ARNOWITT-DESER-MISNER FORMALISM

In the Arnowitt–Deser–Misner formalism the four dimensional metric $g_{\mu\nu}$ is parametrized by the three-metric h_{ij} and the lapse and shift functions N and N^i, which describe the evolution of time-like hypersurfaces,

$$g_{00} = -N^2 + h^{ij} N_i N_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = h_{ij}. \tag{1}$$

The action for the inflaton scalar field with potential $V(\phi)$ in the ADM formalism has the form

$$\begin{aligned} \mathcal{S} &= \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] \\ &= \int d^4x N \sqrt{h} \left[\frac{1}{2\kappa^2} \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right) + \frac{1}{2} \left[(\Pi^\phi)^2 - \phi_{|i} \phi^{|i} \right] - V(\phi) \right], \end{aligned} \tag{2}$$

where $\kappa^2 = 8\pi G = 8\pi / M_{\mathrm{P}}^2$ is the gravitational coupling defining the Planck mass and Π^ϕ is the scalar-field's conjugate momentum

$$\Pi^\phi = \frac{1}{N} (\dot{\phi} - N^i \phi_{|i}) \,. \tag{3}$$

Vertical bars denote three-space-covariant derivatives with connections derived from h_{ij}; ${}^{(3)}R$ is the three-space curvature associated with the metric h_{ij}, and K_{ij} is the extrinsic curvature three-tensor

$$K_{ij} = \frac{1}{2N} (N_{i|j} + N_{j|i} - \dot{h}_{ij}), \tag{4}$$

where a dot denotes differentiation with respect to the time coordinate. The traceless part of a tensor is denoted by an overbar. In particular,

$$\bar{K}_{ij} = K_{ij} - \frac{1}{3} K h_{ij}, \qquad K = K^i_i = \frac{1}{N} \left[N_{i|i} - \partial_t \ln \sqrt{h} \right] . \tag{5}$$

The trace K is a generalization of the Hubble parameter, as will be shown below. Variation of the action with respect to N and N^i yields the energy and momentum constraint equations respectively

$$\begin{aligned} -{}^{(3)}R + \bar{K}_{ij} \bar{K}^{ij} - \frac{2}{3} K^2 + 2\kappa^2 \rho = 0 \,, \\ \bar{K}^j_{i|j} - \frac{2}{3} K_{|i} + \kappa^2 \, \Pi^\phi \phi_{|i} = 0 \,. \end{aligned} \tag{6}$$

equations, which can be separated into the trace and traceless parts

$$\dot{K} - N^i K_{|i} = -\, N^{|i}_{|i} + N\left(\frac{1}{4}{}^{(3)}R + \frac{3}{4}\bar{K}_{ij}\bar{K}^{ij} + \frac{1}{2}K^2 + \frac{\kappa^2}{2}T\right), \quad (7)$$

$$\begin{aligned}\dot{\bar{K}}{}^i{}_j - N^k \bar{K}^i{}_{j|k} + N^i_{|k}\bar{K}^k_j \; - N^k_{|j}\bar{K}^i_k = & \; -N^{|i}_{|j} + \frac{1}{3}N^{|k}_{|k}\delta^i_j \\ & + N\left({}^{(3)}\bar{R}^i_j + K\bar{K}^i_j - \kappa^2\,\bar{T}^i_j\right).\end{aligned} \quad (8)$$

Variation with respect to ϕ gives the scalar-field's equation of motion

$$\frac{1}{N}(\dot{\Pi}^\phi - N^i \Pi^\phi_{|i}) - \; K\Pi^\phi - \frac{1}{N}N_{|i}\phi^{|i} - \phi_{|i}^{\;|i} + \frac{\partial V}{\partial \phi} = 0\,. \quad (9)$$

The energy density on a constant-time hypersurface is

$$\rho = \frac{1}{2}\left[(\Pi^\phi)^2 + \phi_{|i}\phi^{|i}\right] + V(\phi), \quad (10)$$

and the stress three-tensor

$$T_{ij} = \phi_{|i}\phi_{|j} + h_{ij}\left[\frac{1}{2}\left[(\Pi^\phi)^2 - \phi_{|k}\phi^{|k}\right] - V(\phi)\right]. \quad (11)$$

It is extremely difficult to solve these highly nonlinear coupled equations in a cosmological scenario without making some approximations. The usual approach is to assume homogeneity of the fields to give a background solution and then linearize the equations to study deviations from spatial uniformity. The smallness of cosmic microwave background anisotropies gives some justification for this perturbative approach at least in our local part of the Universe. However, there is no reason to believe it will be valid on much larger scales. In fact, the stochastic approach to inflation suggests that the Universe is extremely inhomogeneous on very large scales. Fortunately, in this framework one can coarse-grain over a horizon distance and separate the short- from the long-distance behavior of the fields, where the former communicates with the latter through stochastic forces. The equations for the long-wavelength background fields are obtained by neglecting large-scale gradients, leading to a self-consistent set of equations, as we will discuss in the next section.

SPATIAL GRADIENT EXPANSION

It is reasonable to expand in spatial gradients whenever the forces arising from time variations of the fields are much larger than forces from spatial gradients. In linear perturbation theory one solves the perturbation equations for evolution outside of the horizon: a typical time scale is the Hubble time H^{-1}, which is assumed to exceed the gradient scale a/k, where k is the comoving wave number of the perturbation. Since we are interested in structures on scales larger than the horizon, it is reasonable to expand in $k/(aH)$. In particular, for inflation this is an appropriate parameter of expansion since spatial gradients become exponentially negligible after a few e-folds of expansion beyond horizon crossing, $k = aH$.

It is therefore useful to split the field ϕ into coarse-grained long-wavelength background fields $\phi(t, x^j)$ and residual short-wavelength fluctuating fields $\delta\phi(t, x^j)$. There is a preferred timelike hypersurface within the stochastic inflation approach in which the splitting can be made consistently, but the definition of the background field will depend on the choice of hypersurface, *i.e.* the smoothing is not gauge invariant. For stochastic inflation the natural smoothing scale is the comoving Hubble length $(aH)^{-1}$ and the natural hypersurfaces are those on which aH is constant. In that case a fundamental difference between between ϕ and $\delta\phi$ is that the short-wavelength components are essentially uncorrelated at different times, while long-wavelength components are deterministically correlated through the equations of motion.

In order to solve the equations for the background fields, we will have to make suitable approximations. The idea is to expand in the spatial gradients of ϕ and to treat the terms that depend on the fluctuating fields as stochastic forces describing the connection between short- and long-wavelength components. In this Section we will neglect the stochastic forces due to quantum fluctuations of the scalar fields and will derive the approximate equation of motion for the background fields. We retain only those terms that are at most first order in spatial gradients, neglecting such terms as $\phi_{|i}{}^{|i}$, $\phi_{|i}\phi^{|i}$, ${}^{(3)}R$, ${}^{(3)}R_i^j$, and $\bar{T}_i^j$.

evolution during inflation this is a consequence of the rapid expansion, more than a gauge choice]. The evolution equation (8) for the traceless part of the extrinsic curvature is then $\dot{\bar{K}}{}^i{}_j = NK\bar{K}^i_j$. Using $NK = -\partial_t \ln\sqrt{h}$ from (5), we find the solution $\bar{K}^i_j \propto h^{-1/2}$, where h is the determinant of h_{ij}. During inflation $h^{-1/2} \equiv a^{-3}$, with a the overall expansion factor, therefore $\bar{K}^i_j$ decays extremely rapidly and can be set to zero in the approximate equations. The most general form of the three-metric with vanishing $\bar{K}^i_j$ is

$$h_{ij} = a^2(t,x^k)\,\gamma_{ij}(x^k), \quad a(t,x^k) \equiv \exp[\alpha(t,x^k)], \tag{12}$$

where the time-dependent conformal factor is interpreted as a space-dependent expansion factor. The time-independent three-metric γ_{ij}, of unit determinant, describes the three-geometry of the conformally transformed space. Since $a(t,x^k)$ is interpreted as a scale factor, we can substitute the trace K of the extrinsic curvature for the Hubble parameter

$$H(t,x^i) \equiv \frac{1}{N(t,x^i)}\,\dot{\alpha}(t,x^i) = -\frac{1}{3}K(t,x^i). \tag{13}$$

The energy and momentum constraint equations (6) can now be written as

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}(\Pi^\phi)^2 + V(\phi)\right], \tag{14}$$

$$H_{|i} = -\frac{\kappa^2}{2}\Pi^\phi \phi_{|i}, \tag{15}$$

together with the evolution equation (7)

$$-\frac{1}{N}\dot{H} = \frac{3}{2}H^2 + \frac{\kappa^2}{6}T = \frac{\kappa^2}{2}(\Pi^\phi)^2, \tag{16}$$

where $T = 3\left(\frac{1}{2}(\Pi^\phi)^2 - V(\phi)\right)$.

In general, H is a function of the scalar field and time, $H(t,x^i) \equiv H(\phi(t,x^i),t)$. From the momentum constraint (15) we find that the scalar-field's momentum must obey

$$\Pi^\phi = -\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)_t. \tag{17}$$

$$\frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_x = \Pi^\phi \left(\frac{\partial H}{\partial \phi}\right)_t + \frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_\phi$$
$$= -\frac{\kappa^2}{2}(\Pi^\phi)^2 + \frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_\phi , \tag{18}$$

we find $\left(\frac{\partial H}{\partial t}\right)_\phi = 0$.

In fact, we should not be surprised since this is actually a consequence of the general covariance of the theory.

On the other hand, the scalar field's equation (9) can be written to first order in spatial gradients as

$$\frac{1}{N}\dot{\Pi}^\phi + 3H\Pi^\phi + \frac{\partial V}{\partial \phi} = 0\,. \tag{19}$$

We can also show that the conjugate momentum Π^ϕ does not depend explicitly on time, its only dependence comes through ϕ. For this, differentiate Eq. (14) w.r.t. ϕ to obtain

$$\Pi^\phi \left(\frac{\partial \Pi^\phi}{\partial \phi}\right)_t + 3H\,\Pi^\phi + \frac{\partial V}{\partial \phi} = 0$$

and compare with (19), where

$$\frac{1}{N}\dot{\Pi}^\phi = \Pi^\phi \left(\frac{\partial \Pi^\phi}{\partial \phi}\right)_t + \left(\frac{\partial \Pi^\phi}{\partial t}\right)_\phi , \tag{20}$$

which implies $\left(\frac{\partial \Pi^\phi}{\partial t}\right)_\phi = 0$.

We can now summarise what we have learned. The evolution of a general foliation of space-time in the presence of a scalar field fluid can be described solely in terms of the rate of expansion, which is a function of the scalar field only, $H \equiv H(\phi(t,x^i))$, satisfying the Hamiltonian constraint equation:

$$3H^2(\phi) = \frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)^2 + \kappa^2 V(\phi)\,, \tag{21}$$

together with the momentum constraint and the evolution of the scale factor,

$$\frac{1}{N}\,\dot{\phi} \;=\; -\,\frac{\kappa^2}{2}\left(\frac{\partial H}{\partial \phi}\right) = \Pi^\phi \tag{22}$$

$$\frac{1}{N}\,\dot{\alpha} \;=\; H(\phi)\,, \tag{23}$$

as well as the dynamical gravitational and scalar field evolution equations

$$\frac{1}{N}\,\dot{H} \;=\; -\,\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)^2 \;=\; -\,\frac{\kappa^2}{2}(\Pi^\phi)^2\,, \tag{24}$$

$$\frac{1}{N}\,\dot{\Pi}^\phi \;=\; -\,3H\,\Pi^\phi - V'(\phi)\,. \tag{25}$$

Therefore, $H(\phi)$ is all you need to specify (to second order in field gradients) the evolution of the scale factor and the scalar field during inflation.

These equations are still too complicated to solve for arbitrary potentials $V(\phi)$. In the next section we will find solutions to them in the slow-roll approximation.

SLOW-ROLL APPROXIMATION AND ATTRACTOR

Given the complete set of constraints and evolution equations (21) - (25), we can construct the following parameters,

$$\epsilon \;\equiv\; -\frac{\dot{H}}{H^2} = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = -\frac{\partial \ln H}{\partial \ln a}\,, \tag{26}$$

$$\delta \;\equiv\; -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{2}{\kappa^2}\left(\frac{H''(\phi)}{H(\phi)}\right) = -\frac{\partial \ln H'}{\partial \ln a}\,, \tag{27}$$

in terms of which we can define the number of e-folds N_e as

$$N_e \equiv \ln\frac{a_{\rm end}}{a(t)} = \int_t^{t_{\rm end}} H dt = -\frac{\kappa^2}{2}\int_\phi^{\phi_{\rm end}} \frac{H(\phi)d\phi}{H'(\phi)}\,. \tag{28}$$

In order for inflation to be predictive, you need to ensure that inflation is independent of initial conditions. That is, one should ensure that there is an attractor solution to the dynamics, such that differences between solutions corresponding to different initial conditions rapidly vanish.

Let $H_0(\phi)$ be an exact, particular, solution of the constraint equation (21), either inflationary or not. Add to it a *homogeneous* linear perturbation $\delta H(\phi)$, and substitute into (21). The linear perturbation equation reads $H_0'(\phi)\,\delta H'(\phi) = (3\kappa^2/2)\,H_0\delta H$, whose general solution is

$$\delta H(\phi) = \delta H(\phi_i)\,\exp\left(\frac{3\kappa^2}{2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)d\phi}{H_0'(\phi)}\right) = \delta H(\phi_i)\,\exp(-3\Delta N)\,, \tag{29}$$

where $\Delta N = N_i - N > 0$, and we have used (28) with the particular solution $H_0(\phi)$. This means that very quickly any deviation from the attractor dies away. This ensures that we can effectively reduce our two-dimensional space (ϕ, Π^ϕ) to just a single trajectory in phase space.

As a consequence, regardless of the initial condition, the attractor behaviour implies that late-time solutions are the same *up to a constant time shift*, which cannot be measured.

AN EXAMPLE: POWER-LAW INFLATION

An exponential potential is a particular case where the attractor can be found explicitly and one can study the approach to it, for an arbitrary initial condition.

Consider the inflationary potential

$$V(\phi) = V_0 \, e^{-\beta\kappa\phi} \,, \tag{30}$$

with $\beta \ll 1$ for inflation to proceed. A particular solution to the Hamiltonian constraint equation (21) is

$$H_{\rm att}(\phi) = H_0 \, e^{-\frac{1}{2}\beta\kappa\phi} \,, \tag{31}$$

$$H_0^2 = \frac{\kappa^2}{3} V_0 \left(1 - \frac{\beta^2}{6}\right)^{-1} \,. \tag{32}$$

This model corresponds to an inflationary universe with a scale factor that grows like

$$a(t) \sim t^p \,, \qquad p = \frac{2}{\beta^2} \gg 1 \,. \tag{33}$$

The slow-roll parameters are both constant,

$$\epsilon = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = \frac{\beta^2}{2} = \frac{1}{p} \ll 1 \,, \tag{34}$$

$$\delta = \frac{2}{\kappa^2}\left(\frac{H''(\phi)}{H(\phi)}\right) = \frac{\beta^2}{2} = \frac{1}{p} \ll 1 \,. \tag{35}$$

All trajectories tend to the attractor (31), while we can also write down the solution corresponding to the slow-roll approximation, $\epsilon = \delta = 0$,

$$H_{\rm SR}^2(\phi) = \frac{\kappa^2}{3} V_0 \, e^{-\beta\kappa\phi} \,, \tag{36}$$

which differs from the actual attractor by a tiny constant factor, $3p/(3p-1) \simeq 1$, responsible for a constant time-shift which cannot be measured.

HOMOGENEOUS SCALAR FIELD DYNAMICS

Singlet minimally coupled scalar field ϕ, with effective potential $V(\phi)$

$$\mathcal{S}_{\rm inf} = \int d^4x\, \sqrt{-g}\, \mathcal{L}_{\rm inf}\,, \qquad \mathcal{L}_{\rm inf} = -\frac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,. \tag{1}$$

Its evolution equation in a Friedmann-Robertson-Walker metric:

$$\ddot\phi - \frac{1}{a^2}\nabla^2\phi + 3H\dot\phi + V'(\phi) = 0\,, \tag{2}$$

together with the Einstein equations,

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot\phi^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi)\right]\,, \tag{3}$$

$$\dot H = -\frac{\kappa^2}{2}\,\dot\phi^2\,, \tag{4}$$

where $\kappa^2 \equiv 8\pi G$. The inflation dynamics described as a perfect fluid with a time-dependent pressure and energy density given by

$$\rho = \frac{1}{2}\dot\phi^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi)\,, \tag{5}$$

$$p = \frac{1}{2}\dot\phi^2 - \frac{1}{6a^2}(\nabla\phi)^2 - V(\phi)\,. \tag{6}$$

The field evolution equation (2) implies the energy conservation equation,

$$\dot\rho + 3H(\rho + p) = 0\,. \tag{7}$$

If the potential energy density of the scalar field dominates the kinetic and gradient energy, $V(\phi) \gg \dot\phi^2,\ \frac{1}{a^2}(\nabla\phi)^2$, then

$$p \simeq -\rho \quad \Rightarrow \quad \rho \simeq \text{const.} \quad \Rightarrow \quad H(\phi) \simeq \text{const.}\,, \tag{8}$$

which leads to the solution

$$a(t) \sim \exp(Ht) \quad \Rightarrow \quad \frac{\ddot a}{a} > 0 \quad \text{accelerated expansion}\,. \tag{9}$$

Definition: number of e-folds,

$$N \equiv \ln(a/a_i) \qquad \Rightarrow \qquad a(N) = a_i \exp(N)$$

THE SLOW-ROLL APPROXIMATION

During inflation, the scalar field evolves very slowly down its effective potential. We can then define the slow-roll parameters,

$$\epsilon \;\equiv\; -\frac{\dot H}{H^2} \;=\; \frac{\kappa^2}{2}\frac{\dot\phi^2}{H^2} \;\ll\; 1\,, \tag{10}$$

$$\delta \;\equiv\; -\frac{\ddot\phi}{H\dot\phi} \;\ll\; 1\,. \tag{11}$$

The condition which characterizes inflation is

$$\epsilon < 1 \quad\Longleftrightarrow\quad \frac{\ddot a}{a} > 0\,, \tag{12}$$

i.e. horizon distance $d_{\rm H} \sim H^{-1}$ grows more slowly than scale factor a.

The number of e-folds during inflation:

$$N \;=\; \ln\frac{a_{\rm end}}{a_i} \;=\; \int_{t_i}^{t_e} H dt \;=\; \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon(\phi)}}\,. \tag{13}$$

The evolution equations (2) and (3) become

$$H^2\left(1-\frac{\epsilon}{3}\right) \;\simeq\; H^2 \;=\; \frac{\kappa^2}{3}V(\phi)\,, \tag{14}$$

$$3H\dot\phi\left(1-\frac{\delta}{3}\right) \;\simeq\; 3H\dot\phi \;=\; -V'(\phi)\,. \tag{15}$$

Phase space reduction for single-field inflation, $H(\phi,\dot\phi) \to H(\phi)\,.$

$$\epsilon \;=\; \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \;\ll\; 1\,,$$

$$\eta \;=\; \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} \ll\; 1\,,$$

$$N \;=\; \kappa^2\int_{\phi_i}^{\phi_e}\frac{V(\phi)\,d\phi}{V'(\phi)}\,.$$

GAUGE INVARIANT LINEAR PERTURBATION THEORY

The unperturbed (background) FRW metric can be described by a scale factor $a(t)$ and a homogeneous scalar field $\phi(t)$,

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}\, dx^i dx^j]\,, \tag{16}$$

$$\phi = \phi(\eta)\,, \tag{17}$$

where η is the conformal time $\eta = \int \frac{dt}{a(t)}$
and the background equations of motion can be written as

$$\mathcal{H}^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\phi'^2 + a^2 V(\phi)\right]\,, \tag{18}$$

$$\mathcal{H}' - \mathcal{H}^2 = \frac{\kappa^2}{2}\phi'^2\,, \tag{19}$$

$$\phi'' + 2\mathcal{H}\phi' + a^2 V'(\phi) = 0\,, \tag{20}$$

where $\mathcal{H} = aH$ and $\phi' = a\dot{\phi}$.

During inflation, the quantum fluctuations of the scalar field will induce metric perturbations which will backreact on the scalar field.

The most general line element, in linear perturbation theory, with both scalar and tensor metric perturbations [inflation cannot generate, to linear order, a vector perturbation], is given by

$$\begin{aligned} ds^2 = \ & a^2(\eta)\Big[-(1+2A)d\eta^2 + 2B_{|i}dx^i d\eta \\ & + \{(1+2\mathcal{R})\gamma_{ij} + 2E_{|ij} + 2h_{ij}\}dx^i dx^j\Big]\,, \end{aligned} \tag{21}$$

$$\phi = \phi(\eta) + \delta\phi(\eta, x^i)\,. \tag{22}$$

The indices $\{i, j\}$ label the three-dimensional spatial coordinates with metric γ_{ij}, and the $|i$ denotes covariant derivative with respect to that metric. The gauge-invariant tensor perturbation h_{ij} corresponds to a transverse traceless gravitational wave, $\nabla^i h_{ij} = h^i_i = 0$.

GAUGE INVARIANT GRAVITATIONAL POTENTIALS

The four scalar metric perturbations $(A, B, \mathcal{R}, E)$ and the field perturbation $\delta\phi$ are all *gauge dependent* functions of (η, x^i). The tensor perturbation h_{ij} is *gauge independent.*

Under a general coordinate (gauge) transformation

$$\tilde{\eta} = \eta + \xi^0(\eta, x^i)\,, \tag{23}$$
$$\tilde{x}^i = x^i + \gamma^{ij}\,\xi_{|j}(\eta, x^i)\,, \tag{24}$$

with arbitrary functions (ξ^0, ξ), the scalar and tensor perturbations transform, to linear order, as

$$\tilde{A} = A - \xi^{0\prime} - \mathcal{H}\xi^0\,, \qquad \tilde{B} = B + \xi^0 - \xi'\,, \tag{25}$$
$$\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{H}\xi^0\,, \qquad \tilde{E} = E - \xi\,, \tag{26}$$

$$\tilde{h}_{ij} = h_{ij}\,, \tag{27}$$

(where primes denote derivatives with respect to conformal time).
Possible to construct two gauge-invariant gravitational potentials,

$$\Phi = A + (B - E')' + \mathcal{H}(B - E')\,, \tag{28}$$
$$\Psi = \mathcal{R} + \mathcal{H}(B - E')\,, \tag{29}$$

which are related through the perturbed Einstein equations,

$$\Phi \;=\; -\Psi\,, \tag{30}$$
$$\frac{k^2}{a^2}\Psi \;=\; 4\pi\, G\,\delta\rho\,, \quad \text{(Poisson)} \tag{31}$$

where $\delta\rho$ is the gauge-invariant density perturbation corresponding to the scalar field.

Consider the action (1) with line element

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Phi)d\mathbf{x}^2\right]$$

in the Longitudinal gauge, where Φ is the gauge-invariant gravitational potential (28). Then the gauge-invariant equations for the perturbations on comoving hypersurfaces (constant energy density hypersurfaces) are

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2V'(\phi)\delta\phi]\,, \tag{32}$$

$$-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + \mathcal{H}^2)\Phi = -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2V'(\phi)\delta\phi]\,, \tag{33}$$

$$\Phi' + \mathcal{H}\Phi = \frac{\kappa^2}{2}\phi'\delta\phi\,, \tag{34}$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = 4\phi'\Phi' - 2a^2V'(\phi)\Phi - a^2V''(\phi)\delta\phi\,.$$

This system of equations seem too difficult to solve at first sight. However, there is a gauge invariant combination of Mukhanov variables

$$u \equiv a\delta\phi + z\Phi\,,$$
$$z \equiv a\frac{\phi'}{\mathcal{H}}\,.$$

for which the above equations simplify enormously,

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0\,,$$
$$\nabla^2\Phi = \frac{\kappa^2}{2}\frac{\mathcal{H}}{a^2}(zu' - z'u)\,,$$
$$\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{\kappa^2}{2}zu\,.$$

From these, we can find a solution $u(z)$, which can be integrated to give $\Phi(z)$, and together allow us to obtain $\delta\phi(z)$.

QUANTUM FIELD THEORY IN CURVED SPACE-TIME

We should consider the perturbations Φ and $\delta\phi$ as quantum field fluctuations. Note that the perturbed action for the scalar mode u can be written as

$$\delta S = \frac{1}{2}\int d^3x \, d\eta \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{35}$$

Quantize the field u in the curved background: write the operator

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\eta)\, \hat{a}_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\, \hat{a}_{\mathbf{k}}^\dagger\, e^{-i\,\mathbf{k}\cdot\mathbf{x}}\right], \tag{36}$$

with

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'), \qquad \hat{a}_{\mathbf{k}}|0\rangle = 0\,. \tag{37}$$

Each mode $u_k(\eta)$ decouples in linear perturbation theory,

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0$$

This is a Schrödinger-like equation with potential $\mathcal{U}(\eta) = z''/z$.
We will use the slow-roll parameters (10),

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{\kappa^2}{2}\frac{z^2}{a^2} \simeq \text{const.} \ll 1\,, \tag{38}$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z} \simeq \text{const.} \ll 1\,. \tag{39}$$

In terms of these parameters, the conformal time and the effective potential for the u_k mode can be written as

$$\eta = \frac{-1}{\mathcal{H}}\frac{1}{1-\epsilon}\,, \tag{40}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right), \qquad \text{where} \quad \nu = \frac{1+\epsilon-\delta}{1-\epsilon} + \frac{1}{2}\,. \tag{41}$$

EXACT SOLUTIONS

In the slow-roll approximation, the effective potential is $z''/z \simeq 2\mathcal{H}^2$. The exact solutions are (Hankel: $H^{(1)}_{3/2}(x) = -e^{ix}\sqrt{2/\pi x}(1+i/x)$),

$$u_k(\eta) = \frac{\sqrt{\pi}}{2}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\,(-\eta)^{1/2}\, H^{(1)}_\nu(-k\eta)\,, \tag{42}$$

De Sitter Event horizon H^{-1} sets the physical scale,

$$\begin{aligned} u_k &= \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \qquad &&\text{(Minkowsky vacuum)}\\ u_k &= C_1\, z \qquad\qquad k \ll aH\,. \qquad &&\text{(superhorizon modes)}\end{aligned}$$

In the superhorizon limit $k\eta \to 0$, the solution becomes

$$|u_k| \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu}\,, \qquad \text{for} \quad \epsilon, \delta \ll 1\,. \tag{43}$$

$$\Phi_k = C_1(k)\left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right) + C_2(k)\,\frac{\mathcal{H}}{a^2}\,, \tag{44}$$

$$\delta\phi_k = \frac{C_1(k)}{a^2}\int a^2 d\eta - \frac{C_2(k)}{a^2}\,. \tag{45}$$

where $C_1(k) \Rightarrow$ growing solution, $C_2(k) \Rightarrow$ decaying solution.

For adiabatic perturbations, we can find a gauge invariant quantity that is also *constant* for superhorizon modes,

$$\zeta \equiv \Phi + \frac{1}{\epsilon\mathcal{H}}\,(\Phi' + \mathcal{H}\Phi) = \frac{u}{z} \quad \simeq \quad \mathcal{R}_c\,, \quad \text{for} \quad k \ll aH$$

$\mathcal{R}_c =$ gauge-invariant curvature perturbation on comoving hypersurfaces.

Can evaluate Φ_k when perturbation reenters the horizon during radiation/matter eras in terms of the curvature perturbation $\mathcal{R}_k$ when it left the Hubble scale during inflation,

$$\Phi_k = \left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}_k = \frac{3+3\omega}{5+3\omega}\,\mathcal{R}_k = \begin{cases} \frac{2}{3}\,\mathcal{R}_k & \text{radiation era}\,,\\ \frac{3}{5}\,\mathcal{R}_k & \text{matter era}\,.\end{cases}$$

GRAVITATIONAL WAVE PERTURBATIONS

The action for the tensor perturbation h_{ij} as quantum field

$$\delta S = \frac{1}{2}\int d^3x\, d\eta\, \frac{a^2}{2\kappa^2}\left[(h'_{ij})^2 - (\nabla h_{ij})^2\right], \tag{46}$$

$$\hat{h}_{ij}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1,2}\left[h_k(\eta)\, e_{ij}(\mathbf{k},\lambda)\, \hat{a}_{\mathbf{k},\lambda}\, e^{i\,\mathbf{k}\cdot\mathbf{x}} + h.c.\right], \tag{47}$$

where $e_{ij}(\mathbf{k},\lambda)$ are the two symmetric $e_{ij} = e_{ji}$, transverse $k^i e_{ij} = 0$, and traceless $e_{ii} = 0$ polarization tensors, $e_{ij}(-\mathbf{k},\lambda) = e^*_{ij}(\mathbf{k},\lambda)$, satisfying $\Sigma_\lambda\, e^*_{ij}(\mathbf{k},\lambda)e^{ij}(\mathbf{k},\lambda) = 4$. Redefine the gauge invariant tensor mode,

$$v_k(\eta) = \frac{a}{\sqrt{2}\kappa}\, h_k(\eta)\,, \tag{48}$$

satisfying decoupled evolution equations, in linear perturbation theory,

$$v''_k + \left(k^2 - \frac{a''}{a}\right) v_k = 0\,. \tag{49}$$

For constant slow-roll parameters, the potential becomes

$$\frac{a''}{a} = 2\mathcal{H}^2\left(1 - \frac{\epsilon}{2}\right) = \frac{1}{\eta^2}\left(\mu^2 - \frac{1}{4}\right), \qquad \mu = \frac{1}{1-\epsilon} + \frac{1}{2}\,.$$

We can solve equation (49) exactly in terms of the Hankel function (42) with $\nu \to \mu$. In the two asymptotic regimes,

$$v_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \qquad \text{(Minkowsky vacuum)}$$
$$v_k = C\, a \qquad k \ll aH\,. \qquad \text{(superhorizon modes)}$$

In the superhorizon limit $k\eta \to 0$, the solution becomes

$$|v_k| = \frac{1}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\mu}. \tag{50}$$

Since h_k is constant on superhorizon scales, we can evaluate the tensor metric perturbation when it reentered during the radiation/matter eras directly in terms of its value during inflation.

POWER SPECTRA OF SCALAR AND TENSOR METRIC PERTURBATIONS

Let us consider first the scalar (density) metric perturbations $\mathcal{R}_k$, which enter the horizon at $a = k/H$. Its two-point correlation function is given by

$$\langle 0|\mathcal{R}_k^*\mathcal{R}_{k'}|0\rangle = \frac{|u_k|^2}{z^2}\,\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (51)$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}\,\frac{|u_k|^2}{z^2} = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_S^2\left(\frac{k}{aH}\right)^{n-1} \quad (52)$$

This equation determines the power spectrum in terms of its amplitude at horizon-crossing, A_S, and a tilt,

$$n-1 \equiv \frac{d\ln\mathcal{P}_\mathcal{R}(k)}{d\ln k} = 3-2\nu = 2\left(\frac{\delta-2\epsilon}{1-\epsilon}\right) \simeq 2\eta - 6\epsilon\,, \quad (53)$$

It is possible, in principle, to obtain from inflation a scalar tilt which is either positive ($n > 1$) or negative ($n < 1$). Furthermore, depending on the particular inflationary model, we can have significant departures from scale invariance.

Let us consider now the tensor (gravitational wave) metric perturbation, which enter the horizon at $a = k/H$,

$$\sum_\lambda\langle 0|h_{k,\lambda}^* h_{k',\lambda}|0\rangle = \frac{8\kappa^2}{a^2}\,|v_k|^2\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}') \quad (54)$$

$$\mathcal{P}_g(k) = 8\kappa^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}\,. \quad (55)$$

Therefore, the power spectrum can be approximated by a power-law expression, with amplitude A_T and tilt

$$n_T \equiv \frac{d\ln\mathcal{P}_g(k)}{d\ln k} = 3-2\mu = -\left(\frac{2\epsilon}{1-\epsilon}\right) \simeq -2\epsilon < 0\,, \quad (56)$$

which is always negative. In the slow-roll approximation, $\epsilon \ll 1$, the tensor power spectrum is approximately scale invariant.

MASSLESS MINIMALLY COUPLED SCALAR FIELD FLUCTUATIONS DURING INFLATION

The fluctuations of a massless minimally-coupled scalar field ϕ during inflation (quasi de Sitter) are quantum fields in a curved background. We will redefine $y(\mathbf{x},t) = a(t)\,\delta\phi(\mathbf{x},t)$, whose action is

$$\mathcal{S} = \frac{1}{2}\int d^3\mathbf{x}\,d\eta\,\left[(y')^2 - (\nabla y)^2 + \frac{a''}{a}\,y^2\right]\,, \tag{1}$$

where primes denote derivatives w.r.t. conformal time $\eta = \int dt/a(t) = -1/(aH)$, with H the constant rate of expansion during inflation. Integrating by parts and defining the conjugate momentum as $p = \frac{\partial\mathcal{L}}{\partial y'} = y' - \frac{a'}{a}\,y$, we can write the action and the corresponding Hamiltonian as

$$\mathcal{S} = \frac{1}{2}\int d^3\mathbf{x}\,d\eta\,\left[p^2 - (\nabla y)^2\right]\,, \tag{2}$$

$$\mathcal{H} = \frac{1}{2}\int d^3\mathbf{x}\,\left[p^2 + (\nabla y)^2 + 2\,\frac{a'}{a}\,p\,y\right]\,. \tag{3}$$

We can now Fourier transform all the fields and momenta as:

$$\Phi(\mathbf{k},\eta) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\,\Phi(\mathbf{x},\eta)\,e^{-i\,\mathbf{x}\cdot\mathbf{k}}$$

Since the scalar field is assumed real, we have: $y(\mathbf{k},\eta) = y^\dagger(-\mathbf{k},\eta)$ and $p(\mathbf{k},\eta) = p^\dagger(-\mathbf{k},\eta)$. The Hamiltonian becomes

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\int d^3\mathbf{k}\;\;\Big[&p(\mathbf{k},\eta)\,p^\dagger(\mathbf{k},\eta) + k^2\,y(\mathbf{k},\eta)\,y^\dagger(\mathbf{k},\eta)\\ &+\frac{a'}{a}\,\Big(y(\mathbf{k},\eta)\,p^\dagger(\mathbf{k},\eta) + p(\mathbf{k},\eta)\,y^\dagger(\mathbf{k},\eta)\Big)\Big]\,.\end{aligned} \tag{4}$$

As we will see later, it is the last term which is responsible for squeezing.

The Euler-Lagrange equations for this field can be written in terms of the field eigenmodes as a series of uncoupled oscillator equations:

$$y''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right)\,y(\mathbf{k},\eta) = 0\,. \tag{5}$$

HEISENBERG PICTURE: THE FIELD OPERATORS

We can now treat each mode as a quantum oscillator, and introduce the corresponding creation and annihilation operators:

$$a(\mathbf{k},\eta) = \sqrt{\frac{k}{2}}\, y(\mathbf{k},\eta) + i\,\frac{1}{\sqrt{2k}}\, p(\mathbf{k},\eta)\,, \tag{6}$$

$$a^{\dagger}(-\mathbf{k},\eta) = \sqrt{\frac{k}{2}}\, y(\mathbf{k},\eta) - i\,\frac{1}{\sqrt{2k}}\, p(\mathbf{k},\eta)\,, \tag{7}$$

which can be inverted to give

$$y(\mathbf{k},\eta) = \frac{1}{\sqrt{2k}}\left[a(\mathbf{k},\eta) + a^{\dagger}(-\mathbf{k},\eta)\right], \tag{8}$$

$$p(\mathbf{k},\eta) = -i\sqrt{\frac{k}{2}}\left[a(\mathbf{k},\eta) - a^{\dagger}(-\mathbf{k},\eta)\right]. \tag{9}$$

The usual equal-time commutation relations for fields ($\hbar = 1$ here and throughout),

$$[y(\mathbf{x},\eta),\ p(\mathbf{x}',\eta)] = i\,\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{10}$$

becomes a commutation relation for the creation and annihilation operators,

$$[y(\mathbf{k},\eta),\ p^{\dagger}(\mathbf{k}',\eta)] = i\,\delta^3(\mathbf{k}-\mathbf{k}') \Rightarrow [a(\mathbf{k},\eta),\ a^{\dagger}(\mathbf{k}',\eta)] = \delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{11}$$

In terms of these operators, the Hamiltonian becomes:

$$\mathcal{H} = \frac{1}{2}\int d^3\mathbf{k}\ \ \Big[k\,[a(\mathbf{k},\eta)\,a^{\dagger}(\mathbf{k},\eta) + a^{\dagger}(\mathbf{k},\eta)\,a(\mathbf{k},\eta)] + i\,\frac{a'}{a}[a^{\dagger}(\mathbf{k},\eta)\,a^{\dagger}(\mathbf{k},\eta) + a(\mathbf{k},\eta)\,a(\mathbf{k},\eta)]\Big]. \tag{12}$$

It is the last (non-diagonal) term which is responsible for squeezing.

The evolution equation can be written as

$$\begin{pmatrix} a'(\mathbf{k}) \\ a^{\dagger\prime}(-\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix}\begin{pmatrix} a(\mathbf{k}) \\ a^{\dagger}(-\mathbf{k}) \end{pmatrix}, \tag{13}$$

whose general solution is, in terms of the initial conditions $a(\mathbf{k},\eta_0)$,

$$a(\mathbf{k},\eta) = u_k(\eta)\, a(\mathbf{k},\eta_0) + v_k(\eta)\, a^\dagger(-\mathbf{k},\eta_0)\,, \tag{14}$$

$$a^\dagger(-\mathbf{k},\eta) = u_k^*(\eta)\, a^\dagger(-\mathbf{k},\eta_0) + v_k^*(\eta)\, a(\mathbf{k},\eta_0)\,, \tag{15}$$

which correspond to a Bogoliubov transformation of the creation and annihilation operators, and characterizes the time evolution of the system of harmonic oscillators in the Heisenberg representation.

The commutation relation (11) is preserved under the unitary evolution if

$$|u_k(\eta)|^2 - |v_k(\eta)|^2 = 1\,, \tag{16}$$

which gives a normalization condition for these functions.

We can write the quantum fields y and p in terms of these as,

$$y(\mathbf{k},\eta) \;=\; f_k(\eta)\, a(\mathbf{k},\eta_0) + f_k^*(\eta)\, a^\dagger(-\mathbf{k},\eta_0)\,, \tag{17}$$

$$p(\mathbf{k},\eta) \;=\; -i\,\Big[g_k(\eta)\, a(\mathbf{k},\eta_0) - g_k^*(\eta)\, a^\dagger(-\mathbf{k},\eta_0)\Big]\,, \tag{18}$$

where the functions

$$f_k(\eta) \;=\; \frac{1}{\sqrt{2k}}\,[u_k(\eta) + v_k^*(\eta)]\,, \tag{19}$$

$$g_k(\eta) \;=\; \sqrt{\frac{k}{2}}\,[u_k(\eta) - v_k^*(\eta)]\,, \tag{20}$$

are the field and momentum modes, respectively, satisfying the following equations and initial conditions,

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0\,, \qquad f_k(\eta_0) = \tfrac{1}{\sqrt{2k}}\,, \tag{21}$$

$$g_k = i\left(f_k' - \frac{a'}{a}\, f_k\right)\,, \qquad g_k(\eta_0) = \sqrt{\tfrac{k}{2}}\,, \tag{22}$$

as well as the Wronskian condition,

$$i\,(f_k'\, f_k^* - f_k'^*\, f_k) = g_k\, f_k^* + g_k^*\, f_k = 1\,. \tag{23}$$

Since we have 2 complex functions, f_k and g_k, plus a constraint (23), we can write these in terms of 3 real functions in the standard parametrization for squeezed states,

$$u_k(\eta) = e^{-i\,\theta_k(\eta)} \cosh\, r_k(\eta)\,, \tag{24}$$
$$v_k(\eta) = e^{i\,\theta_k(\eta)+2i\,\phi_k(\eta)} \sinh\, r_k(\eta)\,, \tag{25}$$

where r_k is the squeezing parameter, ϕ_k the squeezing angle, and θ_k the phase.

We can also write its relation to the usual Bogoliubov formalism in terms of the functions $\{\alpha_k,\ \beta_k\}$,

$$u_k = \alpha_k\, e^{-ik\eta}\,, \qquad\qquad v_k^* = \beta_k\, e^{ik\eta}\,, \tag{26}$$

which is useful for the adiabatic expansion, and allows one to write the average number of particles and other quantities,

$$n_k = |\beta_k|^2 = |v_k|^2 = \frac{1}{2k}\,|g_k - k\, f_k|^2 = \sinh^2\, r_k\,, \tag{27}$$
$$\sigma_k = 2\mathrm{Re}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Re}\,(u_k^*\, v_k^*) = \cos 2\phi_k\, \sinh 2r_k\,, \tag{28}$$
$$\tau_k = 2\mathrm{Im}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Im}\,(u_k^*\, v_k^*) = -\sin 2\phi_k\, \sinh 2r_k\,. \tag{29}$$

We can invert these expressions to give $(r_k,\ \theta_k,\ \phi_k)$ as a function of u_k and v_k,

$$\sinh r_k = \sqrt{\mathrm{Re}v_k^2 + \mathrm{Im}v_k^2}\,, \qquad \cosh r_k = \sqrt{\mathrm{Re}u_k^2 + \mathrm{Im}u_k^2}\,, \tag{30}$$
$$\tan\theta_k = -\frac{\mathrm{Im}u_k}{\mathrm{Re}u_k}\,, \qquad \tan 2\phi_k = \frac{\mathrm{Im}v_k\mathrm{Re}u_k + \mathrm{Im}u_k\mathrm{Re}v_k}{\mathrm{Re}v_k\mathrm{Re}u_k - \mathrm{Im}u_k\mathrm{Im}v_k}\,. \tag{31}$$

We can now write Eqs. (17) and (18) in terms of the initial values,

$$y(\mathbf{k},\eta) = \sqrt{2k}\, f_{k1}(\eta)\, y(\mathbf{k},\eta_0) - \sqrt{\frac{2}{k}}\, f_{k2}(\eta)\, p(\mathbf{k},\eta_0)\,, \tag{32}$$
$$p(\mathbf{k},\eta) = \sqrt{\frac{2}{k}}\, g_{k1}(\eta)\, p(\mathbf{k},\eta_0) + \sqrt{2k}\, g_{k2}(\eta)\, y(\mathbf{k},\eta_0)\,, \tag{33}$$

where subindices 1 and 2 correspond to real and imaginary parts, $f_{k1} \equiv \mathrm{Re}\, f_k$ and $f_{k2} \equiv \mathrm{Im}\, f_k$, and similarly for the momentum mode g_k.

SCHRÖDINGER PICTURE: THE VACUUM WAVE FUNCTION

Let us go now from the Heisenger to the Schrödinger picture, and compute the initial state vacuum eigenfunction $\Psi_0(\eta = \eta_0)$. The initial vacuum state $|0, \eta_0\rangle$ is defined through the condition

$$\forall \mathbf{k}\,, \qquad \hat{a}(\mathbf{k}, \eta_0)|0, \eta_0\rangle = \left[\sqrt{\frac{k}{2}}\, \hat{y}_k(\eta_0) + i\, \frac{1}{\sqrt{2k}}\, \hat{p}_k(\eta_0)\right] |0, \eta_0\rangle = 0\,,$$

$$\left[y_k^0 + \frac{1}{k}\,\frac{\partial}{\partial y_k^{0*}}\right] \Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = 0 \;\Rightarrow\; \Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = N_0\, e^{-\frac{k}{2}|y_k^0|^2} \tag{34}$$

where we have used the position representation, $\hat{y}_k(\eta_0) = y_k^0\,,\; \hat{p}_k(\eta_0) = -i\,\frac{\partial}{\partial y_k^{0*}}$, and N_0 gives the corresponding normalization.

We will now study the time evolution of this initial wave function using the unitary evolution operator $S = S(\eta, \eta_0)$, i.e. the state evolves in the Schrödinger picture as $|0, \eta\rangle = S|0, \eta_0\rangle$. Now, inverting (17) and (18)

$$\hat{a}(\mathbf{k}, \eta_0) = g_k^*(\eta)\, \hat{y}(\mathbf{k}, \eta) + i\, f_k^*(\eta)\, \hat{p}(\mathbf{k}, \eta)\,, \tag{35}$$

which, acting on the initial state becomes, $\forall \mathbf{k}\,, \forall \eta$,

$$S\left[\hat{y}(\mathbf{k}, \eta) + i\, \frac{f_k^*(\eta)}{g_k^*(\eta)}\, \hat{p}(\mathbf{k}, \eta)\right] S^{-1} S|0, \eta_0\rangle = 0$$

$$\Rightarrow \qquad \left[\hat{y}_{\mathbf{k}}(\eta_0) + i\, \frac{f_k^*(\eta)}{g_k^*(\eta)}\, \hat{p}_{\mathbf{k}}(\eta_0)\right] |0, \eta\rangle = 0\,,$$

$$\Rightarrow \qquad \Psi_0\left(y_{\mathbf{k}}^0, y_{\mathbf{k}}^{0*}, \eta\right) = \frac{1}{\sqrt{\pi}\,|f_k(\eta)|}\, e^{-\frac{\gamma_k(\eta)}{2}|y_k^0|^2}\,, \tag{36}$$

where

$$\gamma_k(\eta) = \frac{g_k^*(\eta)}{f_k^*(\eta)} = k\,\frac{u_k^* - v_k}{u_k^* + v_k} = \frac{1 - 2i\, F_k(\eta)}{2|f_k(\eta)|^2}\,, \tag{37}$$

$$F_k(\eta) = \mathrm{Im}(f_k^*\, g_k) = \mathrm{Im}(u_k\, v_k) = \frac{1}{2}\, \sin 2\phi_k\, \sinh 2r_k\,. \tag{38}$$

We see that the unitary evolution preserves the Gaussian form of the wave functional. The wave function (36) is called a 2-mode squeezed state.

The normalized probability distribution,

$$P_0\left(y(\mathbf{k},\eta_0),y(-\mathbf{k},\eta_0),\eta\right) = \frac{1}{\pi\,|f_k(\eta)|^2}\,\exp\left(-\frac{|y(\mathbf{k},\eta_0)|^2}{2|f_k(\eta)|^2}\right)\,, \qquad (39)$$

is a Gaussian distribution, with dispersion given by $|f_k|^2$.

In fact, we can compute the vacum expectation values,

$$\langle\Delta y(\mathbf{k},\eta)\,\Delta y^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta y^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |f_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \qquad (40)$$

$$\langle\Delta p(\mathbf{k},\eta)\,\Delta p^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta p^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |g_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \qquad (41)$$

and therefore the Heisenberg uncertainty principle reads

$$\begin{aligned}\Delta y^2(k)\,\Delta p^2(k) = |f_k|^2\,|g_k|^2 &= F_k^2(\eta) + \frac{1}{4}\\ &= \frac{1}{4}\sin^2 2\phi_k\,\sinh^2 2r_k + \frac{1}{4}\,. \end{aligned} \qquad (42)$$

It is clear that for $\eta = \eta_0$, $\gamma_k(\eta_0) = k$ and $F_k(\eta_0) = 0$, and thus we have initially a minimum wave packet, $\Delta y\,\Delta p = \frac{1}{2}$. However, through its unitary evolution, the function F_k grows exponentially, see (38), and we find $\Delta y\,\Delta p \gg \frac{1}{2}$, corresponding to the semiclassical regime, as we will soon demonstrate rigorously.

THE SQUEEZING FORMALISM

Let us now use the squeezing formalism to describe the evolution of the wave function. The equations of motion for the squeezing parameters follow from those of the field and momentum modes,

$$r_k' = \frac{a'}{a}\cos 2\phi_k\,, \tag{43}$$

$$\phi_k' = -k - \frac{a'}{a}\coth 2r_k\,\sin 2\phi_k\,, \tag{44}$$

$$\theta_k' = k + \frac{a'}{a}\tanh 2r_k\,\sin 2\phi_k\,. \tag{45}$$

As we will see, the evolution is driven towards large $r_k \propto N \gg 1$, the number of e-folds during inflation. Thus, in that limit,

$$(\theta_k + \phi_k)' = -\frac{a'}{a}\frac{\sin 2\phi_k}{\sinh 2r_k} \to 0\,,$$

and therefore $\theta_k + \phi_k \to$ const. We can always choose this constant to be zero, so that the real and imaginary components of the field and momentum modes become

$$f_{k1} = \frac{1}{\sqrt{2k}}\,e^{r_k}\,\cos\phi_k\,, \qquad f_{k2} = \frac{1}{\sqrt{2k}}\,e^{-r_k}\,\sin\phi_k\,, \tag{46}$$

$$g_{k1} = \sqrt{\frac{k}{2}}\,e^{-r_k}\,\cos\phi_k\,, \qquad g_{k2} = \sqrt{\frac{k}{2}}\,e^{r_k}\,\sin\phi_k\,. \tag{47}$$

It is clear that, in the limit of large squeezing ($r_k \to \infty$), the field mode f_k becomes purely real, while the momentum mode g_k becomes pure imaginary.

This means that the field (32) and momentum (33) operators become, in that limit,

$$\left.\begin{array}{rcl} \hat{y}(\mathbf{k},\eta) & \to & \sqrt{2k}\,f_{k1}(\eta)\,\hat{y}(\mathbf{k},\eta_0) \\ \hat{p}(\mathbf{k},\eta) & \to & \sqrt{2k}\,g_{k2}(\eta)\,\hat{y}(\mathbf{k},\eta_0) \end{array}\right\} \Rightarrow \hat{p}(\mathbf{k},\eta) \simeq \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\,\hat{y}(\mathbf{k},\eta)\,. \tag{48}$$

As a consequence of this squeezing, information about the initial momentum $\hat{p}_0$ distribution is lost, and the positions (or field amplitudes) at different times commute,

$$[\hat{y}(\mathbf{k},\eta_1),\ \hat{y}(\mathbf{k},\eta_2)] \simeq \frac{1}{2}e^{-2r_k}\cos^2\phi_k \approx 0\,. \tag{49}$$

This result defines what is known as a quantum non-demolition (QND) variable, which means that one can perform succesive measurements of this variable with arbitrary precision without modifying the wave function. Note that $y = a\delta\phi$ is the amplitude of fluctuations produced during inflation, so what we have found is: first, that the amplitude is distributed as a classical Gaussian random field with probability (39); and second that we can measure its amplitude at any time, and as much as we like, without modifying the distribution function.

In a sense, this problem is similar to that of a free non-relativistic quantum particle, described initially by a minimum wave packet, with initial expectation values $\langle x\rangle_0 = x_0$ and $\langle p\rangle_0 = p_0$, which becomes broader by its unitary evolution, and at late times ($t \gg m x_0/p_0$) this Gaussian state becomes an exact WKB state,

$$\Psi(x) = \Omega_R^{-1/2}\,\exp(-\Omega\,x^2/2),$$

with $\mathrm{Im}\Omega \gg \mathrm{Re}\Omega$ (i.e. high squeezing limit). In that limit, $[\hat{x},\ \hat{p}] \approx 0$, and we have lost information about the initial position x_0 (instead of the initial momentum like in the inflationary case), $\hat{x}(t) \simeq \hat{p}(t)\,t/m = p_0\,t/m$ and $\hat{p}(t) = p_0$. Therefore, not only $[\hat{p}(t_1),\ \hat{p}(t_2)] = 0$, but also, at late times, $[\hat{x}(t_1),\ \hat{x}(t_2)] \approx 0$.

THE WIGNER FUNCTION

The Wigner function is the best candidate for a probability density of a quantum mechanical system in phase-space. Of course, we know from QM that such a probability distribution function cannot exist, but the Wigner function is just a good approximation to that distribution. Furthermore, for a Gaussian state, this function is in fact positive definite.

Consider a quantum state described by a density matrix $\hat{\rho}$. Then the Wigner function can be written as

$$W(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) = \int\int \frac{dx_1\, dx_2}{(2\pi)^2}\, e^{-i(p_1 x_1 + p_2 x_2)} \left\langle y - \frac{x}{2}, \eta \middle| \hat{\rho} \middle| y + \frac{x}{2}, \eta \right\rangle .$$

If we substitute for the state our vacuum initial condition $\hat{\rho} = |\Psi_0\rangle\langle\Psi_0|$, with Ψ_0 given by (36), we can perform the integration explicitly to obtain

$$\begin{aligned} W_0(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) &= \frac{1}{\pi^2} \exp\left(-\frac{|y|^2}{|f_k|^2} - 4|f_k|^2 \left| p - \frac{F_k}{|f_k|^2}\, y \right|^2 \right) \\ &\equiv \Phi(y_1,\, p_1)\, \Phi(y_2,\, p_2) \end{aligned} \tag{50}$$

$$\begin{aligned} \Phi(y_1,\, p_1) &= \frac{1}{\pi} \exp\left\{ - \left(\frac{y_1^2}{|f_k|^2} + 4|f_k|^2\, \bar{p}_1^2 \right) \right\} , \\ \bar{p}_1 &\equiv p_1 - \frac{F_k}{|f_k|^2}\, y_1 \,. \end{aligned} \tag{51}$$

In general, W_0 describes an asymmetric Gaussian in phase space, whose 2σ contours satisfy

$$\frac{y_1^2}{|f_k|^2} + 4|f_k|^2\, \bar{p}_1^2 \;\leq\; 1\,. \tag{52}$$

For instance, at time $\eta = \eta_0$, we have $y_1^0 = \frac{1}{\sqrt{2k}} = |f_k(\eta_0)|\,,\ p_1^0 = \sqrt{\frac{k}{2}} = 1/2|f_k(\eta_0)|$, and $F_k(\eta_0) = 0$, so that $\bar{p}_1^0 = p_1^0$, and the 2σ contours become

$$\frac{y_1^2}{y_1^{0\,2}} + \frac{p_1^2}{p_1^{0\,2}} \leq 1\,,$$

which is a circle in phase space.

On the other hand, for time $\eta \gg \eta_0$, we have

$$|f_k| \;\rightarrow\; \frac{1}{\sqrt{2k}}\, e^{r_k} \sim y_k^0\, e^N\,, \qquad \text{growing mode}\,, \tag{53}$$

$$\frac{1}{2|f_k|} \;\rightarrow\; \sqrt{\frac{k}{2}}\, e^{-r_k} \sim p_k^0\, e^{-N}\,, \qquad \text{decaying mode}\,, \tag{54}$$

so that the ellipse (52) becomes highly "squeezed".

Note that Liouville's theorem implies that the volume of phase space is conserved under Hamiltonian (unitary) evolution, so that the area within the ellipse should be conserved. As the probability distribution compresses (squeezes) along the p-direction, it expands along the y-direction. At late times, the Wigner function is highly concentrated around the region

$$\bar{p}^2 = \left(p - \frac{F_k}{|f_k|^2}\, y\right)^2 < \frac{1}{4|f_k|^2} \sim e^{-2N} \ll 1\,. \tag{55}$$

We can thus take the above *squeezing limit* in the Wigner function (50) and write the exponential term as a Dirac delta function,

$$W_0(y,p) \stackrel{r_k\to\infty}{\longrightarrow} \frac{1}{\pi^2}\, \exp\left\{-\frac{|y|^2}{|f_k|^2}\right\}\, \delta\left(p - \frac{F_k}{|f_k|^2}\, y\right)\,. \tag{56}$$

In this limit we have

$$\hat{p}_k(\eta) = \frac{F_k}{|f_k|^2}\, \hat{y}_k(\eta) \;\simeq\; \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\, \hat{y}_k(\eta)\,, \tag{57}$$

so we recover the previous result (48). This explains why we can treat the system as a classical Gaussian random field: the amplitude of the field y is uncertain with probability distribution (39), but once a measurement of y is performed, we can automatically asign to it a *definite* value of the momentum, according to (48).

Note that the condition $F_k^2 \gg 1$ is actually a condition between operators and their commutators/anticommutators. The Heisenberg uncertainty principle states that

$$\Delta_\Psi A \, \Delta_\Psi B \geq \frac{1}{2} \, |\langle\Psi|[A, \; B]|\Psi\rangle|,$$

for any two hermitian operators (observables) in the Hilbert space of the wave function Ψ. In our case, and in Fourier space, this corresponds to (42)

$$\Delta y^2(k) \, \Delta p^2(k) = F_k^2(\eta) + \frac{1}{4} \geq \frac{1}{4} \, |\langle\Psi|[y_k(\eta), \; p_k^\dagger(\eta)]|\Psi\rangle|^2 \,, \tag{58}$$

with $|\Psi\rangle = |0, \eta\rangle$ the evolved wave function.

On the other hand, the phase F_k can be written as

$$\begin{aligned} F_k &= -\frac{i}{2} \, (g_k \, f_k^* - f_k \, g_k^*) = -\frac{i}{2} \left(\frac{g_k}{f_k} \, |f_k|^2 - |f_k|^2 \, \frac{g_k^*}{f_k^*} \right) = \\ &= \frac{1}{2} \, \langle\Psi|p(\mathbf{k}, \eta) \, y^\dagger(\mathbf{k}, \eta) + y(\mathbf{k}, \eta) \, p^\dagger(\mathbf{k}, \eta)|\Psi\rangle \,, \end{aligned} \tag{59}$$

and we have used that, in the semiclassical limit, we can write $\langle\Psi||y_k(\eta)|^2|\Psi\rangle = |f_k|^2$, as well as $p(\mathbf{k}, \eta) = -\, i \, \frac{g_k}{f_k} \, y(\mathbf{k}, \eta)$, see (48).

The above relation just indicates that, for any state Ψ, the condition of classicality ($F_k \gg 1$) is satisfied whenever, for that state,

$$\{y_k(\eta), \; p_k^\dagger(\eta)\} \gg [y_k(\eta), \; p_k^\dagger(\eta)] \,,$$

which is an interesting condition.

MASSLESS SCALAR FIELD FLUCTUATIONS: TENSOR METRIC PERTURBATIONS

The gauge invariant tensor fluctuations (gravitational waves) act as a minimally-coupled massless scalar field during inflation, so we will study here the generation of its fluctuations during quasi de Sitter.

Let us consider here the exact solutions to the equation of motion of a minimally-coupled massless scalar field during inflation or quasi de Sitter, with scale factor $a = -1/H\eta$,

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right) , \tag{60}$$

$$g_k = i \left(f_k' - \frac{a'}{a} f_k\right) = \sqrt{\frac{k}{2}} e^{-ik\eta} , \tag{61}$$

which satisfy the Wronskian condition, $g_k f_k^* + g_k^* f_k = 1$. The eigenmodes become

$$u_k = e^{-ik\eta} \left(1 - \frac{i}{2k\eta}\right) = e^{-ik\eta - i\delta_k} \cosh r_k , \tag{62}$$

$$v_k = e^{ik\eta} \frac{i}{2k\eta} = e^{ik\eta + i\frac{\pi}{2}} \sinh r_k , \tag{63}$$

which comparing with (24) and (25) provides the squeezing parameter, the angle and the phase, as inflation proceeds towards $k\eta \to 0^-$,

$$\sinh r_k = \tan \delta_k = \frac{1}{2k\eta} \to -\infty , \tag{64}$$

$$\theta_k = k\eta + \arctan\frac{1}{2k\eta} \to -\frac{\pi}{2}, \qquad \phi_k = \frac{\pi}{4} - \frac{1}{2}\arctan\frac{1}{2k\eta} \to \frac{\pi}{2} , \tag{65}$$

while the imaginary part of the phase of the wave function becomes

$$F_k(\eta) = \frac{1}{2} \sin 2\phi_k \sinh 2r_k = \frac{1}{2k\eta} \to -\infty . \tag{66}$$

The number of scalar field particles produced during inflation grow exponentially, $n_k = |\beta_k|^2 = \sinh^2 r_k = (2k\eta)^{-2} \to \infty$.

Thus, through unitary evolution, the fluctuations will very soon enter the semiclasical regime due to a highly squeezed wave function. The question which remains is when do fluctuations become classical?

HUBLE CROSSING: SUPERHORIZON MODES

As we will see, the field fluctuation modes will become semiclassical as their wavelength becomes larger than the only physical scale in the problem, the de Sitter horizon scale, $\lambda_{\rm phys} = 2\pi a/k \gg H^{-1}$.

Therefore, let us consider the general solution to Eq. (21) for the superhorizon modes ($k \ll aH$),

$$f_k(\eta) = C_1(k)\, a + C_2(k)\, a \int^\eta \frac{d\eta'}{a^2(\eta')} = C_1(k)\, a - C_2(k)\, \frac{1}{a^2 H}\,. \qquad (67)$$

We can always choose $C_1(k)$ to be real, while $C_2(k)$ will be complex in general. The first term corresponds to the growing mode, while the second term is the decaying mode.

Integrating out g_k from (22), one finds

$$g_k(\eta) = i\, C_2(k)\, \frac{1}{a} - i\, C_1(k)\, k^2\, \frac{1}{a} \int a^2\, d\eta = i\, C_2(k)\, \frac{1}{a} - i\, C_1(k)\, \frac{k^2}{H}\,, \quad (68)$$

where we have added a k^2 term for completeness. To second order in k^2, the Wronskian becomes

$$C_1(k)\, {\rm Im} C_2(k) \left(1 + \frac{k^2}{a^2 H^2}\right) \simeq C_1(k)\, {\rm Im} C_2(k) = -\frac{1}{2}\,. \qquad (69)$$

Comparing with the exact solutions (60), we find, to first order,

$$C_1(k) = \frac{H_k}{\sqrt{2k^3}}\,, \qquad\qquad C_2(k) = -\frac{i\, k^{3/2}}{\sqrt{2}\, H_k}\,, \qquad (70)$$

where H_k is the Hubble rate at horizon crossing, $k\eta = -1$, i.e. when the perturbation's physical wavelength becomes of the same order as the de Sitter horizon size.

We are now prepared to answer the question of classicality of the modes. Let us compute the wave function phase shift

$$|F_k| = |\mathrm{Im}(f_k^*\, g_k)| \; = \; \Bigg| C_1^2(k)\,\frac{k^2 a}{H} + |C_2(k)|^2\,\frac{1}{a^3 H} \tag{71}$$

$$- \, C_1(k)\,\mathrm{Re}C_2(k)\,\left(1 + \frac{k^2}{a^2 H^2}\right)\Bigg| \,. \tag{72}$$

Since only the first term remains after $k\eta \to 0$, we see that $|F_k| \gg 1$ whenever

$$C_1^2(k) = \frac{H_k^2}{2k^3} \gg \frac{H}{k^2 a} \qquad \Rightarrow \qquad \lambda_{\rm phys} = \frac{2\pi a}{k} \gg \lambda_{\rm HC} = \frac{2\pi}{H}\,. \tag{73}$$

Therefore, we confirm that modes that start as Minkowsky vacuum well inside the de Sitter horizon are stretched by the expansion and become semiclassical soon after horizon crossing, and their amplitude can be described as a classical Gaussian random variable.

Furthermore, the fact that the momentum is immediately defined once the amplitude for a given wavelength is known, implies that there is a fixed temporal phase coherence for all perturbations with the same wavelength. As we know, this implies that inflationary perturbations will induce coherent acoustic oscillations in the plasma just before decoupling, which should be seen in the microwave background anisotropies as acoustic peaks in the angular power spectrum.

ANISOTROPIES OF THE MICROWAVE BACKGROUND

The Universe just before recombination is a very tightly coupled fluid, due to the large electromagnetic Thomson cross section. Photons scatter off charged particles (protons and electrons), and carry energy, so they feel the gravitational potential associated with the perturbations imprinted in the metric during inflation. An overdensity of baryons (protons and neutrons) does not collapse under the effect of gravity until it enters the causal Hubble radius. The perturbation continues to grow until radiation pressure opposes gravity and sets up acoustic oscillations in the plasma. Since overdensities of the same size will enter the Hubble radius at the same time, they will oscillate in phase. Moreover, since photons scatter off these baryons, the acoustic oscillations occur also in the photon field and induces a pattern of peaks in the temperature anisotropies in the sky, at different angular scales.

Three different effects determine the temperature anisotropies we observe in the microwave background:

Gravity: photons fall in and escape off gravitational potential wells, characterized by Φ in the comoving gauge, and as a consequence their frequency is gravitationally blue- or red-shifted, $\delta\nu/\nu = \Phi$. If the gravitational potential is not constant, the photons will escape from a larger or smaller potential well than they fell in, so their frequency is also blue- or red-shifted, a phenomenon known as the Rees-Sciama effect.

Pressure: photons scatter off baryons which fall into gravitational potential wells, and radiation pressure creates a restoring force inducing acoustic waves of compression and rarefaction.

Velocity: baryons accelerate as they fall into potential wells. They have minimum velocity at maximum compression and rarefaction. That is, their velocity wave is exactly $90°$ off-phase with the acoustic compression waves. These waves induce a Doppler effect on the frequency of the photons.

The temperature anisotropy induced by these three effects is therefore given by

$$\frac{\delta T}{T}(\mathbf{r}) = \Phi(\mathbf{r}, t_{\rm dec}) + 2\int_{t_{\rm dec}}^{t_0} \dot{\Phi}(\mathbf{r}, t)dt \; + \; \frac{1}{3}\frac{\delta\rho}{\rho}(\mathbf{r}, t_{\rm dec}) \; - \; \frac{\mathbf{r}\cdot\mathbf{v}}{c} \,. \qquad (1)$$

Metric perturbations of different wavelengths enter the horizon at different times. The largest wavelengths, of size comparable to our present horizon, are entering now. There are perturbations with wavelengths comparable to the size of the horizon at the time of last scattering, of projected size about 1° in the sky today, which entered precisely at decoupling. And there are perturbations with wavelengths much smaller than the size of the horizon at last scattering, that entered much earlier than decoupling, during the radiation era, which have gone through several acoustic oscillations before last scattering. All these perturbations of different wavelengths leave their imprint in the CMB anisotropies.

The baryons at the time of decoupling do not feel the gravitational attraction of perturbations with wavelength greater than the size of the horizon at last scattering, because of causality. Perturbations with exactly that wavelength are undergoing their first contraction, or acoustic compression, at decoupling. Those perturbations induce a large peak in the temperature anisotropies power spectrum. Perturbations with wavelengths smaller than these will have gone, after they entered the Hubble scale, through a series of acoustic compressions and rarefactions, which can be seen as secondary peaks in the power spectrum. Since the surface of last scattering is not a sharp discontinuity, but a region of $\Delta z \sim 100$, there will be scales for which photons, travelling from one energy concentration to another, will erase the perturbation on that scale, similarly to what neutrinos or HDM do for structure on small scales. That is the reason why we don't see all the acoustic oscillations with the same amplitude, but in fact they decay exponentialy towards smaller angular scales, an effect known as Silk damping, due to photon diffusion.

THE SACHS-WOLFE EFFECT

The anisotropies corresponding to large angular scales are only generated via gravitational red-shift and density perturbations through the Einstein equations, $\delta\rho/\rho = -2\Phi$ (for adiabatic perturbations); we can ignore the Doppler contribution, since the perturbation is non-causal. In that case, the temperature anisotropy in the sky today is given by

$$\frac{\delta T}{T}(\theta,\phi) = \frac{1}{3}\Phi(\eta_{\rm LS})\, Q(\eta_0,\theta,\phi) + 2\int_{\eta_{\rm LS}}^{\eta_0} dr\, \Phi'(\eta_0 - r)\, Q(r,\theta,\phi)\,, \quad (2)$$

where η_0 is the *coordinate distance* to the surface of last scattering, i.e. the present conformal time, while $\eta_{\rm LS} \simeq 0$ determines its comoving hypersurface. The Sachs-Wolfe effect (2) contains two parts, the intrinsic and the Integrated Sachs-Wolfe (ISW) effect, due to the integration along the line of sight of time variations in the gravitational potential.

In linear perturbation theory, the scalar metric perturbations can be separated into $\Phi(\eta,\mathbf{x}) \equiv \Phi(\eta)\, Q(\mathbf{x})$, where $Q(\mathbf{x})$ are the scalar harmonics, eigenfunctions of the Laplacian in three dimensions,

$$\nabla^2 Q_{klm}(r,\theta,\phi) = -k^2\, Q_{klm}(r,\theta,\phi).$$

These functions have the general form

$$Q_{klm}(r,\theta,\phi) = \Pi_{kl}(r)\, Y_{lm}(\theta,\phi)\,, \quad (3)$$

where $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics, and the radial parts can be written (in a flat Universe) in terms of spherical Bessel functions, $\Pi_{kl}(r) = \sqrt{\frac{2}{\pi}}\, k\, j_l(kr)$. On the other hand, the time evolution of the metric perturbation during the matter era is given by

$$\Phi'' + 3\mathcal{H}\,\Phi' + a^2\Lambda\,\Phi - 2K\,\Phi = 0\,. \quad (4)$$

In the case of a flat universe ($K = 0$) without cosmological constant, the Newtonian potential Φ remains constant during the matter era and only the intrinsic SW effect contributes to $\delta T/T$. In case of a non-vanishing Λ, since its contribution is negligible in the past, most of the photon's trajectory towards us is unperturbed. We will consider here the approximation $\Phi \simeq$ constant during the matter era.

The growing mode solution of the metric perturbation that left the Hubble scale during inflation contributes to the temperature anisotropies on large scales as

$$\frac{\delta T}{T}(\theta,\phi) \;=\; \frac{1}{3}\Phi(\eta_{\rm LS})\,Q \;=\; \frac{1}{5}\,\mathcal{R}\,Q(\eta_0,\theta,\phi) \;\equiv\; \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\,a_{lm}\,Y_{lm}(\theta,\phi)\,, \tag{5}$$

where we have used the fact that, at horizon reentry during the matter era, the gauge-invariant Newtonian potential $\Phi = \frac{3}{5}\mathcal{R}$ is related to the curvature perturbation $\mathcal{R}$ at Hubble-crossing during inflation.

We can now compute the two-point correlation function or angular power spectrum, $C(\theta)$, of the CMB anisotropies on large scales, defined as an expansion in multipole number,

$$C(\theta) = \Big\langle\frac{\delta T^*}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\Big\rangle_{\mathbf{n}\cdot\mathbf{n}'=\cos\theta} = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)\,C_l\,P_l(\cos\theta)\,, \tag{6}$$

where $P_l(z)$ are the Legendre polynomials, and we have averaged over different universe realizations. Since the coefficients a_{lm} are isotropic (to first order), we can compute the $C_l = \langle|a_{lm}|^2\rangle$ as

$$C_l^{(S)} = \frac{4\pi}{25}\int_0^\infty \frac{dk}{k}\,\mathcal{P}_{\mathcal{R}}(k)\,j_l^2(k\eta_0)\,. \tag{7}$$

In the case of scalar metric perturbation produced during inflation, the scalar power spectrum at reentry is given by $\mathcal{P}_{\mathcal{R}}(k) = A_S^2(k\eta_0)^{n-1}$, in the power-law approximation. In that case, one can integrate (7) to give

$$C_l^{(S)} = \frac{2\pi}{25}\,A_S^2\,\frac{\Gamma[\frac{3}{2}]\,\Gamma[1-\frac{n-1}{2}]\,\Gamma[l+\frac{n-1}{2}]}{\Gamma[\frac{3}{2}-\frac{n-1}{2}]\,\Gamma[l+2-\frac{n-1}{2}]}\,, \tag{8}$$

$$\frac{l(l+1)\,C_l^{(S)}}{2\pi} = \frac{A_S^2}{25} \;=\; \text{constant}\,, \qquad \text{for}\;\; n=1\,. \tag{9}$$

This last expression corresponds to what is known as the Sachs-Wolfe plateau, and is the reason why the coefficients C_l are always plotted multiplied by $l(l+1)$.

THE TENSOR PERTURBATION SACHS-WOLFE EFFECT

Tensor metric perturbations also contribute with an approximately constant angular power spectrum, $l(l+1)C_l$. The Sachs-Wolfe effect for a gauge-invariant tensor perturbation is given by

$$\frac{\delta T}{T}(\theta,\phi) = \int_{\eta_{\rm LS}}^{\eta_0} dr\, h'(\eta_0 - r)\, Q_{rr}(r,\theta,\phi)\,, \tag{10}$$

where Q_{rr} is the rr-component of the tensor harmonic along the line of sight. The tensor perturbation $h_k(\eta)$ during the matter era satisfies

$$h_k'' + 3\mathcal{H}\, h_k' + (k^2 + 2K)\, h_k = 0\,, \tag{11}$$

which depends on the wavenumber k, contrary to what happens with the scalar modes, see Eq. (4). For a flat ($K = 0$) universe, the solution to this equation is $h_k(\eta) = h\, G_k(\eta)$, where h is the constant tensor metric perturbation at horizon crossing and $G_k(\eta) = 3\, j_1(k\eta)/k\eta$, normalized so that $G_k(0) = 1$ at the surface of last scattering. The radial part of the tensor harmonic Q_{rr} in a flat universe can be written as

$$Q_{kl}^{rr}(r) = \left[\frac{(l-1)l(l+1)(l+2)}{\pi k^2}\right]^{1/2} \frac{j_l(kr)}{r^2}\,. \tag{12}$$

The tensor angular power spectrum can finally be expressed as

$$C_l^{(T)} = \frac{9\pi}{4}\,(l-1)l(l+1)(l+2)\, \int_0^\infty \frac{dk}{k}\, \mathcal{P}_g(k)\, I_{kl}^2\,, \tag{13}$$

$$I_{kl} = \int_0^{x_0} dx\, \frac{j_2(x_0 - x) j_l(x)}{(x_0 - x)x^2}\,, \tag{14}$$

where $x \equiv k\eta$, and $\mathcal{P}_g(k)$ is the primordial tensor spectrum. For a scale invariant spectrum, $n_T = 0$, we can integrate (13) to give

$$l(l+1)\, C_l^{(T)} = \frac{\pi}{36}(1 + \frac{48\pi^2}{385})\, A_T^2\, B_l\,, \tag{15}$$

with $B_l = (1.1184, 0.8789, \ldots, 1.00)$ for $l = 2, 3, \ldots, 30$. Therefore, $l(l+1)\, C_l^{(T)}$ also becomes constant for large l. Beyond $l \sim 30$, the Sachs-Wolfe expression is not a good approximation and the tensor angular power spectrum decays very quickly at large l.

THE CONSISTENCY CONDITION

In spite of the success of inflation in predicting a homogeneous and isotropic background on which to imprint a scale-invariant spectrum of inhomogeneities, it is difficult to test the idea of inflation. Before the 1980s anyone would have argued that *ad hoc* initial conditions could have been at the origin of the homogeneity and flatness of the universe on large scales, while most cosmologists would have agreed with Harrison and Zel'dovich that the most natural spectrum needed to explain the formation of structure was a scale-invariant spectrum. The surprise was that inflation incorporated an understanding of *both* the globally homogeneous and spatially flat background, and the approximately scale-invariant spectrum of perturbations in the same formalism. But that could have been a coincidence.

What is *unique* to inflation is the fact that inflation determines not just one but *two* primordial spectra, corresponding to the scalar (density) and tensor (gravitational waves) metric perturbations, from a *single* continuous function, the inflaton potential $V(\phi)$. In the slow-roll approximation, one determines, from $V(\phi)$, two continuous functions, $\mathcal{P}_\mathcal{R}(k)$ and $\mathcal{P}_g(k)$, that in the power-law approximation reduces to two amplitudes, A_S and A_T, and two tilts, n and n_T. It is clear that there must be a relation between the four parameters. Indeed, one can see from Eqs. (15) and (9) that the ratio of the tensor to scalar contribution to the angular power spectrum is proportional to the tensor tilt,

$$R \equiv \frac{C_l^{(T)}}{C_l^{(S)}} = \frac{25}{9}(1 + \frac{48\pi^2}{385})\, 2\epsilon \simeq -2\pi\, n_T\,. \tag{16}$$

This is a unique prediction of inflation, which could not have been postulated a priori. If we finally observe a tensor spectrum of anisotropies in the CMB, or a stochastic gravitational wave background in laser interferometers like LIGO or VIRGO, with sufficient accuracy to determine their spectral tilt, one might have some chance to test the idea of inflation, via the consistency relation (16).

For the moment, observations of the microwave background anisotropies suggest that the Sachs-Wolfe plateau exists, but it is still premature to determine the tensor contribution. Perhaps in the near future, from the analysis of polarization as well as temperature anisotropies, with the CMB satellites MAP and Planck, we might have a chance of determining the validity of the consistency relation.

Assuming that the scalar contribution dominates over the tensor on large scales, i.e. $R \ll 1$, one can actually give a measure of the amplitude of the scalar metric perturbation from the observations of the Sachs-Wolfe plateau in the angular power spectrum,

$$\left[\frac{l(l+1)\,C_l^{(S)}}{2\pi}\right]^{1/2} = \frac{A_S}{5} = (1.03 \pm 0.07) \times 10^{-5}\,, \tag{17}$$

$$n = 1.02 \pm 0.12\,. \tag{18}$$

These measurements can be used to normalize the primordial spectrum and determine the parameters of the model of inflation. In the near future these parameters will be determined with much better accuracy.

THE ACOUSTIC PEAKS

Before decoupling, the photons and the baryons are tightly coupled via Thomson scattering. The dynamis of the photon-baryon fluid is described by a forced and damped harmonic oscillator equation for the baryon density contrast,

$$\delta_k'' + \mathcal{H}\,\frac{R}{1+R}\,\delta_k' + k^2\,c_s^2\,\delta_k = F(\Phi_k)\,, \tag{19}$$

where $R = 3\rho_{\rm B}/4\rho_\gamma$ is the baryon-to-photon ratio, $c_s^2 = c^2/3(1+R)$ is the sound speed of the plasma, and $F(\Phi_k)$ is the external force due to the gravitational effect of dark matter and neutrinos. Baryons tend to collapse due to self-gravitation, while radiation pressure provides the restoring force, setting up acoustic oscillations in the plasma. Because of tight coupling, $\delta_k = 3\Theta_0(k,\eta)$, and the baryon oscillations give rise to oscillations in the temperature fluctuations Θ_0. The higher the baryon fraction R, the higher the amplitude of the oscillations. The external gravitational force displaces the zero-point of oscillations, which makes higher the amplitude of compressions versus rarefactions.

At decoupling there is a freeze out of the oscillations. The microwave background is like a snapshot of the instant of last scattering, where each mode k is at a different stage of oscillation,

$$\Theta_0(k,\eta) \propto (1+R)^{-1/4} \begin{cases} \Phi_k(0)\,\cos\,kr_s & \text{adiabatic}\,, \\ S_k(0)\,\sin\,kr_s & \text{isocurvature}\,, \end{cases} \tag{20}$$

where $r_s = \int_0^{\eta_{\rm dec}} c_s d\eta \simeq c_s\,\eta_{\rm dec}$ is the sound horizon at decoupling. These fluctuations induce acoustic peaks in the Angular Power Spectrum that correspond to maxima and minima of oscillations. For adiabatic and isocurvature perturbations, the harmonic peaks appear at wavenumber $k_n^{(A)} = n\pi/c_s\eta_{\rm dec}$ and $k_n^{(I)} = (n+1/2)\pi/c_s\eta_{\rm dec}$, respectively. In particular, the angle subtended by the sound horizon at decoupling, $\theta_s = r_s/d_A$, corresponds to a multipole number (e.g. for adiabatic perturbations)

$$l_n \approx \frac{n\pi}{\theta_s} = \frac{n\pi}{2c_s}\left(\frac{\Omega_{\rm M}}{|\Omega_{\rm K}|}\right)^{1/2} {\rm sinn}\int_0^{z_{\rm dec}} \frac{(1+z_{\rm dec})^{1/2}|\Omega_{\rm K}|^{1/2}dz}{[\Omega_\Lambda + \Omega_{\rm M}(1+z)^3 + \Omega_{\rm K}(1+z)^2]^{1/2}}$$

For the moment, observations of the microwave background anisotropies suggest that the Sachs-Wolfe plateau exists, but it is still premature to determine the tensor contribution. Perhaps in the near future, from the analysis of polarization as well as temperature anisotropies, with the CMB satellites MAP and Planck, we might have a chance of determining the validity of the consistency relation. Assuming that the scalar contribution dominates over the tensor on large scales, i.e. $r \ll 1$, one can actually give a measure of the amplitude of the scalar metric perturbation from the observations of the Sachs-Wolfe plateau in the angular power spectrum,

$$\left[\frac{l(l+1)\,C_l^{(S)}}{2\pi}\right]^{1/2} = \frac{A_S}{5} = (1.03 \pm 0.07) \times 10^{-5}\,, \tag{1}$$

$$n = 0.96 \pm 0.05\,. \tag{2}$$

These measurements can be used to normalize the primordial spectrum and determine the parameters of a particular model of inflation. In the near future these parameters will be determined with much better accuracy, to less than a percent.

In the next sections we will consider specific models of inflation. The formulae we will be using are

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 \qquad \eta = \frac{1}{\kappa^2}\left(\frac{V''}{V}\right) \tag{3}$$

$$N = \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} \tag{4}$$

together with the formula for the amplitude and tilt of scalar and tensor anisotropies

$$A_S = \frac{\kappa}{\sqrt{2\epsilon}}\frac{H}{2\pi}\,, \qquad n = 1 + 2\eta - 6\epsilon \tag{5}$$

$$A_T = \frac{\sqrt{2}}{\pi}\kappa H\,, \qquad n_T = -2\epsilon\,, \qquad r = -2\pi\, n_T \tag{6}$$

POWER-LAW INFLATION

$$V(\phi) = V_0 e^{-\beta\kappa\phi} \qquad \beta \ll 1 \quad \text{for} \quad \text{inflation} \tag{7}$$

$$3H^2(\phi) = \frac{2}{\kappa^2}\left(\frac{\partial H}{\partial\phi}\right)^2 + \kappa^2 V(\phi) \tag{8}$$

$$H(\phi) = H_0 e^{-\frac{1}{2}\beta\kappa\phi} \qquad \Rightarrow \qquad \frac{1}{H}\frac{\partial H}{\partial\phi} = -\frac{1}{2}\beta\kappa = \text{const} \tag{9}$$

$$H_0^2 = \frac{\kappa^2}{3}V_0\left(1 - \frac{\beta^2}{6}\right)^{-1} \qquad \text{where} \qquad V_0 \equiv M^4 \tag{10}$$

$$\epsilon \;=\; \frac{2}{\kappa^2}\left(\frac{H'}{H}\right)^2 = \frac{1}{2}\beta^2 < 1 \tag{11}$$

$$\delta \;=\; \frac{2}{\kappa^2}\left(\frac{H''}{H}\right)^2 = \frac{1}{2}\beta^2 < 1 \tag{12}$$

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\beta^2 \qquad \Rightarrow \qquad a \propto t^p \;\xrightarrow{\epsilon=1/p}\; p = \frac{2}{\beta^2} \tag{13}$$

$$\epsilon = \delta = \frac{1}{p} = \text{const} \tag{14}$$

$$N = \int_\phi^{\phi_{\rm end}} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \frac{\kappa}{\beta}(\phi_{\rm end} - \phi) = 65 \tag{15}$$

$$A_S = \frac{\kappa}{\sqrt{2\epsilon}}\frac{H}{2\pi} = 5\times 10^{-5} \qquad \Rightarrow \qquad M \simeq 10^{-3}M_P \tag{16}$$

$$n - 1 = 2\left(\frac{\delta - 2\epsilon}{1 - \epsilon}\right) = -\frac{2}{p-1} \tag{17}$$

$$|n-1| < 0.05 \qquad \Rightarrow \qquad p > 41 \tag{18}$$

$$n_T = -\frac{2\epsilon}{1-\epsilon} = -\frac{2}{p-1}\,, \qquad r = -2\pi\, n_T < \frac{\pi}{10} = 0.314 \tag{19}$$

CHAOTIC INFLATION ($m^2\phi^2$)

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad H^2 \simeq \frac{\kappa^2}{6}m^2\phi^2 \tag{20}$$

$$\epsilon \;=\; \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 = \frac{2}{\kappa^2\phi^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{M_P}{2\sqrt{\pi}} \simeq \frac{M_P}{3.5} \tag{21}$$

$$\eta \;=\; \frac{1}{\kappa^2}\left(\frac{V''}{V}\right) = \frac{2}{\kappa^2\phi^2} = \epsilon = \frac{1}{2N} \tag{22}$$

$$N \;=\; \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \left(\frac{\kappa\phi}{2}\right)^2\Bigg|_{\phi_{\rm end}}^{\phi} \approx \frac{\kappa^2\phi^2}{4} \quad \Rightarrow \quad \phi_{65} = 6.4 M_P \tag{23}$$

$$A_S = \frac{\kappa m}{\sqrt{6}}\,\frac{\kappa^2\phi^2}{4\pi} \;=\; N\sqrt{\frac{4}{3\pi}}\,\frac{m}{M_P} = 5\times 10^{-5} \quad \Rightarrow \tag{24}$$

$$m \;=\; 1.2\times 10^{-6} M_P = 1.4\times 10^{13}\,{\rm GeV} \tag{25}$$

$$n = 1 + 2\eta - 6\epsilon \;=\; 1 - \frac{2}{N} \approx 0.97 \tag{26}$$

$$A_T \;=\; \frac{4}{\sqrt{\pi}}\,\frac{H}{M_P} < 10^{-5} \tag{27}$$

$$n_T \;=\; -2\epsilon = -\frac{1}{N} \simeq -0.016 \tag{28}$$

$$r = \frac{C_l^T}{C_l^S} = \frac{25}{9}\left(1 + \frac{48\pi^2}{385}\right)2\epsilon \simeq -2\pi\, n_T = \frac{2\pi}{N} \simeq 0.1 \tag{29}$$

CHAOTIC INFLATION ($\lambda\phi^4$)

$$V(\phi) = \frac{1}{4}\lambda\phi^4 \quad \Rightarrow \quad H^2 \simeq \frac{\kappa^2}{12}\,\lambda\phi^4 \tag{30}$$

$$\epsilon \;=\; \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 = \frac{8}{\kappa^2\phi^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{M_P}{\sqrt{\pi}} \simeq \frac{M_P}{1.8} \tag{31}$$

$$\eta \;=\; \frac{1}{\kappa^2}\left(\frac{V''}{V}\right) = \frac{12}{\kappa^2\phi^2} = \frac{3\epsilon}{2} = \frac{3}{2N} \tag{32}$$

$$N \;=\; \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \left(\frac{\kappa\phi}{8}\right)^2\Big|_{\phi_{\rm end}}^{\phi} \approx \frac{\kappa^2\phi^2}{8} \quad \Rightarrow \quad \phi_{65} = 4.5 M_P \tag{33}$$

$$A_S = \sqrt{\frac{\lambda}{3}}\,\frac{\kappa^3\phi^3}{16\pi} \;=\; \sqrt{\frac{\lambda}{3}}\,\frac{(8N)^{3/2}}{16\pi} = 5\times10^{-5} \quad \Rightarrow \tag{34}$$

$$\lambda \;=\; 1.3\times10^{-13} \tag{35}$$

$$n = 1 + 2\eta - 6\epsilon \;=\; 1 - \frac{3}{N} \approx 0.95 \tag{36}$$

$$A_T \;=\; \frac{4}{\sqrt{\pi}}\,\frac{H}{M_P} < 10^{-5} \tag{37}$$

$$n_T \;=\; -2\epsilon = -\frac{2}{N} \simeq -0.03 \tag{38}$$

$$r = \frac{C_l^T}{C_l^S} = \frac{25}{9}\left(1 + \frac{48\pi^2}{385}\right)2\epsilon \simeq -2\pi\, n_T = \frac{4\pi}{N} \simeq 0.2 \tag{39}$$

NATURAL INFLATION

$$V(\phi) = M^2 f^2 \left(1 - \cos\frac{\phi}{f}\right) = 2M^2 f^2 \sin^2\frac{\phi}{2f} \tag{40}$$

$$= \frac{1}{2} M^2\phi^2 - \frac{1}{4!}\frac{M^2}{f^2}\phi^4 + \mathcal{O}(\phi^6) \tag{41}$$

$$\epsilon = \frac{1}{2\kappa^2 f^2}\left(\frac{\sin\frac{\phi}{f}}{1-\cos\frac{\phi}{f}}\right)^2 = \frac{\cot^2\frac{\phi}{2f}}{2\kappa^2 f^2} \ll 1 \tag{42}$$

$$\eta = \frac{1}{\kappa^2 f^2}\frac{\cos\frac{\phi}{f}}{1-\cos\frac{\phi}{f}} = \epsilon - \frac{1}{2\kappa^2 f^2} \ll 1 \tag{43}$$

$$N = 2\kappa^2 f^2 \int_{x_{\rm end}}^{x} dx\, \tan x = -2\kappa^2 f^2 \ln\cos\frac{\phi}{2f}\Big|_{\phi_{\rm end}}^{\phi} \tag{44}$$

$$\epsilon = 1 \qquad \Rightarrow \qquad \cos\frac{\phi_{\rm end}}{2f} = \left(\frac{2\kappa^2 f^2}{1+2\kappa^2 f^2}\right)^{1/2} < 1 \tag{45}$$

$$\cos\frac{\phi}{2f} = \left(\frac{2\kappa^2 f^2}{1+2\kappa^2 f^2}\right)^{1/2} e^{-\frac{N}{2\kappa^2 f^2}} \tag{46}$$

$$\epsilon_{65} = \frac{1}{2\kappa^2 f^2}\left(e^{\frac{N}{\kappa^2 f^2}} - 1\right)^{-1} \ll \frac{1}{2\kappa^2 f^2} \tag{47}$$

$$\eta_{65} = \epsilon_{65} - \frac{1}{2\kappa^2 f^2} \quad \Rightarrow \quad n \simeq 1 - \frac{1}{\kappa^2 f^2} \tag{48}$$

$$A_S = \sqrt{\frac{2}{3}}\frac{\kappa M}{2\pi}(2\kappa^2 f^2)\sinh\frac{N}{2\kappa^2 f^2} \tag{49}$$

$$\text{If} \quad f = M_P \Rightarrow M = 9\times 10^{-7} M_P = 10^{13}\,\text{GeV} \tag{50}$$

$$n = 1 - \frac{1}{8\pi} = 0.96 \tag{51}$$

$$r = -2\pi\, n_T = \frac{2\pi}{\kappa^2 f^2}\left(e^{\frac{N}{\kappa^2 f^2}} - 1\right)^{-1} \simeq 0.02 \tag{52}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R = \kappa^2\,\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{6M^2}\,{}^{(1)}H_{\mu\nu} + \frac{1}{H_0^2}\,{}^{(3)}H_{\mu\nu}\,, \tag{53}$$

$$ {}^{(1)}H_{\mu\nu} = 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2)R + 2R\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 \tag{54}$$

$$ {}^{(3)}H_{\mu\nu} = R_\mu^{\ \lambda}R_{\lambda\nu} - \frac{2}{3}R\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma} + \frac{1}{4}g_{\mu\nu}R^2 \tag{55}$$

Substituting FRW metric and using the Slow Roll Approximation,

$$\dot{H} = -\frac{M^2}{6}\left(1 - \frac{H^2}{H_0^2}\right)\,. \tag{56}$$

At first stage: $H_0^2 \gg M^2 \;\Rightarrow\; -\dot{H} < M^2/6 \ll H_0^2 \;\Rightarrow\; H \approx H_0 =$ const. However, H grows and becomes unstable. When $H \sim M$ inflation ends. Alternatively, one can study the evolution in the effective action formalism, including higher derivatives,

$$\mathcal{S}_g = \int d^4x\,\sqrt{-g}\,\frac{1}{2\kappa^2}\left(R - \frac{R^2}{6M^2}\right) \equiv \int d^4x\,\sqrt{-g}\,\frac{1}{2\kappa^2}\,f(R) \tag{57}$$

which gives rise to Eq. (54). One can then write this action as the usual Einstein-Hilbert action plus a scalar field, making use of the conformal transformation

$$\tilde{g}_{\mu\nu} = F(R)\,g_{\mu\nu} \equiv e^{\alpha\kappa\phi}g_{\mu\nu} \quad\Rightarrow\quad \sqrt{-\tilde{g}} = e^{2\alpha\kappa\phi}\sqrt{-g} \tag{58}$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{\alpha\kappa}{2}\left(g_{\mu\nu}\nabla^2\phi + 2\nabla_\mu\nabla_\nu\phi\right) \tag{59}$$

$$\tilde{R} = e^{-\alpha\kappa\phi}\left[R - 3\alpha\kappa\,\nabla^2\phi - \frac{3}{2}\alpha^2\kappa^2(\partial\phi)^2\right] \tag{60}$$

The scalar field ϕ will have canonical kinetic term for $\alpha^2 = 2/3$. From the equations of motion one finds the relationship $F(R) = f'(R)$, and therefore the effective scalar potential becomes

$$V(\phi) = \frac{1}{2\kappa^2}\,\frac{f(R) - R\,f'(R)}{(f'(R))^2} = \frac{R^2}{12\kappa^2M^2}\left(1 - \frac{R}{3M^2}\right)^{-2} \tag{61}$$

$$V(\phi) = \frac{3M^2}{4\kappa^2}\left(1 - e^{-\alpha\kappa\phi}\right)^2 = \frac{1}{2}M^2\phi^2\left(1 + \alpha\kappa\phi + \ldots\right) \tag{62}$$

$$\begin{aligned}
\epsilon &= \frac{2\alpha^2}{(e^{\alpha\kappa\phi}-1)^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{\sqrt{3}M_P}{4\sqrt{\pi}}\,\ln(1+\frac{2}{\sqrt{3}}) \simeq \frac{M_P}{5.33} \\
\eta &= \frac{2\alpha^2(2-e^{\alpha\kappa\phi})}{(e^{\alpha\kappa\phi}-1)^2} = 0 \quad \Rightarrow \quad \phi_* = \frac{\sqrt{3}M_P}{4\sqrt{\pi}}\,\ln 2 \simeq \frac{M_P}{5.90} < \phi_{\rm end} \\
N &= \left.\frac{e^{\alpha\kappa\phi}-\alpha\kappa\phi}{2\alpha^2}\right|_{\phi_{\rm end}}^{\phi} \simeq \frac{3}{4}\,e^{\alpha\kappa\phi} \quad \Rightarrow \quad \phi_{65} = 1.09M_P
\end{aligned} \tag{63}$$

$$\epsilon_{65} \simeq \frac{1}{2\alpha^2N^2} \qquad\qquad \eta_{65} \simeq -\frac{1}{N} \tag{64}$$

$$\alpha\kappa\phi_{65} = 4.46 \gg 1 \;\Rightarrow\; V(\phi_{65}) \simeq \frac{M^2}{2\alpha^2\kappa^2} \;\Rightarrow\; H_{65} \simeq \frac{M}{2} \tag{65}$$

$$A_S = \frac{\alpha N}{2\pi}\,\kappa H = 5\times 10^{-5} \quad \Rightarrow \quad M \simeq 2.4\times 10^{-6}M_P \tag{66}$$

$$n = 1 - \frac{2}{N} \simeq 0.97 \tag{67}$$

$$A_T = \frac{\sqrt{2}}{\pi}\frac{H}{M_P} = \frac{2}{\sqrt{\pi}}\frac{M}{M_P} = 2.7\times 10^{-6} \tag{68}$$

$$n_T = -2\epsilon \simeq -\frac{3}{2N^2} = -1.6\times 10^{-4} \tag{69}$$

$$r = -2\pi\, n_T \simeq 10^{-3} \tag{70}$$

HYBRID INFLATION

$$V(\phi,\chi) = \frac{\lambda}{4}(\chi^2 - v^2)^2 + \frac{1}{2}g^2\phi^2\chi^2 + \frac{1}{2}m^2\phi^2 \tag{71}$$

The effective Higgs mass in the false vacuum ($\chi = 0$):

$$m_\chi^2 \equiv \frac{\partial^2 V}{\partial \chi^2} = g^2\phi^2 - \lambda v^2 = 0 \quad \Rightarrow \quad \phi_c \equiv \frac{M}{g} = \frac{\sqrt{\lambda} v}{g} \tag{72}$$

For large values of the inflaton, the Higgs has a large mass and sits at its minimum, and therefore the ffective potential during inflation is

$$V(\phi) \;=\; V_0 + \frac{1}{2}m^2\phi^2 \equiv V_0(1+\xi) \simeq V_0 = \text{const.} \tag{73}$$

$$H_0 \;\simeq\; \sqrt{\frac{2\pi}{3}\frac{Mv}{M_P}} \tag{74}$$

$$\epsilon \;=\; \frac{m^2}{\kappa^2 V_0}\,\xi \;\ll\; \eta = \frac{m^2}{\kappa^2 V_0} \quad \Rightarrow \quad n = 1 + \frac{2m^2}{\kappa^2 V_0} > 1 \tag{75}$$

$$N \;=\; \frac{\kappa^2 V_0}{m^2}\ln\frac{\phi}{\phi_c} \quad \Rightarrow \quad \phi = \phi_c\, e^{\eta N} \tag{76}$$

Inflation ends not because of the end of slow-roll ($\epsilon = 1$) but because of symmetry breaking by the Higgs

$$A_S \;=\; \frac{H^2}{2\pi\dot{\phi}} = \frac{gH}{2\pi\eta\, M}\, e^{-\eta N} = 5\times 10^{-5} \quad \Rightarrow \tag{77}$$

$$g \;=\; \sqrt{\frac{3\pi}{8}}\,(n-1)\,10^{-4}\,\frac{M_P}{v}\, e^{-(n-1)\frac{N}{2}} \tag{78}$$

Negligible gravitational waves:

$$r = -2\pi\, n_T = 4\pi\epsilon \;\ll\; 2\pi\,(n-1) \tag{79}$$

Many possibilities of scales of inflation: e.g. GUT scale,

$$v \;=\; 10^{-3}M_P\,,\; \lambda = 0.1\,,\; g = 0.01\,, \quad \Rightarrow \quad n - 1 = 0.035\,, \tag{80}$$

$$M \;=\; 4\times 10^{15}\,\text{GeV}\,,\; m = 1.3\times 10^{12}\,\text{GeV}\,,\; r = 5\times 10^{-4} \tag{81}$$

RADIATIVE CORRECTIONS

Coleman-Weinberg potential

$$V_{1-\mathrm{loop}} = \frac{1}{64\pi^2}\sum_i (-1)^{F_i}\, m_i^4\, \ln\frac{m_i^2}{\Lambda^2} \tag{82}$$

Supergravity hybrid model (units $\kappa = 1$)

$$W = \sqrt{\lambda}\,\Phi(\bar\Sigma\Sigma - v^2) \tag{83}$$

$$V = \lambda\left|\bar\sigma\sigma - v^2\right|^2 + \frac{\lambda}{2}\phi^2(|\sigma|^2 + |\bar\sigma|^2) + \mathrm{D-term} \tag{84}$$

where $\phi = \sqrt{2}\,\Phi$ is the canonically normalized field. The absolute minimum appears at $\phi = 0,\ \sigma = \bar\sigma = v$. For $\phi > \phi_c = \sqrt{2}\,v$, the fields $\sigma, \bar\sigma$ have a positive mass squared and stay at the origin. Inflation takes place along that "flat" direction, which is lifted by radiative corrections. The masses of bosons are $m_B^2 = \frac{1}{2}\lambda(\phi^2 \pm 2v^2)$, while that of the fermion is $m_F^2 = \frac{1}{2}\lambda\phi^2$. The loop corrected potential along the flat direction is

$$\begin{aligned} V_{1-\mathrm{loop}}(\phi) &= \frac{\lambda^2}{128\pi^2}\left[(\phi^2 - 2v^2)^2\,\ln\left(\frac{\phi^2 - 2v^2}{\Lambda^2}\right)\right. \\ &+ \left.(\phi^2 + 2v^2)^2\,\ln\left(\frac{\phi^2 + 2v^2}{\Lambda^2}\right) - 2\phi^4\,\ln\left(\frac{\phi^2}{\Lambda^2}\right)\right] \end{aligned} \tag{85}$$

$$\Rightarrow\ V(\phi) \simeq \lambda\, v^4\left(1 + \frac{\lambda}{8\pi^2}\ln\frac{\phi}{\phi_c}\right), \qquad \phi \gg \phi_c \tag{86}$$

$$\epsilon = \frac{\lambda^2}{128\pi^4\phi^2} = \frac{\lambda}{32\pi^2 N}, \qquad \eta = -\frac{\lambda}{8\pi^2\phi^2} = -\frac{1}{2N} \tag{87}$$

$$N = \int\frac{d\phi}{\sqrt{2\epsilon}} = \frac{4\pi^2\phi^2}{\lambda} \tag{88}$$

$$A_S = \sqrt{\frac{N}{3}}\,16\pi\,\frac{v^2}{M_P^2} = 5\times 10^{-5} \quad\Rightarrow\quad v = 5.6\times 10^{15}\,\mathrm{GeV} \tag{89}$$

$$n = 1 - \frac{1}{N} = 0.98\,, \qquad r = -2\pi\, n_T = 4\pi\epsilon = \frac{\lambda}{8\pi N} \ll 1 \tag{90}$$

CERN

May 27-29, 2002

PREHEATING

I. Tkachev

CERN

Outline:

- Basics of inflation
- Particle creation in classical backgrounds
  - General Theory
  - Some analytical solutions
    - ⋆ Parametric resonance
    - ⋆ Parabolic cylinder functions
- Applications to cosmology
  - Gravitational particle creation in expanding Universe.
  - Efficiency of particle creation as function of coupling and mass. Hartree approximation.
  - Stochastic resonance
- Transition to classical regime.
  - General Theory.

- Lattice results.
  - Efficiency of particle creation
  - Non-thermal phase transitions.
- Thermalization. Turbulence.
- Fermions.
  - General Theory.
  - Comparison of Fermi and Bose cases.
- Some possible extra discussion
  - Possibility of baryogenesis.
  - Generation of gravitational waves.
  - Tachionic preheating.
  - Particle creation during inflation.

QFT in time-dependent background

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \,,$$

Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \,,$$

where

$$\pi(\mathbf{x}, t) = \frac{\delta\mathcal{L}}{\delta\dot{\phi}(\mathbf{x}, t)} = \dot{\phi}(\mathbf{x}, t) \,,$$

and

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y}) \,. \tag{1}$$

With

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3}\int d^3k\phi_{\mathbf{k}}(t)\mathrm{e}^{i\mathbf{k}\mathbf{x}} \,.$$

equation of motion reduces to

$$\ddot{\phi}_{\mathbf{k}} + \omega_k^2\phi_{\mathbf{k}} = 0 \,,$$

where

$$\omega_k^2 = \mathbf{k}^2 + m^2 \,.$$

Constraint $\phi_{\mathbf{k}} = \phi^*_{-\mathbf{k}}$ can be solved explicitly by

$$\phi_{\mathbf{k}}(t) \equiv \frac{(2\pi)^{3/2}}{\sqrt{2\omega_k}} \left(a_{\mathbf{k}}(t) + a^\dagger_{-\mathbf{k}}(t) \right) . \qquad (2)$$

Now we want to substitute the pair $\{\phi, \pi\}$ by the pair $\{a, a^\dagger\}$. Decomposition for π which complements (2) is

$$\pi(\mathbf{x}, t) = i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} (a^\dagger_{-\mathbf{k}} - a_{\mathbf{k}}) \mathrm{e}^{i\mathbf{kx}} , \qquad (3)$$

and canonical commuitation relations (1) will be satisfied if

$$\left[a_{\mathbf{k}}(t), a^\dagger_{\mathbf{p}}(t) \right] = \delta(\mathbf{k} - \mathbf{p}) .$$

The Hamiltonian in terms of the $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ operators becomes

$$H = \frac{1}{2} \int d^3k \omega_k (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}}) .$$

This can be written as $H \equiv H_{\mathrm{part}} + H_{\mathrm{vac}}(t)$, where

$$H_{\mathrm{part}} \equiv \int d^3k \omega_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} ,$$

$$H_{\mathrm{vac}}(t) \equiv \frac{V}{(2\pi)^3} \int d^3k \frac{\omega_k}{2} .$$

The Fock space

Let us introduce the vacuum state $|0_t\rangle$ which has the property

$$a_{\mathbf{k}}(t)|0_t\rangle = 0\,,$$

and $\langle 0_t|0_t\rangle = 1$. Here t is some specified (but arbitrary at this point) moment of time.

The state

$$|n_k\rangle = (a^\dagger_{\mathbf{k}})^{n_k}|0_t\rangle$$

can be interpreted as a state which contains n_k particles, each with energy ω_k. Indeed

$$H_{\mathrm{part}}|n_k\rangle = n_k\omega_k|n_k\rangle\,.$$

and

$$N = \int d^3p a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

counts the number of particles, $N|n_k\rangle = n_k|n_k\rangle$.

In the vacuum state, $|0_t\rangle$, the energy takes its lowest possible value at this moment of time

$$H_{\mathrm{vac}}(t) \equiv \langle 0_t|H|0_t\rangle\,.$$

All this procedure goes through even if ω is time dependent.

Equations of motion

$$\frac{da_{\mathbf{k}}}{dt} = \frac{\partial a_{\mathbf{k}}}{\partial t} + i[H, a_{\mathbf{k}}]\,. \tag{4}$$

Let us invert relations (2) and (3)

$$a_{\mathbf{k}} = \frac{1}{\sqrt{2}} \int \frac{d^3x}{(2\pi)^{3/2}}\, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left(\sqrt{\omega_k}\phi + i\frac{\pi}{\sqrt{\omega_k}} \right),$$

$$a^{\dagger}_{-\mathbf{k}} = \frac{1}{\sqrt{2}} \int \frac{d^3x}{(2\pi)^{3/2}}\, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left(\sqrt{\omega_k}\phi - i\frac{\pi}{\sqrt{\omega_k}} \right).$$

The original canonical variables $\{\phi, \pi\}$ do not have explicit time dependence, $\partial\phi/\partial t = \partial\pi/\partial t = 0$, but the ω_k can be time-dependent. We find

$$\frac{da_{\mathbf{k}}}{dt} = -i\omega_k a_{\mathbf{k}} + \frac{1}{2}\frac{\dot{\omega}_k}{\omega_k}\, a^{\dagger}_{-\mathbf{k}}\,. \tag{5}$$

We see that the solution of the equations of motion for operators can be parametrized as

$$\begin{aligned} a_{\mathbf{k}}(t) &= \alpha_k(t) a_{\mathbf{k}}(0) + \beta_k(t) a^{\dagger}_{-\mathbf{k}}(0)\,, \\ a^{\dagger}_{\mathbf{k}}(t) &= \alpha^*_k(t) a^{\dagger}_{\mathbf{k}}(0) + \beta^*_k(t) a_{-\mathbf{k}}(0)\,, \end{aligned} \tag{6}$$

with the initial conditions $\alpha_k(0) = 1$, $\beta_k(0) = 0$. The commutation relations should be satisfied at any moment of time, therefore α and β obey the constraint

$$|\alpha_k|^2 - |\beta_k|^2 = 1\,. \tag{7}$$

An immediate consequence of the relations (6) is that the system which was in vacuum initially, $a_{\mathbf{k}}(0)|0\rangle = 0$[1], will not remain in vacuum as the time goes by

$$a_{\mathbf{k}}(t)|0\rangle = \beta_k(t)a^{\dagger}_{-\mathbf{k}}(0)|0\rangle \neq 0$$

In particular, the number density of particles created from the vacuum is

$$n(t) = \frac{1}{V}\langle 0|N|0\rangle = \frac{1}{(2\pi)^3}\int d^3k|\beta_k(t)|^2 \,, \qquad (8)$$

To find these quantities explicitly, we need to know the functions $\beta_k(t)$. To get this function we need only to substitute Eq. (6) into Eq. (5), which gives

$$\begin{aligned} \dot{\alpha_k} &= -i\omega_k\alpha_k + \frac{1}{2}\frac{\dot{\omega_k}}{\omega_k}\,\beta_k^* \,, \\ \dot{\beta_k} &= -i\omega_k\beta_k + \frac{1}{2}\frac{\dot{\omega_k}}{\omega_k}\,\alpha_k^* \,. \end{aligned} \qquad (9)$$

With intial conditions $\alpha_k(0) = 1$, $\beta_k(0) = 0$ and the function $\omega_k(t)$ being given explicitly, one can directly solve this system of four ordinary differential equations (say on the computer).

This is it for the general theory !

---

[1]In situations like this, when the initial state of the system was specified to be the vacuum, I omit corresponding subscript for the vacuum vector, i.e. $|0\rangle \equiv |0_{t=0}\rangle$.

Adiabaticity condition

The number of particles created during the time $\Delta t \sim \omega_k^{-1}$ is

$$\left|\, \Delta |\beta_k|^2 \,\right| < \frac{1}{4} \left(\frac{\dot{\omega}_k}{\omega_k^2} \right)^2 \, .$$

The particle number is conserved approximately if

$$\left| \frac{\dot{\omega}_k}{\omega_k^2} \right| \ll 1 \, . \qquad (10)$$

Such approximately conserved quantities are called adiabatic invariants.

In and *out* states

One can do field decomposition over time independent operators as well

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(g_k(t)\, a_{\mathbf{k}}(0)\, \mathrm{e}^{i\mathbf{kx}} + \mathrm{h.c.}\right) \,.$$

Equation of motion for the mode functions

$$\ddot{g}_k + \omega_k^2 g_k = 0\,.$$

Comparing to decomposition of $\phi(\mathbf{x},t)$ over $a(t)$ we find immideately

$$\begin{aligned} \beta_k^* &= \frac{\omega_k g_k - i\dot{g}_k}{\sqrt{2\omega_k}}\,, \\ \alpha_k &= \frac{\omega_k g_k + i\dot{g}_k}{\sqrt{2\omega_k}}\,. \end{aligned} \tag{11}$$

This gives in particular

$$|\beta_k|^2 = \frac{|\dot{g}_k|^2 + \omega_k^2|g_k|^2}{2\omega} - \frac{1}{2}\,. \tag{12}$$

Diagonalization of the Hamiltonian

$$H = \int d^3k\,[\,E_k(t)\left(a^\dagger_{\mathbf{k}}(0)a_{\mathbf{k}}(0) + a_{\mathbf{k}}(0)a^\dagger_{\mathbf{k}}(0)\right)$$

$$+F_k(t)a_{\mathbf{k}}(0)a_{-\mathbf{k}}(0) + F^*_k(t)a^\dagger_{\mathbf{k}}(0)a_{-\mathbf{k}}(0)^\dagger\,]\,,$$

where

$$E_k(t) = \frac{1}{2}\left(|\dot{g}_k|^2 + \omega_k^2|g_k|^2\right)\;,$$

$$F_k(t) = \frac{1}{2}\left(\dot{g}_k^2 + \omega_k^2 g_k^2\right)\;.$$

Bogolyubov's transformation:

$$a_{\mathbf{k}} = \alpha_k b_{\mathbf{k}} + \beta_k b^\dagger_{-\mathbf{k}}\,,$$

$$a^\dagger_{\mathbf{k}} = \alpha^*_k b^\dagger_{\mathbf{k}} + \beta^*_k b_{-\mathbf{k}}\,.$$

$$|\beta_k|^2 = \frac{2E_k - \omega_k}{2\omega_k}\,. \qquad (13)$$

Examples. Parametric resonance.

$$V_{\rm int}(\phi, \varphi) = \frac{1}{2} g^2 \varphi^2 \phi^2 \,,$$

$$\varphi(t) = \varphi_0 \cos(Mt) \,,$$

$$\omega_k^2(t) = \mathbf{k}^2 + m^2 + \frac{1}{2} g^2 \varphi_0^2 + \frac{1}{2} g^2 \varphi_0^2 \cos(2Mt)$$

Equation for the mode functions can be reduced to the standard form of the Mathieu equation

$$g_{\mathbf{k}}'' + [A_k - 2q \cos 2z] g_{\mathbf{k}} = 0 \,, \tag{14}$$

where

$$q \equiv \frac{g^2 \varphi_0^2}{4M^2} \,,$$

$$A_k \equiv \frac{k^2 + m^2}{M^2} + 2q \,.$$

$$A > 2q \tag{15}$$

Examples.
Parabolic Cylinder Functions.

Analytical solutions of a large class of problems of particle creation in time varying background can be expressed in terms of the well studied parabolic cylinder functions. These are solutions of the equation

$$\frac{d^2y}{d\tau^2} + (\frac{1}{4}\tau^2 + \nu)y = 0\,. \tag{16}$$

0.1 Particle creation during "short" non-adiabatic interval.

Assume $\omega(t)$ goes through a minimum.

$$\omega_k^2(t) = \omega_k^2(t_*) + \frac{1}{2}\omega_k^2{}''(t_*)(t-t_*)^2 + \dots$$

Let us change the time variable to

$$\tau \equiv \left[2\omega_k^2{}''(t_*)\right]^{\frac{1}{4}}\,(t-t_*)\,.$$

Equation for mode functions reduces to Eq. (16) with

$$\nu_k \equiv \frac{\omega_k^2(t_*)}{\sqrt{2\omega_k^2{}''(t_*)}}\,.$$

The answer:

$$|\beta_k|^2 = \mathrm{e}^{-2\pi\nu_k}\,.$$

0.2 Gravitational particle production in an expanding Friedmann Universe

In conformal reference frame

$$ds^2 = a^2(t)(d\tau^2 - d\mathbf{x}^2)$$

$$\phi \equiv \chi/a$$

the frequency is

$$\omega_k^2 = k^2 + m^2a^2 - \frac{a''}{a}(1 - 6\xi)\,,$$

where ξ is coupling to curvature, $\frac{1}{2}\xi R\phi$.

In radiation dominated universe $a'' = 0$ and $a(\eta) = H_0\tau$ (assuming $a(\tau_0) = 1$). We again obtain equation for Parabolic Cylinder Functions with

$$\nu_k \equiv \frac{k^2}{2mH_0}\,.$$

Therefore

$$n = 1.495 \times 10^{-3}\,\frac{(mH_0)^{\frac{3}{2}}}{a^3}\,.$$

The adiabatic regime starts at $\tau = \tau_* = 1/\sqrt{mH_0}$. Therefore, particles are created when $H > m$. In general,

$$n = \frac{m^3}{a^3}C\,.$$

Schroedinger picture of the evolution

Find $U(t)$ such that

$$a_k(t) = U^\dagger(t) a_k(0) U(t)\,.$$

Solution of the Schroedinger equations of motion

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle\,.$$

(Note, $|\psi(t)\rangle$ is called Squeezed state.)
Clearly, vacuum at time t is given by

$$|0_t\rangle = U^\dagger(t)|0\rangle\,.$$

Schroedinger picture of the evolution

Since we know $a(t)$, we can also find

$$U a_k(0) U^\dagger = \alpha_k^*\, a_{\mathbf{k}}(0) - \beta_k a^\dagger_{-\mathbf{k}}(0)\,.$$

This product annihilates $|\psi(t)\rangle$. Substituting a for the field and the conjugate momenta gives

$$U a_k(0) U^\dagger = \frac{(\alpha_{\mathbf{k}}^* - \beta_{\mathbf{k}})\omega_k \phi_{\mathbf{k}} + i(\alpha_k^* + \beta_k)\pi_{\mathbf{k}}}{\sqrt{2\omega_k}}\,.$$

(note: $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ should be taken at the initial moment of time.)

Therefore, $|\psi(t)\rangle$ satisfies Schroedinger equation

$$(\Omega_k \phi_{\mathbf{k}} + i\pi_{\mathbf{k}})\,|\psi(t)\rangle = 0\,,$$

where

$$\Omega_k \equiv \frac{\alpha_k^* - \beta_k}{\alpha_k^* + \beta_k}\omega_k\,,$$

and

$$\pi_{\mathbf{k}} = -i\frac{\partial}{\partial \phi_{-\mathbf{k}}}\,,$$

This equation is easy to solve

$$\psi(\phi_{\mathbf{k}}, t) = \mathrm{e}^{-\Omega_k \phi_{-\mathbf{k}} \phi_{\mathbf{k}}} = \mathrm{e}^{-\Omega_k |\phi_{\mathbf{k}}|^2}\,.$$

In particular, this gives for the probability distribution of field values

$$P(\phi_{\mathbf{k}}, t) = |\psi(\phi_{\mathbf{k}}, t)|^2 = \mathrm{e}^{-|\phi_{\mathbf{k}}|^2/|g_k|^2}\,.$$

Fermions

Heisenberg equations of motion give

$$\dot{\alpha}_k = -i\omega_k \alpha_k + \frac{k\dot{m}}{2\omega_k^2}\beta_k^* \ ,$$

$$\dot{\beta}_k = -i\omega_k \beta_k - \frac{k\dot{m}}{2\omega_k^2}\alpha_k^* \ .$$

In terms of mode functions:

$$\ddot{U}_\pm + (\omega_k^2 \pm i\dot{m})U_\pm = 0\,.$$

we have

$$|\beta_k|^2 = \frac{\omega_k \pm m + \mathrm{Im}(U_\pm^* \dot{U}_\pm)}{2\omega_k} \ .$$

Examples. Coupling to the classical scalar field.

$$\mathcal{L}_Y = g\phi\bar{\psi}\psi.$$

The effective mass of the fermion field

$$m_{\rm eff}(t) = m_\psi + g\phi(t)\,.$$

Since creation occurs at $m_{\rm eff} = m_X + g\phi(\eta) = 0$, we can disregard details of evolution and write

$$m_{\rm eff} = g{\phi_*}'\,\eta$$

Equation for mode function reduces to

$$u'' + (p^2 - i + \tau^2)u = 0$$

where $p \equiv k/\sqrt{g\phi'_*}$, $\tau \equiv \eta\sqrt{g\phi'_*}$, and $' = d/d\tau$. Solutions are Parabolic Cylinder functions. Using its properties we find

$$n(k) = \exp\left(-\pi k^2/g\phi'_*\right)$$

E.g., for harmonic oscillations in flat space-time this gives

$$n(k) = \exp\left(\frac{-\pi k^2}{m_\phi^2\sqrt{4q - m_X^2/m_\phi^2}}\right)$$

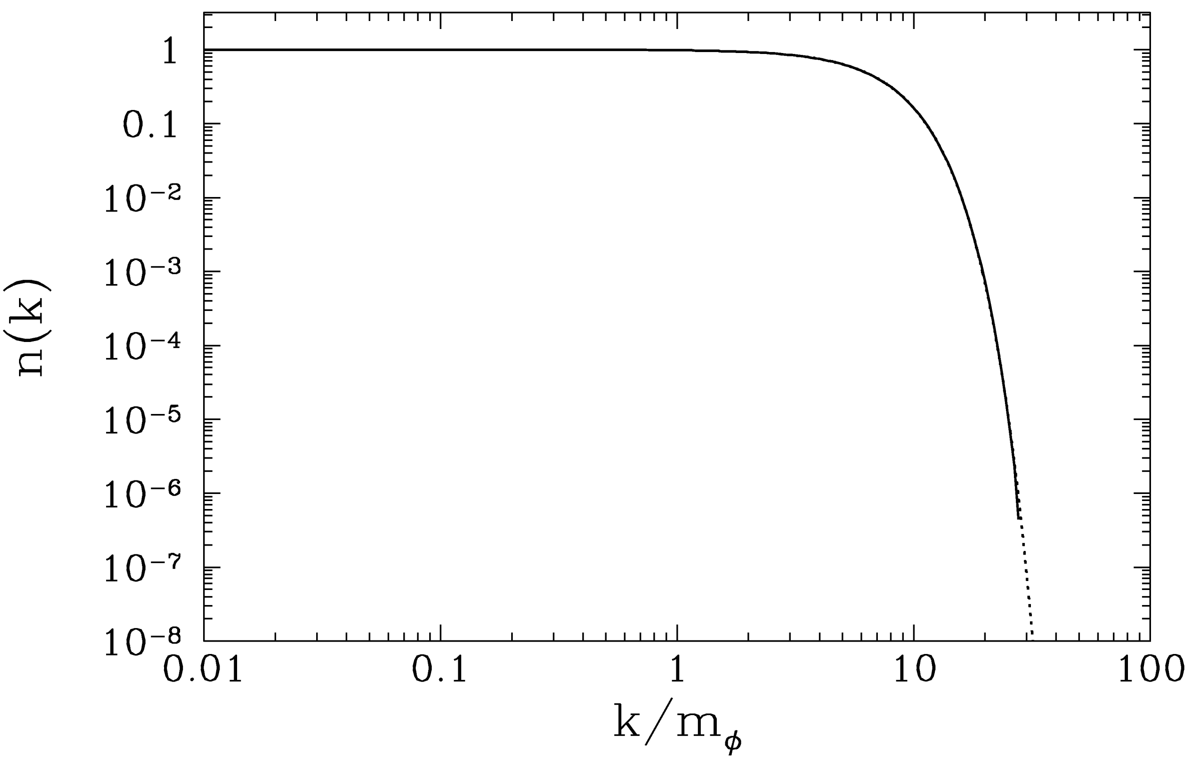

Figure 1: Solid line: numerical integration of complete problem with $q = 10^4$, $m_X/m_\phi = 100$. Dotted line: analytical approximation based on Parabolic Cylinder functions.

$$q \equiv \frac{g^2\phi^2(0)}{4m_\phi^2}\,.$$

REHEATING AFTER INFLATION

One of the fundamental quests of cosmology is to understand the origin of all the matter and radiation present in the universe today. We have seen how inflation produces a homogeneous and flat background space-time, and imprints on top of it a set of scalar and tensor quantum fluctuations that become classical Gaussian random fields outside the horizon, with an approximately scale invariant spectrum.

Inflation also dilutes any relic species left from a hypothetical earlier period of the universe, such that at the end of inflation there remains only a homogeneous zero mode of the inflaton field with tiny fluctuations on the homogeneous metric. That is, the universe is empty and very cold: the entropy of the universe is exponentially small and the temperature can be taken to be zero, $S = T = 0$.

Therefore we are left with the puzzle: How does the large entropy and energy of our present horizon, $S \sim 10^{89}$ and $M \sim 10^{23} M_\odot$, arise from such a cold and empty universe? The answer seems to lie in the process by which the large potential energy density present during inflation gets converted into radiation at the end of inflation, a process known as *reheating of the universe*.

This process was studied soon after the first models of inflation were proposed and considered the *perturbative* decay of the inflaton field into quanta of other fields to which it coupled, e.g. fermions, gauge fields, and other scalars. Such couplings exist during inflation but play no role (except for inducing radiative corrections, as we will discuss later), because even if those particles were produced during inflation the exponential expansion would dilute them almost instantaneously, and nothing would be left at the end of inflation.

Let us write down the most general Lagrangian with couplings of the inflaton to other fields and among themselves,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}\xi\chi^2 R \\ &+ \bar\psi(i\gamma^\mu\partial_\mu - m_\psi)\psi - h\phi\bar\psi\psi - \frac{1}{2}g^2\phi^2\chi^2 - g^2\sigma\phi\chi^2\,, \end{aligned} \tag{1}$$

where g, h, ξ, etc. are small couplings (to avoid large radiative corrections during inflation); σ is the possibly finite vev of the inflaton, and we have shifted the inflaton potential by $\phi - \sigma \to \phi$, such that the minimum is at $\phi = 0$ and the potential can be expanded around the minimum as

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \mathcal{O}(\phi^4)\,, \tag{2}$$

where m is the mass of the inflaton at the minimum. In chaotic inflation of the type $m^2\phi^2$ or $\lambda\phi^4$, this mass and self-coupling are bounded by observations of the CMB to be

$$m \sim 10^{13}\ \text{GeV}\,, \qquad \lambda \lesssim 10^{-13}\,. \tag{3}$$

We will consider this mass to be much larger than that of the other fields to which it couples: $m^2 \gg m_\chi^2, m_\psi^2 \gg g^2\sigma\phi, h\phi$. Also, the end of inflation occurs in these models when $H \sim m$, and subsequently, the rate of expansion decays as $H \sim 1/t < m$.

Let us compute the evolution of the inflaton after inflation, whose amplitude satisfies the equation (we are neglecting here the couplings to other fields, but we will consider them later)

$$\ddot{\phi} + 3H(t)\dot{\phi} + m^2\phi = 0\,, \tag{4}$$

whose solution is oscillatory,

$$\phi(t) = \Phi(t)\,\sin\,mt\,, \tag{5}$$

with the amplitude of oscillations decaying like $\Phi \sim a^{-3/2}$, as we will prove now. Consider the average energy density and pressure of the homogeneous inflaton field over one period of oscillations,

$$\langle\rho\rangle = \frac{1}{2}\langle\dot{\phi}^2\rangle + \frac{1}{2}m^2\langle\phi^2\rangle = \frac{1}{2}m^2\Phi^2(t)(\langle\cos^2 mt\rangle + \langle\sin^2 mt\rangle)\,, \tag{6}$$

$$\langle p\rangle = \frac{1}{2}\langle\dot{\phi}^2\rangle - \frac{1}{2}m^2\langle\phi^2\rangle = \frac{1}{2}m^2\Phi^2(t)(\langle\cos^2 mt\rangle - \langle\sin^2 mt\rangle) \simeq 0 \tag{7}$$

where we have neglected the change in $\Phi(t)$ due to the condition $m \gg H$ during reheating. The fact that an oscillating homogeneous scalar field behaves like a pressureless fluid means that the universe during that period expands like a matter dominated universe,

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad \Rightarrow \qquad \rho = \frac{1}{2}m^2\Phi^2(t) \sim a^{-3}\,, \tag{8}$$

and therefore $\Phi \sim a^{-3/2} \sim t^{-1}$. That is, a homogeneous scalar field oscillating with frequency equal to its mass can be considered as a coherent wave of ϕ particles with zero momenta and particle density

$$n_\phi = \rho_\phi/m = \frac{1}{2}m\Phi^2 \sim a^{-3}\,, \tag{9}$$

oscillating coherently with the same phase.

Until now we have considered only the effects of expansion, and ignored the effects due to the production of particles from the inflaton. This can be accounted for by including, in the equation of motion, the denominator of the QFT propagator,

$$\ddot{\phi} + 3H(t)\dot{\phi} + (m^2 + \Pi(\omega))\phi = 0\,, \tag{10}$$

where $\Pi(\omega)$ is the Minkowski space polarization operator for ϕ with four-momentum $k^\mu = (\omega, 0, 0, 0)$, where $\omega = m$. The real part of the polarization operator can be neglected (due to the small couplings), $\mathrm{Re}\,\Pi(\omega) \ll m^2$. However, due to phase space, if the frequency of oscillations satisfies $\omega \gg \min(2m_\chi, 2m_\psi)$, then the polarization operator acquires an imaginary part,

$$\mathrm{Im}\,\Pi(m) = m\,\Gamma_\phi\,, \tag{11}$$

where Γ_ϕ is the total decay rate of the inflaton, and we have used the optical theorem (i.e. unitarity) to relate both quantities at the physical pole, $\omega = m$.

The total decay rate can be written as a sum over partial decays,

$$\Gamma_\phi = \sum_i \Gamma(\phi \to \chi_i\chi_i) + \sum_i \Gamma(\phi \to \bar{\psi}_i\psi_i)\,, \tag{12}$$

$$\Gamma(\phi \to \chi_i\chi_i) = \frac{g_i^4 v^2}{8\pi m}\,, \qquad \Gamma(\phi \to \bar{\psi}_i\psi_i) = \frac{h_i^2 m}{8\pi}\,, \tag{13}$$

$$\Gamma_\phi \equiv \frac{h_{\rm eff}^2 m}{8\pi} \ll m\,, \qquad h_{\rm eff}^2 = \sum_i (h_i^2 + \frac{g_i^4 v^2}{m^2}) \tag{14}$$

The evolution of the inflaton during the period of oscillations after inflation can be described through the phenomenological equation

$$\ddot{\phi} + 3H(t)\dot{\phi} + \Gamma_\phi \dot{\phi} + m^2\phi = 0\,, \tag{15}$$

which includes the decay rate Γ_ϕ as a friction term giving rise to the damping of the oscillations due to inflaton particle decay. It assumes the inflaton condensate (the homogeneous zero mode) is composed of very many inflaton particles, each of these decaying into other particles to which it couples. The solution to this equation is given by (5) with

$$\Phi(t) = \Phi_0\, e^{-\frac{1}{2}\int 3H dt}\, e^{-\frac{1}{2}\Gamma_\phi t} = \frac{\Phi_0}{t}\, e^{-\frac{1}{2}\Gamma_\phi t}\,, \tag{16}$$

where we have used $H = 2/3t$.

We can now compute the evolution of the energy and number density of the inflaton field under the effect of particle production,

$$\frac{d}{dt}(\rho_\phi a^3) = -\Gamma_\phi\, \rho_\phi a^3\,, \tag{17}$$

$$\frac{d}{dt}(n_\phi a^3) = -\Gamma_\phi\, n_\phi a^3\,, \tag{18}$$

which simply states the usual exponential decay law for particles with decay rate Γ. Initially, the total decay rate is much smaller than the rate of expansion, $\Gamma_\phi \ll 3H = 2/t \ll m$, and the total comoving energy and total number of inflaton particles is conserved, their energy and number densities decaying like a matter fluid, $\rho_\phi \simeq m\, n_\phi \sim a^{-3}$.

Eventually, the universe expands sufficiently (this may take many many inflaton oscillations) that the decay rate becomes larger than the rate of expansion, or alternatively, the inflaton life-time, $\tau_\phi = \Gamma_\phi^{-1}$, becomes smaller than the age of the universe, $\tau_\phi < t_U = H^{-1}$, and the inflaton decays suddenly (in less than one Hubble time), releasing *all* its energy density ρ_ϕ into relativistic particles χ and ψ, in an exponential burst of energy. Subsequently, the produced particles interact among themselves and soon thermalize to a common temperature. This process is responsible for the present abundance of matter and radiation energy, and could be associated with the Big Bang of the "old" cosmology.

At first sight it may seem paradoxical that the universe may have to "wait" until it is old enough for the inflaton to decay, because we are accustomed to very rapid decays in our particle physics detectors, where life-times of order 10^{-17}s are possible, while our universe is 10^{+17}s old! However, if inflation took place at energy densities of order the GUT scale, the Hubble time of a causal domain at the end of inflation would be of order 10^{-35}s, which is many orders of magnitude smaller than even the fastest decays of the inflaton, $\sim 10^{-25}$s. So the probability that the inflaton decays in such a short Hubble time is negligible, and the universe has to wait until it is old enough that there is any probability of decay of a single inflaton particle. Eventually, of course, once the universe is older than the inflaton life-time, it (the inflaton) will decay exponentially fast due to its *constant* decay rate Γ_ϕ.

Let us now compute the reheating temperature of the universe that arises from the thermalization of the products of decay of the inflaton. Note that the process of reheating, once possible, is essentially instantaneous and therefore the energy density at reheating can be estimated as that corresponding to a rate of expansion $H = \Gamma_\phi$. Since all that energy density will be quickly converted into a plasma of relativistic particles, we can estimate

$$\rho(t_{\rm rh}) = \frac{3\Gamma_\phi^2 M_{\rm P}^2}{8\pi} = \frac{\pi^2}{30}\, g(T_{\rm rh})\, T_{\rm rh}^4\,, \tag{19}$$

$$\Rightarrow \qquad T_{\rm rh} \simeq 0.1\,\sqrt{\Gamma_\phi M_{\rm P}} \qquad , \tag{20}$$

where we have assumed $g(T_{\rm rh}) \sim 10^2 - 10^3$. Let us estimate this temperature. If we substitute $\Gamma_\phi = h_{\rm eff}^2 m/8\pi$ with $m \sim 10^{13}$ GeV, we find

$$T_{\rm rh} \simeq 2\times 10^{14}\, h_{\rm eff}\ {\rm GeV} \lesssim 10^{11}\ {\rm GeV}\,, \tag{21}$$

where we have imposed the constraint $h_{\rm eff} \lesssim 10^{-3}$ from radiative corrections in chaotic type models. Let us estimate it: if we consider the quantum loop corrections to the inflaton potential due to its coupling to other fields like in the Lagrangian described above, we find

$$V(\phi) = \frac{1}{2}m^2\phi^2\,(1 + \frac{3g^4}{16\pi^4\lambda}) + \frac{\lambda}{4}\phi^4\,(1 + \frac{3g^4}{16\pi^4\lambda} - \frac{h^4}{16\pi^4\lambda}) + \dots \tag{22}$$

Therefore, the couplings of the inflaton to other fields cannot be very large otherwise they would modify the amplitude of CMB anisotropies. If we impose that the mass and self-coupling of the inflaton satisfy (3), then the other couplings are bound to

$$3g^4\,,h^4 < 16\pi^2\lambda \qquad \Rightarrow \qquad g\,,h \lesssim 10^{-3}\,. \tag{23}$$

For completeness, let us mention that in theories with only gravitational interactions, like e.g. in Starobinsky model, the decay of the inflaton is induced via gravity only and

$$\Gamma_{\rm grav} \sim \frac{m^3}{M_{\rm P}^2} \sim 10^{-18}\,M_{\rm P} \qquad \Rightarrow \qquad T_{\rm rh} \sim 10^9\ {\rm GeV}\,. \tag{24}$$

All this indicates that, although the energy density at the end of inflation may be large, the *final* reheating temperature $T_{\rm rh}$ may not be higher than 10^{12} GeV, and thus the usually assumed thermal phase transition at Grand Unification, which was the basis for most of the early universe phenomenology, like production of topological defects, GUT baryogenesis, etc., could not have taken place.

We will see shortly that such phenomenology may be resucitated in the context of preheating and non-thermal phase transitions, but for the moment let us focus our attention onto two well differentiated and concrete cases of ordinary reheating:

Reheating in chaotic inflation models

Consider a $m^2\phi^2$ model of inflation, for which the value of the inflaton at the end of inflation is $\phi_{\rm end} = M_{\rm P}/2\sqrt{\pi}$, and the corresponding energy density

$$\rho_{\rm end} = \frac{3}{2}V(\phi_{\rm end}) = \frac{3m^2M_{\rm P}^2}{16\pi} = (6.5\times 10^{15}\ {\rm GeV})^4\,. \tag{25}$$

On the other hand, CMB anisotropies require

$$A_S = N\sqrt{\frac{4}{3\pi}\frac{m}{M_{\rm P}}} = 5\times 10^{-5} \qquad \Rightarrow \qquad m \simeq 1.4\times 10^{13}\ {\rm GeV}\,, \tag{26}$$

while radiative corrections impose the constraint $h_{\rm eff} \lesssim 10^{-3}$.

We are thus left with three time scales:

$$\left.\begin{array}{r} t_{\rm osc} \sim m^{-1} \sim 10^{-36}\,{\rm s} \\ t_{\rm exp} \gtrsim H_{\rm end}^{-1} \sim 10^{-35}\,{\rm s} \\ t_{\rm dec} \sim \Gamma_\phi^{-1} \sim 10^{-25}\,{\rm s} \end{array}\right\} \Rightarrow \qquad t_{\rm osc} \ll t_{\rm exp} \ll t_{\rm dec}\,, \tag{27}$$

so there are several oscillations per Hubble time, and we also expect many oscillations of the inflaton field before it decays. This result is typical of most high-scale models of inflation.

Reheating in low-scale hybrid inflation models

In this case, reheating occurs in very different circumstances. Most models of inflation occur at scales of order the GUT scale, because their parameters are fixed by the amplitude of CMB anisotropies, $\delta T/T \sim m/M_{\rm P} \sim 10^{-5}$. However, in models of hybrid inflation, which end due to the symmetry breaking of a field coupled to the inflaton, and not because of the end of slow-roll, it is possible to decouple the amplitude of CMB fluctuations from the scale of inflation. For instance, consider a hybrid model at the electroweak scale, where the symmetry breaking field is the SM Higgs field, with a vev $v = 246$ GeV, a relatively strong coupling to the inflaton, $g = 0.4$, and a Higgs self-coupling $\lambda = 0.12$, giving rise to the following masses in the true vacuum

$$m_{\rm inf} = g\,v \sim 100 \text{ GeV}\,, \qquad\qquad m_{_{\rm H}} = \sqrt{2\lambda}\,v \sim 120 \text{ GeV}\,, \tag{28}$$

which are much larger than the rate of expansion at the end of inflation

$$H_{\rm end} = \sqrt{\frac{\pi}{3}\,\frac{m_{_{\rm H}} v}{M_{\rm P}}} \sim 2\times 10^{-5} \text{ eV} \ll m_{_{\rm H}}\,, \tag{29}$$

and therefore we can neglect it during the oscillations of the inflaton and Higgs fields around the minimum of their potential.

The energy density at the end of inflation is

$$\rho_{\rm end} = \frac{1}{8} m_{_{\rm H}}^2 v^2 \sim (10^2 \text{ GeV})^4\,, \tag{30}$$

which is very low indeed.

The couplings of the Higgs to matter could be large, e.g. the top quark Yukawa $h_t \sim 1$, although for such a low mass Higgs there is no phase space for top perturbative production. On the other hand, the inflaton may couple to other particles, so it is expected that their decay widths be similar and both of order $\Gamma \sim 1$ GeV. Naively, using (20), one would thus expect that the reheating temperature be $T_{\rm rh} \sim 10^9$ GeV, but that is impossible because it would correspond to an energy density during inflation much above the actual false vacuum energy, $\rho_{\rm end} \sim (10^2 \text{ GeV})^4$.

Actually, since the rate of expansion is so low compared with the other scales, we can ignore the decay in energy due to the expansion of the universe, which was so important during chaotic inflation, and use energy conservation to estimate

$$\rho_{\rm end} = \frac{\lambda v^4}{4} = \frac{\pi^2}{30} g_* T_{\rm rh}^4 \quad \Rightarrow \quad T_{\rm rh} \simeq \left(\frac{15\lambda}{2\pi^2 g_*}\right)^{1/4} v \sim 42 \text{ GeV}\,, \qquad (31)$$

where we have used $g_* = 106.75$ as the effective number of degrees of freedom of the SM particles. Note that this temperature is rather low, but in fact we have no observational evidence that the universe has actually gone through a thermal period with a temperature above this.

We are thus left with three time scales:

$$\left.\begin{aligned} t_{\rm osc} &\sim m_{\rm H}^{-1} \sim 10^{-27}\,\text{s} \\ t_{\rm exp} &\sim H^{-1} \sim 10^{-10}\,\text{s} \\ t_{\rm dec} &\sim \Gamma^{-1} \sim 10^{-23}\,\text{s} \end{aligned}\right\} \Rightarrow \qquad t_{\rm osc} \ll t_{\rm dec} \ll t_{\rm exp}\,, \qquad (32)$$

so there are many oscillations per Hubble time, but contrary to the case of chaotic inflation models, here the decay time is much smaller than the expansion time, because the universe is already quite old, so once the inflaton and Higgs start oscillating they decay very soon via their usual perturbative decay.

PREHEATING

The previous discussion falls under the name of *perturbative reheating*, because it assumes that the coherently oscillating inflaton will decay as if it were composed on individual inflaton quanta, each one decaying as described by ordinary QFT, with the perturbative decay rate computed above. This was the standard lore during at least a decade since it was first proposed in 1982. However, it was soon realized that the inflaton at the end of inflation is actually a coherent wave, a zero mode, a condensate made out of many inflaton quanta, all oscillating with the same phase, and non-perturbative effects associated with this condensate were bound to be important for the problem of reheating. In fact, a few years ago, in a seminal paper, Linde, Kofman and Starobinsky proposed a new picture of reheating, which has become known as *preheating*. I will describe these new developments in the following sections. They make use of the well studied problem of particle production in the presence of strong background fields, whose formalism we have already encountered for the analysis of the generation of metric fluctuations during inflation. In this case, instead of a quantum field evolving in a rapidly changing gravitational field (like during inflation), we have a field coupled to the inflaton, which has a rapidly changing frequency or mass due to the inflaton oscillations.

We will first describe the Bogolyubov formalism for a single scalar field with a time-dependent mass and then particularize to the case of the inflaton oscillations after inflation. Later on, we will also extended the formalism to fermions, which can also be produced at preheating.

Consider a massive scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\dot{\phi} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\,, \tag{33}$$

which gives a canonically conjugate momentum $\pi = \dfrac{\delta\mathcal{L}}{\delta\dot{\phi}} = \dot{\phi}$, and the Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,. \tag{34}$$

We can treat the fields as quantum fields and define the usual equal time commutation relation

$$[\phi(\mathbf{x},t),\ \pi(\mathbf{x}',t)] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,, \tag{35}$$

as well as expand in Fourier components,

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\phi_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{36}$$

The field mode $\phi_{\mathbf{k}}(t)$ satisfies the harmonic oscillator equation

$$\ddot{\phi}_k + \omega_k^2\,\phi_k = 0\,, \tag{37}$$

$$\omega_k^2(t) = k^2 + m^2(t)\,, \tag{38}$$

where the time dependence of the oscillation frequency comes through that of the mass. We will assume that the field is real, so we should impose the constraint $\phi_{\mathbf{k}}(t) = \phi^*_{-\mathbf{k}}(t)$. Following the quantization condition (35), we can write the field and momentum operators in terms of time-dependent creation and annihilation operators,

$$\begin{aligned}\phi_{\mathbf{k}}(t) &= \frac{1}{\sqrt{2\omega_k}}\Big(a_{\mathbf{k}}(t) + a^\dagger_{-\mathbf{k}}(t)\Big)\,,\\ \pi_{\mathbf{k}}(t) &= -i\sqrt{\frac{\omega_k}{2}}\Big(a_{\mathbf{k}}(t) + a^\dagger_{-\mathbf{k}}(t)\Big)\,,\end{aligned} \tag{39}$$

satisfying the usual commutation relation, $\forall t$,

$$[a_{\mathbf{k}}(t),\ a^\dagger_{\mathbf{k}'}(t)] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,$$

and in terms of which the Hamiltonian becomes

$$\begin{aligned}H &= \frac{1}{2}\int d^3k\,[\pi_{\mathbf{k}}\pi^\dagger_{\mathbf{k}} + \omega_k^2\,\phi_{\mathbf{k}}\phi^\dagger_{\mathbf{k}}]\\ &= \frac{1}{2}\int d^3k\,\omega_k\,(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}}) \equiv H_{\rm part} + H_{\rm vac}(t)\,,\end{aligned} \tag{40}$$

where

$$H_{\rm part} = \int d^3k\,\omega_k\,a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\,, \tag{41}$$

$$H_{\rm vac}(t) = \frac{V}{(2\pi)^3}\int d^3k\,\frac{\omega_k}{2}\,. \tag{42}$$

We can then define a number operator for these fields

$$N = \int d^3k\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\,, \tag{43}$$

and a Fock space with vacuum state defined as

$$a_{\mathbf{k}}(t)|0_t\rangle = 0\,, \qquad\qquad \langle 0_t|0_t\rangle = 1\,, \tag{44}$$

and particle states $|n_k\rangle \propto (a^\dagger_{\mathbf{k}})^n|0_t\rangle$ satisfying

$$H_{\rm part}|n_k\rangle \;=\; n_k\omega_k|n_k\rangle \equiv E_k|n_k\rangle\,, \tag{45}$$

$$N|n_k\rangle \;=\; n_k|n_k\rangle\,. \tag{46}$$

In the vacuum state $|0_t\rangle$, the energy takes its lowest possible value, $H_{\rm vac}(t) = \langle 0_t|H|0_t\rangle$.

We can compute the equations of motion as usual with

$$\frac{d}{dt}a_{\mathbf{k}} = \frac{\partial a_{\mathbf{k}}}{\partial t} + i\,[H, a_{\mathbf{k}}]$$

where we can invert the relations (39)

$$\begin{aligned} a_{\mathbf{k}}(t) &= \sqrt{\frac{\omega_k}{2}}\phi_{\mathbf{k}}(t) + \frac{i}{\sqrt{2\omega_k}}\pi_{\mathbf{k}}(t)\,, \\ a^\dagger_{-\mathbf{k}}(t) &= \sqrt{\frac{\omega_k}{2}}\phi_{\mathbf{k}}(t) - \frac{i}{\sqrt{2\omega_k}}\pi_{\mathbf{k}}(t)\,. \end{aligned} \tag{47}$$

In the Heisenberg picture, the original canonical operators $\{\phi_k, \pi_k\}$ may have no explicit time-dependence, but ω_k is indeed time-dependent, so

$$\frac{d}{dt}a_{\mathbf{k}} = -i\omega_k a_{\mathbf{k}} + \frac{\dot\omega_k}{2\omega_k}a^\dagger_{-\mathbf{k}}\,. \tag{48}$$

The solution to the equations of motion is

$$\begin{pmatrix} a_{\mathbf{k}}(t) \\ a^\dagger_{-\mathbf{k}}(t) \end{pmatrix} = \begin{pmatrix} u_k(t) & v_k(t) \\ v^*_k(t) & u^*_k(t) \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}}(0) \\ a^\dagger_{-\mathbf{k}}(0) \end{pmatrix} \tag{49}$$

The unitary evolution preserves the commutation relation (35) iff

$$|u_k|^2 - |v_k|^2 = 1\,, \tag{50}$$

$$\text{with initial condition :} \qquad |u_k|^2 = 1\,, \qquad |v_k|^2 = 0\,. \tag{51}$$

If the initial state is the vacuum, $|0\rangle \equiv |0_{t=0}\rangle$, then

$$a_{\mathbf{k}}(0)|0\rangle = 0 \qquad \Rightarrow \qquad a_{\mathbf{k}}(t)|0\rangle = v_k(t)\, a^\dagger_{-\mathbf{k}}(0)|0\rangle \neq 0 \tag{52}$$

In particular, the number density of particles created from the vacuum is

$$n(t) = \frac{1}{V}\langle 0|N|0\rangle = \int \frac{d^3k}{(2\pi)^3}\, |v_k|^2(t)\,. \tag{53}$$

In order to find the function $n(t)$ explictly, we have to solve for u_k and v_k as a solution of

$$\begin{pmatrix} \dot{u}_k(t) \\ \dot{v}^*_k(t) \end{pmatrix} = \begin{pmatrix} -i\omega_k & \dfrac{\dot{\omega}_k}{2\omega_k} \\ \dfrac{\dot{\omega}_k}{2\omega_k} & i\omega_k \end{pmatrix} \begin{pmatrix} u_k(t) \\ v^*_k(t) \end{pmatrix} \tag{54}$$

It is customary to write the mode functions u_k and v_k is terms of the usual Bogolyubov coefficients, $\{\alpha_k,\ \beta_k\}$,

$$u_k = \alpha_k\, e^{-i\int^t \omega_k dt}\,, \qquad v^*_k = \beta_k\, e^{+i\int^t \omega_k dt}\,, \tag{55}$$

then the evolution equations (54) become

$$\begin{aligned} \dot{\alpha}_k &= \frac{\dot{\omega}_k}{2\omega_k}\,\beta_k\, e^{+2i\int^t \omega_k dt}\,, \\ \dot{\beta}_k &= \frac{\dot{\omega}_k}{2\omega_k}\,\alpha_k\, e^{-2i\int^t \omega_k dt}\,, \end{aligned} \tag{56}$$

which can be integrated in the adiabatic approximation, to give

$$n(t) = \int \frac{d^3k}{(2\pi)^3}\, n_k(t) = \int \frac{d^3k}{(2\pi)^3}\, |\beta_k|^2(t)\,, \tag{57}$$

the number density of particles produced due to the time-dependent background field.

Alternatively, one can introduce the $|in\rangle$ and $|out\rangle$ states, and make the field decomposition over time-independent creation and annihilation operators $\{a_{\mathbf{k}},\ a^\dagger_{-\mathbf{k}}\}$,

$$\begin{aligned}
\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^{3/2}}[f_k(t)\, a_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.] \\
&= \int \frac{d^3k}{(2\pi)^{3/2}} \left(f_k(t)\, a_{\mathbf{k}} + f_k^*(t)\, a^\dagger_{-\mathbf{k}}\right) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \qquad (58)\\
\pi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^{3/2}} \left(g_k(t)\, a_{\mathbf{k}} + g_k^*(t)\, a^\dagger_{-\mathbf{k}}\right) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \qquad (59)
\end{aligned}$$

where the mode functions $f_k(t)$ and $g_k(t)$ depend only on the modulus $k = |\mathbf{k}|$, thanks to the homogeneity and isotropy of the background fields. These functions satisfy the equations of motion

$$\ddot{f}_k + \omega_k^2\, f_k = 0\,, \qquad g_k = i\dot{f}_k\,. \qquad (60)$$

Comparing with the former decomposition (39), we find the relation

$$\begin{aligned}
u_k &= \frac{1}{\sqrt{2\omega_k}}(\omega_k f_k + g_k)\,,\\
v_k &= \frac{1}{\sqrt{2\omega_k}}(\omega_k f_k - g_k)\,,
\end{aligned} \qquad (61)$$

and viceversa

$$\begin{aligned}
f_k &= \frac{1}{\sqrt{2\omega_k}}(u_k + v_k^*)\,,\\
g_k &= \sqrt{\frac{\omega_k}{2}}(u_k - v_k^*)\,,
\end{aligned} \qquad (62)$$

which gives for the occupation number

$$n_k(t) = |\beta_k|^2 = \frac{1}{2\omega_k}|\dot{f}_k|^2 + \frac{\omega_k}{2}|f_k|^2 - \frac{1}{2}\,, \qquad (63)$$

where we have used the Wronskian

$$i(\dot{f}_k f_k^* - \dot{f}_k^* f_k) = 2\,\mathrm{Re}\,(f_k^* g_k) = 1 \quad \Leftrightarrow \quad |u_k|^2 - |v_k|^2 = 1\,. \qquad (64)$$

DIAGONALIZATION OF THE HAMILTONIAN

With the above decomposition, we can write the Hamiltonian as

$$H = \int d^3k \, \Big[E_k(t)(a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}} + a_{\mathbf{k}}\, a^\dagger_{\mathbf{k}}) \\ + F_k(t)\, a_{\mathbf{k}}\, a_{-\mathbf{k}} + F^*_k(t)\, a^\dagger_{\mathbf{k}}\, a^\dagger_{-\mathbf{k}}\Big] \,, \tag{65}$$

where

$$E_k(t) \;=\; \frac{1}{2}(|\dot f_k|^2 + \omega_k^2 |f_k|^2) = \omega_k \left(n_k + \frac{1}{2}\right) \,, \tag{66}$$

$$F_k(t) \;=\; \frac{1}{2}(\dot f_k^2 + \omega_k^2 f_k^2) \,, \tag{67}$$

$$E_k^2(t) \;-\; |F_k(t)|^2 = \frac{\omega_k^2}{4} \,. \tag{68}$$

Let us now introduce a canonical Bogolyubov transformation

$$\begin{pmatrix} a_{\mathbf{k}} \\ a^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} u_k(t) & v_k(t) \\ v^*_k(t) & u^*_k(t) \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ b^\dagger_{-\mathbf{k}} \end{pmatrix} \tag{69}$$

Then

$$n_k = |\beta_k|^2 \;=\; \frac{2E_k - \omega_k}{2\omega_k} \,, \tag{70}$$

$$\frac{u_k}{v_k} \;=\; \frac{2E_k + \omega_k}{2F^*_k} \,, \tag{71}$$

and the Hamiltonian becomes diagonal

$$H = \int d^3k \, \frac{\omega_k}{2} \, (b^\dagger_{\mathbf{k}}\, b_{\mathbf{k}} + b_{\mathbf{k}}\, b^\dagger_{\mathbf{k}}) \,, \tag{72}$$

which can be decomposed into $H_{\rm part}$ and $H_{\rm vac}$, as before, see (41).

THE SCHRÖDINGER PICTURE

We can define the unitary evolution operator $U^\dagger(t) = U^{-1}(t)$, where $i\hbar\,\partial_t U(t) = H\,U(t)$, such that time evolution determines

$$a_{\mathbf{k}}(t) = U^\dagger(t) a_{\mathbf{k}}(0) U(t)\,. \tag{73}$$

The solution of the Schrödinger equation is the squeezed state

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle\,. \tag{74}$$

The vacuum at time t is given by $|0_t\rangle = U^\dagger(t)|0\rangle$. Let $|\psi(0)\rangle$ be the initial vacuum state $|0\rangle$. Then the operator

$$b_{\mathbf{k}}(t) = U(t) a_{\mathbf{k}}(0) U^\dagger(t) = u_k^*\, a_{\mathbf{k}}(0) - v_k\, a^\dagger_{-\mathbf{k}}(0) \tag{75}$$

annihilates the state $|\psi(t)\rangle$. Now let us use (47) to evaluate

$$a_{\mathbf{k}}(0) = \sqrt{\frac{\omega_k}{2}}\,\phi_k(0) + \frac{i}{\sqrt{2\omega_k}}\,\pi_k(0)\,, \tag{76}$$

and substitute into $b_{\mathbf{k}}(t)|\psi(t)\rangle = 0$,

$$\frac{1}{\sqrt{2\omega_k}}\left[(u_k^* - v_k)\omega_k\,\phi_k(0) + i(u_k^* + v_k)\,\pi_k(0)\right]|\psi(t)\rangle = 0.$$

Therefore, the evolved state satisfies the Schrödinger equation

$$[\Omega_k(t)\phi_k(0) + i\pi_k(0)]|\psi(t)\rangle = 0\,, \tag{77}$$

$$\Omega_k(t) \equiv \omega_k \frac{u_k^* - v_k}{u_k^* + v_k} = \frac{g_k^*}{f_k^*} = \frac{1 - 2iF_k(t)}{2|f_k(t)|^2}\,, \tag{78}$$

where we have used $g_k^* f_k = \mathrm{Re}\,(g_k^* f_k) - i\,\mathrm{Im}\,(f_k^* g_k) = \frac{1}{2} - i\,\mathrm{Re}\,(f_k^* \dot{f}_k)$. Using the operator definition $\pi_k = -i\frac{\partial}{\partial\phi_{-\mathbf{k}}} = -i\frac{\partial}{\partial\phi_k^*}$, we find the solution

$$\psi(\phi_k, \phi_k^*, t) \sim e^{-\Omega_k(t)|\phi_k|^2}\,, \tag{79}$$

$$P(\phi_k, \phi_k^*, t) = |\psi(\phi_k, \phi_k^*, t)|^2 \sim e^{-\frac{1}{|f_k(t)|^2}|\phi_k|^2}\,. \tag{80}$$

The phase $F_k(t) = \mathrm{Re}\,(f_k^* \dot{f}_k) \gg 1$ quickly becomes very large during preheating, which ensures that the state becomes a squeezed state, with large occupation numbers, described by the Gaussian distribution (80).

PARAMETRIC RESONANCE

We will consider here the case of a scalar field χ coupled to the inflaton ϕ with coupling $\frac{1}{2}g^2\phi^2\chi^2$, which induces an oscillating mass term

$$m_\chi^2(t) = m_\chi^2 + g^2\phi^2(t)\,. \tag{81}$$

The inflaton is assumed to oscillate like (5) with a frequency given by its mass m, not necessarily much larger than the "bare" mass of the field χ. In that case, the frequency can be written as

$$\omega_k^2(t) = k^2 + m_\chi^2 + g^2\Phi^2(t)\sin^2 mt\,, \tag{82}$$

and the mode equation (60) can be written as a Mathieu equation, where $z = mt$, and primes denote differentiation w.r.t. z,

$$f_k'' + (A_k - 2q\,\cos\,2z)\,f_k = 0\,, \tag{83}$$

$$A_k = \frac{k^2 + m_\chi^2}{m^2} + 2q\,, \qquad\qquad q = \frac{g^2\Phi^2(t)}{4m^2}\,. \tag{84}$$

The Mathieu equation is part of a large class of Hill equations (which includes also the Lamé equation and many others) that present unstable solutions for certain values of the momenta for a given set of parameters $\{A_k, q\}$, with $A \geq 2q$. These solutions fall into bands of instability that are narrow for small values of the resonant parameter $q \leq 1$, but can be very broad for larger values of q.

The solutions to the Mathieu eq. have the form $f_k(z) = e^{\mu_k\,z}\,p(z)$, with μ_k the Floquet index, characterizing the exponential instability, and typically much smaller than one, although it could be as large as $\mu_{\rm max} = 0.28055$; and where $p(z)$ is a periodic function of z. The occupation number can then be computed to be

$$n_k(t) \sim e^{2\mu_k\,mt}\,, \tag{85}$$

which can grow significantly in a few oscillations, if the growth index μ_k is not totally negligible.

The effect of parametric resonance is similar to the lasing effect (or light amplification by stimulated emission of radiation), where a large population of particles is produced from oscillations of a coherent source.

NARROW RESONANCE

Let us consider first the case where $m_\chi,\ g\Phi \ll m$, or $q \ll 1$. Then the Mathieu equation instability chart shows that the resonance occurs only in some narrow bands around $A_k \simeq l^2\,,\ l = 1, 2, \ldots$, with widths in momentum space of order $\Delta k \sim m\,q^l$; so, for $q < 1$, the most important band is the first one, $A_k \sim 1 \pm q$, centered around $k = m/2$.

The growth factor μ_k for the first instability band is given by

$$\mu_k = \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{2k}{m} - 1\right)^2}\,. \tag{86}$$

The resonance occurs for k within the range $\frac{m}{2}(1 \pm \frac{q}{2})$. The index μ_k vanishes at the edges of the resonance band and takes its maximum value $\mu_k = \frac{q}{2}$ at $k = \frac{m}{2}$. The corresponding modes grow at a maximal rate $\chi_k \sim \exp(qz/2)$. This leads to a growth of the occupation numbers (85) as $n_k \sim \exp(qmt)$.

We can interpret this as follows. In the limit $q \ll 1$, the effective mass of the χ particles is much smaller than m, and each decaying ϕ particle creates two χ particles with momentum $k \sim m/2$. The difference with respect to the perturbative decay $\Gamma(\phi \to \chi\chi)$ is that, in the regime of parametric resonance, the rate of production of χ particles is proportional to the number of particles produced earlier (which gives rise to an exponential growth in time). This is a non-perturbative effect, as we will discuss later, and we could not have obtained it by using the methods described in the previous section, at any finite order of perturbation theory with respect to the interaction term $g^2\Phi^2\sin^2 mt$. It is by solving the mode equation (83) *exactly* that we have found this result.

Note that only a very narrow range of modes grow exponentially with time, so the spectrum of particles is dominated by these modes, while the rest are still in the vacuum, produced only through the ordinary perturbative decay process. Of course, the exponential production does not last for ever: the universe expansion is going to affect the resonant production of particles in two ways, leading to the end of the narrow resonance regime.

First, the time-dependent amplitude of oscillations $\Phi(t)$, which determines q, see (84), not only decays ($\sim t^{-1}$) due to the expansion of the universe, but also due to the perturbative decay of the inflaton field, $\Phi(t) \sim \exp(\Gamma_\phi t/2)$. Therefore, the narrow resonance will end when the usual perturbative decay becomes important, i.e. when $q\, m < \Gamma_\phi$.

Second, in the evolution equation (83), the momenta k are actually physical momenta, which redshift with the scale factor as $k_{\rm phys} = k/a$, and therefore, even if a given mode is initially within the narrow band, $\Delta k \sim q\, m$, it will very quickly redshift away from it, within the time scale $\Delta t \sim q\, H^{-1}$, preventing its occupation numbers (85) from growing exponentially. Thus, the narrow resonance will end when $q^2\, m < H$.

Therefore, if the amplitude of inflaton oscillations decays like $\Phi \sim 1/t$, there will always be a time (typically a dozen oscillations) for which one of the two conditions above will hold and the narrow resonance will end.

BROAD RESONANCE

If the *initial* amplitude of oscillations is very large, like in models of chaotic inflation, in which $\Phi_0 \sim M_{\rm P}/10$ and $m \sim 10^{-6} M_{\rm P}$, then the initial q-parameter could be very large,

$$q_0 = \frac{g^2\Phi_0^2}{4m^2} \sim g^2\, 10^{10} \lesssim 10^4\,, \tag{87}$$

where we have used the constraint due to radiative corrections (23). In this case, the χ particle production due to stimulated emission by the oscillating inflaton field can be very efficient as it enters into the broad resonance regime.

Particles are produced only at the instances of maximum acceleration of the inflaton field, when $\phi(t) \sim 0$, and

$$\left|\frac{\dot\omega_k}{\omega_k^2}\right| \gg 1\,, \tag{88}$$

a relation known as the *non-adiabaticity condition*. When it holds, we cannot define a proper Fock space for the χ particles, and the occupation numbers of those particles grow very quickly. We thus associate (88) with particle production.

We will now describe how to compute the growth of modes and the Floquet index in this regime, using the formalism developed above. We can expand the quantum field χ in Fourier components f_k satisfying the mode equation (60) with time-dependent frequency (82) and initial conditions

$$f_k(0) = \frac{1}{\sqrt{2\omega_k}}\, e^{-i\omega_k t}\,, \qquad g_k(0) = i\dot{f}_k(0) = \omega_k\, f_k(0)\,, \tag{89}$$

whose evolution in terms of the Bogolyubov coefficients is

$$f_k(t) = \frac{\alpha_k(t)}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k(t)}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{90}$$

$$\alpha_k(0) = 1\,, \qquad \beta_k(0) = 0\,. \tag{91}$$

And the occupation numbers are

$$n_k(t) = |\beta_k(t)|^2 = \frac{1}{2\omega_k}|\dot{f}_k|^2 + \frac{\omega_k}{2}|f_k|^2 - \frac{1}{2}\,. \tag{92}$$

The inflaton field has maximum acceleration at $t = t_j = j\pi/m$, such that $\sin mt_j = 0$. Between t_j and t_{j+1}, the amplitude $\phi(t) \approx \phi_0 = \text{const}$, so that the frequency $\omega_k(t)$ is approximately constant between succesive zeros of the inflaton, and we can properly define a Fock space for χ. At t_j, the amplitude changes rapidly, such that (88) is satisfied and we cannot define an adiabatic invariant like the occupation number (92). Therefore, let us study the behaviour of the modes χ_k precisely at those instances $t = t_j$. We can expand the time-dependent frequency (82) around those points (where the frequency has a minimum) as

$$\omega_k^2(t) = \omega_k^2(t_j) + \frac{1}{2}\omega_k^{2\prime\prime}(t_j)(t-t_j)^2 + \cdots \tag{93}$$

and make the change of variables

$$\tau \equiv [2\omega_k^{2\prime\prime}(t_j)]^{1/4}(t-t_j)\,, \tag{94}$$

$$\kappa^2 \equiv \frac{\omega_k^2(t_j)}{\sqrt{2\omega_k^{2\prime\prime}(t_j)}} = \frac{k^2+m_\chi^2}{2gm\Phi} = \frac{A_k - 2q}{4\sqrt{q}}\,. \tag{95}$$

The mode equation (60) around $t = t_j$ then becomes

$$\frac{d^2 f_k}{d\tau^2} + \left(\kappa^2 + \frac{\tau^2}{4}\right) f_k = 0\,, \tag{96}$$

which can be interpreted as a Schrödinger equation for a wave function scattering in an inverted parabolic potential. The exact solutions are parabolic cylinder functions, $W(-\kappa^2, \pm\tau)$, whose asymptotic expressions are well known. Thus we have substituted the problem of parametric resonance after chaotic inflation with that of partial waves scattering off successive inverted parabolic potentials.

Let the wave $f_k(t)$ have the form of the adiabatic solution (90) before scattering at t_j,

$$f_k^j(t) = \frac{\alpha_k^j}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k^j}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{97}$$

where the coefficients $\{\alpha_k^j,\ \beta_k^j\}$ are constant, for $t_{j-1} < t < t_j$. After scattering off the potential at t_j, the wave $f_k(t)$ takes the form

$$f_k^{j+1}(t) = \frac{\alpha_k^{j+1}}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k^{j+1}}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{98}$$

where the coefficients $\{\alpha_k^{j+1},\ \beta_k^{j+1}\}$ are again constant, for $t_j < t < t_{j+1}$. These are essentially the asymptotic expressions for the incoming and the outgoing waves, scattered at t_j. Therefore, the outgoing amplitudes $\{\alpha_k^{j+1},\ \beta_k^{j+1}\}$ can be expressed in terms of the incoming amplitudes $\{\alpha_k^j,\ \beta_k^j\}$ with the help of the reflection R_k and transmission D_k coefficients of scattering at t_j,

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\theta_k^j} \\ \\ \beta_k^{j+1} e^{+i\theta_k^j} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_k} & \frac{R_k^*}{D_k^*} \\ \\ \frac{R_k}{D_k} & \frac{1}{D_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \\ \beta_k^j e^{+i\theta_k^j} \end{pmatrix} \tag{99}$$

where $\theta_k^j = \int_0^{t_j} \omega_k(t) dt$, and

$$\left.\begin{aligned} R_k &= -i\, e^{-i\phi_k}\, [1 + e^{2\pi\kappa^2}]^{-1/2}\,, \\ D_k &= e^{-i\phi_k}\, [1 + e^{-2\pi\kappa^2}]^{-1/2}\,, \end{aligned}\right\} \qquad |R_k|^2 + |D_k|^2 = 1\,. \tag{100}$$

The k-dependent angle of scattering is

$$\phi_k = \mathrm{Arg}\,\Gamma[\frac{1}{2} + i\kappa^2] + \kappa^2(1 - \ln\kappa^2)\,. \tag{101}$$

Simplifying (99), we find

$$\begin{pmatrix} \alpha_k^{j+1} \\ \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} [1+e^{-2\pi\kappa^2}]^{1/2} e^{i\phi_k} & i\,e^{-\pi\kappa^2+2i\theta_k} \\ \\ -i\,e^{-\pi\kappa^2-2i\theta_k} & [1+e^{-2\pi\kappa^2}]^{1/2} e^{-i\phi_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \\ \beta_k^j \end{pmatrix} \tag{102}$$

and therefore, using $n_k^j = |\beta_k^j|^2$ and $|\alpha_k^j|^2|\beta_k^j|^2 = n_k^j(n_k^j+1)$, we have

$$\begin{aligned} n_k^{j+1} &= e^{-2\pi\kappa^2} + (1+2e^{-2\pi\kappa^2})\,n_k^j \\ &- 2e^{-\pi\kappa^2}[1+e^{-2\pi\kappa^2}]^{1/2}[n_k^j(n_k^j+1)]^{1/2}\,\sin\theta_{\rm tot}^j\,, \end{aligned} \tag{103}$$

where $\theta_{\rm tot}^j = 2\theta_k^j - \phi_k + \mathrm{Arg}\,\beta_k^j - \mathrm{Arg}\,\alpha_k^j$.

This expression is very enlightening. Let us describe its properties:

- **Step-like**. The number of created particles is a step-like function of time. The occupation number between successive scatterings is constant. In the first scattering (when $n_k^0 = 0$), we have

$$n_k = e^{-2\pi\kappa^2} = e^{-\frac{\pi k^2}{gm\Phi}} < 1\,. \tag{104}$$

- **Non-perturbative**. The occupation number (104) cannot be expanded perturbatively, for small coupling, because the function $e^{-1/g}$ is non-analytical at $g=0$. This is the form that most non-perturbative effects take in quantum field theory.

- **Infrared effect**. For large momenta, the occupation number decays exponentially, so even if there are bands at low momenta, i.e. in the IR region, the high momentum modes will not be populated,

$$\kappa^2 \gg \pi^{-1} \quad \Rightarrow \quad n_k^{j+1} \simeq n_k^j \simeq 0\,. \tag{105}$$

- **Non-linear**. For small momenta one may have production of particles with mass *greater* than that of the inflaton:

$$\kappa^2 = \frac{k^2+m_\chi^2}{2gm\Phi_0} \lesssim \pi^{-1} \;\Rightarrow\; n_k \text{ large if } m^2 < m_\chi^2 \ll gm\Phi_0 \tag{106}$$

- **Exponential boson production**. In the case of bosons (we will discuss the fermionic case later), the occupation number can grow exponentially due to Bose-Einstein statistics, $n_k \sim \exp(2\mu_k z) \gg 1$,

  $$n_k^{j+1} \simeq [(1+2e^{-2\pi\kappa^2}) - 2e^{-\pi\kappa^2}[1+e^{-2\pi\kappa^2}]^{1/2} \sin\theta^j_{\rm tot}]\, n_k^j \equiv e^{2\pi\mu_k^j}\, n_k^j$$

  which allows one to estimate the Floquet index μ_k.

- **Resonant production**. Valid only for periodic sources. If scattering occurs in phase, the incoming and outgoing waves add up constructively, and we can have resonant effects. This occurs when $\theta^j_{\rm tot}$ is a semi-integer multiple of π. In that case, it is possible that, for some modes, $n_k^{j+1} > n_k^j$. This gives rise to a particular band structure.

- **Stochastic preheating**. It may happen that the phase a mode has acquired in a given scattering exactly compensates for the universe expansion in that interval and the phases destructively interfere, *decreasing* the number of particles in that mode. This gives rise to a stochastic growth of particles, where approximately 3/4 of the time the particle number increases.

- **Band structure**. Different models of inflation give rise to different evolution laws for the amplitude of inflaton oscillations, and therefore to different mode equations (60). The corresponding Hill equations (linear second order differential equations with periodic coefficients) can have quite different band structures, e.g. those of Mathieu or Lamé equations.

Even if we compute the complete band structure of the Mathieu or Lamé equation and we determine the growth factors μ_k with great accuracy, the universe expansion will shift any given mode from one band to the next, as the mode redshifts and the amplitude of inflaton oscillations decreases: A mode starts in a given band, its occupation numbers increase exponentially through several oscillations, and suddenly it falls out of the band, until the expansion makes it fall into the next band, and so on until it reaches the narrow resonance regime described above.

FERMIONIC PREHEATING

Not only bosons can be produced from the coherent oscillations of the inflaton field. Although their occupation numbers cannot grow very large because of the Pauli exclusion principle, the resonant production of fermions at preheating can nevertheless create a wide spectrum of particles with occupation number $n_k < 1$ but a wide range $\Delta k \gg m$. We will describe here the formalism of fermion production during preheating. Consider a fermionic particle ψ with mass

$$m_{\rm F} = m_\psi + h\,\phi(t)\,, \tag{107}$$

satisfying the Dirac equation, in conformal time $d\eta = dt/a$,

$$(i\,\gamma^\mu\nabla_\mu - m_{\rm F})\psi = \left(\frac{i}{a}\,\gamma^\mu\partial_\mu + i\frac{3a'}{2a}\gamma^0 - m_{\rm F}\right)\psi = 0\,, \tag{108}$$

where γ^μ are the Dirac matrices in Minkowsky spacetime.
Let us now redefine $\Psi = a^{3/2}\psi$ to obtain the familiar form ($m \equiv m_{\rm F}$)

$$(i\,\gamma^\mu\partial_\mu - am)\Psi = 0\,. \tag{109}$$

We can then expand in Fourier components,

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_r [u_r(k,\eta)\,a_r(\mathbf{k}) + v_r(k,\eta)\,b_r^\dagger(-\mathbf{k})]\,e^{i\mathbf{k}\cdot\mathbf{x}} \tag{110}$$

where the sum is over spin components, and $v_r(\mathbf{k}) = C\,\bar{u}_r^T(-\mathbf{k})$ for the antiparticle amplitude. We can then impose canonical equal time anticommutator relations,

$$\{a_r(\mathbf{k}),\ a_s^\dagger(\mathbf{k}')\} = \{b_r(\mathbf{k}),\ b_s^\dagger(\mathbf{k}')\} = \delta_{rs}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{111}$$

together with a fermionic vacuum defined by $a_r(\mathbf{k})|0\rangle = b_r(\mathbf{k})|0\rangle = 0$. The spinors are normalized such that

$$u_r^\dagger(k,\eta)\,u_s(k,\eta) = v_r^\dagger(k,\eta)\,v_s(k,\eta) = 2\delta_{rs}\,, \tag{112}$$

$$u_r^\dagger(k,\eta)\,v_s(k,\eta) = 0\,, \tag{113}$$

is preserved by the unitary evolution.

In the representation in which

$$\gamma^0 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}, \quad \text{and} \quad u \equiv \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{114}$$

the equation of motion (109) becomes

$$u''_\pm + [\omega_k^2 \pm i(am)']u_\pm(k) = 0\,, \tag{115}$$

$$\omega_k^2 = k^2 + a^2m^2\,, \tag{116}$$

$$k\, u_\mp = iu'_\pm \mp am\, u_\pm\,, \tag{117}$$

where we have chosen $\mathbf{k}$ along the third axis.

We can then write the Hamiltonian as

$$\begin{aligned} H(\eta) &= \frac{1}{a}\int d^3x\, \Psi^\dagger i\partial_0\Psi = \frac{1}{a}\int d^3k \sum_r \Big[E_k(\eta)(a_r^\dagger(\mathbf{k})a_r(\mathbf{k}) - b_r(\mathbf{k})b_r^\dagger(\mathbf{k})) \\ &+ F_k(\eta)b_r(-\mathbf{k})a_r(\mathbf{k}) + F_k^*(\eta)a_r^\dagger(\mathbf{k})b_r^\dagger(-\mathbf{k})\Big]\,, \end{aligned} \tag{118}$$

$$E_k = k\,\mathrm{Re}\,(u_+^*u_-) - am(1 - u_+^*u_+) = -\mathrm{Im}\,(u_+^*u'_+) - am \tag{119}$$

$$F_k = \frac{k}{2}(u_+^2 - u_-^2) - am\, u_+u_- = \frac{1}{2k}({u'_+}^2 + \omega_k^2u_+^2) \tag{120}$$

$$E_k^2 + |F_k|^2 = \omega_k^2\,. \tag{121}$$

To provide the "quasi-particle" interpretation, we have to diagonalize the Hamiltonian with a time-dependent canonical transformation, and redefine new creation and annihilation operators,

$$\begin{aligned} \bar{a}(\mathbf{k},\eta) &= \alpha_k(\eta)\, a(\mathbf{k}) + \beta_k(\eta)\, b^\dagger(-\mathbf{k})\,, \\ \bar{b}^\dagger(\mathbf{k},\eta) &= -\beta_k^*(\eta)\, a(\mathbf{k}) + \alpha_k^*(\eta)\, b^\dagger(-\mathbf{k})\,. \end{aligned} \tag{122}$$

Imposing canonical anticommutation relations on $\bar{a}$ and $\bar{b}$, we find

$$|\beta_k|^2 = \frac{|F_k|^2}{2\omega_k(\omega_k + E_k)} = \frac{\omega_k - E_k}{2\omega_k}\,, \tag{123}$$

$$\frac{\alpha_k}{\beta_k} = \frac{E_k + \omega_k}{F_k^*} = \frac{-F_k}{E_k - \omega_k}\,, \tag{124}$$

$$|\alpha_k|^2 + |\beta_k|^2 = 1\,, \tag{125}$$

The initial conditions

$$u_\pm(0) \;=\; \sqrt{1 \pm \frac{am}{\omega_k}}\,, \tag{126}$$

$$u'_\pm(0) \;=\; -ik\,u_\mp(0) \mp i\,am\,u_\pm(0)\,, \tag{127}$$

correspond to $E_k = \omega_k,\ F_k = 0$, and therefore to the no-particle vacuum.

With the Bogolyubov transformation (122), the (diagonal) normal-ordered Hamiltonian becomes

$$H(\eta) = \frac{1}{a}\int d^3k \sum_r \omega_k(\eta)\,\left[\bar{a}_r^\dagger(\mathbf{k})\,\bar{a}_r(\mathbf{k}) + \bar{b}_r^\dagger(\mathbf{k})\,\bar{b}_r(\mathbf{k})\right]\,. \tag{128}$$

We can define the "quasi-particle" vacuum by $\bar{a}(\mathbf{k})|0_\eta\rangle = \bar{b}(\mathbf{k})|0_\eta\rangle = 0$, and then the number density of produced particles becomes

$$n_{\rm F}(\eta) = \frac{1}{V}\langle 0|N|0\rangle = \frac{1}{2\pi^2 a^3(\eta)}\int_0^\infty dk\,k^2\,|\beta_k|^2\,. \tag{129}$$

We will now use this formalism to study fermion production at preheating.

PARAMETRIC PREHEATING OF FERMIONS

Consider the case of fermions ψ coupled to the inflaton field at the end of inflation, in which the fermion mass (107) changes with time due to the inflaton oscillations (5). Let us parametrise these oscillations as $\phi(t) = \phi_0\,f(t)$, where f is a periodic function. Then, defining the resonance parameter $\sqrt{q} = h\phi_0/m$, we find

$$u''_k + [\omega_k^2 - i\sqrt{q}\,f']u_k = 0\,, \tag{130}$$

$$\omega_k^2 = k^2 + (\bar{m}_\psi + \sqrt{q}\,f)^2\,, \tag{131}$$

where $\bar{m}_\psi = m_\psi/m$, primes denote derivatives w.r.t. $z = mt$, and the momentum k is actually k/m, i.e. in units of the inflaton mass m.

This equation is very similar to that of the scalar field modes, except that it has an imaginary potential. In fact, we can study the parametric resonance regime here too by analysing the scattering of partial waves on complex parabolic potentials.

Let us expand the frequency ω_k around the zeros of the inflaton oscillation, i.e. at t_j, as in Eq. (93), and make the change of variables

$$\tau \equiv [2\omega_k^{2\prime\prime}(t_j)]^{1/4}(t-t_j)\,, \tag{132}$$

$$\kappa^2 \equiv \frac{k^2}{2\sqrt{q-\bar{m}_\psi^2}}\,, \qquad \text{sign}\, f'(t_j) = (-1)^j\,. \tag{133}$$

The equation of motion then becomes

$$\frac{d^2u_k}{d\tau^2} + \left(\kappa^2 + \frac{\tau^2}{4} - \frac{i}{2}(-1)^j\right) u_k = 0\,, \tag{134}$$

whose solutions are parabolic cylinder functions with complex argument, $W(-\kappa^2 + \frac{i}{2}(-1)^j, \pm\tau)$. The analysis of the scattering amplitudes performed for the scalar particles can be used here by simply analytically continuing

$$\kappa^2 \to \bar{\kappa}^2 = \kappa^2 - \frac{i}{2}(-1)^j\,, \tag{135}$$

with which the scattering matrix (99) becomes

$$\begin{pmatrix} \alpha_k^{j+1} \\ \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} [1-e^{-2\pi\kappa^2}]^{1/2} e^{i\phi_k} & (-1)^{j+1}\, e^{-\pi\kappa^2+2i\theta_k} \\ \\ (-1)^j\, e^{-\pi\kappa^2-2i\theta_k} & [1-e^{-2\pi\kappa^2}]^{1/2} e^{-i\phi_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \\ \beta_k^j \end{pmatrix} \tag{136}$$

where the scattering angle is given by

$$\phi_k = \text{Arg}\,\Gamma[\frac{1}{2} + i\bar{\kappa}^2] + \bar{\kappa}^2(1-\ln\bar{\kappa}^2)\,, \tag{137}$$

and therefore, using $n_k^j = |\beta_k^j|^2$ and $|\alpha_k^j|^2|\beta_k^j|^2 = n_k^j(n_k^j-1)$, we find

$$\begin{aligned} n_k^{j+1} &= e^{-2\pi\kappa^2} + (1-2e^{-2\pi\kappa^2})\, n_k^j \\ &- 2(-1)^j\, e^{-\pi\kappa^2}[1-e^{-2\pi\kappa^2}]^{1/2}[n_k^j(n_k^j-1)]^{1/2}\sin\theta_{\text{tot}}^j\,, \end{aligned} \tag{138}$$

where

$$\theta_{\text{tot}}^j = 2\theta_k^j - \phi_k + \text{Arg}\,\beta_k^j - \text{Arg}\,\alpha_k^j\,.$$

Note from (138) that the occupation numbers cannot be larger than 1, although in the first scattering, the occupation number jumps exactly by the same amount as the bosons, $n_k = e^{-2\pi\kappa^2}$.