot{x}^2 + 2V(x)\right) = 0.$$

If we identify $F = -dV/dx$ and $T = m\dot{x}^2/2$, we may write this as

$$(F - m\ddot{x})(T + V) = 0$$

So, this is saying that, at all times, either the difference between F and ma is zero, *or* the sum of kinetic and potential energy is zero.

Problem 1.19

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole).

Let d be the height of m_2 above its lowest possible position, so that $d = 0$ when the string is fully extended beneath the table and m_1 is just about to fall through the hole. Also, let θ be the angular coordinate of m_1 on the table. Then the kinetic energy of m_2 is just $m_2\dot{d}^2/2$, while the kinetic energy of m_1 is $m_1\dot{d}^2/2 + m_1 d^2\dot{\theta}^2/2$, and the potential energy of the system is just the gravitational potential energy of m_2, $U = m_2 g d$. Then the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_1 d^2\dot{\theta}^2 - m_2 g d$$

and the Euler-Lagrange equations are

$$\frac{d}{dt}(m_1 d^2\dot{\theta}) = 0$$
$$(m_1 + m_2)\ddot{d} = -m_2 g + m_1 d\dot{\theta}^2$$

From the first equation we can identify a first integral, $m_1 d^2\dot{\theta} = l$ where l is a constant. With this we can substitute for $\dot{\theta}$ in the second equation:

$$(m_1 + m_2)\ddot{d} = -m_2 g + \frac{l^2}{m_1 d^3}$$

Because the sign of the two terms on the RHS is different, this is saying that, if l is big enough (if m_1 is spinning fast enough), the centrifugal force of m_1 can balance the downward pull of m_2, and the system can be in equilibrium.

Problem 1.20

Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig. 1-4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Taking the origin at the fulcrum of the first pendulum, we can write down the coordinates of the first mass point:

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ y_1 &= -l_1 \cos\theta_1 \end{aligned}$$

The coordinates of the second mass point are defined relative to the coordinates of the first mass point by exactly analogous expressions, so relative to the coordinate origin we have

$$\begin{aligned} x_2 &= x_1 + l_2 \sin\theta_2 \\ y_2 &= y_1 - l_2 \cos\theta_2 \end{aligned}$$

Differentiating and doing a little algebra we find

$$\begin{aligned} \dot{x}_1^2 + \dot{y}_1^2 &= l_1^2\dot{\theta}_1^2 \\ \dot{x}_2^2 + \dot{y}_2^2 &= l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 - 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \end{aligned}$$

The Lagrangian is

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 - m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_2\cos\theta_1 + m_2gl_2\cos\theta_2$$

with equations of motion

$$\frac{d}{dt}\Big[(m_1+m_2)l_1^2\dot{\theta}_1 - m_2l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2)\Big] = -(m_1+m_2)gl_2\sin\theta_1$$

and

$$\frac{d}{dt}\Big[l_2\dot{\theta}_2 - l_1\dot{\theta}_1\cos(\theta_1-\theta_2)\Big] = -g\sin\theta_2.$$

If $\dot{\theta}_1 = 0$, so that the fulcrum for the second pendulum is stationary, then the second of these equations reduces to the equation we derived in Problem 1.17.

Problem 1.21

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\Psi(\mathbf{r}, t),$$
$$\Phi \rightarrow \Phi - \frac{1}{c}\frac{\partial\Psi}{\partial t},$$

where Ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

The Lagrangian for a particle in an electromagnetic field is

$$L = T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t)$$

If we make the suggested gauge transformation, this becomes

$$\begin{aligned}
&\rightarrow T - q\left[\Phi(\mathbf{x}(t)) - \frac{1}{c}\left.\frac{\partial\Psi}{\partial t}\right|_{\mathbf{x}=\mathbf{x}(t)}\right] + \frac{q}{c}\left[\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \mathbf{v} \cdot \nabla\Psi(\mathbf{x}(t))\right] \\
&= T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \frac{q}{c}\left[\frac{\partial\Psi}{\partial t} + \mathbf{v} \cdot \nabla\Psi(\mathbf{x}(t))\right] \\
&= T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \frac{q}{c}\frac{d}{dt}\Psi(\mathbf{x}(t)) \\
&= L + \frac{q}{c}\frac{d}{dt}\Psi(\mathbf{x}(t)).
\end{aligned}$$

So the transformed Lagrangian equals the original Lagrangian plus a total time derivative. But we proved in Problem 1.15 that adding the total time derivative of any function to the Lagrangian does not affect the equations of motion, so the motion of the particle is unaffected by the gauge transformation.

Problem 1.22

Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for fall from rest is $v = mg/k$.

The Lagrangian for the particle is

$$L = \frac{1}{2}m\dot{z}^2 - mgz$$

and the dissipation function is $k\dot{z}^2/2$, so the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \frac{\partial F}{\partial \dot{z}} \rightarrow m\ddot{z} = mg - k\dot{z}.$$

This says that the acceleration goes to zero when $mg = k\dot{z}$, or $\dot{z} = mg/k$, so the velocity can never rise above this terminal value (unless the initial value of the velocity is greater than the terminal velocity, in which case the particle will slow down to the terminal velocity and then stay there).

Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

Homer Reid

December 1, 2001

Chapter 3

Problem 3.1

> A particle of mass m is constrained to move under gravity without friction on the inside of a paraboloid of revolution whose axis is vertical. Find the one-dimensional problem equivalent to its motion. What is the condition on the particle's initial velocity to produce circular motion? Find the period of small oscillations about this circular motion.

We'll take the paraboloid to be defined by the equation $z = \alpha r^2$. The kinetic and potential energies of the particle are

$$\begin{aligned} T &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + 4\alpha^2 r^2 \dot{r}^2) \\ V &= mgz = mg\alpha r^2. \end{aligned}$$

Hence the Lagrangian is

$$L = \frac{m}{2}\left[(1 + 4\alpha^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2\right] - mg\alpha r^2.$$

This is cyclic in θ, so the angular momentum is conserved:

$$l = mr^2\dot{\theta} = \text{constant}.$$

For r we have the derivatives

$$\begin{aligned}\frac{\partial L}{\partial r} &= 4\alpha^2 mr\dot{r}^2 + mr\dot{\theta}^2 - 2mg\alpha r\\ \frac{\partial L}{\partial \dot{r}} &= m(1+4\alpha^2 r^2)\dot{r}\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 8m\alpha^2 r\dot{r}^2 + m(1+4\alpha^2r^2)\ddot{r}.\end{aligned}$$

Hence the equation of motion for r is

$$8m\alpha^2 r\dot{r}^2 + m(1+4\alpha^2r^2)\ddot{r} = 4\alpha^2 mr\dot{r}^2 + mr\dot{\theta}^2 - 2mg\alpha r$$

or

$$m(1+4\alpha^2r^2)\ddot{r} + 4m\alpha^2 r\dot{r}^2 - mr\dot{\theta}^2 + 2mg\alpha r = 0.$$

In terms of the constant angular momentum, we may rewrite this as

$$m(1+4\alpha^2r^2)\ddot{r} + 4m\alpha^2 r\dot{r}^2 - \frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

So this is the differential equation that determines the time evolution of r.

If initially $\dot{r} = 0$, then we have

$$m(1+4\alpha^2r^2)\ddot{r} + -\frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

Evidently, $\ddot{r}$ will then vanish—and hence $\dot{r}$ will remain 0, giving circular motion—if

$$\frac{l^2}{mr^3} = 2mg\alpha r$$

or

$$\dot{\theta} = \sqrt{2g\alpha}.$$

So if this condition is satisfied, the particle will execute circular motion (assuming its initial r velocity was zero). It's interesting to note that the condition on $\dot{\theta}$ for circular motion is independent of r.

Problem 3.2

A particle moves in a central force field given by the potential

$$V = -k\frac{e^{-ar}}{r},$$

where k and a are positive constants. Using the method of the equivalent one-dimensional potential discuss the nature of the motion, stating the ranges of l and E appropriate to each type of motion. When are circular orbits possible? Find the period of small radial oscillations about the circular motion.

The Lagrangian is

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + k\frac{e^{-ar}}{r}.$$

As usual the angular momentum is conserved:

$$l = mr^2\dot{\theta} = \text{constant}.$$

We have

$$\begin{aligned}\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - k\left(1 + ar\right)\frac{e^{-ar}}{r^2}\\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r}\end{aligned}$$

so the equation of motion for r is

$$\begin{aligned}\ddot{r} &= r\dot{\theta}^2 - \frac{k}{m}\left(1 + ar\right)\frac{e^{-ar}}{r^2}\\ &= \frac{l^2}{m^2r^3} - \frac{k}{m}\left(1 + ar\right)\frac{e^{-ar}}{r^2}. \qquad (1)\end{aligned}$$

The condition for circular motion is that this vanish, which yields

$$\dot{\theta} = \sqrt{\frac{k}{m}(1 + ar_0)}\frac{e^{-ar_0/2}}{r_0^{3/2}}. \qquad (2)$$

What this means is that that if the particle's initial θ velocity is equal to the above function of the starting radius r_0, then the second derivative of r will remain zero for all time. (Note that, in contrast to the previous problem, in this case the condition for circular motion *does* depend on the starting radius.)

To find the frequency of small oscillations, let's suppose the particle is executing a circular orbit with radius r_0 (in which case the θ velocity is given by (2)), and suppose we nudge it slightly so that its radius becomes $r = r_0 + x$, where x is small. Then (1) becomes

$$\ddot{x} = \frac{k}{m}\big(1 + ar_0\big)\frac{e^{-ar_0}}{r_0^2} - \frac{k}{m}\big(1 + a[r_0 + x]\big)\frac{e^{-a[r_0+x]}}{[r_0 + x]^2} \qquad (3)$$

Since x is small, we may write the second term approximately as

$$\begin{aligned}
&\approx \frac{k}{m}\frac{e^{-ar_0}}{r_0^2}(1+ar_0+ax)(1-ax)\left(1-2\frac{x}{r_0}\right)\\
&\approx \frac{k}{m}(1+ar_0)\frac{e^{-ar_0}}{r_0^2}+\frac{k}{m}\frac{e^{-ar_0}}{r_0^2}\left(a-a(1+ar_0)-2\frac{(1+ar_0)}{r_0}\right)x\\
&\approx \frac{k}{m}(1+ar_0)\frac{e^{-ar_0}}{r_0^2}-\frac{k}{m}\frac{e^{-ar_0}}{r_0^2}\left(2a+\frac{2}{r_0}+a^2r_0\right)x.
\end{aligned}$$

The first term here just cancels the first term in (**??**), so we are left with

$$\ddot{x}=\frac{k}{m}\frac{e^{-ar_0}}{r_0^2}\left(2a+\frac{2}{r_0}+a^2r_0\right)x$$

The problem is that the RHS here has the wrong sign—this equation is satisfied by an x that grows (or decays) exponentially, rather than oscillates. Somehow I messed up the sign of the RHS, but I can't find where–can anybody help?

Problem 3.3

Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ. Their motion is suddenly stopped, and they are then released and allowed to fall into each other. Prove that they collide after a time $\tau/4\sqrt{2}$.

Since we are dealing with gravitational forces, the potential energy between the particles is

$$U(r)=-\frac{k}{r}$$

and, after reduction to the equivalent one-body problem, the Lagrangian is

$$\mathcal{L}=\frac{\mu}{2}[\dot{r}^2+r^2\dot{\theta}^2]+\frac{k}{r}$$

where μ is the reduced mass. The equation of motion for r is

$$\mu\ddot{r}=\mu r\dot{\theta}^2-\frac{k}{r^2}. \tag{4}$$

If the particles are to move in circular orbits with radius r_0, (4) must vanish at $r=r_0$, which yields a relation between r_0 and $\dot{\theta}$:

$$\begin{aligned}
r_0&=\left(\frac{k}{\mu\dot{\theta}^2}\right)^{1/3}\\
&=\left(\frac{k\tau^2}{4\pi^2\mu}\right)^{1/3} \qquad (5)
\end{aligned}$$

where we used the fact that the angular velocity in the circular orbit with period τ is $\dot{\theta} = 2\pi/\tau$.

When the particles are stopped, the angular velocity goes to zero, and the first term in (4) vanishes, leaving only the second term:

$$\ddot{r} = -\frac{k}{\mu r^2}. \tag{6}$$

This differential equation governs the evolution of the particles after they are stopped. We now want to use this equation to find r as a function of t, which we will then need to invert to find the time required for the particle separation r to go from r_0 to 0.

The first step is to multiply both sides of (6) by the integrating factor $2\dot{r}$:

$$2\dot{r}\ddot{r} = -\frac{2k}{\mu r^2}\dot{r}$$

or

$$\frac{d}{dt}\left(\dot{r}^2\right) = +\frac{d}{dt}\left(\frac{2k}{\mu r}\right)$$

from which we conclude

$$\dot{r}^2 = \frac{2k}{\mu r} + C. \tag{7}$$

The constant C is determined from the boundary condition on $\dot{r}$. This is simply that $\dot{r} = 0$ when $r = r_0$, since initially the particles are not moving at all. With the appropriate choice of C in (7), we have

$$\begin{aligned} \dot{r} = \frac{dr}{dt} &= \left(\frac{2k}{\mu}\right)^{1/2} \sqrt{\frac{1}{r} - \frac{1}{r_0}} \\ &= \left(\frac{2k}{\mu}\right)^{1/2} \sqrt{\frac{r_0 - r}{r r_0}}. \end{aligned} \tag{8}$$

We could now proceed to solve this differential equation for $r(t)$, but since in fact we're interested in solving for the time difference corresponding to given boundary values of r, it's easier to invert (8) and solve for $t(r)$:

$$\begin{aligned} \Delta t &= \int_{r_0}^{0} \left(\frac{dt}{dr}\right) dr \\ &= \int_{r_0}^{0} \left(\frac{dr}{dt}\right)^{-1} dr \\ &= \left(\frac{\mu}{2k}\right)^{1/2} \int_{r_0}^{0} \left(\frac{r r_0}{r_0 - r}\right)^{1/2} dr \end{aligned}$$

We change variables to $u = r/r_0, du = dr/r_0$:

$$= \left(\frac{\mu}{2k}\right)^{1/2} r_0^{3/2} \int_1^0 \left(\frac{u}{1-u}\right)^{1/2} du$$

Next we change variables to $u = \sin^2 x, du = 2\sin x \cos x\, dx$:

$$\begin{aligned} &= 2\left(\frac{\mu}{2k}\right)^{1/2} r_0^{3/2} \int_{\pi/2}^0 \sin^2 x\, dx \\ &= \left(\frac{\mu}{2k}\right)^{1/2} r_0^{3/2} \frac{\pi}{4}. \end{aligned}$$

Now plugging in (5), we obtain

$$\begin{aligned} \Delta t &= \left(\frac{\mu}{2k}\right)^{1/2} \left(\frac{k\tau^2}{4\pi^2\mu}\right)^{1/2} \left(\frac{\pi}{4}\right) \\ &= \frac{\tau}{4\sqrt{2}} \end{aligned}$$

as advertised.

Problem 3.6

(a) Show that if a particle describes a circular orbit under the influence of an attractive central force directed at a point on the circle, then the force varies as the inverse fifth power of the distance.

(b) Show that for the orbit described the total energy of the particle is zero.

(c) Find the period of the motion.

(d) Find $\dot{x}$, $\dot{y}$, and v as a function of angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

Let's suppose the center of force is at the origin, and that the particle's orbit is a circle of radius R centered at $(x = R, y = 0)$ (so that the leftmost point of the particle's origin is the center of force). The equation describing such an orbit is

$$r(\theta) = \sqrt{2}R(1 + \cos 2\theta)^{1/2}$$

so

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{\sqrt{2}R(1 + \cos 2\theta)^{1/2}}. \tag{9}$$

Differentiating,

$$\begin{aligned}
\frac{du}{d\theta} &= \frac{\sin 2\theta}{\sqrt{2}R(1+\cos 2\theta)^{3/2}} \\
\frac{du}{d\theta} &= \frac{1}{\sqrt{2}R}\left[\frac{2\cos 2\theta}{(1+\cos 2\theta)^{3/2}} + 3\frac{\sin^2 2\theta}{(1+\cos 2\theta)^{5/2}}\right] \\
&= \frac{1}{2\sqrt{2}R}\frac{1}{(1+\cos 2\theta)^{5/2}}\left[2\cos 2\theta + 2\cos^2 2\theta + 3\sin^2 2\theta\right]. \qquad (10)
\end{aligned}$$

Adding (9) and (10),

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{1}{\sqrt{2}R(1+\cos 2\theta)^{5/2}}\left[(1+\cos 2\theta)^2 + 2\cos 2\theta + 2\cos^2 2\theta + 3\sin^2 2\theta\right] \\
&= \frac{1}{\sqrt{2}R(1+\cos 2\theta)^{5/2}}\left[4 + 4\cos 2\theta\right] \\
&= \frac{4}{\sqrt{2}R(1+\cos 2\theta)^{3/2}} \\
&= 8R^2u^3. \qquad (11)
\end{aligned}$$

The differential equation for the orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right) \qquad (12)$$

Plugging in (11), we have

$$8R^2u^3 = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

so

$$V\left(\frac{1}{u}\right) = -\frac{2l^2R^2}{m}u^4 \qquad \longrightarrow \qquad V(r) = -\frac{2l^2R^2}{mr^4} \qquad (13)$$

so

$$f(r) = -\frac{8l^2R^2}{mr^5} \qquad (14)$$

which is the advertised r dependence of the force.

(b) The kinetic energy of the particle is

$$T = \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2]. \qquad (15)$$

We have

$$\begin{aligned} r &= \sqrt{2}R(1+\cos 2\theta)^{1/2} \\ r^2 &= 2R^2(1+\cos 2\theta) \\ \dot{r} &= \sqrt{2}R\frac{\sin 2\theta}{(1+\cos 2\theta)^{1/2}}\,\dot{\theta} \\ \dot{r}^2 &= 2R^2\frac{\sin^2 2\theta}{1+\cos 2\theta}\,\dot{\theta}^2 \end{aligned}$$

Plugging into (15),

$$\begin{aligned} T &= mR^2\dot{\theta}^2\left[\frac{\sin^2 2\theta}{1+\cos 2\theta}+1+\cos 2\theta\right] \\ &= mR^2\dot{\theta}^2\left[\frac{\sin^2 2\theta+1+2\cos 2\theta+\cos^2 2\theta}{1+\cos\theta}\right] \\ &= 2mR^2\dot{\theta}^2 \end{aligned}$$

In terms of $l = mr^2\dot{\theta}$, this is just

$$= \frac{2R^2l^2}{mr^4}$$

But this is just the negative of the potential energy, (13); hence the total particle energy $T + V$ is zero.

(c) Suppose the particle starts out at the furthest point from the center of force on its orbit, i.e the point $x = 2R, y = 0$, and that it moves counter-clockwise from this point to the origin. The time required to undergo this motion is half the period of the orbit, and the particle's angle changes from $\theta = 0$ to $\theta = \pi/2$. Hence we can calculate the period as

$$\begin{aligned} \tau &= 2\int_0^{\pi/2}\frac{dt}{d\theta}d\theta \\ &= 2\int_0^{\pi/2}\frac{d\theta}{\dot{\theta}} \end{aligned}$$

Using $\dot{\theta} = l/mr^2$, we have

$$\begin{aligned} &= 2\frac{m}{l}\int_0^{\pi/2} r^2(\theta)\,d\theta \\ &= \frac{4R^2m}{l}\int_0^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta \\ &= \frac{4R^2m}{l}\cdot\frac{3\pi}{4} \\ &= \frac{3\pi R^2m}{l}. \end{aligned}$$

Problem 3.8

(a) For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that the perihelion distance of the parabola is one half the radius of the circle.

(b) Prove that in the same central force as in part (a) the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

(a) The equations describing the orbits are

$$r = \begin{cases} \frac{l^2}{mk} & \text{(circle)} \\ \frac{l^2}{mk}\left(\frac{1}{1+\cos\theta}\right) & \text{(parabola.)} \end{cases}$$

Evidently, the perihelion of the parabola occurs when $\theta = 0$, in which case $r = l^2/2mk$, or one-half the radius of the circle.

(b) For the parabola, we have

$$\begin{aligned} \dot{r} &= \frac{l^2}{mk}\left(\frac{\sin\theta}{(1+\cos\theta)^2}\right)\dot{\theta} \qquad (16) \\ &= r\dot{\theta}\frac{\sin\theta}{1+\cos\theta} \end{aligned}$$

so

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ &= r^2\dot{\theta}^2\left[\frac{\sin^2\theta}{(1+\cos\theta)^2} + 1\right] \\ &= r^2\dot{\theta}^2\left[\frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)^2}\right] \\ &= 2r^2\dot{\theta}^2\left[\frac{1}{1+\cos\theta}\right] \\ &= \frac{2mkr^3\dot{\theta}^2}{l^2} \\ &= \frac{2k}{mr} \qquad (17) \end{aligned}$$

in terms of the angular momentum $l = mr^2\dot{\theta}^2$. On the other hand, for the circle $\dot{r} = 0$, so

$$v^2 = r^2\dot{\theta}^2 = \frac{l^2}{m^2r^2} = \frac{k}{mr} \qquad (18)$$

where we used that fact that, since this is a circular orbit, the condition $k/r = l^2/mr^2$ is satisfied. Evidently (17) is twice (18) for the same particle at the same point, so the unsquared speed in the parabolic orbit is $\sqrt{2}$ times that in the circular orbit at the same point.

Problem 3.12

At perigee of an elliptic gravitational orbit a particle experiences an impulse S (cf. Exercise 9, Chapter 2) in the radial direction, sending the particle into another elliptic orbit. Determine the new semimajor axis, eccentricity, and orientation of major axis in terms of the old.

The orbit equation for elliptical motion is

$$r(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta-\theta_0)}. \tag{19}$$

For simplicity we'll take $\theta_0 = 0$ for the initial motion of the particle. Then perigee happens when $\theta = 0$, which is to say the major axis of the orbit is on the x axis.

Then at the point at which the impulse is delivered, the particle's momentum is entirely in the y direction: $\mathbf{p}_i = p_i\hat{\mathbf{j}}$. After receiving the impulse S in the radial (x) direction, the particle's y momentum is unchanged, but its x momentum is now $p_x = S$. Hence the final momentum of the particle is $\mathbf{p}_f = S\hat{\mathbf{i}} + p_i\hat{\mathbf{j}}$. Since the particle is in the same location before and after the impulse, its potential energy is unchanged, but its kinetic energy is increased due to the added momentum:

$$E_f = E_i + \frac{S^2}{2m}. \tag{20}$$

Hence the semimajor axis length shrinks accordingly:

$$a_f = -\frac{k}{2E_f} = -\frac{k}{2E_i + S^2/m} = \frac{a_i}{1 + S^2/(2mE_i)}. \tag{21}$$

Next, since the impulse is in the same direction as the particle's distance from the origin, we have $\Delta\mathbf{L} = \mathbf{r}\times\Delta\mathbf{p} = 0$, i.e. the impulse does not change the particle's angular momentum:

$$L_f = L_i \equiv L. \tag{22}$$

With (20) and (22), we can compute the change in the particle's eccentricity:

$$\begin{aligned}\epsilon_f &= \sqrt{1 + \frac{2E_f L^2}{mk^2}} \\ &= \sqrt{1 + \frac{2E_i L^2}{mk^2} + \frac{L^2 S^2}{m^2 k^2}}.\end{aligned} \tag{23}$$

What remains is to compute the constant θ_0 in (19) for the particle's orbit after the collision. To do this we need merely observe that, since the location of the particle is unchanged immediately after the impulse is delivered, expression (19) must evaluate to the same radius at $\theta = 0$ with both the "before" and "after" values of a and ϵ:

$$\frac{a_i(1-\epsilon_i^2)}{1+\epsilon_i} = \frac{a_f(1-\epsilon_f^2)}{1+\epsilon_f\cos\theta_0}$$

or

$$\cos\theta_0 = \frac{1}{\epsilon_f}\left\{\frac{a_f(1-\epsilon_f^2)}{a_i(1-\epsilon_i)} - 1\right\}.$$

Problem 3.13

A uniform distribution of dust in the solar system adds to the gravitational attraction of the sun on a planet an additional force

$$\mathbf{F} = -mC\mathbf{r}$$

where m is the mass of the planet, C is a constant proportional to the gravitational constant and the density of the dust, and $\mathbf{r}$ is the radius vector from the sun to the planet (both considered as points). This additional force is very small compared to the direct sun-planet gravitational force.

(a) Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.

(b) Calculate the period of radial oscillations for slight disturbances from this circular orbit.

(c) Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession the same or opposite direction to the orbital angular velocity?

(a) The equation of motion for r is

$$\begin{aligned} m\ddot{r} &= \frac{l^2}{mr^3} + f(r) \\ &= \frac{l^2}{mr^3} - \frac{k}{r^2} - mCr. \end{aligned} \tag{24}$$

For a circular orbit at radius r_0 this must vanish:

$$0 = \frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - mCr_0 \tag{25}$$

$$\longrightarrow l = \sqrt{mkr_0 + m^2Cr_0^4}$$

$$\begin{aligned}\longrightarrow \dot{\theta} = \frac{l}{mr_0^2} &= \frac{1}{mr_0^2}\sqrt{mkr_0 + m^2Cr_0^4} \\ &= \sqrt{\frac{k}{mr_0^3}}\sqrt{1 + \frac{mCr_0^3}{k}} \\ &\approx \sqrt{\frac{k}{mr_0^3}}\left[1 + \frac{mCr_0^3}{2k}\right]\end{aligned}$$

Then the period is

$$\begin{aligned}\tau = \frac{2\pi}{\dot{\theta}} &\approx 2\pi r_0^{3/2}\sqrt{\frac{m}{k}}\left[1 - \frac{mCr_0^3}{2k}\right] \\ &= \tau_0\left[1 - \frac{C\tau_0^2}{8\pi^2}\right]\end{aligned}$$

where $\tau_0 = 2\pi r_0^{3/2}\sqrt{m/k}$ is the period of circular motion in the absence of the perturbing potential.

(b) We return to (24) and put $r = r_0 + x$ with $x \ll r_0$:

$$\begin{aligned}m\ddot{x} &= \frac{l^2}{m(r_0+x)^3} - \frac{k}{(r_0+x)^2} - mC(r_0 + x) \\ &\approx \frac{l^2}{mr_0^3}\left(1 - 3\frac{x}{r_0}\right) - \frac{k}{r_0^2}\left(1 - 2\frac{x}{r_0}\right) - mCr_0 - mCx\end{aligned}$$

Using (25), this reduces to

$$m\ddot{x} = \left[-\frac{3l^2}{mr_0^4} + \frac{2k}{r_0^3} - mC\right]x$$

or

$$\ddot{x} + \omega^2 x = 0$$

with

$$\begin{aligned}\omega &= \left[\frac{3l^2}{m^2r_0^4} - \frac{2k}{mr_0^3} - C\right]^{1/2} \\ &= \left[\frac{2l^2}{m^2r_0^4} - \frac{k}{mr_0^3}\right]^{1/2}\end{aligned}$$

where in going to the last line we used (25) again.

Problem 3.14

Show that the motion of a particle in the potential field

$$V(r) = -\frac{k}{r} + \frac{h}{r^2}$$

is the same as that of the motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of force. For negative total energy show that if the additional potential term is very small compared to the Kepler potential, then the angular speed of precession of the elliptical orbit is

$$\dot{\Omega} = \frac{2\pi m h}{l^2 \tau}.$$

The perihelion of Mercury is observed to precess (after corrections for known planetary perturbations) at the rate of about 40″ of arc per century. Show that this precession could be accounted for classically if the dimensionless quantity

$$\eta = \frac{k}{ka}$$

(which is a measure of the perturbing inverse square potential relative to the gravitational potential) were as small as 7×10^{-8}. (The eccentricity of Mercury's orbit is 0.206, and its period is 0.24 year).

The effective one-dimensional equation of motion is

$$\begin{aligned} m\ddot{r} &= \frac{L^2}{mr^3} - \frac{k}{r^2} + \frac{2h}{r^3} \\ &= \frac{L^2 + 2mh}{mr^3} + \frac{k}{r^2} \\ &= \frac{L^2 + 2mh + (mh/L)^2 - (mh/L)^2}{mr^3} + \frac{k}{r^2} \\ &= \frac{[L + (mh/L)]^2 - (mh/L)^2}{mr^3} + \frac{k}{r^2} \end{aligned}$$

If $mh \ll L^2$, then we can neglect the term $(mh/L)^2$ in comparison with L^2, and write

$$m\ddot{r} = \frac{[L + (mh/L)]^2}{mr^3} + \frac{k}{r^2} \tag{26}$$

which is just the normal equation of motion for the Kepler problem, but with the angular momentum L augmented by the additive term $\Delta L = mh/L$.

Such an augmentation of the angular momentum may be accounted for by augmenting the angular velocity:

$$L = mr^2\dot{\theta} \qquad \longrightarrow \qquad L\left(1+\frac{mh}{L^2}\right) = mr^2\dot{\theta}\left(1+\frac{mh}{L^2}\right)$$
$$= mr^2\dot{\theta} + mr^2\dot{\Omega}$$

where

$$\dot{\Omega} = \frac{mh}{L^2\dot{\theta}} = \frac{2\pi mh}{L^2\tau}$$

is a precession frequency. If we were to go back and work the problem in the reference frame in which everything is precessing with angular velocity $\dot{\Omega}$, but there is no term h/r^2 in the potential, then the equations of motion would come out the same as in the stationary case, but with a term $\Delta L = mr^2\dot{\Omega}$ added to the effective angular momentum that shows up in the equation of motion for r, just as we found in (26).

To put in the numbers, we observe that

$$\begin{aligned}
\dot{\Omega} &= \left(\frac{2\pi}{\tau}\right)\left(\frac{m}{L^2}\right)(h) \\
&= \left(\frac{2\pi}{\tau}\right)\left(\frac{mka}{L^2}\right)\left(\frac{h}{ka}\right) \\
&= \left(\frac{2\pi}{\tau}\right)\left(\frac{1}{1-e^2}\right)\left(\frac{h}{ka}\right)
\end{aligned}$$

so

$$\begin{aligned}
\frac{h}{ka} &= (1-e^2)\frac{\tau\dot{\Omega}}{2\pi} \\
&= (1-e^2)\tau f_{\text{prec}}
\end{aligned}$$

where in going to the third-to-last line we used Goldstein's equation (3-62), and in the last line I put $f_{\text{prec}} = \dot{\Omega}/2\pi$. Putting in the numbers, we find

$$\begin{aligned}
\frac{h}{ka} &= (1-.206^2)\cdot\big(0.24 \text{ yr}\big)\cdot 40''\left(\frac{1^\circ}{3600''}\right)\left(\frac{1 \text{ revolution}}{360^\circ}\right)\left(\frac{1 \text{ century}^{-1}}{100 \text{ yr}^{-1}}\right) \text{yr}^{-1} \\
&= 7.1\cdot 10^{-8}.
\end{aligned}$$

Problem 3.22

In hyperbolic motion in a $1/r$ potential the analogue of the eccentric anomaly is F defined by

$$r = a(e\cosh F - 1),$$

where $a(1-e)$ is the distance of closest approach. Find the analogue to Kepler's equation giving t from the time of closest approach as a function of F.

We start with Goldstein's equation (3.65):

$$\begin{aligned} t &= \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{k}{r} - \frac{l^2}{2mr^2} + E}} \\ &= \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{Er^2 + kr - \frac{l^2}{2m}}}. \end{aligned} \tag{27}$$

With the suggested substitution, the thing under the radical in the denominator of the integrand is

$$\begin{aligned} Er^2 + kr - \frac{l^2}{2m} &= Ea^2(e^2\cosh^2 F - 2e\cosh F + 1) + ka(e\cosh F - 1) - \frac{l^2}{2m} \\ &= Ea^2e^2\cosh^2 F + ae(k - 2Ea)\cosh F + \left(Ea^2 - ka - \frac{l^2}{2m}\right) \end{aligned}$$

It follows from the orbit equation that, if $a(e-1)$ is the distance of closest approach, then $a = k/2E$. Thus

$$\begin{aligned} &= \frac{k^2e^2}{4E}\cosh^2 F - \frac{k^2e^2}{4E} - \frac{l^2}{2m} \\ &= \frac{k^2}{4E}\left\{e^2\cosh^2 F - \left(1 + \frac{2El^2}{mk^2}\right)\right\} \\ &= \frac{k^2e^2}{4E}\left[\cosh^2 F - 1\right] = \frac{k^2e^2}{4E}\sinh^2 F = a^2e^2E\sinh^2 F. \end{aligned}$$

Plugging into (27) and observing that $dr = ae\sinh F\,dF$, we have

$$t = \sqrt{\frac{m}{2E}} \int_{F_0}^{F} a(e\cosh F - 1)\,dF \quad = \sqrt{\frac{ma^2}{2E}}\left[e(\sinh F - \sinh F_0) - (F - F_0)\right]$$

and I suppose this equation could be a jumping-off point for numerical or other investigations of the time of travel in hyperbolic orbit problems.

Problem 3.26

Examine the scattering produced by a repulsive central force $f = kr^{-3}$. Show that the differential cross section is given by

$$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2 \sin \pi x}$$

where x is the ratio Θ/π and E is the energy.

The potential energy is $U = k/2r^2 = ku^2/2$, and the differential equation for the orbit reads

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dU}{du} = -\frac{mk}{l^2}u$$

or

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{mk}{l^2}\right)u = 0$$

with solution

$$u = A\cos\gamma\theta + B\sin\gamma\theta \tag{28}$$

where

$$\gamma = \sqrt{1 + \frac{mk}{l^2}}. \tag{29}$$

We'll set up our coordinates in the way traditional for scattering experiments: initially the particle is at angle $\theta = \pi$ and a great distance from the force center, and ultimately the particle proceeds off to $r = \infty$ at some new angle θ_s. The first of these observations gives us a relation between A and B in the orbit equation (28):

$$u(\theta = \pi) = 0 \qquad \longrightarrow \qquad A\cos\gamma\pi + B\sin\gamma\pi = 0$$

$$\longrightarrow \qquad A = -B\tan\gamma\pi. \tag{30}$$

The condition that the particle head off to $r = \infty$ at angle $\theta = \theta_s$ yields the condition

$$A\cos\gamma\theta_s + B\sin\gamma\theta_s = 0.$$

Using (30), this becomes

$$-\cos\gamma\theta_s\tan\gamma\pi + \sin\gamma\theta_s = 0$$

or

$$\begin{aligned} -\cos\gamma\theta_s\sin\gamma\pi + \sin\gamma\theta_s\cos\gamma\pi &= 0 \\ \longrightarrow \qquad \sin\gamma(\theta_s - \pi) &= 0 \\ \longrightarrow \qquad \gamma(\theta_s - \pi) &= \pi \end{aligned}$$

or, in terms of Goldstein's variable $x = \theta/\pi$,

$$\gamma = \frac{1}{x-1}. \tag{31}$$

Plugging in (29) and squaring both sides, we have

$$1 + \frac{mk}{l^2} = \frac{1}{(x-1)^2}.$$

Now $l = mv_0 s = (2mE)^{1/2}s$ with s the impact parameter and E the particle energy. Thus the previous equation is

$$1 + \frac{k}{2Es^2} = \frac{1}{(x-1)^2}$$

or

$$s^2 = -\frac{k}{2E}\left[\frac{(x-1)^2}{x(x-2)}\right].$$

Taking the differential of both sides,

$$\begin{aligned} 2s\,ds &= -\frac{k}{2E}\left[\frac{2(x-1)}{x(x-2)} - \frac{(x-1)^2}{x^2(x-2)} - \frac{(x-1)^2}{x(x-2)^2}\right]dx \\ &= -\frac{k}{2E}\left[\frac{2x(x-1)(x-2) - (x-1)^2(x-2) - x(x-1)^2}{x^2(x-2)^2}\right] \\ &= -\frac{k}{2E}\left[\frac{2(1-x)}{x^2(x-2)^2}\right]. \end{aligned} \tag{32}$$

The differential cross section is given by

$$\sigma(\theta)d\Omega = \frac{|\,s\,ds\,|}{\sin\theta}.$$

Plugging in (32), we have

$$\sigma(\theta)d\Omega = \frac{k}{2E}\left[\frac{(1-x)}{x^2(x-2)^2\sin\theta}\right]dx$$

as advertised.

Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

Homer Reid

April 21, 2002

Chapter 7

Problem 7.2

> Obtain the Lorentz transformation in which the velocity is at an infinitesimal angle $d\theta$ counterclockwise from the z axis, by means of a similarity transformation applied to Eq. (7-18). Show directly that the resulting matrix is orthogonal and that the inverse matrix is obtained by substituting $-v$ for v.

We can obtain this transformation by first applying a pure rotation to rotate the z axis into the boost axis, then applying a pure boost along the (new) z axis, and then applying the inverse of the original rotation to bring the z axis back in line with where it was originally. Symbolically we have $\mathbf{L} = \mathbf{R}^{-1}\mathbf{K}\mathbf{R}$ where R is the rotation to achieve the new z axis, and K is the boost along the z axis.

Goldstein tells us that the new z axis is to be rotated $d\theta$ counterclockise from the original z axis, but he doesn't tell us *in which plane*, i.e. we know θ but not ϕ for the new z axis in the unrotated coordinates. We'll assume the z axis is rotated around the x axis, in a sense such that if you're standing on the positive x axis, looking toward the negative x axis, the rotation appears to be counterclockwise, so that the positive z axis is rotated toward the negative y

axis. Then, using the real metric,

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d\theta & \sin d\theta & 0 \\ 0 & -\sin d\theta & \cos d\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d\theta & -\sin d\theta & 0 \\ 0 & \sin d\theta & \cos d\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d\theta & \sin d\theta & 0 \\ 0 & -\sin d\theta & \cos d\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\, d\theta & -\sin d\theta & 0 \\ 0 & \gamma \sin d\theta & \gamma\cos d\theta & -\beta\gamma \\ 0 & -\beta\gamma\sin d\theta & -\beta\gamma\cos d\theta & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\, d\theta + \gamma\sin^2 d\theta & (\gamma-1)\sin d\theta\cos d\theta & -\beta\gamma\sin d\theta \\ 0 & (\gamma-1)\sin d\theta\cos d\theta & \sin^2 d\theta + \gamma\cos^2 d\theta & -\beta\gamma\cos d\theta \\ 0 & -\beta\gamma\sin d\theta & -\beta\gamma\cos d\theta & \gamma \end{pmatrix}.$$

Problem 7.4

> A rocket of length l_0 in its rest system is moving with constant speed along the z axis of an inertial system. An observer at the origin observes the apparent length of the rocket at any time by noting the z coordinates that can be seen for the head and tail of the rocket. How does this apparent length vary as the rocket moves from the extreme left of the observer to the extreme right?

Let's imagine a coordinate system in which the rocket is at rest and centered at the origin. Then the world lines of the rocket's top and bottom are

$$x_\mu^{t\prime} = \{0, 0, +L_0/2, \tau\} \qquad x_\mu^{b\prime} = \{0, 0, -L_0/2, \tau\}.$$

where we are parameterizing the world lines by the proper time τ. Now, the rest frame of the observer is moving in the negative z direction with speed $v = \beta c$ relative to the rest frame of the rocket. Transforming the world lines of the rocket's top and bottom to the rest frame of the observer, we have

$$x_\mu^t = \{0, 0, \gamma(L_0/2 + v\tau), \gamma(\tau + \beta L_0/2c)\} \tag{1}$$

$$x_\mu^b = \{0, 0, \gamma(-L_0/2 + v\tau), \gamma(\tau - \beta L_0/2c)\}. \tag{2}$$

Now consider the observer. At any time t in his own reference frame, he is receiving light from two events, namely, the top and bottom of the rocket moving past imaginary distance signposts that we pretend to exist up and down the z axis. He sees the top of the rocket lined up with one distance signpost and the bottom of the rocket lined up with another, and from the difference between the two signposts he computes the length of the rocket. Of course, the light that he sees was emitted by the rocket some time in the past, and, moreover, the light signals from the top and bottom of the rocket that the observer receives simultaneously at time t were in fact emitted at different proper times τ in the rocket's rest frame.

First consider the light received by the observer at time t_0 coming from the bottom of the rocket. Suppose in the observer's rest frame this light were emitted at time $t_0 - \Delta t$, i.e. Δt seconds before it reaches the observer at the origin; then the rocket bottom was passing through $z = -c\Delta t$ when it emitted this light. But then the event identified by $(z, t) = (-c\Delta t, t_0 - \Delta_t)$ must lie on the world line of the rocket's bottom, which from (2) determines both Δt and the proper time τ at which the light was emitted:

$$\begin{aligned} -c\Delta t &= \gamma(-L_0/2 + v\tau) \\ t_0 - \Delta_t &= \gamma(\tau + \beta L_0/2c) \end{aligned} \qquad \Longrightarrow \qquad \tau = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} t_0 - \frac{L_0}{2c} \quad \equiv \tau_b(t_0).$$

We use the notation $\tau_b(t_0)$ to indicate that this is the proper time at which the bottom of the rocket emits the light that arrives at the observer's origin at the observer's time t_0. At this proper time, from (2), the position of the bottom of the rocket in the observer's reference frame was

$$\begin{aligned} z_b(\tau_b(t_0)) &= -\gamma L_0/2 + v\gamma\tau_b(t_0) \\ &= -\gamma L_0/2 + v\gamma\left\{\left(\frac{1+\beta}{1-\beta}\right)^{1/2} t_0 - \frac{L_0}{2c}\right\} \end{aligned} \tag{3}$$

Similarly, for the top of the rocket we have

$$\tau_t(t_0) = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} t_0 + \frac{L_0}{2c}$$

and

$$z_t(\tau_t(t_0)) = \gamma L_0/2 + v\gamma\left\{\left(\frac{1+\beta}{1-\beta}\right)^{1/2} t_0 + \frac{L_0}{2c}\right\} \tag{4}$$

Subtracting (3) from (4), we have the length for the rocket computed by the observer from his observations at time t_0 in his reference frame:

$$\begin{aligned} L(t_0) &= \gamma(1+\beta)L_0 \\ &= \left(\frac{1+\beta}{1-\beta}\right)^{1/2} L_0. \end{aligned}$$

Problem 7.17

Two particles with rest masses m_1 and m_2 are observed to move along the observer's z axis toward each other with speeds v_1 and v_2, respectively. Upon collision they are observed to coalesce into one particle of rest mass m_3 moving with speed v_3 relative to the observer. Find m_3 and v_3 in terms of m_1, m_2, v_1, and v_2. Would it be possible for the resultant particle to be a photon, that is $m_3 = 0$, if neither m_1 nor m_2 are zero?

Equating the 3rd and 4th components of the initial and final 4-momentum of the system yields

$$\gamma_1 m_1 v_1 - \gamma_2 m_2 v_2 = \gamma_3 m_3 v_3$$
$$\gamma_1 m_1 c + \gamma_2 m_2 c = \gamma_3 m_3 c$$

Solving the second for m_3 yields

$$m_3 = \frac{\gamma_1}{\gamma_3} m_1 + \frac{\gamma_2}{\gamma_3} m_2 \tag{5}$$

and plugging this into the first yields v_3 in terms of the properties of particles 1 and 2:

$$v_3 = \frac{\gamma_1 m_1 v_1 - \gamma_2 m_2 v_2}{\gamma_1 m_1 + \gamma_2 m_2}$$

Then

$$\begin{aligned}
\beta_3 = \frac{v_3}{c} &= \frac{\gamma_1 m_1 \beta_1 - \gamma_2 m_2 \beta_2}{\gamma_1 m_1 + \gamma_2 m_2} \\
1 - \beta_3^2 &= \frac{\gamma_1^2 m_1^2 + 2\gamma_1\gamma_2 m_1 m_2 + \gamma_2^2 m_2^2 - [\gamma_1^2 m_1^2 \beta_1^2 + \gamma_2^2 m_2^2 \beta_2^2 - 2\gamma_1\gamma_2 m_1 m_2 \beta_1 \beta_2]}{(\gamma_1 m_1 + \gamma_2 m_2)^2} \\
&= \frac{\gamma_1^2 m_1^2 (1-\beta_1^2) + \gamma_2^2 m_2^2 (1-\beta_2^2) + 2\gamma_1\gamma_2 m_1 m_2 (1-\beta_1\beta_2)}{(\gamma_1 m_1 + \gamma_2 m_2)^2} \\
&= \frac{m_1^2 + m_2^2 + 2\gamma_1\gamma_2 m_1 m_2 (1-\beta_1\beta_2)}{(\gamma_1 m_1 + \gamma_2 m_2)^2}
\end{aligned}$$

and hence

$$\gamma_3^2 = \frac{1}{1-\beta_3^2} = \frac{(\gamma_1 m_1 + \gamma_2 m_2)^2}{m_1^2 + m_2^2 + 2\gamma_1\gamma_2 m_1 m_2 (1-\beta_1\beta_2)}. \tag{6}$$

Now, (5) shows that, for m_3 to be zero when either m_1 or m_2 is zero, we must have $\gamma_3 = \infty$. That this condition cannot be met for nonzero m_1, m_2 is evident from the denominator of (6), in which all terms are positive (since $\beta_1\beta_2 < 1$ if m_1 or m_2 is nonzero).

Problem 7.19

A meson of mass π comes to rest and disintegrates into a meson of mass μ and a neutrino of zero mass. Show that the kinetic energy of motion of the μ meson (i.e. without the rest mass energy) is

$$\frac{(\pi-\mu)^2}{2\pi}c^2.$$

Working in the rest frame of the pion, the conservation relations are

$$\begin{aligned} \pi c^2 &= (\mu^2 c^4 + p_\mu^2 c^2)^{1/2} + p_\nu c &\qquad \text{(energy conservation)} \\ 0 &= \mathbf{p}_\mu + \mathbf{p}_\nu &\qquad \text{(momentum conservation).} \end{aligned}$$

From the second of these it follows that the muon and neutrino must have the same momentum, whose magnitude we'll call p. Then the energy conservation relation becomes

$$\begin{aligned} \pi c^2 &= (\mu^2 c^4 + p^2 c^2)^{1/2} + pc \\ \longrightarrow \qquad (\pi c - p)^2 &= \mu^2 c^2 + p^2 \\ \longrightarrow \qquad p &= \frac{\pi^2 - \mu^2}{2\pi} c. \end{aligned}$$

Then the total energy of the muon is

$$\begin{aligned} E_\mu &= (\mu^2 c^4 + p^2 c^2)^{1/2} \\ &= c^2 \left(\mu^2 + \frac{(\pi^2 - \mu^2)^2}{4\pi^2} \right)^{1/2} \\ &= \frac{c^2}{2\pi} \left(4\pi^2\mu^2 + (\pi^2 - \mu^2)^2 \right)^{1/2} \\ &= \frac{c^2}{2\pi}(\pi^2 + \mu^2) \end{aligned}$$

Then subtracting out the rest energy to get the kinetic energy, we obtain

$$\begin{aligned} K = E_\mu - \mu c^2 &= \frac{c^2}{2\pi}(\pi^2 + \mu^2) - \mu c^2 \\ &= \frac{c^2}{2\pi}(\pi^2 + \mu^2 - 2\pi\mu) \\ &= \frac{c^2}{2\pi}(\pi - \mu)^2 \end{aligned}$$

as advertised.

Problem 7.20

A π^+ meson of rest mass 139.6 MeV collides with a neutron (rest mass 939.6 MeV) stationary in the laboratory system to produce a K^+ meson (rest mass 494 MeV) and a Λ hyperon (rest mass 1115 MeV). What is the threshold energy for this reaction in the laboratory system?

We'll put $c = 1$ for this problem. The four-momenta of the pion and neutron before the collision are

$$p_{\mu,\pi} = (\mathbf{p}_\pi, \gamma_\pi m_\pi), \qquad p_{\mu,n} = (0, m_n)$$

and the squared magnitude of the initial four-momentum is thus

$$\begin{aligned} p_{\mu,T} p^\mu_T &= -|\mathbf{p}_\pi|^2 + (\gamma_\pi m_\pi + m_n)^2 \\ &= -|\mathbf{p}_\pi|^2 + \gamma_\pi^2 m_\pi^2 + m_n^2 + 2\gamma_\pi m_\pi m_n \\ &= m_\pi^2 + m_n^2 + 2\gamma_\pi m_\pi m_n \\ &= (m_\pi + m_n)^2 + 2(\gamma_\pi - 1) m_\pi m_n \end{aligned} \tag{7}$$

The threshold energy is the energy needed to produce the K and Λ particles at rest in the COM system. In this case the squared magnitude of the four-momentum of the final system is just $(m_K + m_\Lambda)^2$, and, by conservation of momentum, this must be equal to the magnitude of the four-momentum of the initial system (7):

$$(m_K + m_\Lambda)^2 = (m_\pi + m_n)^2 + 2(\gamma_\pi - 1) m_\pi m_n$$

$$\Longrightarrow \gamma_\pi = 1 + \frac{(m_K + m_\Lambda)^2 - (m_\pi + m_n)^2}{2 m_\pi m_n} = 6.43$$

Then the total energy of the pion is $T = \gamma_\pi m_\pi = (6.43 \cdot 139.6 \text{ MeV}) = 898$ MeV, while its kinetic energy is $K = T - m = 758$ MeV.

The above appears to be the correct solution to this problem. On the other hand, I first tried to do it a different way, as below. This way yields a different and hence presumably incorrect answer, but I can't figure out why. Can anyone find the mistake?

The K and Λ particles must have, between them, the same total momentum in the direction of the original pion's momentum as the original pion had. Of course, the K and Λ may also have momentum in directions transverse to the original pion momentum (if so, their transverse momenta must be equal and opposite). But any transverse momentum just increases the energy of the final system, which increases the energy the initial system must have had to produce the final system. Hence the minimum energy situation is that in which the K and Λ both travel in the direction of the original pion's motion. (This is equivalent to Goldstein's conclusion that, just at threshold, the produced particles are at rest in the COM system). Then the momentum conservation relation becomes simply

$$p_\pi = p_K + p_\lambda \tag{8}$$

and the energy conservation relation is (with $c = 1$)

$$(m_\pi^2 + p_\pi^2)^{1/2} + m_n = (m_K^2 + p_K^2)^{1/2} + (m_\Lambda^2 + p_\Lambda^2)^{1/2}. \tag{9}$$

The problem is to find the minimum value of p_π that satisfies (9) subject to the constraint (8).

To solve this we must first resolve a subquestion: for a given p_π, what is the relative allocation of momentum to p_K and p_Λ that minimizes (9) ? Minimizing

$$E_f = (m_K^2 + p_K^2)^{1/2} + (m_\Lambda^2 + p_\Lambda^2)^{1/2}.$$

subject to $p_K + p_\Lambda = p_\pi$, we obtain the condition

$$\frac{p_K}{(m_K^2 + p_K^2)^{1/2}} = \frac{p_\Lambda}{(m_\Lambda^2 + p_\Lambda^2)^{1/2}} \qquad \Longrightarrow \qquad p_K = \frac{m_K}{m_\Lambda} p_\Lambda \tag{10}$$

Combining this with (8) yields

$$p_\Lambda = \frac{m_\Lambda}{m_K + m_\Lambda} p_\pi \qquad p_K = \frac{m_K}{m_K + m_\Lambda} p_\pi. \tag{11}$$

For a given total momentum p_π, the minimum possible energy the final system can have is realized when p_π is partitioned between p_K and p_Λ according to (11). Plugging into (8), the relation defining the threshold momentum is

$$(m_\pi^2 + p_\pi^2)^{1/2} + m_n = \left(m_K^2 + \left(\frac{m_K}{m_K + m_\Lambda} \right)^2 p_\pi^2 \right)^{1/2} + \left(m_\Lambda^2 + \left(\frac{m_\Lambda}{m_K + m_\Lambda} \right)^2 p_\pi^2 \right)^{1/2}$$

Solving numerically yields $p_\pi \approx 655$ MeV/c, for a total pion energy of about 670 MeV.

Problem 7.21

A photon may be described classically as a particle of zero mass possessing nevertheless a momentum $h/\lambda = h\nu/c$, and therefore a kinetic energy $h\nu$. If the photon collides with an electron of mass m at rest it will be scattered at some angle θ with a new energy $h\nu'$. Show that the change in energy is related to the scattering angle by the formula

$$\lambda' - \lambda = 2\lambda_c \sin^2 \frac{\theta}{2},$$

where $\lambda_c = h/mc$, known as the Compton wavelength. Show also that the kinetic energy of the recoil motion of the electron is

$$T = h\nu \frac{2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\frac{\theta}{2}}{1 + 2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\theta/2}.$$

Let's assume the photon is initially travelling along the z axis. Then the sum of the initial photon and electron four-momenta is

$$p_{\mu,i} = p_{\mu,\gamma} + p_{\mu,e} = \begin{pmatrix} 0 \\ 0 \\ h/\lambda \\ h/\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ mc \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h/\lambda \\ mc + h/\lambda \end{pmatrix}. \qquad (12)$$

Without loss of generality we may assume that the photon and electron move in the xz plane after the scatter. If the photon's velocity makes an angle θ with the z axis, while the electron's velocity makes an angle ϕ, the four-momentum after the collision is

$$p_{\mu,f} = p_{\mu,\gamma} + p_{\mu,e} = \begin{pmatrix} (h/\lambda')\sin\theta \\ 0 \\ (h/\lambda')\cos\theta \\ h/\lambda' \end{pmatrix} + \begin{pmatrix} p_e \sin\phi \\ 0 \\ p_e\cos\phi \\ \sqrt{m^2c^2 + p_e^2} \end{pmatrix} = \begin{pmatrix} (h/\lambda')\sin\theta + p_e\sin\phi \\ 0 \\ (h/\lambda')\cos\theta + p_e\cos\phi \\ (h/\lambda') + \sqrt{m^2c^2+p_e^2} \end{pmatrix}. \qquad (13)$$

Equating (12) and (13) yields three separate equations:

$$(h/\lambda')\sin\theta + p_e\sin\phi = 0 \qquad (14)$$

$$(h/\lambda')\cos\theta + p_e\cos\phi = h/\lambda \qquad (15)$$

$$h/\lambda' + \sqrt{m^2c^2 + p_e^2} = mc + h/\lambda \qquad (16)$$

From the first of these we find

$$\sin\phi = -\frac{h}{\lambda' p_e}\sin\theta \Longrightarrow \cos\phi = \left[1 + \left(\frac{h}{\lambda' p_e}\right)^2 \sin^2\theta\right]^{1/2}$$

and plugging this into (15) we find

$$p_e^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos\theta. \tag{17}$$

On the other hand, we can solve (16) to obtain

$$p_e^2 = h^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)^2 + 2mch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right).$$

Comparing these two determinations of p_e yields

$$\cos\theta = 1 - \frac{mc}{h}\left(\lambda' - \lambda\right)$$

or

$$\sin^2\frac{\theta}{2} = \frac{mc}{2h}\left(\lambda' - \lambda\right) = \frac{1}{2\lambda_c}(\lambda' - \lambda)$$

so this is advertised result number 1.

Next, to find the kinetic energy of the electron after the collision, we can write the conservation of energy equation in a slightly different form:

$$\begin{aligned}
mc + \frac{h}{\lambda} &= \gamma mc + \frac{h}{\lambda'} \\
\Longrightarrow (\gamma - 1)mc = K &= h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \\
&= h\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right) \\
&= h\left(\frac{2\lambda_c\sin^2(\theta/2)}{\lambda[\lambda + 2\lambda_c\sin^2(\theta/2)]}\right) \\
&= \frac{h}{\lambda}\left(\frac{2\chi\sin^2(\theta/2)}{1 + 2\chi\sin^2(\theta/2)}\right)
\end{aligned}$$

where we put $\chi = \lambda_c/\lambda$.

Problem 7.22

A photon of energy $\mathcal{E}$ collides at angle θ with another photon of energy E. Prove that the minimum value of $\mathcal{E}$ permitting formation of a pair of particles of mass m is

$$\mathcal{E}_{th} = \frac{2m^2c^4}{E(1 - \cos\theta)}.$$

We'll suppose the photon of energy E is traveling along the positive z axis, while that with energy $\mathcal{E}$ is traveling in the xz plane (i.e., its velocity has spherical polar angles θ and $\phi = 0$). Then the 4-momenta are

$$\begin{aligned} p_1 &= \left(0, 0, \frac{E}{c}, \frac{E}{c}\right) \\ p_2 &= \left(\frac{\mathcal{E}}{c}\sin\theta, 0, \frac{\mathcal{E}}{c}\cos\theta, \frac{\mathcal{E}}{c}\right) \\ p_t = p_1 + p_2 &= \left(\frac{\mathcal{E}}{c}\sin\theta, 0, \frac{E + \mathcal{E}\cos\theta}{c}, \frac{E+\mathcal{E}}{c}\right) \end{aligned}$$

It's convenient to rotate our reference frame to one in which the space portion of the composite four-momentum of the two photons is all along the z direction. In this frame the total four-momentum is

$$p_t' = \left(0, 0, \frac{1}{c}\sqrt{\mathcal{E}^2 + E^2 + 2E\mathcal{E}\cos\theta}, \frac{E+\mathcal{E}}{c}\right). \tag{18}$$

At threshold energy, the two produced particles have the same four-momenta:

$$p_3 = p_4 = \left(0, 0, p, (m^2c^2 + p^2)^{1/2}\right) \tag{19}$$

and 4-momentum conservation requires that twice (19) add up to (18), which yields two conditions:

$$\begin{aligned} 2p &= \tfrac{1}{c}\sqrt{\mathcal{E}^2 + E^2 + 2E\mathcal{E}\cos\theta} &\longrightarrow \qquad p^2c^2 &= \tfrac{1}{4}(\mathcal{E}^2 + E^2 + 2E\mathcal{E}\cos\theta) \\ 2\sqrt{m^2c^2 + p^2} &= \tfrac{E+\mathcal{E}}{c} &\longrightarrow \quad m^2c^4 + p^2c^2 &= \tfrac{1}{4}(\mathcal{E}^2 + E^2 + 2E\mathcal{E}) \end{aligned}$$

Subtracting the first of these from the second, we obtain

$$m^2c^4 = \frac{E\mathcal{E}}{2}(1 - \cos\theta)$$

or

$$\mathcal{E} = \frac{2m^2c^4}{E(1 - \cos\theta)}$$

as advertised.

Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

Homer Reid

October 29, 2002

Chapter 9

Problem 9.1

One of the attempts at combining the two sets of Hamilton's equations into one tries to take q and p as forming a complex quantity. Show directly from Hamilton's equations of motion that for a system of one degree of freedom the transformation

$$Q = q + ip, \qquad P = Q^*$$

is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates Q', P' that are related to Q, P by a change of scale only, and that are canonical?

Generalizing a little, we put

$$Q = \mu(q + ip), \qquad P = \nu(q - ip). \tag{1}$$

The reverse transformation is

$$q = \frac{1}{2}\left(\frac{1}{\mu}Q + \frac{1}{\nu}P\right), \qquad p = \frac{1}{2i}\left(\frac{1}{\mu}Q - \frac{1}{\nu}P\right).$$

The direct conditions for canonicality, valid in cases (like this one) in which the transformation equations do not depend on the time explicitly, are

$$\begin{aligned}\frac{\partial Q}{\partial q} &= \frac{\partial p}{\partial P} \\ \frac{\partial Q}{\partial p} &= -\frac{\partial q}{\partial P} \\ \frac{\partial P}{\partial q} &= -\frac{\partial p}{\partial Q} \\ \frac{\partial P}{\partial p} &= \frac{\partial q}{\partial Q}.\end{aligned} \tag{2}$$

When applied to the case at hand, all four of these yield the same condition, namely

$$\mu = -\frac{1}{2i\nu}.$$

For $\mu = \nu = 1$, which is the case Goldstein gives, these conditions are clearly not satisfied, so (1) is not canonical. But putting $\mu = 1, \nu = -\frac{1}{2i}$ we see that equations (1) *are* canonical.

Problem 9.2

(a) For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2},$$

show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht.$$

(b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n},$$

where $\mathbf{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\mathbf{p}\cdot\mathbf{r}}{n} - Ht.$$

(c) The transformation $Q = \lambda q, p = \lambda P$ is obviously canonical. However, the same transformation with t time dilatation, $Q = \lambda q, p = \lambda P, t' = \lambda^2 t$, is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part (a) are invariant under the transformation. The constant of the motion D is said to be associated with this invariance.

(a) The equation of motion for the quantity D is

$$\frac{dD}{dT} = \{D, H\} + \frac{\partial D}{\partial t}$$

The Poisson bracket of the second term in D clearly vanishes, so we have

$$\begin{aligned} &= \frac{1}{2}\{pq, H\} - H \\ &= \frac{1}{4}\{pq, p^2\} - \frac{1}{4}\left\{pq, \frac{1}{q^2}\right\} - H. \end{aligned} \tag{3}$$

The first Poisson bracket is

$$\begin{aligned} \{pq, p^2\} &= \frac{\partial(pq)}{\partial q}\frac{\partial(p^2)}{\partial p} - \frac{\partial(pq)}{\partial p}\frac{\partial(p^2)}{\partial q} \\ &= (p)(2p) - 0 \\ &= 2p^2 \end{aligned} \tag{4}$$

Next,

$$\left\{pq, \frac{1}{q^2}\right\} = \frac{\partial(pq)}{\partial q}\frac{\partial\left(\frac{1}{q^2}\right)}{\partial p} - \frac{\partial(pq)}{\partial p}\frac{\partial\left(\frac{1}{q^2}\right)}{\partial q}$$
$$= 0 - \left(-\frac{2}{q^3}\right)q$$
$$= \frac{2}{q^2} \tag{5}$$

Plugging (4) and (5) into (3), we obtain

$$\frac{dD}{dt} = \frac{p^2}{2} - \frac{1}{2q^2} - H$$
$$= 0.$$

(b) We have

$$H = (p_1^2 + p_2^2 + p_3^2)^{n/2} - a(x_1^2 + x_2^2 + x_3^2)^{-n/2}$$

so

$$\frac{\partial H}{\partial x_i} = anx_i(x_1^2 + x_2^2 + x_3^2)^{-n/2-1}$$
$$\frac{\partial H}{\partial p_i} = 2np_i(p_1^2 + p_2^2 + p_3^2)^{n/2-1}.$$

Then

$$\{\mathbf{p}\cdot\mathbf{r}, H\} = \sum_i \left\{\frac{\partial(p_1x_1 + p_2x_2 + p_3x_3)}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial(p_1x_1 + p_2x_2 + p_3x_3)}{\partial p_i}\frac{\partial H}{\partial x_i}\right\}$$
$$= \sum_i \left\{np_i^2(p_1^2 + p_2^2 + p_3^2)^{n/2-1} - anx_i^2(x_1^2 + x_2^2 + x_3^2)^{-n/2-1}\right\}$$
$$= n(p_1^2 + p_2^2 + p_3^2)^{n/2} - an(x_1^2 + x_2^2 + x_3^2)^{-n/2} \tag{6}$$

so if we define $D = \mathbf{p}\cdot\mathbf{r}/n - Ht$, then

$$\frac{dD}{dT} = \{D, H\} - \frac{\partial D}{\partial t}$$
$$= \frac{1}{n}\{\mathbf{p}\cdot\mathbf{r}, H\} - \frac{\partial D}{\partial t}$$

Substituting in from (6),

$$= |p|^n - ar^{-n} - H$$
$$= 0.$$

(c) We put

$$Q(t') = \lambda q\left(\frac{t'}{\lambda^2}\right), \qquad P(t') = \frac{1}{\lambda}p\left(\frac{t'}{\lambda^2}\right). \tag{7}$$

Since q and p are the original canonical coordinates, they satisfy

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial q} = \frac{1}{q^3}. \end{aligned} \tag{8}$$

On the other hand, differentiating (7), we have

$$\begin{aligned} \frac{dQ}{dt'} &= \frac{1}{\lambda}\dot{q}\left(\frac{t'}{\lambda^2}\right) \\ &= \frac{1}{\lambda}p\left(\frac{t'}{\lambda^2}\right) \\ &= P(t') \\ \frac{dP}{dt'} &= \frac{1}{\lambda^3}\dot{p}\left(\frac{t'}{\lambda^2}\right) \\ &= \frac{1}{\lambda^3}\frac{1}{q\left(\frac{t'}{\lambda^2}\right)} \\ &= \frac{1}{Q^3(t')} \end{aligned}$$

which are the same equations of motion as (8).

Problem 9.4

Show directly that the transformation

$$Q = \log\left(\frac{1}{p}\sin p\right), \qquad P = q\cot p$$

is canonical.

The Jacobian of the transformation is

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\mathbf{M}}\mathbf{J}\mathbf{M} &= \left(\begin{array}{cc} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)\left(\begin{array}{cc} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{array}\right) \\
&= \left(\begin{array}{cc} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{array}\right)\left(\begin{array}{cc} \cot p & -q\csc^2 p \\ \frac{1}{q} & -\cot p \end{array}\right) \\
&= \left(\begin{array}{cc} 0 & \csc^2 p - \cot^2 p \\ \cot^2 p - \csc^2 p & 0 \end{array}\right) \\
&= \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \\
&= \mathbf{J}
\end{aligned}$$

so the symplectic condition is satisfied.

Problem 9.5

Show directly for a system of one degree of freedom that the transformation

$$Q = \arctan\frac{\alpha q}{p}, \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical, where α is an arbitrary constant of suitable dimensions.

The Jacobian of the transformation is

$$\begin{aligned}
\mathbf{M} &= \left(\begin{array}{cc} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{array}\right) \\
&= \left(\begin{array}{cc} \left(\frac{\alpha}{p}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & -\left(\frac{\alpha q}{p^2}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} \\ \alpha q & \frac{p}{\alpha} \end{array}\right).
\end{aligned}$$

so

$$\begin{aligned}
\tilde{\mathbf{M}}\mathbf{J}\mathbf{M} &= \left(\begin{array}{cc} \left(\frac{\alpha}{p}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & \alpha q \\ -\left(\frac{\alpha q}{p^2}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & \frac{p}{\alpha} \end{array}\right)\left(\begin{array}{cc} \alpha q & \frac{p}{\alpha} \\ -\left(\frac{\alpha}{p}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & +\left(\frac{\alpha q}{p^2}\right)\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} \end{array}\right) \\
&= \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \\
&= \mathbf{J}
\end{aligned}$$

so the symplectic condition is satisfied.

Problem 9.6

The transformation equations between two sets of coordinates are

$$Q = \log(1 + q^{1/2}\cos p)$$
$$P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p$$

(a) Show directly from these transformation equations that Q, P are canonical variables if q and p are.

(b) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

(a) The Jacobian of the transformation is

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + 2\cos p \sin p & 2q^{1/2}\cos p + 2q\cos^2 p - 2q\sin^2 p \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\tilde{\mathbf{M}}\mathbf{J}\mathbf{M} &= \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & q^{-1/2}\sin p + \sin 2p \\ -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix} \\
&\quad \times \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{\cos^2 p + \sin^2 p + q^{1/2}\cos p\cos 2p + q^{1/2}\sin p \sin 2p}{1+q^{1/2}\cos p} \\ -\frac{\cos^2 p + \sin^2 p + q^{1/2}\cos p\cos 2p + q^{1/2}\sin p \sin 2p}{1+q^{1/2}\cos p} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \mathbf{J}
\end{aligned}$$

so the symplectic condition is satisfied.

(b) For an F_3 function the relevant relations are $q = -\partial F/\partial p$, $P = -\partial F/\partial Q$. We have

$$F_3(p, Q) = -(e^Q - 1)^2 \tan p$$

so

$$P = -\frac{\partial F_3}{\partial Q} = 2e^Q(e^Q - 1)\tan p$$
$$q = -\frac{\partial F_3}{\partial p} = (e^Q - 1)^2 \sec^2 p.$$

The second of these may be solved to yield Q in terms of q and p:

$$Q = \log(1 + q^{1/2}\cos p)$$

and then we may plug this back into the equation for P to obtain

$$P = 2q^{1/2}\sin p + q\sin 2p$$

as advertised.

Problem 9.7

(a) If each of the four types of generating functions exist for a given canonical transformation, use the Legendre transformation to derive relations between them.

(b) Find a generating function of the F_4 type for the identity transformation and of the F_3 type for the exchange transformation.

(c) For an orthogonal point transformation of q in a system of n degrees of freedom, show that the new momenta are likewise given by the orthogonal transformation of an $n-$dimensional vector whose components are the old momenta plus a gradient in configuration space.

Problem 9.8

Prove directly that the transformation

$$Q_1 = q_1, \qquad P_1 = p_1 - 2p_2$$
$$Q_2 = p_2, \qquad P_2 = -2q_1 - q_2$$

is canonical and find a generating function.

After a little hacking I came up with the generating function

$$F_{13}(p_1, Q_1, q_2, Q_2) = -(p_1 - 2Q_2)Q_1 + q_2 Q_2$$

which is of mixed F_3, F_1 type. This is Legendre-transformed into a function of the F_1 type according to

$$F_1(q_1, Q_1, q_2, Q_2) = F_{13} + p_1 q_1.$$

The least action principle then says

$$p_1\dot{q}_1 + p_2\dot{q}_2 - H(q_i, p_i) = P_1\dot{Q}_1 + P_2\dot{Q}_2 - K(Q_i, P_i) + \frac{\partial F_{13}}{\partial p_1}\dot{p}_1 + \frac{\partial F_{13}}{\partial Q_1}\dot{Q}_1 + \frac{\partial F_{13}}{\partial q_2}\dot{q}_2 + \frac{\partial F_{13}}{\partial Q_2}\dot{Q}_2 + p_1\dot{q}_1 + q_1\dot{p}_1$$

whence clearly

$$\begin{aligned}
q_1 &= -\frac{\partial F_{13}}{\partial p_1} = Q_1 & \checkmark \\
P_1 &= -\frac{\partial F_{13}}{\partial Q_1} = -p_1 - 2Q_2 \\
&= -p_1 - 2p_2 & \checkmark \\
p_2 &= \frac{\partial F_{13}}{\partial q_2} = Q_2 & \checkmark \\
P_2 &= -\frac{\partial F_{13}}{\partial Q_2} = -2Q_1 - q_2 \quad = -2q_1 - q_2 & \checkmark.
\end{aligned}$$

Problem 9.14

By any method you choose show that the following transformation is canonical:

$$\begin{aligned}
x &= \frac{1}{\alpha}(\sqrt{2P_1}\sin Q_1 + P_2), \qquad & p_x &= \frac{\alpha}{2}(\sqrt{2P_1}\cos Q_1 - Q_2) \\
y &= \frac{1}{\alpha}(\sqrt{2P_1}\cos Q_1 + Q_2), \qquad & p_y &= -\frac{\alpha}{2}(\sqrt{2P_1}\sin Q_1 - P_2)
\end{aligned}$$

where α is some fixed parameter.
Apply this transformation to the problem of a particle of charge q moving in a plane that is perpendicular to a constant magnetic field $\mathbf{B}$. Express the Hamiltonian for this problem in the (Q_i, P_i) coordinates, letting the parameter α take the form

$$\alpha^2 = \frac{qB}{c}.$$

From this Hamiltonian obtain the motion of the particle as a function of time.

We will prove that the transformation is canonical by finding a generating function. Our first step to this end will be to express everything as a function of some set of four variables of which two are old variables and two are new. After some hacking, I arrived at the set $\{x, Q_1, p_y, Q_2\}$. In terms of this set, the remaining quantities are

$$y = \left(\frac{1}{2}x - \frac{1}{\alpha^2}p_y\right)\cot Q_1 + \frac{1}{\alpha}Q_2 \tag{9}$$

$$p_x = \left(\frac{\alpha^2}{4}x - \frac{1}{2}p_y\right)\cot Q_1 - \frac{\alpha}{2}Q_2 \tag{10}$$

$$P_1 = \left(\frac{\alpha^2 x^2}{8} - \frac{1}{2}xp_y + \frac{1}{2\alpha^2}p_y^2\right)\csc^2 Q_1 \tag{11}$$

$$P_2 = \frac{\alpha}{2}x + \frac{1}{\alpha}p_y \tag{12}$$

We now seek a generating function of the form $F(x, Q_1, p_y, Q_2)$. This is of mixed type, but can be related to a generating function of pure F_1 character according to

$$F_1(x, Q_1, y, Q_2) = F(x, Q_1, p_y, Q_2) - yp_y.$$

Then the principle of least action leads to the condition

$$p_x\dot{x} + p_y\dot{y} = P_1\dot{Q}_1 + P_2\dot{Q}_2 + \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial p_y}\dot{p}_y + \frac{\partial F}{\partial Q_1}\dot{Q}_1 + \frac{\partial F}{\partial Q_2}\dot{Q}_2 + y\dot{p}_y + p_y\dot{y}$$

from which we obtain

$$p_x = \frac{\partial F}{\partial x} \tag{13}$$

$$y = -\frac{\partial F}{\partial p_y} \tag{14}$$

$$P_1 = -\frac{\partial F}{\partial Q_1} \tag{15}$$

$$P_2 = -\frac{\partial F}{\partial Q_2}. \tag{16}$$

Doing the easiest first, comparing (12) and (16) we see that F must have the form

$$F(x, Q_1, p_y, Q_2) = -\frac{\alpha}{2}xQ_2 - \frac{1}{\alpha}p_yQ_2 + g(x, Q_1, p_y). \tag{17}$$

Plugging this in to (14) and comparing with (14) we find

$$g(x, Q_1, p_y) = \left(-\frac{1}{2}xp_y + \frac{1}{2\alpha^2}p_y^2\right)\cot Q_1 + \psi(x, Q_1). \tag{18}$$

Plugging (17) and (18) into (13) and comparing with (10), we see that

$$\frac{\partial \psi}{\partial x} = \frac{\alpha^2}{4}x\cot Q_1$$

or

$$\psi(x, Q_1) = \frac{\alpha^2 x^2}{8} \cot Q_1. \tag{19}$$

Finally, combining (19), (18), (17), and (15) and comparing with (11) we see that we may simply take $\phi(Q_1) \equiv 0$. The final form of the generating function is then

$$F(x, Q_1, p_y, Q_2) = -\left(\frac{\alpha}{2}x + \frac{1}{\alpha}p_y\right) Q_2 + \left(\frac{\alpha^2 x^2}{8} - \frac{1}{2}x p_y + \frac{1}{2\alpha^2}p_y^2\right) \cot Q_1$$

and its existence proves the canonicality of the transformation.

Turning now to the solution of the problem, we take the **B** field in the z direction, i.e. $\mathbf{B} = B_0\hat{\mathbf{k}}$, and put

$$\mathbf{A} = \frac{B_0}{2}\Big(- y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}\Big).$$

Then the Hamiltonian is

$$\begin{aligned}
H(x, y, p_x, p_y) &= \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 \\
&= \frac{1}{2m}\left[\left(p_x + \frac{qB_0}{2c}y\right)^2 + \left(p_y - \frac{qB_0}{2c}x\right)^2\right] \\
&= \frac{1}{2m}\left[\left(p_x + \frac{\alpha^2}{2}y\right)^2 + \left(p_y - \frac{\alpha^2}{2}x\right)^2\right]
\end{aligned}$$

where we put $\alpha^2 = qB/c$. In terms of the new variables, this is

$$\begin{aligned}
H(Q_1, Q_2, P_1, P_2) &= \frac{1}{2m}\left[\left(\alpha\sqrt{2P_1}\cos Q_1\right)^2 + \left(\alpha\sqrt{2P_1}\sin Q_1\right)^2\right] \\
&= \frac{\alpha^2}{m}P_1 \\
&= \omega_c P_1
\end{aligned}$$

where $\omega_c = qB/mc$ is the cyclotron frequency. From the Hamiltonian equations of motion applied to this Hamiltonian we see that Q_2, P_1, and P_2 are all constant, while the equation of motion for Q_1 is

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = \omega_c \qquad \longrightarrow \qquad Q_1 = \omega_c t + \phi$$

for some phase ϕ. Putting $r = \sqrt{2P_1}/\alpha$, $x_0 = P_2/\alpha$, $y_0 = Q_2/\alpha$ we then have

$$\begin{aligned}
x &= r(\sin\omega_c t + \phi) + x_0, \qquad & p_x &= \frac{m\omega_c}{2}\left[r\cos(\omega_c t + \phi) - y_0\right] \\
y &= r(\cos\omega_c t + \phi) + y_0, \qquad & p_y &= \frac{m\omega_c}{2}\left[r\sin(\omega_c t + \phi) + x_0\right]
\end{aligned}$$

in agreement with the standard solution to the problem.

Solutions to Problems in Goldstein,
Classical Mechanics, Second Edition

Homer Reid

August 10, 2003

Chapter 10

Problem 10.3

Solve the problem of the motion of a point projectile in a vertical plane, using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle α with the horizontal.

The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

so the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy + \frac{\partial S}{\partial t} = 0. \tag{1}$$

We seek a solution of the form

$$S(x, \gamma, y, E, t) = \gamma x + f(y, E) - Et \tag{2}$$

where γ and E are to be the (constant) transformed momenta. With this ansatz for S, (1) becomes

$$\frac{\gamma^2}{2m} + \frac{1}{2m}\left(\frac{\partial f}{\partial y}\right)^2 + mgy = E$$

or

$$\frac{\partial f}{\partial y} = \sqrt{2mE - \gamma^2 - 2m^2 gy}.$$

Integrating,

$$\begin{aligned} f(y) &= \int^{y} dy' \sqrt{2mE - \gamma^2 - 2m^2 g y'} \\ &= -\frac{1}{3m^2 g}\left[2mE - \gamma^2 - 2m^2 g y\right]^{3/2}. \end{aligned}$$

Then Hamilton's principal function (2) is

$$S = \gamma x - \frac{1}{3m^2 g}\left[2mE - \gamma^2 - 2m^2 g y\right]^{3/2} - Et.$$

The (constant) transformed coordinates conjugate to the constant transformed momenta E and γ are

$$\begin{aligned} \beta_1 &= \frac{\partial S}{\partial E} \\ &= -\frac{1}{mg}\left[2mE - \gamma^2 - 2m^2 g y\right]^{1/2} - t \qquad (3) \\ \beta_2 &= \frac{\partial S}{\partial \gamma} \\ &= x + \frac{\gamma}{m^2 g}\left[2mE - \gamma^2 - 2m^2 g y\right]^{1/2} \qquad (4) \end{aligned}$$

Turning these inside out to obtain x and y as functions of time and the constants, we find

$$\begin{aligned} y &= \frac{E}{mg} - \frac{\gamma^2}{2m^2 g} - \frac{g}{2}(t+\beta_1)^2 \\ x &= \beta_2 + \frac{\gamma}{m}(t+\beta_1) \end{aligned}$$

Finally, from the given initial conditions we obtain the following equations for the constants $E, \gamma, \beta_1, \beta_2$:

$$\begin{aligned} y(t=0) = 0 \quad &\Longrightarrow \quad \frac{E}{mg} - \frac{\gamma^2}{2m^2 g} - \frac{g\beta_1^2}{2} = 0 \\ x(t=0) = 0 \quad &\Longrightarrow \quad \beta_2 + \frac{\gamma}{m}\beta_1 = 0 \\ \dot{y}(t=0) = v_0 \sin\alpha \quad &\Longrightarrow \quad -g\beta_1 = v_0 \sin\alpha \\ \dot{x}(t=0) = v_0 \cos\alpha \quad &\Longrightarrow \quad \frac{\gamma}{m} = v_0 \cos\alpha \end{aligned}$$

we obtain

$$\gamma = mv_0 \cos\alpha$$
$$\beta_1 = -\frac{v_0}{g}\sin\alpha$$
$$\beta_2 = \frac{v_0^2}{g}\cos\alpha\sin\alpha$$
$$E = \frac{mv_0^2}{2}$$

and the solutions for $x(t)$ and $y(t)$ become

$$y(t) = v_0 \sin\alpha t - \frac{gt^2}{2}$$
$$x(t) = v_0 \cos\alpha t.$$

Problem 10.6

A charged particle is constrained to move in a plane under the influence of a central force potential (nonelectromagnetic) $V = \frac{1}{2}kr^2$, and a constant magnetic field $\mathbf{B}$ perpendicular to the plane, so that

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}.$$

Set up the Hamilton-Jacobi equation for Hamilton's characteristic function in plane polar coordinates. Separate the equation and reduce it to quadratures. Discuss the motion if the canonical momentum p_θ is zero at time $t = 0$.

I got a little confused on the introduction of the polar coordinates in this problem and found it useful to start with the Lagrangian in Cartesian coordinates:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\dot{\mathbf{x}}\cdot\mathbf{A}) - \frac{k}{2}(x^2+y^2)$$

Inserting the vector potential $\mathbf{A} = \frac{1}{2}B(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$,

$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x}) - \frac{k}{2}(x^2+y^2).$$

Now we go over to polar coordinates according to

$$x = r\cos\theta, \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$y = r\sin\theta, \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

and obtain

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{qB}{2c}r^2\dot{\theta} - \frac{k}{2}r^2.$$

To go over to the Hamiltonian we introduce the canonical momenta:

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{qB}{2c}r^2. \end{aligned}$$

Then the Hamiltonian is

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} - L \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}kr^2 \\ &= \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}\left(p_\theta - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2. \end{aligned}$$

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta} - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2 - \frac{\partial S}{\partial t} = 0. \tag{5}$$

Since θ is cyclic its corresponding conjugate momentum must be constant, and we look for a solution of the form

$$S(r, \theta, E, \alpha, t) = f(r, E, \alpha) + \alpha\theta - Et. \tag{6}$$

Equation (5) becomes

$$\frac{1}{2m}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\alpha - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2 = E$$

with formal solution

$$f(r) = \int^r dr' \sqrt{2mE - mkr'^2 - \frac{1}{r'^2}\left(\alpha - \frac{qB}{2c}r'^2\right)^2}.$$

If $\alpha = 0$ this simplifies to

$$f(r) = \int^r dr' \sqrt{2mE - m^2(\omega_0^2 + \omega_c^2)r'^2}$$

and the problem becomes just that of the normal harmonic oscillator, with frequency

$$\omega = \sqrt{\omega_0^2 + \omega_c^2}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

is the natural frequency of the particle in the nonelectromagnetic potential well, and

$$\omega_c = \frac{qB}{2mc}$$

is half the cyclotron frequency of the particle in the given magnetic field. (Why does *half* the cyclotron frequency enter the problem?)

Problem 10.7

(a) A single particle moves in space under a conservative potential. Set up the Hamilton-Jacobi equation in ellipsoidal coordinates u, v, ϕ defined in terms of the usual cylindrical coordinates r, z, ϕ by the equations

$$r = a \sinh v \sin u, \qquad z = a \cosh v \cos u.$$

For what forms of $V(u, v, \phi)$ is the equation separable?

(b) Use the results of part (a) to reduce to quadratures the problem of a point particle of mass m moving in the gravitational field of two unequal mass points fixed on the z axis a distance $2a$ apart.

(a) In cylindrical polar coordinates, the Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - V(r, z, \phi). \tag{7}$$

We have

$$\begin{aligned} \dot{r} &= a\dot{v}\cosh v \sin u + a\dot{u}\sinh v \cos u \\ \dot{z} &= a\dot{v}\sinh v \cos u - a\dot{u}\cosh v \sin u. \end{aligned}$$

Plugging into (7),

$$L = \frac{ma^2}{2}\left[(\dot{v}^2 + \dot{u}^2)\cosh 2v + \dot{\phi}^2 \sinh^2 v \sin^2 u\right] - V(u, v, \phi). \tag{8}$$

The conjugate momenta are

$$\begin{aligned} p_v &= \frac{\partial L}{\partial \dot{v}} = ma^2\dot{v}\cosh 2v \\ p_u &= \frac{\partial L}{\partial \dot{u}} = ma^2\dot{u}\cosh 2v \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = ma^2\dot{\phi}\sinh^2 v \sin^2 u \end{aligned}$$

and the Hamiltonian is

$$\begin{aligned}
H &= \sum_i p_i \dot{q}_i - L \\
&= \frac{ma^2}{2}\Big[(\dot{v}^2 + \dot{u}^2)\cosh 2v + \dot{\phi}^2 \sinh^2 v \sin^2 u\Big] + V(u,v,\phi) \\
&= \frac{p_v^2}{2ma^2\cosh 2v} + \frac{p_u^2}{2ma^2\cosh 2v} + \frac{p_\phi^2}{2ma^2\sinh^2 v\sin^2 u} + V(u,v,\phi).
\end{aligned}$$

This is of the form in Goldstein's (10-44) with $\mathbf{a} = 0$ and the $\mathbf{T}$ matrix defined by

$$\mathbf{T}^{-1} = \frac{1}{ma^2}\begin{pmatrix} \text{sech } 2v & 0 & 0 \\ 0 & \text{sech } 2v & 0 \\ 0 & 0 & (\text{csch } v \csc u)^2 \end{pmatrix}. \tag{9}$$

Then the form of the potential necessary for separability is, from Goldstein (**??**),

$$V(u,v,\phi) = f(v) + \frac{g(u)}{\cosh 2v} + \frac{h(\phi)}{\sinh^2 v \sin^2 u}.$$

To write down the three separated Hamilton-Jacobi equations we need to find a matrix ϕ satisfying

$$\phi_{1j}^{-1} = \left[\mathbf{T}^{-1}\right]_{jj}, \qquad j = 1,2,3 \tag{10}$$

and with the additional condition that first, second, and third rows of ϕ depend only on v, u, and ϕ respectively.

To find such a matrix, we postulate the form

$$\phi = ma^2 \begin{pmatrix} f_1(v) & f_2(v) & f_3(v) \\ 0 & g_1(u) & g_2(u) \\ 0 & 0 & 1 \end{pmatrix}$$

with inverse

$$\phi^{-1} = \frac{1}{ma^2 f_1(v) g_1(u)} \begin{pmatrix} g_1(u) & -f_2(v) & f_2(v)g_2(u) - f_3(v)g_1(u) \\ * & * & * \\ * & * & * \end{pmatrix}$$

where $*$ denotes entries about which we don't care. From (10) and (9) we obtain the conditions

$$\begin{aligned}
\frac{1}{ma^2 f_1(v)} &= \frac{1}{ma^2\cosh 2v} \\
\frac{-f_2(v)}{ma^2 f_1(v) g_1(u)} &= \frac{1}{ma^2\cosh 2v} \\
\frac{f_2(v)g_2(u) - f_3(v)g_1(u)}{ma^2 f_1(v) g_1(u)} &= \frac{1}{ma^2 \sinh^2 v \sin^2 u}
\end{aligned}$$

A little inspection shows that there is no solution, which means the form we postulated for ϕ doesn't work, so we need to go back and try something else, except this is the point at which I decided I would temporarily shelve this problem.

(b) The potential energy is

$$\begin{aligned} V(r,z) &= -G\frac{mm_1}{\sqrt{r^2+(z-a)^2}} - G\frac{mm_2}{\sqrt{r^2+(z+a)^2}} \\ &= -\frac{Gm}{a}\left[\frac{m_1}{\cosh v - \cos u} + \frac{m_2}{\cosh v + \cos u}\right] \end{aligned}$$

which doesn't appear to be of the form required for

Problem 10.8

> Suppose the potential in a problem of one degree of freedom is linearly dependent on time, such that the Hamiltonian has the form
>
> $$H = \frac{p^2}{2m} - mAtx,$$
>
> where A is a constant. Solve the dynamical problem by means of Hamilton's principal function, under the initial conditions $t = 0$, $x = 0$, $p = mv_0$.

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mAtx + \frac{\partial S}{\partial t} = 0. \tag{11}$$

We postulate a solution of the form

$$S(x,t) = f(t)x + g(t).$$

Then (11) becomes

$$\frac{1}{2m}f^2(t) - mAtx + f'(t)x + g'(t) = 0.$$

Matching powers of x, we obtain

$$f'(t) = mAt \qquad \Longrightarrow \qquad f(t) = \frac{1}{2}mAt^2 + f_0$$

and

$$\begin{aligned} g'(t) &= -\frac{1}{2m}f^2(t) \\ &= -\frac{1}{2m}\left[\frac{1}{2}mAt^2 + f_0\right]^2 \\ &= -\frac{1}{2m}\left[\frac{m^2A^2}{4}t^4 + mAf_0t^2 + f_0^2\right] \end{aligned}$$

so

$$g(t) = -\frac{mA^2}{40}t^5 + \frac{Af_0}{6}t^3 + \frac{f_0^2}{2m}t + g_0.$$

The Hamilton's principal function is

$$S(x, t, f_0) = \frac{1}{2}mAt^2x + f_0x - \frac{mA^2}{40}t^5 - \frac{Af_0}{6}t^3 - \frac{f_0^2}{2m}t.$$

The constant transformed coordinate conjugate to f_0 is

$$\frac{\partial S}{\partial f_0} = \beta = x - \frac{A}{6}t^3 - \frac{f_0 t}{m}.$$

Turning this inside-out,

$$x = \beta + \frac{A}{6}t^3 + \frac{f_0 t}{m}.$$

To to satisfy the conditions $x(t=0) = 0$, $\dot{x}(t=0) = v_0$, we must take $\beta = 0$ and $f_0 = mv_0$, and we then have

$$x = v_0 t + \frac{A}{6}t^3.$$

Problem 10.13

> A particle moves in periodic motion in one dimension under the influence of a potential $V(x) = F|x|$, where F is a constant. Using action-angle variables find the period of the motion as a function of the particle's energy.

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + F|x| = E$$

The action integral is

$$\begin{aligned} J &= \oint \frac{\partial W}{\partial x}dx \qquad (12) \\ &= 4\int_0^{x_0} \sqrt{2mE - 2mF|x|}dx \end{aligned}$$

(where $\pm x_0$ are the extremes of the particle's orbit, and where we have restricted the integral over the whole period to an integral over the first quarter-period and multiplied by 4 to compensate)

$$= -\frac{4}{3mF}\left\{[2mE - 2mFx_0]^{3/2} - [2mE]^{3/2}\right\}$$

but the first term here vanishes since $E = Fx_0$, so

$$J = \frac{4}{3mF}\left[2mE\right].^{3/2} \tag{13}$$

Expressing the Hamiltonian in terms of J, we then obtain

$$E = \frac{1}{2m}\left(\frac{3mF}{4}\right)^{2/3} J^{2/3}.$$

The frequency is

$$\begin{aligned}\nu &= \frac{\partial E}{\partial J}\\ &= \frac{1}{3m}\left(\frac{3mF}{4}\right)^{2/3} J^{-1/3}\end{aligned}$$

Plugging in from (13),

$$\begin{aligned}&= \frac{1}{3m}\left(\frac{3mF}{4}\right)^{2/3}\left(\frac{4}{3mF}\right)^{-1/3}\frac{1}{\sqrt{2mE}}\\ &= \frac{F}{4\sqrt{2mE}}. \qquad (14)\end{aligned}$$

On the other hand, on the basis of elementary considerations we could reason as follows: If the particle starts out with momentum $p = \sqrt{2mE}$, and it is always under the influence of the constant force $F = dp/dt$, then the time it takes for the particle's initial momentum to decay to zero, which is one-fourth the total period of the motion, is

$$\tau_{1/4} = \frac{p}{dp/dt} = \frac{\sqrt{2mE}}{F}.$$

The total period is just $4\tau_{1/4}$ and the frequency is

$$\nu = 1/4\tau_{1/4} = \frac{F}{4\sqrt{2mE}}$$

in accordance with (14).

Problem 10.14

A particle of mass m moves in one dimension under a potential $V = -k/|x|$. For energies that are negative the motion is bounded and oscillatory. By the method of action-angle variables find an expression for the period of motion as a function of the particle's energy.

The Hamilton-Jacobi equation is

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{k}{|x|} + \frac{\partial S}{\partial t} = 0.$$

We seek a solution of the form $S = W(x, E) - Et$, in which case

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 - \frac{k}{|x|}$$

or

$$\begin{aligned}\frac{\partial W}{\partial x} &= \sqrt{2m}\left[E + \frac{k}{|x|}\right]^{1/2} \\ \frac{\partial W}{\partial x} &= \sqrt{2m}\left[\frac{k}{|x|} - |E|\right]^{1/2},\end{aligned}$$

since we know the energy is negative. Then the turning points of the motion are at $x = \pm k/|E|$, and the action variable is

$$\begin{aligned}J &= \oint p dq \qquad (15) \\ &= 4\sqrt{2m}\int_0^{k/|E|}\left[\frac{k}{x} - E\right]^{1/2} dx \\ &= 4k\sqrt{\frac{2m}{E}}\int_0^1\left[\frac{1}{u} - 1\right]^{1/2} du. \qquad (16)\end{aligned}$$

We have

$$\int_0^1\left[\frac{1}{u} - 1\right]^{1/2} du = \int_0^1\sqrt{\frac{1-u}{u}}du$$

Change variables to $\alpha = u^{1/2}, 2d\alpha = du/\sqrt{u}$:

$$= 2\int_0^1\sqrt{1-\alpha^2}\,d\alpha$$

Now change to $\alpha = \sin x, d\alpha = \cos x dx$:

$$\begin{aligned}&= 2\int_0^{\pi/2}\cos^2 x dx \\ &= \frac{\pi}{2}\end{aligned}$$

so (16) becomes

$$J = 2\pi k\sqrt{\frac{2m}{|E|}}.$$

Then the (constant) Hamiltonian expressed in terms of the action variable is

$$H = E = \frac{8m\pi^2 k^2}{J^2}.$$

The frequency associated with the corresponding angle variable is

$$\nu = \frac{\partial H}{\partial J} = \frac{16m\pi^2 k^2}{J^3}$$
$$= \frac{|E|^{3/2}}{\pi k\sqrt{2m}}. \tag{17}$$

It's interesting to compare this with the solution we obtained to Problem 3.3, since in that case the motion of the particles (after the orbital motion is stopped) is one-dimensional with the kind of force law considered here. In that problem the energy of the two-particle system would be

$$E = \frac{\mu}{2}r_0^2\dot\theta^2 - \frac{k}{r_0}$$

Plugging in our expression (equation (5) in solutions to Chapter 3) for r_0 in terms of τ, we find

$$E = \left(\frac{k\pi\sqrt{m}}{\tau\sqrt{2}}\right)^{2/3}.$$

Then the frequency of the motion, from (17), is $\nu = \frac{1}{2\tau}$. The time for the particles to fall into each other would be just a quarter-period, so on this theory we would get

$$\text{time for particles to fall into each other} = \frac{1}{4\nu} = \frac{\tau}{2}$$

whereas the answer we obtained in Chapter 3 was $\tau/4\sqrt{2}$. But I can't figure out where I am making a mistake.

Problem 10.16

A particle of mass m is constrained to move on a curve in the vertical plane defined by the parametric equations

$$y = l(1-\cos 2\phi)$$
$$x = l(2\phi + \sin 2\phi).$$

There is the usual constant gravitational force acting in the vertical y direction. By the method of action-angle variables find the frequency of oscillation for all initial conditions such that the maximum of ϕ is less than or equal to $\pi/4$.

The kinetic energy is

$$T = \frac{m}{2}(\dot x^2 + \dot y^2)$$
$$= \frac{ml^2}{2}\left[(2\dot\phi\sin 2\phi)^2 + (2\dot\phi + 2\dot\phi\cos\phi)^2\right]$$
$$= 8ml^2\cos^2\phi\dot\phi^2.$$

The potential energy is

$$\begin{aligned} V = mgy &= mgl(1 - \cos 2\phi) \\ &= 2mgl \sin^2 \phi. \end{aligned} \tag{18}$$

Then the Lagrangian is

$$L = 8ml^2 \cos^2 \phi \dot{\phi}^2 - 2mgl \sin^2 \phi$$

so the canonical momentum conjugate to ϕ is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 16ml^2 \cos^2 \phi \dot{\phi}$$

and the Hamiltonian is

$$\begin{aligned} H = E &= p_\phi \dot{\phi} - L \\ &= 8ml^2 \cos^2 \phi \dot{\phi}^2 + 2mgl \sin^2 \phi \\ &= \frac{p_\phi^2}{32ml^2 \cos^2 \phi} + 2mgl \sin^2 \phi. \end{aligned}$$

Solving for p_ϕ as a function of E and ϕ, we have

$$p_\phi = 4l\sqrt{2mE} \cos \phi \left[1 - \frac{2mgl}{E} \sin^2 \phi \right]^{1/2}.$$

The action variable for periodic motion is

$$\begin{aligned} J &= \oint p_\phi d\phi \\ &= 16l\sqrt{2mE} \int_0^{\phi_0} \cos \phi \left[1 - \frac{2mgl}{E} \sin^2 \phi \right]^{1/2} d\phi \end{aligned}$$

Put $u = \sqrt{\frac{2mgl}{E}} \sin \phi$:

$$= 16E\sqrt{\frac{l}{g}} \int_0^{u_0} \sqrt{1 - u^2}\, du.$$

Here the upper integration limit is

$$u_0 = \sqrt{\frac{2mgl}{E}} \sin \phi_0$$

where ϕ_0 is the maximum value attained by ϕ. On the other hand, referring back to (18), $2mgl \sin^2 \phi_0$ is just the potential energy of the particle at its maximum height, which is just its total energy since it has zero kinetic energy at that point. Hence

$$u_0 = \sqrt{\frac{E}{E}} = 1.$$

The action integral is then

$$\begin{aligned} J &= 16E\sqrt{\frac{l}{g}} \int_0^1 \sqrt{1-u^2}\,du \\ &= 8E\sqrt{\frac{l}{g}} \left| u\sqrt{1-u^2} + \sin^{-1} u \right|_0^1 \\ &= 4\pi E\sqrt{\frac{l}{g}}. \end{aligned}$$

Then the Hamiltonian expressed in terms of the action variable is

$$H = E = \frac{1}{4\pi}\sqrt{\frac{g}{l}} J$$

and the frequency is just

$$\nu = \frac{\partial H}{\partial J} = \frac{1}{4\pi}\sqrt{\frac{g}{l}}.$$

Problem 10.17

A three-dimensional harmonic oscillator has the force constant k_1 in the x and y directions and k_3 in the z direction. Using cylindrical coordinates (with the axis of the cylinder in the z direction) describe the motion in terms of the corresponding action-angle variables, showing how the frequencies can be obtained. Transform to the "proper" action-angle variables to eliminate degenerate frequencies.

The Lagrangian is

$$\frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right] - \frac{1}{2}k_1 r^2 - \frac{1}{2}k_3 z^2.$$

with canonical momenta

$$\begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \\ p_z &= m\dot{z}. \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H &= \sum p_i \dot{q}_i - L \\ &= \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right] + \frac{1}{2}k_1 r^2 + \frac{1}{2}k_3 z^2 \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_z^2}{2m} + \frac{1}{2}k_1 r^2 + \frac{1}{2}k_3 z^2. \end{aligned}$$