Wenming Zou

Sign-Changing Critical Point Theory

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I dedicate this book to my wife and son

Qinying Wang and Yuezhang Zou

for their love and support

Contents

Preface

There has been increasing interest in recent years to develop a critical point theory by which one can obtain additional information on the critical points of a differentiable functional. What I mean by additional information is the locations of the critical points related to closed convex subsets in Banach spaces. This is the theme of the current book.

This book mainly reflects a significant part of my research activity during recent years. Except for the last chapter, it is constructed based on the results obtained myself or through direct cooperation with other mathematicians. On the whole, the readers will observe that the main abstract existence theorems of critical points in classical minimax theory are generalized to the cases of sign-changing critical points. Hence, a new theory is built. To the best of my knowledge, no book on sign-changing critical point theory has ever been published.

The material covered in this book is for advanced graduate and PhD students or anyone who wishes to seek an introduction into sign-changing critical point theory. The chapters are designed to be as self-contained as possible.

I have had the good fortune to teach at the University of California at Irvine and to work with Martin Schechter for the years 2001 to 2004. During that period, some results of the current book were obtained. M. Schechter has had a profound influence on me not only by his research, but also by his writing and his generosity. I am grateful to T. Bartsch and Z. Q. Wang for sending me their interesting papers and enlightening discussions with Wang when I visited Utah. Thanks also go to A. Szulkin and M. Willem for inviting me to visit their prestigious departments years ago. Special thanks are also given to S. Li who first introduced me into the variational and topological methods ten years ago. I wish to thank the University of California at Irvine for providing me a favorable environment during the period 2001 to 2004. This book is supported by the NSFC (No. 10001019 & 10571096), the SRF-ROCS-SEM, the Program of the Education Ministry in P. R. China for New Century Excellent Talents in Universities of China.

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Tsinghua University, Beijing Wenming Zou

Introduction

A theory is the more impressive,
the simpler are its premises,
the more distinct are the things it connects,
and the broader is its range of applicability.

Albert Einstein

Many nonlinear problems in physics, engineering, biology, and social sciences can be reduced to finding critical points (minima, maxima, and minimax points) of real-valued functions on various spaces. The first class of critical points to be studied were minima and maxima and much of the activity in the calculus of variations has been devoted to finding such points. A more difficult problem is to find critical points that are neither maxima nor minima. So far we may say, to some extent, that there is an organized procedure for producing such critical points and these methods are called global variational and topological methods. Roughly speaking, the modern variational and topological methods consist of the following two parts.

Minimax Methods. Ljusternik and Schnirelman [214] in 1929 mark the beginning of global analysis, by which some earlier mathematicians no longer consider only the minima or maxima of variational integrals. In 1934, Ljusternik and Schnirelman [215] developed a method that seeks to get information concerning the number of critical points of a functional from topological data. These ideas are referred to as the Ljusternik–Schnirelman theory. One celebrated and important result in the last 30 years has been the mountain pass theorem due to Ambrosetti and Rabinowitz [15] in 1973. Since then, a series of new theorems in the form of minimax have appeared via various linking, category, and index theories. Now these results in fact become a wonderful tool in studying the existence of solutions to differential equations with variational structures. We refer readers to the books (or surveys) due to Brézis and Nirenberg [71], Nirenberg [232, 233, 235], Rabinowitz [255], Schechter [275], Struwe [313], Willem [335], Mawhin and Willem [225], and Zou and Schechter [351], among others.

Morse Theory. This approach towards a global theory of critical points was pursued by Morse [229] in 1934. It reveals a deep relation between the topology of spaces and the number and types of critical points of any function defined on it. This theory was highly successful in topology in the 1950s due to the efforts of Milnor [226] and Smale [303]. In the works of Palais [239], Smale [304], and Rothe [264, 263], Morse theory was generalized to infinite-dimensional spaces. By then it was recognized as a useful approach in dealing with differential equations and in particular, in finding the existence of multiple solutions (see Chang [92, 94]). The critical group and Morse index also can be derived in some cases. Although there are some profound works on Morse theory and related topics, the applications are somewhat limited by the smoothness and nondegeneracy assumptions on the functionals. Readers may consult Mawhin and Willem [225], Conley [106], and Benci [51], among others.

However, both minimax theory and Morse theory essentially give answers on the existence of (multiple) critical points of a functional. They usually cannot provide many more additional properties of the critical points except some special profiles such as the Morse index, critical groups, and so on. I make no attempt here to give an exhaustive account of the field or a complete survey of the literature.

There has been increasing interest in recent years to develop a theory by which one can obtain much more information on critical points. The central theme of the current volume is the theory of finding sign-changing critical points. The book is organized as follows.

In Chapter 1, we provide some prerequisites for this book such as degree theory, Sobolev space, and so on. Basically, these theories are relatively mature and readily available in many existing books. However, we still spend some pages on the flows of the ODEs in Banach spaces which play important roles in this book. Well-trained readers may skip over this chapter to the next parts.

In Chapter 2, we establish the relation between linking and the sign-changing critical point. The linking introduced by Schechter is more general and realistic. We say that a set A links another set B if they do not intersect and A cannot be continuously shrunk to a point without intersecting B. This kind of linking includes the original ones. But more examples can be found. We show how the new linking produces sign-changing critical points.

We devote Chapter 3 to the sign-changing saddle point theory. The saddle point theory can be traced back to Rabinowitz's theory 30 years ago, which gives the sufficient conditions on the existence of a saddle point. But it never excludes the triviality of that point, nor the sign-changingness of it. We solve this question.

Essentially, in Chapter 4, we generalize the Brezis–Nirenberg critical point theorem obtained in 1991 by judging the location and nodal structure of the (PS) sequences and critical points.

Chapter 5 is about the even functionals. We obtain the relationship between the classical symmetric mountain pass theorem and the sign-changing critical points.

Chapter 6 discusses the parameter dependence of sign-changing critical points. This theory is independent of the (PS) compactness condition.

In Chapter 7 we provide sign-changing critical point theories due Bartsch, Chang and Wang, and Bartsch and Weth. The Morse index and the number of nodal domains are included.

In each chapter, based on the new abstract sign-changing critical point theory, applications are considered mainly on Schrödinger equations or Dirichlet boundary value problems.

This book mainly consists of the results of my recent research. It is not intended and nor is it possible to be complete. In fact, many other results on sign-changing solutions of elliptic equations in recently years are not in this book. I just cite them in the bibliography or quote some lemmas from them. We refer the readers to the references in the bibliography written by T. Bartsch, A. Castro, G. Cerami, K. C. Chang, M. Clapp, V. Coti-Zelati, E. N. Dancer, Y. Du, N. Ghoussoub, F. A. van Heerden, N. Hirano, S. Li, J. Q. Liu, Z. Liu, P. H. Rabinowitz, S. Solimini, M. Struwe, Z. Q. Wang, T. Weth, C. Yuan, et al. for other interesting results on concrete elliptic equations. Finally, although Chapter 7 involves some theories due to Bartsch and others, I would like to mention the following additional topics due to them: symmetry results for sign-changing solutions, in particular for the least energy nodal solution; upper estimates on the number of nodal domains; and some discussions of singularly perturbed equations and multiple nodal solutions without oddness of the nonlinearity.

Chapter 1
Preliminaries

For readers' convenience, we collect in this chapter some classical results on nonlinear functional analysis and the elementary theory of partial differential equations. Some of them are well known and their proofs are omitted. For others, although their proofs may be found in many existing books, we make no apology for repeating them.

1.1 Partition of Unity

Let E be a metric space with a distance function $\mathrm{dist}(\cdot,\cdot)$ on it. Let $A \subset E$ and $\mathcal{O}$ be a family of open subsets of E. If each point of A belongs to at least one member of $\mathcal{O}$, then $\mathcal{O}$ is called an open covering of A.

Definition 1.1. Let $\mathcal{O}$ be an open covering of a subset A of E. $\mathcal{O}$ is called locally finite if for any $u \in A$, there is an open neighborhood U such that $u \in U$ and that U intersects only finitely many elements of $\mathcal{O}$.

A well-known result on this line is the underlying proposition due to Stone [308].

Proposition 1.2. *Any metric space E is paracompact; that is, every open covering $\mathcal{O}$ of E has an open, locally finite refinement Θ. That is, Θ is a locally finite covering of E and for any V_i in Θ, we can find a U_i in $\mathcal{O}$ such that $V_i \subset U_i$.*

Proposition 1.3. *Assume that E is a metric space with the distance function* $\mathrm{dist}(\cdot,\cdot)$. *Let $\mathcal{O}$ be an open covering of E. Then $\mathcal{O}$ admits a locally finite partition of unity $\{\lambda_i\}_{i\in J}$ subordinate to it satisfying*

(1) $\lambda_i : E \to [0,1]$ is Lipschitz continuous.
(2) $\{u \in E : \lambda_i(u) \neq 0\}_{i\in J}$ is a locally finite covering of E.

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(3) For each V_i, there is a $U_i \in \mathcal{O}$ such that $V_i \subset U_i$.
(4) $\sum_{i\in J} \lambda_i(u) = 1, \forall u \in E$,

where J is the index set.

Proof. Because (E, dist) is a metric space with an open covering $\mathcal{O}$, by Proposition 1.2, there is an open, locally finite refinement Θ; that is, Θ is locally finite and for any V_i of Θ, we can find a U_i of $\mathcal{O}$ such that $V_i \subset U_i$. Define

$$\rho_i(u) = \mathrm{dist}(u, E\backslash V_i), \qquad i \in J.$$

Then ρ_i is locally Lipschitz. Let

$$\lambda_i(u) = \frac{\rho_i(u)}{\sum_{j\in J} \rho_j(u)}, \qquad i \in J.$$

Then $\{\lambda_i\}_{i\in J}$ is what we want. □

1.2 Ekeland's Variational Principle

We recall Ekeland's variational principle (see Ekeland [137]).

Lemma 1.4. *Let E be a complete metric space with a metric* dist *and $I : E \to \mathbf{R}$ be a lower semicontinuous functional that is bounded below. For any $T > 0, \varepsilon > 0$, let $u_1 \in E$ be such that $I(u_1) \le \inf_E I + \varepsilon$. Then there exists a $v_1 \in E$ such that*

$$I(v_1) \le I(u_1), \tag{1.1}$$

$$\mathrm{dist}(u_1, v_1) \le 1/T, \tag{1.2}$$

$$I(v_1) < I(w) + \varepsilon T \mathrm{dist}(v_1, w), \qquad \textit{for all } w \neq v_1. \tag{1.3}$$

Proof. Define a partial order $\preceq$ in E as the following.

$$u \preceq v \Leftrightarrow I(u) \le I(v) - \varepsilon T \mathrm{dist}(v, u).$$

Then obviously,

$$\begin{aligned} &u \preceq u, && \text{for all } u \in E,\\ &u \preceq v, v \preceq u \Rightarrow u = v, && \text{for all } u, v \in E,\\ &u \preceq v, v \preceq w \Rightarrow u \preceq w, && \text{for all } u, v, w \in E. \end{aligned}$$

Let $C_1 := \{u \in E : u \preceq u_1\}$ and let $u_2 \in C_1$ be such that

$$I(u_2) \le \inf_{C_1} I + \frac{\varepsilon}{2^2}.$$

Then, let $C_2 := \{u \in E : u \preceq u_2\}$. Inductively,

$$u_{n+1} \in C_n := \{u \in E : u \preceq u_n\}, \qquad I(u_{n+1}) \le \inf_{C_n} I + \frac{\varepsilon}{2^{n+1}}.$$

By the lower semicontinuity of I and the continuity of $\mathrm{dist}(\cdot,\cdot)$, we see that C_n is closed. Moreover,

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots,$$

$$\cdots \preceq u_n \preceq \cdots \preceq u_2 \preceq u_1.$$

For any $v \in C_n$, then

$$I(v) \le I(u_n) - \varepsilon T \mathrm{dist}(v, u_n). \tag{1.4}$$

Note that $v \in C_{n-1}$; we have

$$I(u_n) \le \inf_{C_{n-1}} I + \frac{\varepsilon}{2^n} \le I(v) + \frac{\varepsilon}{2^n}. \tag{1.5}$$

Combine Equations (1.4) and (1.5); we have that $\mathrm{dist}(v, u_n) \le (1/T2^n)$. Because $v \in C_n$ is arbitrary, we know that the diameter of C_n is less than or equal to $(1/T2^{n-1})$, hence, approaches zero. Therefore,

$$\bigcap_{n=1}^{\infty} C_n = \{v_1\}.$$

We claim that v_1 is what we want. Indeed, $v_1 \in C_1$ implies that

$$I(v_1) \le I(u_1) - \varepsilon T \mathrm{dist}(u_1, v_1) \le I(u_1).$$

For any $w \ne v_1$, we observe that we cannot have $w \preceq v_1$, otherwise $w \in \bigcap_{n=1}^{\infty} C_n$ hence $w = v_1$. That is, we must have

$$I(w) > I(v_1) - \varepsilon T \mathrm{dist}(w, v_1).$$

Finally, noting that

$$\mathrm{dist}(u_1, u_n) \le \sum_{i=1}^{n-1} \mathrm{dist}(u_i, u_{i+1}) \le \sum_{i=1}^{n-1} \frac{1}{T2^i} \le \frac{1}{T}$$

and that $\lim_{n\to\infty} u_n = v_1$, then we get that $\mathrm{dist}(u_1, v_1) \le 1/T$. Thus, v_1 satisfies Equations (1.1) to (1.3). This completes the proof. □

Notes and Comments. Readers may consult Ekeland [138], de Figueiredo [147], Ghoussoub [156], Grossinho and Tersian [162], Mawhin and Willem [225], Struwe [313], and Willem [335] for the variants of Ekeland's variational principle and their applications. Ghoussoub [156] contains the Borwein and Preiss principle and also the mountain pass principle which is presented as a "multidimensional extension" of the Ekeland variational principle. A simple and elegant generalization of Ekeland's variational principle to a general form on ordered sets was obtained in Brézis and Browder [66]. It was applied to nonlinear semigroups and to derive diverse results from nonlinear analysis including the variational principle and one of its equivalent forms, the Bishop–Phelps theorem. Some other generalizations of Ekeland's variational principle can also be found in Li and Shi [203] and Zhong [339, 340].

1.3 Sobolev Spaces and Embedding Theorems

Let Ω be an open subset of $\mathbf{R}^N, N \in \mathbf{N}$. Denote

$$L^p(\Omega) := \{u : \Omega \to \mathbf{R} \text{ is Lebesgue measurable, } \|u\|_{L^p(\Omega)} < \infty\},$$

where

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p}, \qquad 1 \le p < +\infty.$$

If $p = +\infty$,

$$\|u\|_{L^\infty(\Omega)} = ess \sup_\Omega |u| := \inf_{A \subset \Omega, \operatorname{meas}(A)=0} \sup_{\Omega \setminus A} |u|,$$

where meas denotes the Lebesgue measure. If $\|u\|_{L^\infty(\Omega)} < \infty$, we say that u is essentially bounded on Ω. Let

$$L^p_{loc}(\Omega) := \{u : \Omega \to \mathbf{R}, u \in L^p(V) \text{ for each } V \subset\subset \Omega\},$$

where $V \subset\subset \Omega \Leftrightarrow V \subset \bar{V} \subset \Omega$ and $\bar{V}$ is compact. Sometimes in this book we denote $\|u\|_{L^p(\Omega)}$ by $\|u\|_p$ or $|u|_p$.

We denote by $\operatorname{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}$ the support of $u : \Omega \to \mathbf{R}$. Let $\mathbf{C}_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\phi : \Omega \to \mathbf{R}$ with compact support in Ω. For each $\phi \in \mathbf{C}_c^\infty(\Omega)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ with order $|\alpha| := \alpha_1 + \cdots + \alpha_N$, we denote

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \phi.$$

Definition 1.5. Suppose $u, v \in L^1_{loc}(\Omega)$. We say that v is the αth-weak partial derivative of u, written $D^\alpha u = v$ provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx$$

for all $\phi \in \mathbf{C}_c^\infty(\Omega)$.

It is easy to check that the αth-weak partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.

Let $\mathbf{C}^m(\Omega)$ be the set of functions having derivatives of order $\leq m$ being continuous in Ω ($m = integer \geq 0$ or $m = \infty$). Let $\mathbf{C}^m(\bar{\Omega})$ be the set of functions in $\mathbf{C}^m(\Omega)$ all of whose derivatives of order $\leq m$ have continuous extension to $\bar{\Omega}$.

Definition 1.6. Fix $p \in [1, +\infty]$ and $k \in \mathbf{N} \cup \{0\}$. The Sobolev space

$$W^{k,p}(\Omega)$$

consists of all $u : \Omega \to \mathbf{R}$ which has αth-weak partial derivative $D^\alpha u$ for each multi-index α with $|\alpha| \leq k$ and $D^\alpha u \in L^p(\Omega)$.

If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega), \qquad k = 0, 1, 2, \ldots.$$

Note that $H^0(\Omega) = L^2(\Omega)$. We henceforth identify functions in $W^{k,p}(\Omega)$ which agree a.e

Definition 1.7. If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p}, & p \in [1, +\infty), \\ \sum_{|\alpha| \leq k} ess\sup_\Omega |D^\alpha u|, & p = +\infty. \end{cases}$$

Definition 1.8. We denote $W_0^{k,p}(\Omega)$ the closure of $\mathbf{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to its norm defined in Definition 1.7. It is customary to write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

and denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$.

The following results can be found in Evans [141] and Adams and Fournier [2].

Proposition 1.9. *For each $k = 1, 2, \ldots$ and $1 \leq p \leq +\infty$, the Sobolev space*

$$(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$$

is a Banach space and so is $W_0^{k,p}(\Omega)$. In particular, $H^k(\Omega), H_0^k(\Omega)$ are Hilbert spaces; $W_0^{k,p}(\mathbf{R}^N) = W^{k,p}(\mathbf{R}^N)$.

Definition 1.10. Let $(E, \|\cdot\|_E)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, $E \subset Y$. We say that E is continuously embedded in Y (denoted by $E \hookrightarrow Y$) if the identity $\mathbf{id} : E \to Y$ is a linear bounded operator; that is, there is a constant $C > 0$ such that $\|u\|_Y \leq C\|u\|_E$ for all $u \in E$. In this case, the constant $C > 0$

is called the embedding constant. If moreover, each bounded sequence in E is precompact in Y, we say the embedding is compact, written $E \hookrightarrow\hookrightarrow Y$.

Definition 1.11. A function $u : \Omega \subset \mathbf{R}^N \to \mathbf{R}$ is Hölder continuous with exponent $\gamma > 0$ if

$$[u]^{(\gamma)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < \infty.$$

Definition 1.12. The Hölder space $\mathbf{C}^{k,\gamma}(\bar{\Omega})$ consists of all functions $u \in \mathbf{C}^k(\bar{\Omega})$ for which the norm

$$\|u\|_{\mathbf{C}^{k,\gamma}}(\bar{\Omega}) := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\mathbf{C}(\bar{\Omega})} + \sum_{\||\alpha|=k} [D^\alpha u]^{(\gamma)}$$

is finite. It is a Banach space. We set $\mathbf{C}^{k,0}(\bar{\Omega}) = \mathbf{C}^k(\bar{\Omega})$.

We have the following embedding results; see Adams [1], Adams and Fournier [2], Evans [141], and Gilbarg and Trudinger [160].

Proposition 1.13. *If Ω is a bounded domain in $\mathbf{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \leq q \leq Np/(N - kp); \\ \mathbf{C}^{m,\alpha}(\bar{\Omega}), & 0 \leq \alpha \leq k - m - N/p\ , \\ & 0 \leq m < k - N/p < m + 1. \end{cases}$$

Proposition 1.14. *If Ω is a bounded domain in $\mathbf{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \leq q < Np/(N - kp); \\ \mathbf{C}^{m,\alpha}(\bar{\Omega}), & 0 \leq \alpha < k - m - N/p\ , \\ & 0 \leq m < k - N/p < m + 1. \end{cases}$$

In general, $W_0^{k,p}(\Omega)$ cannot be replaced by $W^{k,p}(\Omega)$ in Proposition 1.13. However, this replacement can be made for a large class of domains, which includes, for example, domains with a smooth boundary.

Definition 1.15. A bounded domain $\Omega \subset \mathbf{R}^N$ with boundary $\partial\Omega$. Let k be a nonnegative integer and $\alpha \in [0, 1]$. Ω is called $\mathbf{C}^{k,\alpha}$ if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and one-to-one mapping φ from B onto $D \subset \mathbf{R}^N$ such that

(1) $\varphi(B \cap \Omega) \subset \mathbf{R}^N_+ := \{x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : x_N > 0\}$.
(2) $\varphi(B \cap \partial\Omega) \subset \partial\mathbf{R}^N_+ := \{x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : x_N = 0\}$.
(3) $\varphi \in \mathbf{C}^{k,\alpha}(B), \varphi^{-1} \in \mathbf{C}^{k,\alpha}(D)$.

The following proposition is due to Gilbarg and Trudinger [160, Theorem 7.26].

Proposition 1.16. *Let Ω be a $\mathbf{C}^{0,1}$ domain in $\mathbf{R}^N$. Then*

(1) If $kp < N$, then $W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $p^* = Np/(N-kp)$; and $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < p^*$.
(2) If $0 \le m < k - N/p < m+1$, then $W^{k,p}(\Omega) \hookrightarrow \mathbf{C}^{m,\alpha}(\bar{\Omega}), \alpha = k - N/p - m$; and $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathbf{C}^{m,\beta}(\bar{\Omega})$ for any $\beta < \alpha$.

The following proposition can be found in Brezis [64] and Willem [335].

Proposition 1.17. *The following embeddings are continuous.*

$$H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N), \qquad 2 \le p < \infty, N = 1, 2,$$

$$H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N), \qquad 2 \le p \le 2^*, N \ge 3,$$

where $2^* := 2N/(N-2)$ if $N \ge 3$; $2^* = +\infty$ if $N = 1, 2$, is called a critical exponent.

For $N \ge 3$, let

$$S := \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$$

be the best Sobolev constant. Then, by Talenti's [321] result,

$$S = \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2},$$

where

$$U^*(x) = \frac{(N(N-2))^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}.$$

Note that if $\mathbf{R}^N$ is replaced by a bounded domain, S is never achieved. We frequently use the following Gagliardo–Nirenberg inequality, see Chabrowski [88], Evans [141], and Nirenberg [231].

Proposition 1.18. *For every $v \in H^1(\mathbf{R}^N)$,*

$$\|v\|_p \le c\|\nabla v\|_2^\gamma \|v\|_q^{1-\gamma}$$

with

$$\frac{N}{p} = \gamma\frac{N-2}{2} + (1-\gamma)\frac{N}{q}, \qquad q \ge 1, \gamma \in [0,1],$$

where c is a constant depending on p, γ, q, N.

Note. In this book, from time to time the letter c is indiscriminately used to denote various constants when the exact values are irrelevant.

The following concentration-compactness lemma due to Lions [196] is also a powerful tool in dealing with Schrödinger equations.

Lemma 1.19. *Let $r > 0$ and $q \in [2, 2^*)$. For any bounded sequence $\{w_n\}$ of $E := H^1(\mathbf{R}^N)$, if*

$$\sup_{y \in \mathbf{R}^N} \int_{B(y,r)} |w_n|^q dx \to 0, \qquad n \to \infty,$$

where $B(y, r) := \{u \in E : \|u - y\| \leq r\}$; then $w_n \to 0$ in $L^p(\mathbf{R}^N)$ for $q < p < 2^*$.

Proof. We only consider $N \geq 3$. Choose $p_1, p_2, t > 1, t' > 1$ such that

$$p_1 t = q, \qquad p_2 t' = 2^*, \qquad 1/t + 1/t' = 1, \qquad p_1 + p_2 = p.$$

By the Hölder inequality and Proposition 1.14, we have

$$\begin{aligned}
&\int_{B(y,r)} |w_n|^p dx \\
&\quad \leq \left(\int_{B(y,r)} |w_n|^{p_1 t} dx \right)^{1/t} \left(\int_{B(y,r)} |w_n|^{p_2 t'} dx \right)^{1/t'} \\
&\quad \leq c \left(\int_{B(y,r)} |w_n|^{p_1 t} dx \right)^{1/t} \quad \|w_n\|_{2^*}^{p_2} \\
&\quad \leq c \left(\int_{B(y,r)} |w_n|^{p_1 t} dx \right)^{1/t} \left(\int_{B(y,r)} (w_n^2 + |\nabla w_n|^2) dx \right)^{p_2/2} \\
&\quad \leq c \left(\int_{B(y,r)} |w_n|^{p_1 t} dx \right)^{1/t} \left(\int_{B(y,r)} (w_n^2 + |\nabla w_n|^2) dx \right)^{p_2/2}.
\end{aligned}$$

Covering $\mathbf{R}^N$ by balls of radius r in such a way that each point of $\mathbf{R}^N$ is contained in at most $N + 1$ balls, we have

$$\int_{\mathbf{R}^N} |w_n|^p dx \leq (N + 1) c \sup_{y \in \mathbf{R}^N} \left(\int_{B(y,r)} |w_n|^q dx \right)^{1/t},$$

which implies the conclusion. □

1.4 Differentiable Functionals

Let E be a Banach space with the norm $\| \cdot \|$. Let $U \subset E$ be an open set of E. The conjugate (or dual) space of E is denoted by E'; that is, E' denotes the set of all bounded linear operators on E. Consider a functional $G : U \to \mathbf{R}$.

Definition 1.20. The functional G has a Fréchet derivative $F \in E'$ at $u \in U$ if

$$\lim_{h\in E, h\to 0} \frac{G(u+h) - G(u) - F(h)}{\|h\|} = 0.$$

We denote $G'(u) = F$ or $\nabla G(u) = F$ and sometimes say the gradient of G at u. Usually, $G'(\cdot)$ is a nonlinear operator. We use $\mathbf{C}^1(U, \mathbf{R})$ to denote the set of all functionals G that have a continuous Fréchet derivative on U. A point $u \in U$ is called a critical point of a functional $G \in \mathbf{C}^1(U, \mathbf{R})$ if $G'(u) = 0$.

Definition 1.21. The functional G has a Gateaux derivative $I \in E'$ at $u \in U$ if, for every $h \in E$,

$$\lim_{t\to 0} \frac{G(u+th) - G(u)}{t} = I(h).$$

The Gateaux derivative at $u \in U$ is denoted by $DG(u)$. Obviously, if G has a Fréchet derivative $F \in E'$ at $u \in U$, then G has a Gateaux derivative $I \in E'$ at $u \in U$ and $G'(u) = DG(u)$. Unfortunately, the converse is not true. However, if G has Gateaux derivatives at every point of some neighborhood of $u \in U$ such that $DG(u)$ is continuous at u, then G has a Fréchet derivative and $G'(u) = DG(u)$. This is a straightforward consequence of the mean value theorem.

Sometimes, we use the concepts of the second-order Fréchet and Gateaux derivatives.

Definition 1.22. The functional $G \in \mathbf{C}^1(U, \mathbf{R})$ has a second-order Fréchet derivative at $u \in U$ if there is an L, which is a linear bounded operator from E to E', such that

$$\lim_{h\in E, h\to 0} \frac{G'(u+h) - G'(u) - Lh}{\|h\|} = 0;$$

we denote $G''(u) = L$.

We say that $G \in \mathbf{C}^2(U, \mathbf{R})$ if the second-order Fréchet derivative of G exists and is continuous on U.

Definition 1.23. The functional $G \in \mathbf{C}^1(U, \mathbf{R})$ has a second-order Gateaux derivative at $u \in U$ if there is an L, which is a linear bounded operator from E to E', such that

$$\lim_{t\to 0} \frac{(G'(u+th) - G'(u) - Lth)v}{t} = 0, \qquad \forall h, v \in E.$$

We denote $D^2G(u) = L$.

Evidently, any second-order Fréchet derivative of G is a second-order Gateaux derivative. Using the mean value theorem, if G has a continuous second-order Gateaux derivative on U, then $G \in \mathbf{C}^2(U, \mathbf{R})$.

Definition 1.24. Let $f(x,t)$ be a function on $\Omega \times \mathbf{R}$, where Ω is either bounded or unbounded. We say that f is a Carathéodory function if $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbf{R}$.

Lemma 1.25. *Assume $p \geq 1, q \geq 1$. Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbf{R}$ and satisfy*

$$|f(x,t)| \leq a + b|t|^{p/q}, \qquad \forall (x,t) \in \Omega \times \mathbf{R},$$

where $a, b > 0$ and Ω is either bounded or unbounded. Define a Carathéodory operator by

$$Ou := f(x, u(x)), \qquad u \in L^p(\Omega).$$

Let $\{w_k\}_{k=0}^{\infty} \subset L^p(\Omega)$. If $\|w_k - w_0\|_p \to 0$ as $k \to +\infty$, then

$$\|Ow_k - Ow_0\|_q \to 0$$

as $k \to \infty$. In particular, if Ω is bounded, then O is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a = 0$.

Proof. Note that

$$w_k(x) \to w_0(x), \qquad \text{a.e. } x \in \Omega. \tag{1.6}$$

Because f is a Carathéodory function,

$$Ow_k(x) \to Ow_0(x), \qquad \text{a.e. } x \in \Omega. \tag{1.7}$$

Let

$$v_k(x) := a + b|w_k(x)|^{p/q}, \qquad k = 0, 1, 2, \ldots. \tag{1.8}$$

Then by (1.6)–(1.8),

$$|Ow_k(x)| \leq v_k(x) \quad \text{for all } x \in \Omega; \qquad v_k(x) \to v_0(x) \ \text{ a.e. } x \in \Omega. \tag{1.9}$$

Because

$$|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p| \geq 0,$$

by Fatou's theorem, we have

$$\begin{aligned} &\int_\Omega \liminf_{k \to +\infty} (|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p|) dx \\ &\quad \leq \liminf_{k \to +\infty} \int_\Omega (|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p|) dx. \end{aligned} \tag{1.10}$$

Combining (1.6)–(1.10), thus we see that

$$\lim_{k\to+\infty}\int_\Omega ||w_k|^p-|w_0|^p|dx=0. \tag{1.11}$$

It follows that

$$\int_\Omega |v_k-v_0|^q dx\le b^q\int_\Omega ||w_k|^p-|w_0|^p|dx\to 0 \tag{1.12}$$

as $k\to\infty$. Because there is a constant $C>0, C_1>0$ such that

$$\begin{aligned}&|Ow_k-Ow_0|^q\\ &\quad\le C(|Ow_k|^q+|Ow_0|^q)\\ &\quad\le C(|v_k|^q+|v_0|^q)\\ &\quad\le C_1(|v_k-v_0|^q+|v_0|^q)\end{aligned}$$

a.e. $x\in\Omega$, then by Fatou's theorem,

$$\begin{aligned}&\int_\Omega \liminf_{k\to+\infty}(C_1(|v_k-v_0|^q+|v_0|^q)-|Ow_k-Ow_0|^q|)dx\\ &\quad\le \liminf_{k\to+\infty}\int_\Omega (C_1(|v_k-v_0|^q+|v_0|^q)-|Ow_k-Ow_0|^q)dx.\end{aligned} \tag{1.13}$$

By (1.7), (1.8), (1.12), and (1.13), we have

$$\|Ow_k-Ou_0\|_q\to 0.$$

Finally, if Ω is bounded, then for any $u\in L^p(\Omega)$, evidently we have

$$\|Ou\|_q\le c+c\|u\|_p^{p/q}, \tag{1.14}$$

where $c>0$ is a constant. Equation (1.14) remains true if Ω is unbounded and $a=0$. Therefore, O is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a=0$. □

The following lemma comes from Willem [335].

Lemma 1.26. *Assume $p_1,p_2,q_1,q_2\ge 1$. Let $f(x,t)$ be a Carathéodory function on $\Omega\times\mathbf{R}$ and satisfy*

$$|f(x,t)|\le a|t|^{p_1/q_1}+b|t|^{p_2/q_2},\qquad \forall (x,t)\in\Omega\times\mathbf{R},$$

where $a,b\ge 0$ and Ω is either bounded or unbounded. Define a Carathéodory operator by

$$Ou:=f(x,u(x)),\qquad u\in\mathcal{H}:=L^{p_1}(\Omega)\cap L^{p_2}(\Omega).$$

Define the space

$$\mathcal{E}_0 := L^{q_1}(\Omega) + L^{q_2}(\Omega)$$

with a norm

$$\|u\|_{\mathcal{E}_0} = \inf\{\|v\|_{L^{q_1}(\Omega)} + \|w\|_{L^{q_2}(\Omega)} : u = v + w \in \mathcal{E}_0, v \in L^{q_1}(\Omega), w \in L^{q_2}(\Omega)\}.$$

Then $O = O_1 + O_2$, where O_i is bounded continuous from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega), i = 1, 2$. In particular, O is a bounded continuous mapping from $\mathcal{H}$ to $\mathcal{E}_0$.

Proof. Let $\xi : \mathbf{R} \to [0, 1]$ be a smooth function such that $\xi(t) = 1$ for $t \in (-1, 1)$; $\xi(t) = 0$ for $t \notin (-2, 2)$. Let

$$\phi(x, t) = \xi(t) f(x, t), \qquad \psi(x, t) = (1 - \xi(t)) f(x, t).$$

We may assume that $p_1/q_1 \leq p_2/q_2$. Then there are two constants $d > 0$, $m > 0$ such that

$$|\phi(x, t)| \leq d|t|^{p_1/q_1}, \qquad |\psi(x, t)| \leq m|t|^{p_2/q_2}.$$

Define

$$O_1 u = \phi(x, u), \qquad u \in L^{p_1}(\Omega);$$
$$O_2 u = \psi(x, u), \qquad u \in L^{p_2}(\Omega).$$

Then by Lemma 1.25, O_i is bounded continuous from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega), i = 1, 2$. It is readily seen that $O = O_1 + O_2$ is a bounded continuous mapping from $\mathcal{H}$ to $\mathcal{E}_0$. □

The following theorem and its idea of proof are enough for us to see those functionals encountered in this book are of $\mathbf{C}^1$.

Theorem 1.27. *Assume $\kappa \geq 0, p \geq 0$. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbf{R}$ and satisfy*

$$|f(x, t)| \leq a|t|^\kappa + b|t|^p, \qquad \forall (x, t) \in \Omega \times \mathbf{R}, \tag{1.15}$$

where $a, b > 0$ and Ω is either bounded or unbounded. Define a functional

$$J(u) := \int_\Omega F(x, u)dx, \qquad \text{where } F(x, u) = \int_0^u f(x, s)ds.$$

Assume $(E, \|\cdot\|)$ is a Sobolev Banach space such that $E \hookrightarrow L^{p+1}(\Omega)$ and $E \hookrightarrow L^{\kappa+1}(\Omega)$; then $J \in \mathbf{C}^1(E, \mathbf{R})$ and

$$J'(u)h := \int_\Omega f(x, u)h dx, \qquad \forall\ h \in E.$$

Moreover, if $E \hookrightarrow\hookrightarrow L^{\kappa+1}, E \hookrightarrow\hookrightarrow L^{p+1}$, *then* $J' : E \to E'$ *is compact.*

Proof. Because $E \hookrightarrow L^{\kappa+1}(\Omega)$ and $E \hookrightarrow L^{p+1}(\Omega)$, we may find a constant $C_0 > 0$ such that

$$\|w\|_{\kappa+1} \le C_0\|w\|, \qquad \|w\|_{p+1} \le C_0\|w\|, \qquad \forall w \in E. \tag{1.16}$$

Recall the Young inequality and

$$(|s| + |t|)^\tau \le 2^{\tau-1}(|s|^\tau + |t|^\tau), \qquad \tau \ge 1, s, t \in \mathbf{R}.$$

Combining the assumptions on f, for any $\gamma \in [0, 1]$, it is easy to check that

$$|f(x, u + \gamma h)h| \le C_1(|u|^{(p+1)} + |h|^{(p+1)} + |u|^{\kappa+1} + |h|^{\kappa+1}),$$

where C_1 is a constant independent of γ. Therefore, for any $u, h \in E$, by the mean value theorem and Lebesgue theorem,

$$\begin{aligned} &\lim_{t\to 0} \frac{J(u + th) - J(u)}{t} \\ &\quad = \lim_{t\to 0} \int_\Omega f(x, u + \theta th)hdx \\ &\quad = \int_\Omega f(x, u)hdx \\ &\quad =: T_0(u, h), \end{aligned} \tag{1.17}$$

where $\theta \in [0, 1]$ depending on u, h, t. Obviously, $T_0(u, h)$ is linear in h. Furthermore, by (1.16),

$$\begin{aligned} &|T_0(u, h)| \\ &\quad \le \int_\Omega |f(x, u)h|dx \\ &\quad \le c(\|u\|^\kappa_{\kappa+1}\|h\|_{\kappa+1} + \|u\|^p_{p+1}\|h\|_{p+1}) \\ &\quad \le c(\|u\|^\kappa + \|u\|^p)\|h\|. \end{aligned}$$

It follows that $T_0(u, h)$ is linear bounded in h. Therefore, $DJ(u) = T_0(u, \cdot) \in E'$ is the Gateaux derivative of J at u. Next, we show that $DJ(u)$ is continuous in u. Let $Ou := f(x, u), u \in E$. By Lemma 1.26, $O = O_1 + O_2$, where O_1 is bounded continuous from $L^{\kappa+1}(\Omega)$ to $L^{(\kappa+1)/\kappa}(\Omega)$ and O_2 is bounded continuous from $L^{p+1}(\Omega)$ to $L^{(p+1)/\kappa}(\Omega)$. For any $v, h \in E$,

$$\begin{aligned} &|(DJ(u) - DJ(v))h| \\ &\quad = \left|\int_\Omega (f(x, u) - f(x, v))hdx\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \int_\Omega (Ou - Ov) h dx \right| \\
&= \left| \int_\Omega (O_1 u + O_2 u - O_1 v - O_2 v) h dx \right| \\
&\le \int_\Omega |O_1 u - O_1 v| |h| dx + \int_\Omega |O_2 u - O_2 v| |h| dx \\
&\le C_0 \|O_1 u - O_1 v\|_{(\kappa+1)/\kappa} \|h\| + C_0 \|O_2 u - O_2 v\|_{(p+1)/p} \|h\|.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
(1.18) \qquad & \|DJ(u) - DJ(v)\|_{E'} \\
& \le C_0 (\|O_1 u - O_1 v\|_{(\kappa+1)/\kappa} + \|O_2 u - O_2 v\|_{(p+1)/p}),
\end{aligned}
$$

where $\|\cdot\|_{E'}$ is the norm in E'. If $v_k \to u$ in $E \subset L^{\kappa+1}(\Omega) \cap L^{p+1}(\Omega)$, then

$$
\begin{aligned}
&\|O_1 v_k - O_1 u\|_{(\kappa+1)/\kappa} \to 0, \\
&\|O_2 v_k - O_2 u\|_{(p+1)/p} \to 0.
\end{aligned}
$$

Therefore, $DJ(v_k) \to DJ(u)$. This means $DJ(u)$ is continuous in u. Hence, $J'(u) = DJ(u)$; that is, $J \in \mathbf{C}^1(E, \mathbf{R})$. Furthermore, if $E \hookrightarrow\hookrightarrow L^{p+1}$, $E \hookrightarrow\hookrightarrow L^{\kappa+1}$, then any bounded sequence $\{u_k\}$ in E has a subsequence denoted by $\{u_k\}$ that converges to u_0 in $L^{p+1}(\Omega)$ and in $L^{\kappa+1}(\Omega)$. Hence, $O_1(u_k) \to O_1(u_0)$ in $L^{(\kappa+1)/\kappa}(\Omega)$; $O_2(u_k) \to O_2(u_0)$ in $L^{(p+1)/p}(\Omega)$. Finally, $DJ(u_k) \to DJ(u_0)$ in E'; that is, J' is compact in E. $\square$

1.5 The Topological Degree

Since the invention of Brouwer's degree in 1912, topological degree has become an eternal topic of every book on nonlinear functional analysis. Therefore, we just outline the main ideas and results and omit the proofs. Readers may consult the books of Berger [57], Chang [91], Deimling [134], Mawhin [224], Nirenberg [234], and Zeidler [337] (also Brézis and Nirenberg [72] for applications).

Definition 1.28. Let $W \subset X := \mathbf{R}^N$ $(N \ge 1)$ be an open subset and a mapping $J \in \mathbf{C}^1(W, X)$. A point $u \in W$ is called a regular point and $J(u)$ is a regular value if $J'(u) : X \to X$ is surjective. Otherwise, u is called a critical point and $J(u)$ is the critical value.

To construct the degree theory, we need a simplified Sard's theorem. Refer to Sard [266].

Theorem 1.29. *Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be an open subset and $J \in \mathbf{C}^1(W, X)$. Then the set of all critical values of J has zero Lebesgue measure in X.*

Definition 1.30 (Brouwer's degree). Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}^2(\bar{W}, X)$, $p \in X \backslash J(\partial W)$.

(1) If p is a regular value of J, define the Brouwer degree by

$$\deg(J, W, p) := \sum_{v \in J^{-1}(p)} \text{sign det} J'(v),$$

where det denotes the determinant.

(2) If p is a critical value of J, choose p_1 to be a regular value (by Sard's theorem) such that $\|p - p_1\| < \text{dist}(p, J(\partial W))$ and define the Brouwer degree by

$$\deg(J, W, p) := \deg(J, W, p_1).$$

In item (1), $J^{-1}(p)$ is a finite set when p is a regular value. In item (2), the degree is independent of the choice of p_1.

If $J \in \mathbf{C}(\bar{W}, X)$, we may find by Weierstrass's theorem an approximation of J via a smooth function.

Definition 1.31 (Brouwer's degree). Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}(\bar{W}, X)$, $p \in X \backslash J(\partial W)$. Choose $\tilde{J} \in \mathbf{C}^2(\bar{W}, X)$ such that

$$\sup_{u \in W} \|J(u) - \tilde{J}(u)\| < \text{dist}(p, J(\partial W))$$

and define Brouwer's degree by

$$\deg(J, W, p) := \deg(\tilde{J}, W, p),$$

which is independent of the choice of $\tilde{J}$.

Proposition 1.32. *Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}(\bar{W}, X)$, $p \in X \backslash J(\partial W)$.*

(1)

$$\deg(\mathbf{id}, W, p) = \begin{cases} 1, & p \in W, \\ 0, & p \notin \bar{\Omega}, \end{cases}$$

where **id** *is the identity.*

(2) Let W_1, W_2 be two disjoint open subsets of W, $p \notin J(\bar{W} \backslash (W_1 \cup W_2))$; then

$$\deg(J, W, p) = \deg(J, W_1, p) + \deg(J, W_2, p).$$

(3) Let $H \in \mathbf{C}([0,1] \times \bar{W}, \mathbf{R}^N)$, $p \in \mathbf{C}([0,1], \mathbf{R}^N)$ and $p(t) \notin H(t, \partial W)$. Then $\deg(H(t, \cdot), W, p(t))$ is independent of $t \in [0,1]$.

(4) (Kronecker's theorem) If $\deg(J, W, p) \neq 0$, *then there exists a* $u \in W$ *such that* $J(u) = p$.

Theorem 1.33 (Borsuk–Ulam theorem). *Let* W *be an open bounded symmetric neighborhood of* 0 *in* $\mathbf{R}^N$. *Every continuous odd map* $f : \partial W \to \mathbf{R}^{N-1}$ *has a zero.*

Brouwer's degree can be extended to infinite-dimensional spaces. This is the Leray–Schauder degree for a compact perturbation of the identity.

Definition 1.34. Let E be a Banach space; $M \subset E$. A mapping $J : M \to E$ is called compact if $\overline{J(S)}$ is compact for any bounded subset S of E. Furthermore, if J is continuous, we say that J is completely continuous. In this case, $\mathbf{id} - J$ is called a completely continuous field.

Theorem 1.35. *Let* E *be a Banach space and* $M \subset E$ *be a bounded closed subset. Let* $J : M \to E$ *be a continuous mapping. Then* J *is completely continuous if and only if, for any* $\varepsilon > 0$, *there exists a finite-dimensional subspace* E_n *of* E *and a bounded continuous mapping* $J_n : M \to E_n$ *such that*

$$\sup_{u \in D} \|J(u) - J_n(u)\| < \varepsilon.$$

Let E be a Banach space and $W \subset E$ be a bounded open subset. Let $J : \bar{W} \to E$ be completely continuous and $f = \mathbf{id} - J$. If $p \in E \backslash f(\partial W)$, then by Theorem 1.35, there exists a finite-dimensional subspace E_n of E and a bounded continuous mapping $J_n : \bar{W} \to E_n$ such that

$$\sup_{u \in W} \|J(u) - J_n(u)\| < \operatorname{dist}(p, f(\partial W)).$$

Denote $W_n = E_n \cap W$; $f_n(u) = u - J_n(u)$; then $f_n \in \mathbf{C}(\bar{W}_n, E_n), p \in E_n \backslash f_n(\partial W_n)$. Hence, $\deg(f_n, W_n, p_n)$ is well defined.

Definition 1.36 (Leray–Schauder degree). Let f be the completely continuous field defined as above. Define the Leray–Schauder degree of f at $p \in E \backslash f(\partial W)$ by

$$\deg(f, W, p) = \deg(f_n, W_n, p),$$

which is independent of the choice of E_n, p, J_n.

Proposition 1.37. *Let* $W \subset E$ *be a bounded open subset of the Banach space* E; $f = \mathbf{id} - J$ *is a completely continuous field,* $p \in E \backslash f(\partial W)$.

(1)

$$\deg(\mathbf{id}, W, p) = \begin{cases} 1, & p \in W, \\ 0, & p \notin \bar{W}. \end{cases}$$

(2) Let W_1, W_2 *be two disjoint open subsets of* W, $p \notin f(\bar{W} \backslash (W_1 \cup W_2))$*; then*

$$\deg(f, W, p) = \deg(f, W_1, p) + \deg(f, W_2, p).$$

(3) *Let* $H \in \mathbf{C}([0,1]\times\bar{W}, E)$ *be completely continuous,* $h_t(u) = u - H(t,u)$, $p \in \mathbf{C}([0,1], E)$, *and* $p(t) \notin h_t(\partial W)$ *for each* $t \in [0,1]$. *Then*

$$\deg(h_t(\cdot), W, p(t))$$

is independent of $t \in [0,1]$.

(4) *(Kronecker's theorem) If* $\deg(f, W, p) \neq 0$, *then there exists a* $u \in W$ *such that* $f(u) = p$.

Theorem 1.38 (Borsuk–Ulam theorem). *Let* W *be an open bounded symmetric neighborhood of* 0 *in a Banach space* E. *A completely continuous field* $f = \mathbf{id} - J : \bar{W} \to E$, *where* J *is odd on* ∂W; $p \in E\backslash f(\partial W)$; *then* $\deg(f, W, p)$ *is an odd number.*

1.6 The Global Flow

Let $(E, \|\cdot\|)$ be a Banach space. Consider the following Cauchy initial value problem of the ordinary differential equation.

$$\begin{cases} \dfrac{d\sigma}{dt} = W(\sigma(t,u_0)), \\ \sigma(0,u_0) = u_0 \in E, \end{cases} \tag{1.19}$$

where W is a potential function. We are interested in the existence of a solution to (1.19), which plays an important role in the following chapters. First, we prepare two auxiliary results.

Lemma 1.39 (Gronwall's inequality). *If* $\kappa \geq 0, \gamma > 0$ *and* $f \in \mathbf{C}([0,T], \mathbf{R}^+)$ *satisfies*

$$f(t) \leq \kappa + \gamma \int_0^t f(s)ds, \qquad \forall t \in [0,T], \tag{1.20}$$

then $f(t) \leq \kappa e^{\gamma t}$ *for all* $t \in [0,T]$.

Proof. By (1.20), we observe that $(d/dt)\big(e^{-\gamma t}\int_0^t f(s)ds\big) \leq \kappa e^{-\gamma t}$. Integrating both sides on $[0,t]$, we get the conclusion. □

Lemma 1.40 (Banach's fixed point theorem). *Let* $D \subset E$ *be closed. Let* $H : D \to D$ *satisfy*

$$\|Hu - Hv\| \leq k\|u - v\| \qquad \text{for some } k \in (0,1) \text{ and all } u, v \in D. \tag{1.21}$$

Then there exists a unique u^* *such that* $Hu^* = u^*$.

Proof. Let $u_{n+1} = Hu_n \quad (n = 0, 1, 2, \ldots)$ with $u_0 \in D$. Using (1.21) repeatedly, we have $\|u_{n+m+1} - u_n\| \le (1-k)^{-1} k^n \|u_1 - u_0\| \to 0$ as $n \to +\infty$. Therefore, $\{u_n\}$ is a Cauchy sequence. The conclusion follows from the continuity of H. $\square$

We assume

(O) $W : E \to E$ is a locally Lipschitz continuous mapping; that is, for any $u \in E$, there exists a ball $B(u, r) := \{w \in E : \|w - u\| < r\}$ with radius r and a constant $\rho > 0$ depending on r and u such that

$$\|W(w) - W(v)\| \le \rho \|w - v\|, \qquad \forall w, v \in B(u, r).$$

Moreover, $\|W(u)\| \le a + b\|u\|$ for all $u \in E$, where $a, b > 0$ are constants.

Theorem 1.41. *Assume* (O)*. Then for any $u \in E$, Cauchy problem (1.19) has a unique solution $\sigma(t, u)$ (called the flow or trajectory) defined in a maximal interval* $[0, +\infty)$ *of t.*

Proof. For any fixed $u_0 \in E$, by condition (O), we find a ball $B(u_0, r) := \{w \in E : \|w - u_0\| < r\}$ with radius r and a constant $\rho > 0$ depending on r and u_0 such that

$$\|W(w) - W(u_0)\| \le \rho \|w - u_0\|, \qquad \forall w \in B(u_0, r).$$

Let $\Lambda := \sup_{B(u_0,r)} \|W\|$. Then $\Lambda < +\infty$. Choose $\varepsilon > 0$ such that $\varepsilon\rho < 1$, $\varepsilon\Lambda \le r$. Consider the Banach space

$$\hat{E} := \mathbf{C}([0, \varepsilon], E) := \{u : [0, \varepsilon] \to E \text{ is a continuous function}\}$$

with the norm $\|u\|_{\hat{E}} := \max_{t \in [0,\varepsilon]} \|u(t)\|$ for each $u \in \hat{E}$. Let $D := \{u \in \hat{E} : \|u - u_0\|_{\hat{E}} \le r\}$. Define a mapping $H : \hat{E} \to \hat{E}$ by

$$Hu := u_0 + \int_0^t W(u(s))ds, \qquad u \in \hat{E}.$$

For any $u, w \in D$ we have

$$\|Hu - u_0\|_{\hat{E}} \le \int_0^t \|W(u(s))\|_{\hat{E}} ds \le \Lambda\varepsilon \le r$$

and

$$\|Hu - Hw\|_{\hat{E}} \le \max_{t \in [0,\varepsilon]} \int_0^t \|W(u) - W(w)\|_{\hat{E}} ds \le \rho\varepsilon \|u - w\|_{\hat{E}}.$$

Therefore, $H : D \to D$ satisfies all conditions of Lemma 1.40. Hence, H has a unique fixed point $u^* \in D$, which is a solution of Cauchy problem (1.19).

On the other hand, assume that $u(t)$ and $v(t)$ are solutions of the Cauchy problem (1.19) corresponding to the initial data u_0 and v_0, respectively. Then

$$\begin{aligned}
&\|u(t)-v(t)\|_{\hat{E}} \\
&\quad \leq \|u_0-v_0\|_{\hat{E}} + \int_0^t \|W(u(s)) - W(v(s))\|_{\hat{E}} ds \\
&\quad \leq \|u_0-v_0\|_{\hat{E}} + \rho \int_0^t \|u(s)-v(s)\|_{\hat{E}} ds.
\end{aligned}$$

By Lemma 1.39, $\|u(t)-v(t)\|_{\hat{E}} \leq \|u_0-v_0\|_{\hat{E}} e^{\rho t}$. This proves the continuous dependence on the initial data of solution of (1.19).

Summing up, (1.19) has a unique solution $u(t)$ on the maximal existence interval $[0,\varrho]$ which is continuously dependent on the initial data. Next, we just show that $\varrho = +\infty$. Assume that $\varrho < +\infty$; then

$$u(t) = u_0 + \int_0^t W(u(s))ds.$$

Then by (O),

$$\|u(t)\| \leq \|u_0\| + a\varrho + b\int_0^t \|u(s)\|ds.$$

Lemma 1.39 implies that there is a constant C_1 dependent on u_0, ϱ, a, and b such that $\|u(t)\| \leq C_1$. It follows that

$$\|u(t)-u(s)\| \leq C_2|t-s|.$$

This implies that the limit $\lim_{t\to\varrho-} u(t) = u_1$ exists. Consider the following Cauchy initial value problem.

$$\begin{cases} \dfrac{d\sigma}{dt} = W(\sigma(t,u)), \\ \sigma(0,u_1) = u_1 \in E. \end{cases} \tag{1.22}$$

Similarly, it has a unique solution $\bar{u}(t)$ on a maximal interval $[0,\varrho_1]$ with initial data $u_1 = u(\varrho-0)$. Let

$$v(t) = \begin{cases} u(t), & t\in[0,\varrho], \\ \bar{u}(t-\varrho), & t\in[\varrho,\varrho+\varrho_1], \end{cases}$$

Then $v(t)$ is also a solution of (1.19) with the initial data u_0 on the maximal interval $[0,\varrho+\varrho_1]$. This is a contradiction. □

1.7 The Local Flow

Let $(E, \|\cdot\|)$ be a Banach space. Sometimes, we cannot expect the global existence of the flow. But we have the following local results of the flow.

Theorem 1.42. *Let $u_0 \in E, R > 0, B(u_0, R) := \{w \in E : \|w - u_0\| < R\}$. Assume that $W : B(u_0, R) \to E$ is Lipschitz continuous:*

$$\|W(u) - W(v)\| \le K\|u - v\| \quad \text{for all } u, v \in B(u_0, R).$$

Then the following initial value problem

$$\frac{d\sigma(t, u_0)}{dt} = -W(\sigma(t, u_0)), \qquad \sigma(0, u_0) = u_0, \tag{1.23}$$

has a unique solution $\sigma : [-\delta, \delta] \to B(u_0, R)$, where

$$0 < \delta < \min\{R/M', 1/K\}, \qquad M' := \sup_{u \in B(u_0, R)} \|W(u)\|.$$

Proof. First, we note that

$$\begin{aligned}\|W(u)\| &\le \|W(u) - W(u_0)\| + \|W(u_0)\| \\ &\le K\|u - u_0\| + \|W(u_0)\| \\ &< \infty,\end{aligned}$$

thus it follows that $M' < \infty$. Define

$$\mathbf{C}([-\delta, \delta], E) := \{u(t) : [-\delta, \delta] \to E \text{ is continuous}\}.$$

Then $\mathbf{C}([-\delta, \delta], E)$ is a Banach space with the norm $\|u\|_{\mathbf{C}} := \max_{t \in [-\delta, \delta]} \|u(t)\|$. Let

$$D := \{u(t) \in \mathbf{C}([-\delta, \delta], E) : \|u(t) - u_0\| \le R, \ \forall\, t \in [-\delta, \delta]\}.$$

Then D is a closed subset of $\mathbf{C}([-\delta, \delta], E)$ with respect to the norm $\|\cdot\|_{\mathbf{C}}$. Define

$$Fu(t) = u_0 - \int_0^t W(u(t))dt.$$

Then the solution of (1.23) is equivalent to the fixed point of F. For each $u(t) \in D$, we observe that

$$\|Fu(t) - u_0\| = \left\| \int_0^t W(u(t))dt \right\| \le M'\delta < R; \tag{1.24}$$

it follows that F maps D into D. For any $u(t), v(t) \in D$,

$$\begin{aligned}
&\|Fu(t) - Fv(t)\| \\
&\quad \le \left\| \int_0^t (W(u) - W(v))dt \right\| \\
&\quad \le K\delta \left(\max_{t \in [-\delta,\delta]} \|u(t) - v(t)\| \right) \\
&\quad \le K\delta \|u - v\|_{\mathbf{C}};
\end{aligned}$$

then

$$\|Fu - Fv\|_{\mathbf{C}} \le K\delta \|u - v\|_{\mathbf{C}}.$$

Note that $K\delta < 1$; then by Lemma 1.40, F has only one fixed point $\sigma(t, u_0)$ in D. By (1.24), $\sigma(t, u_0) \in B(u_0, R)$. □

Theorem 1.43. *Let U be an open subset of E and $W : U \to E$ is locally Lipschitz continuous. Then the following initial value problem*

$$\frac{d\sigma(t, u_0)}{dt} = -W(\sigma(t, u_0)), \qquad \sigma(0, u_0) = u_0 \in U, \tag{1.25}$$

has a unique solution $\sigma : [0, T(u_0)) \to U$, where $T(u_0) \in (0, \infty]$ is the maximal time of the existence of the flow with initial data u_0. If $T(u_0) < +\infty$ and $\lim_{t \to T(u_0)-0} \sigma(t, u_0) = u_0^*$, then $u_0^* \in \partial U$.

Proof. By Theorem 1.42, there exists a $\delta_1 > 0$ such that (1.25) has a unique solution $\sigma_1(t, u_0) : [0, \delta_1] \to U$. Let $w_0 = \sigma_1(\delta_1, u_0) \in U$. Because W is locally Lipschitz continuous at w_0, we may find a $\delta_2 > 0$ such that the problem

$$\frac{d\sigma(t, w_0)}{dt} = -W(\sigma(t, w_0)), \qquad \sigma(0, w_0) = w_0, \tag{1.26}$$

has a unique solution $\sigma_2(t, w_0) : [0, \delta_2] \to U$. Define

$$\sigma(t, u_0) := \begin{cases} \sigma_1(t, u_0), & t \in [0, \delta_1], \\ \sigma_2(t - \delta_1, \sigma_1(\delta_1, u_0)), & t \in [\delta_1, \delta_1 + \delta_2]. \end{cases}$$

Then $\sigma(t, u_0) : [0, \delta_1 + \delta_2] \to U$ is also a solution of (1.25); that is, $\sigma_1(t, u_0)$ can be extended to $\sigma(t, u_0)$ from $[0, \delta_1]$ to $[0, \delta_1 + \delta_2]$. Keep going. We may assume that $\sigma_1(t, u_0)$ is extended to an interval $[0, \lambda)$ on the right-hand side of 0.

Next, we show that the solution of (1.25) on $[0, \lambda)$ is unique. Assume by negation that (1.25) has two solutions $\xi_1(t, u_0)$ and $\xi_2(t, u_0)$ on $[0, \lambda)$. Let

$$O = \{t \in (0, \lambda) : \xi_1(t, u_0) = \xi_2(t, u_0)\}.$$

By Theorem 1.42, there exists a $\delta > 0$ and a ball B_{u_0} centered at u_0 such that (1.25) has a unique solution $\sigma^*(t, u_0) : [-\delta, \delta] \to B_{u_0}$. If necessary, we may choose δ so small that $\xi_1(t, u_0), \xi_2(t, u_0) \in B_{u_0}$ for all $t \in [0, \delta]$. For $i = 1, 2$, define

$$\eta_i(t, u_0) := \begin{cases} \sigma^*(t, u_0), & t \in [-\delta, 0), \\ \xi_i(t, u_0), & t \in [0, \delta]. \end{cases}$$

Then $\eta_i(t, u_0) : [-\delta, \delta] \to B_{u_0}$ $(i = 1, 2)$ are two solutions of (1.25). By Theorem 1.42, we know that

$$\xi_1(t, u_0) = \xi_2(t, u_0), \qquad \forall\, t \in [0, \delta].$$

Therefore, O is a nonempty closed subset of $[0, \lambda)$. Take any $t_0 \in O$; then $\xi_1(t_0, u_0) = \xi_2(t_0, u_0) := h$. By Theorem 1.42, there exists a $\delta_3 > 0$ and a ball B_h centered at h such that (1.25) has a unique solution $\sigma^{**}(t, h) : [-\delta_3, \delta_3] \to B_h$. But, we know that $\xi_1(t+t_0, u_0), \xi_2(t+t_0, u_0)$ for all $t \in [-\delta_3, \delta_3]$ are also solutions of (1.25) if δ_3 is small enough; we must have

$$\xi_1(t + t_0, u_0) = \xi_2(t + t_0, u_0), \qquad \forall\, t \in [-\delta_3, \delta_3].$$

That is, $(t_0 - \delta_3, t_0 + \delta_3) \subset O$. Therefore, O is not only closed but also open in $[0, \lambda)$. Therefore, $O = [0, \lambda)$. This shows that the solution of (1.25) on $[0, \lambda)$ is unique. Obviously, by the above arguments, we may get a maximal interval $[0, T(u_0))$ of the right-hand side of 0 for the unique solution. Otherwise, we can continue to extend the solution. Here $T(u_0) \leq +\infty$. Finally, if $T(u_0) < +\infty$ and $\lim_{t \to T(u)-0} \sigma(t, u_0) = u_0^*$, then $u_0^* \in \partial U$. Indeed, if $u_0^* \notin \partial U$, then $u_0^* \in U$. Because U is open and W is locally Lipschitz continuous, by Theorem 1.42, we may get the solution starting from u_0^* defined on a small interval $[0, \varepsilon)$. Pasting the solutions defined on $[0, T(u_0))$ and on $[0, \varepsilon)$, we get a solution defined on $[0, T(u_0) + \varepsilon)$. This contradicts the fact that $[0, T(u_0))$ is maximal. □

The following theorem reveals the continuous dependence of the solution of (1.25) on the initial data.

Theorem 1.44. *Let U be an open subset of E and $W : U \to E$ is locally Lipschitz continuous. Let $\sigma : [0, T(u_0)) \to U$ be the unique solution of the initial value problem (1.25) with initial data $u_0 \in U$, where $T(u_0) \in (0, \infty]$ is the maximal time of the existence of the flow with initial data u_0. Then, for each $\lambda \in (0, T(u_0))$ and $\varepsilon > 0$, there exists an $r > 0$ such that for each $v \in B(u_0, r) := \{w \in E : \|w - u_0\| < r\}$, $T(v)$, the maximal time of the existence of the flow with initial data v, is greater than λ and*

$$\|\sigma(t, u_0) - \sigma(t, v)\| \leq \varepsilon, \qquad \forall t \in [0, \lambda];$$

hence, $\sigma(t, v) \to \sigma(t, u_0)$ uniformly for $t \in [0, \lambda]$ as $v \to u_0$.

Proof. Because $W : U \to E$ is locally Lipschitz, there is an $R > 0, K > 0$ such that

$$\|W(u) - W(v)\| \leq K\|u - v\|$$

for all $u, v \in B(u_0, R) := \{w \in W : \|w - u_0\| < R\}$. By Theorem 1.42, the initial value problem

$$\frac{d\sigma(t, u_0)}{dt} = -W(\sigma(t, u_0)), \qquad \sigma(0, u_0) = u_0, \tag{1.27}$$

has a unique solution $\sigma : [-\delta, \delta] \to E$, $\sigma(t, u_0) \in B(u_0, R)$ for all $t \in [-\delta, \delta]$, where

$$0 < \delta := \delta(R) < \min\{R/M', 1/K\}, \qquad M' := \sup_{u \in B(u_0, R)} \|W(u)\|. \tag{1.28}$$

The above arguments remain true if we replace R by $R/5$ and in this case, δ becomes $\delta/5$. This observation is useful in the following step.

Step 1. Take any $t_1 \in [-\delta/5, \delta/5]$; let $u_1 = \sigma(t_1, u_0)$. We show that for any u_1^* with $\|u_1^* - u_1\| < R/5$, the solution $\sigma(t, u_1^*)$ with initial data u_1^* exists on $[-\delta/5, \delta/5]$. Moreover,

$$\|\sigma(t, u_1^*) - \sigma(t + t_1, u_0)\| \leq 3\|u_1^* - u_1\| \qquad \forall t \in \left[-\frac{\delta}{5}, \frac{\delta}{5}\right].$$

In fact, when $t_1 \in [-\delta/5, \delta/5]$ and $\|u_1^* - u_1\| < R/5$, note that $\sigma(t_1, u_0) \in B(u_0, R/5)$, then $\|u_1^* - u_0\| \leq R/5 + \|u_1 - u_0\| \leq R/5 + R/5 = 2R/5$. Let

$$B(u_1^*, R/5) := \{w \in E : \|w - u_1^*\| \leq R/5\}.$$

Then $B(u_1^*, R/5) \subset B(u_0, 3R/5) \subset B(u_0, R)$. Therefore, $\sigma(t, u_1^*)$ exists on $[-\delta/5, \delta/5]$. Because

$$\sigma(t, u_1) = u_1 - \int_0^t W(\sigma(s, u_1))ds,$$

$$\sigma(t, u_1^*) = u_1^* - \int_0^t W(\sigma(s, u_1^*))ds,$$

for all $t \in [-\delta/5, \delta/5]$, we see that

$$\begin{aligned} &\|\sigma(t, u_1) - \sigma(t, u_1^*)\| \\ &\quad \leq \|u_1 - u_1^*\| + K \int_0^t \|\sigma(s, u_1) - \sigma(s, u_1^*)\| ds \end{aligned}$$

for all $t \in [0, \delta/5]$. By Lemma 1.39 (Gronwall's inequality), we have that

$$\|\sigma(t, u_1) - \sigma(t, u_1^*)\| \leq e^{Kt}\|u_1 - u_1^*\|$$

for all $t \in [0, \delta/5]$. In a similar way, this is also true for $t \in [-\delta/5, 0]$. Hence,

$$\|\sigma(t, u_1) - \sigma(t, u_1^*)\| \leq e^{2\delta K/5}\|u_1 - u_1^*\| \tag{1.29}$$

for all $t \in [-\delta/5, \delta/5]$. Finally,

$$\sigma(t, u_1) \equiv \sigma(t + t_1, u_0), \qquad \forall t \in [-\delta/5, \delta/5].$$

Combining this with (1.29) and (1.28), we get the conclusion stated in the beginning of Step 1.

Step 2. Take $\lambda \in (0, T(u_0))$ and $\varepsilon > 0$. Any $t^* \in [0, \lambda]$ and let $u^* = \sigma(t^*, u_0)$. By Step 1, there are $\delta(t^*) > 0, R(t^*) > 0$ such that for any $s^* \in [-\delta(t^*), \delta(t^*)]$, $\sigma(s^*, u^*) = \sigma(t^* + s^*, u_0)$, as long as $\|w - \sigma(t^* + s^*, u_0)\| < R(t^*)$, the solution $\sigma(t, w)$ exists on $[-\delta(t^*), \delta(t^*)]$ and

$$\|\sigma(t, w) - \sigma(t + t^* + s^*, u_0)\| \leq 3\|w - \sigma(t^* + s^*, u_0)\| \tag{1.30}$$

for all $t \in [-\delta(t^*), \delta(t^*)]$. Consider the covering of $[0, \lambda]$:

$$\{(t^* - \delta(t^*), t^* + \delta(t^*)) : t^* \in [0, \lambda]\}.$$

Then we have a finite covering of $[0, \lambda]$. We denote it by

$$(t_i^* - \delta(t_i^*), t_i^* + \delta(t_i^*)), \qquad i = 1, 2, \ldots, m.$$

Without loss of generality, we assume that

$$0 = t_1^* < t_2^* < \cdots < t_m^* = \lambda; \tag{1.31}$$

$$t_1^* + \delta(t_1^*) \in (t_2^* - \delta(t_2^*), t_2^* + \delta(t_2^*)); \tag{1.32}$$

$$t_1^* + \delta(t_1^*) + \delta(t_2^*) \in (t_3^* - \delta(t_3^*), t_3^* + \delta(t_3^*)); \tag{1.33}$$

$$t_1^* + \delta(t_1^*) + \delta(t_2^*) + \delta(t_3^*) \in (t_4^* - \delta(t_4^*), t_4^* + \delta(t_4^*)); \tag{1.34}$$

$$\cdots$$

$$\delta(t_1^*) + \delta(t_2^*) + \delta(t_3^*) + \cdots + \delta(t_m^*) > \lambda. \tag{1.35}$$

Let

$$r = \frac{1}{3^{m+3}} \min\{R(t_1^*), \ldots, R(t_m^*), \varepsilon\}. \tag{1.36}$$

Now, for each $v \in B(u_0, r)$, let $s_1^* = 0$; then $\|v - u_0\| = \|v - \sigma(t_1^* + s_1^*, u_0)\| < r < R(t_1^*)$. The argument for (1.30) implies that $\sigma(t, v)$ exists on $[-\delta(t_1^*), \delta(t_1^*)]$ and that

$$\|\sigma(t, v) - \sigma(t + t_1^* + s_1^*, u_0)\| \leq 3\|v - \sigma(t_1^* + s_1^*, u_0)\| < 3r < \varepsilon \tag{1.37}$$

for all $t \in [-\delta(t_1^*), \delta(t_1^*)]$. Hence,

$$\|\sigma(\delta(t_1^*), v) - \sigma(\delta(t_1^*) + t_1^* + s_1^*, u_0)\| < 3r. \tag{1.38}$$

Choose $s_2^* = \delta(t_1^*) + t_1^* + s_1^* - t_2^*$; then by (1.32), we see that $s_2^* \in (-\delta(t_2^*), \delta(t_2^*))$. Thus, (1.38) becomes

$$\|\sigma(\delta(t_1^*), v) - \sigma(t_2^* + s_2^*, u_0)\| < 3r < R(t_2^*). \tag{1.39}$$

The argument for (1.30) implies that $\sigma(t, \sigma(\delta(t_1^*), v))$ exists on $[-\delta(t_2^*), \delta(t_2^*)]$ and

$$\begin{aligned}
&\|\sigma(t, \sigma(\delta(t_1^*), v)) - \sigma(t + t_2^* + s_2^*, u_0)\| \\
&\quad = \|\sigma(t + \delta(t_1^*), v) - \sigma(t + t_2^* + s_2^*, u_0)\| \\
&\quad \leq 3\|\sigma(\delta(t_1^*), v) - \sigma(t_2^* + s_2^*, u_0)\| \\
&\quad = 3\|\sigma(\delta(t_1^*), v) - \sigma(\delta(t_1^*) + t_1^* + s_1^*, u_0)\| \\
&\quad < 3^2 r
\end{aligned} \tag{1.40}$$

for all $t \in [-\delta(t_2^*), \delta(t_2^*)]$. This is also true if $v = u_0$; that is,

$$\|\sigma(t + \delta(t_1^*), u_0) - \sigma(t + t_2^* + s_2^*, u_0)\| < 3^2 r. \tag{1.41}$$

By Equations (1.40) and (1.41), we have

$$\|\sigma(t + \delta(t_1^*), v) - \sigma(t + \delta(t_1^*), u_0)\| < 2 \cdot 3^2 r < \varepsilon \tag{1.42}$$

for all $t \in [-\delta(t_2^*), \delta(t_2^*)]$.

In (1.42), we choose $t = \delta(t_2^*)$ and $s_3^* = \delta(t_2^*) + t_2^* + s_2^* - t_3^*$; then by (1.33), $s_3^* \in [-\delta(t_3^*), \delta(t_3^*)]$. Moreover,

$$\|\sigma(\delta(t_2^*) + \delta(t_1^*), v) - \sigma(\delta(t_3^* + s_3^*), u_0)\| < 3^2 r < R(t_3^*). \tag{1.43}$$

The argument for (1.30) implies that

$$\sigma(t, \sigma(\delta(t_1^*) + \delta(t_2^*), v)) = \sigma(t + \delta(t_1^*) + \delta(t_2^*), v)$$

exists on $[-\delta(t_3^*), \delta(t_3^*)]$ and

$$\begin{aligned}
&\|\sigma(t, \sigma(\delta(t_1^*) + \delta(t_2^*), v)) - \sigma(t + t_3^* + s_3^*, u_0)\| \\
&\quad = \|\sigma(t + \delta(t_1^*) + \delta(t_2^*), v) - \sigma(t + t_3^* + s_3^*, u_0)\|
\end{aligned} \tag{1.44}$$

$$\leq 3\|\sigma(\delta(t_1^*) + \delta(t_2^*), v) - \sigma(\delta(t_2^*) + t_2^* + s_2^*, u_0)\|$$
$$< 3^3 r$$

for all $t \in [-\delta(t_3^*), \delta(t_3^*)]$. This is also true if $v = u_0$; that is,

$$\|\sigma(t + \delta(t_1^*) + \delta(t_2^*), u_0) - \sigma(t + t_3^* + s_3^*, u_0)\| < 3^3 r. \tag{1.45}$$

By (1.44) and (1.45), we have

$$\|\sigma(t + \delta(t_1^*) + \delta(t_2^*), v) - \sigma(t + \delta(t_1^*) + \delta(t_2^*), u_0)\| < 2 \cdot 3^3 r < \varepsilon \tag{1.46}$$

for all $t \in [-\delta(t_3^*), \delta(t_3^*)]$. Combine (1.37), (1.42), and (1.46). We know that $\sigma(t, v)$ exists on $[0, \delta(t_1^*) + \delta(t_2^*) + \delta(t_3^*)]$ and

$$\|\sigma(t, v) - \sigma(t, u_0)\| < \varepsilon, \qquad t \in [0, \delta(t_1^*) + \delta(t_2^*) + \delta(t_3^*)].$$

We continue this procedure, we observe that $\sigma(t, v)$ exists on

$$[0, \delta(t_1^*) + \cdots + \delta(t_m^*)]$$

and

$$\|\sigma(t, v) - \sigma(t, u_0)\| < \varepsilon, \qquad t \in [0, \delta(t_1^*) + \cdots + \delta(t_m^*)].$$

This completes the proof of the theorem because $\delta(t_1^*) + \cdots + \delta(t_m^*) > \lambda$. □

Remark 1.45. From the proof of Theorem 1.44, it is easily seen that the unique solution $\sigma(t, u)$ to (1.19) obtained in Theorem 1.41 is continuously dependent on the initial data u. Hence, $\sigma \in \mathbf{C}^1([0, +\infty) \times E, E)$.

Sometimes, we have to consider the local flow that starts from a point in a closed subset and does not leave that set. We first have a necessary condition for the existence of this kind of local flow.

Lemma 1.46. *Assume that $\mathcal{M}$ is a closed subset of E. Let $W \in \mathbf{C}(\mathcal{M}, E)$, $u_0 \in \mathcal{M}$. There exists a $\delta > 0$ such that the initial value problem*

$$\begin{cases} \dfrac{d\sigma(t, u_0)}{dt} = W(\sigma(t, u_0)) \\ \sigma(0, u_0) = u_0 \end{cases} \tag{1.47}$$

has a solution $\sigma(t, u_0) : [0, \delta] \to \mathcal{M}$. Then

$$\lim_{\lambda \to 0^+} \frac{\mathrm{dist}\,(u_0 + \lambda W(u_0), \mathcal{M})}{\lambda} = 0. \tag{1.48}$$

Proof. Note that

$$\sigma(\lambda, u_0) = u_0 + \lambda \sigma'(0, u_0) + o(\lambda), \qquad \lambda \to 0^+.$$

If $\lambda \to 0^+$, then $\sigma(\lambda, u_0) \in \mathcal{M}$. Hence,

$$\begin{aligned}\frac{\operatorname{dist}(u_0 + \lambda W(u_0), \mathcal{M})}{\lambda} &\\ &\le \frac{\|u_0 + \lambda W(u_0) - \sigma(\lambda, u_0)\|}{\lambda} \\ &= \frac{o(\lambda)}{\lambda},\end{aligned}$$

which implies the conclusion of the lemma. □

We observe that (1.48) holds automatically if u_0 is an interior point of $\mathcal{M}$. This means that (1.48) is actually a boundary condition. The next result is about the existence of a polygonal approximation of the solution of (1.47).

Lemma 1.47. *Assume that $\mathcal{M}$ is a closed subset of E. Let $W \in \mathbf{C}(\mathcal{M}, E)$, and $u_0 \in \mathcal{M}$. There exist $M' > 0, b > 0$ such that $\|W(u)\| \le M'$ for all $u \in \mathcal{M} \cap B(u_0, b)$, where $B(u_0, b) := \{u \in E : \|u - u_0\| \le b\}$. Moreover, assume that*

$$\lim_{\lambda \to 0^+} \frac{\operatorname{dist}(u + \lambda W(u), \mathcal{M})}{\lambda} = 0, \qquad \forall u \in \mathcal{M}. \tag{1.49}$$

Let $\varepsilon_n \in (0,1), \varepsilon_n \to 0$ as $n \to \infty$. Then for each n there is a

$$\sigma_n \in \mathbf{C}\left(\left[0, \frac{b}{M'+1}\right], B(u_0, b)\right)$$

and a sequence $0 = t_0^n < t_1^n < t_2^n < \cdots < t_i^n < \cdots \to b/(M'+1)$ as $i \to \infty$ such that

(1) $t_i^n - t_{i-1}^n \le \varepsilon_n, \quad i = 1, 2, \ldots;$
(2) $\sigma_n(0, u_0) = u_0, \|\sigma_n(t, u_0) - \sigma_n(s, u_0)\| \le (M'+1)|t - s|, \quad \forall s, t \in \left[0, \frac{b}{M'+1}\right];$
(3) $\sigma_n(t_{i-1}^n, u_0) \in \mathcal{M} \cap B(u_0, b)$ *and* $\sigma_n(t, u_0)$ *is linear on* $[t_{i-1}^n, t_i^n]$ *for* $i = 1, 2, \ldots;$
(4) $\left\| \frac{\sigma_n(t_i^n, u_0) - \sigma_n(t_{i-1}^n, u_0)}{t_i^n - t_{i-1}^n} - W(\sigma_n(t_{i-1}^n, u_0)) \right\| \le \varepsilon_n;$
(5) If $u \in \mathcal{M} \cap B(u_0, b)$ *with* $\|u - \sigma_n(t_{i-1}^n, u_0)\| \le (M'+1)(t_i^n - t_{i-1}^n)$, *then* $\|W(u) - W(\sigma_n(t_{i-1}^n, u_0))\| \le \varepsilon_n$.

Proof. For each fixed $n \ge 1$, we prove the lemma by induction. Obviously, we may find a $t_1^n > 0$ and construct a $\sigma_n(t, u_0)$ on $[0, t_1^n]$ that satisfies (1)–(5). Suppose that $\sigma_n(t, u_0)$ is well defined on $[0, t_{i-1}^n]$ ($t_{i-1}^n < b/(M'+1)$) and satisfies (1)–(5) above. Now we find $t_i^n > 0$ and define $\sigma_n(t, u_0)$ on $[t_{i-1}^n, t_i^n]$. Let $\gamma_i \in [0, \varepsilon_n]$ denote the maximal number that has the following properties.

(i) $t_{i-1}^n + \gamma_i \le \frac{b}{M'+1}$;
(ii) If $u \in \mathcal{M} \cap B(u_0, b)$ with $\|u - \sigma_n(t_{i-1}^n, u_0)\| \le (M'+1)\gamma_i$, then

$$\|W(u) - W(\sigma_n(t_{i-1}^n, u_0))\| \le \varepsilon_n;$$

(iii) $\operatorname{dist}\big(\sigma_n(t_{i-1}^n, u_0) + \gamma_i W(\sigma_n(t_{i-1}^n, u_0)), \mathcal{M}\big) \le \varepsilon_n \gamma_i / 2$.

Here, the existence of the maximal γ_i is obvious. In fact, (iii) comes from (1.49). Now we choose $t_i^n = t_{i-1}^n + \gamma_i$. By (iii), we take $z \in \mathcal{M}$ such that

$$\|\sigma_n(t_{i-1}^n, u_0) + (t_i^n - t_{i-1}^n) W(\sigma_n(t_{i-1}^n, u_0)) - z\| \le \varepsilon_n (t_i^n - t_{i-1}^n). \tag{1.50}$$

Define

$$\sigma_n(t, u_0) = \frac{z - \sigma_n(t_{i-1}^n, u_0)}{t_i^n - t_{i-1}^n}(t - t_{i-1}^n) + \sigma_n(t_{i-1}^n, u_0), \qquad \forall t \in [t_{i-1}^n, t_i^n]. \tag{1.51}$$

Then $\sigma_n(t_i^n, u_0) = z \in \mathcal{M}$. By (1.50), we have

$$\left\| \frac{\sigma_n(t_i^n, u_0) - \sigma_n(t_{i-1}^n, u_0)}{(t_i^n - t_{i-1}^n)} - W(\sigma_n(t_{i-1}^n, u_0)) \right\| \le \varepsilon_n; \tag{1.52}$$

this is (4). By it, we have

$$\begin{aligned}
\|\sigma_n(t, u_0) - \sigma_n(s, u_0)\| &= \left\| \frac{\sigma_n(t_i^n, u_0) - \sigma_n(t_{i-1}^n, u_0)}{t_i^n - t_{i-1}^n} \right\| |t - s| \\
&\le |t - s| (\|W(\sigma_n(t_{i-1}^n, u_0))\| + \varepsilon_n) \\
&\le (M'+1)|t - s|.
\end{aligned}$$

Therefore, (2) holds on $[0, t_i^n]$. Furthermore,

$$\begin{aligned}
&\|\sigma_n(t_i^n, u_0) - u_0\| \\
&\quad \le \sum_{j=1}^{i} \|\sigma_n(t_j^n, u_0) - \sigma_n(t_{j-1}^n, u_0)\| \\
&\quad \le (M'+1) \sum_{j=1}^{i} (t_j^n - t_{j-1}^n) \\
&\quad = (M'+1)(t_i^n) \\
&\quad \le b;
\end{aligned}$$

that is, $\sigma_n(t_i^n, u_0) \in B(u_0, b)$. It follows (3). In view of (ii), (5) is evident. If $t_i^n \to b/(M'+1)$ as $i \to \infty$, then, by (2) there exists $\lim_{t \to b/(M'+1)} \sigma_n(t, u_0) = \bar{u} \in B(u_0, b)$. Let $\sigma_n(b/(M'+1), u_0) = \bar{u}$. Then $\sigma_n(t, u_0)$ is what we want. Therefore, to finish the proof of the lemma, it suffices to show that $t_i^n \to b/(M'+1)$ as $i \to \infty$. By negation, if $t_i^n \to s_0 < b/(M'+1)$ as $i \to \infty$. By (2), we know that there exists a limit:

$$\lim_{i\to\infty} \sigma_n(t_i^n, u_0) = z^* \in \mathcal{M} \cap B(u_0, b).$$

By the continuity of W, we have a $\rho > 0$ such that $\|W(w) - W(z^*)\| \le \varepsilon_n/3$ as long as $\|w - z^*\| \le 2(M'+1)\rho$. By (1.49), there exists an $\varepsilon_0 > 0$ small enough such that

$$\operatorname{dist}(z^* + \varepsilon_0 W(z^*), \mathcal{M}) \le \varepsilon_0\varepsilon_n/3. \tag{1.53}$$

Choose i_0 large enough such that

$$s_0 - t_i^n < \varepsilon_0, \qquad \|z^* - \sigma_n(t_i^n, u_0)\| \le \varepsilon_0(M'+1), \qquad i \ge i_0. \tag{1.54}$$

Thus, for $t \in [t_i^n, t_i^n + \varepsilon_0]$, $i \ge i_0$, $w \in \mathcal{M} \cap B(u_0, b)$, and $\|w - \sigma_n(t_i^n, u_0)\| \le (M'+1)\varepsilon_0$, then

$$\begin{aligned}\|w - z^*\| &\le \|w - \sigma_n(t_i^n, u_0)\| + \|\sigma_n(t_i^n, u_0) - z^*\| \\ &\le 2\varepsilon_0(M'+1) < 2\rho(M'+1)\end{aligned}$$

because $\varepsilon_0 > 0$ was taken small enough. It follows that

$$\begin{aligned}&\|W(w) - W(\sigma_n(t_i^n, u_0))\| \\ &\quad\le \|W(w) - W(z^*)\| + \|W(z^*) - W(\sigma(t_i^n, u_0))\| \\ &\quad\le \frac{2}{3}\varepsilon_n.\end{aligned}$$

In items (i)–(iii), if we replace γ_{i+1} by ε_0, (i)–(ii) are still true. By (1.54), we see that $\varepsilon_0 > \gamma_{i+1}$ ($i \ge i_0$). Because γ_{i+1} is maximal, then (iii) is not true for ε_0. That is,

$$\operatorname{dist}(\sigma_n(t_i^n, u_0) + \varepsilon_0 W(\sigma_n(t_i^n, u_0)), \mathcal{M}) > \varepsilon_n\varepsilon_0/2, \qquad i \ge i_0.$$

Passing the limit as $i \to \infty$, we get that

$$\operatorname{dist}(z^* + \varepsilon_0 W(z^*), \mathcal{M}) > \varepsilon_n\varepsilon_0/2,$$

which contradicts (1.53). □

Theorem 1.48. *Assume that $\mathcal{M}$ is a closed subset of E. Let $W \in \mathbf{C}(\mathcal{M}, E)$, $u_0 \in \mathcal{M}$. There exists an $M' > 0, b > 0$ such that $\|W(u)\| \le M'$ for all $u \in \mathcal{M} \cap B(u_0, b)$, where $B(u_0, b) := \{u \in E : \|u - u_0\| \le b\}$. Moreover, assume that*

$$\lim_{\lambda\to 0^+}\frac{\operatorname{dist}(u+\lambda W(u),\mathcal{M})}{\lambda}=0,\qquad \forall u\in\mathcal{M}. \tag{1.55}$$

Let $g\in\mathbf{C}([0,\infty),\mathbf{R})$ *be a nondecreasing function with* $g(0)=0$ *and the following ODE*

$$u'(t)=g(u(t)),\ \ u(0)=0 \tag{1.56}$$

has only the trivial solution 0. Furthermore, assume that

$$\|W(u)-W(v)\|\le g(\|u-v\|),\qquad u,v\in\mathcal{M}\cap B(u_0,b). \tag{1.57}$$

Then the initial value problem

$$\begin{cases}\dfrac{d\sigma(t,u_0)}{dt}=W(\sigma(t,u_0)),\\ \sigma(0,u_0)=u_0,\end{cases} \tag{1.58}$$

has a unique solution

$$\sigma(t,u_0):\left[0,\frac{b}{M'+1}\right]\to\mathcal{M}\cap B(u_0,b)\subset\mathcal{M}.$$

Proof. By Lemma 1.47, for each $n\ge 1$, there is a

$$\sigma_n(t,u_0)\in\mathbf{C}\left(\left[0,\frac{b}{M'+1}\right],B(u_0,b)\right)$$

that satisfies (1)–(5). Now we show that the sequence $\{\sigma_n(t,u_0)\}$ is convergent uniformly in $[0,b/(M'+1)]$. Let

$$\pi(t):=\|\sigma_n(t,u_0)-\sigma_m(t,u_0)\|,\qquad t\in\left[0,\frac{b}{M'+1}\right].$$

For each $t\in[0,b/(M'+1)]$, we may assume that $t\in(t_i^n,t_{i+1}^n)\cap(t_j^m,t_{j+1}^m)$ for some i and j. Then by Lemma 1.47, we may estimate the Dini derivative $D^+\pi(t)$ of $\pi(t)$ as the following.

$$\begin{aligned}D^+\pi(t)&=\limsup_{h\to 0^+}\frac{\pi(t+h)-\pi(t)}{h}\\&\le\|\sigma_n'(t,u_0)-\sigma_m'(t,u_0)\|\\&=\left\|\frac{\sigma_n(t_{i+1}^n,u_0)-\sigma_n(t_i^n,u_0)}{t_{i+1}^n-t_i^n}-\frac{\sigma_m(t_{j+1}^m,u_0)-\sigma_m(t_j^m,u_0)}{t_{j+1}^m-t_j^m}\right\|\end{aligned}$$

$$\le \left\| \frac{\sigma_n(t_{i+1}^n, u_0) - \sigma_n(t_i^n, u_0)}{t_{i+1}^n - t_i^n} - W(\sigma_n(t_i^n, u_0)) \right\|$$
$$+ \|W(\sigma_n(t_i^n, u_0)) - W(\sigma_m(t_j^m, u_0))\|$$
$$+ \left\| \frac{\sigma_m(t_{j+1}^m, u_0) - \sigma_m(t_j^m, u_0)}{t_{j+1}^m - t_j^m} - W(\sigma_m(t_j^m, u_0)) \right\|$$
$$\le g(\|\sigma_n(t_i^n, u_0) - \sigma_m(t_j^m, u_0)\|) + (\varepsilon_n + \varepsilon_m). \tag{1.59}$$

Note that

$$\|\sigma_n(t_i^n, u_0) - \sigma_m(t_j^m, u_0)\|$$
$$\le \|\sigma_n(t_i^n, u_0) - \sigma_n(t, u_0)\| + \|\sigma_n(t, u_0) - \sigma_m(t, u_0)\|$$
$$+ \|\sigma_m(t, u_0) - \sigma_m(t_j^m, u_0)\|$$
$$\le (M' + 1)(\varepsilon_n + \varepsilon_m) + \pi(t)$$
$$:= \kappa(t).$$

Keeping in mind (1.59), we have

$$D^+\kappa(t) \le g(\kappa(t)) + (M' + 1)(\varepsilon_n + \varepsilon_m).$$

It is routine to show that

$$\pi(t) \le \kappa(t) \le \xi_{nm}(t), \qquad t \in \left[0, \frac{b}{M' + 1}\right], \tag{1.60}$$

where $\xi_{nm}(t)$ is the maximal solution of the ODE

$$u'(t) = g(u) + (M' + 1)(\varepsilon_n + \varepsilon_m), \qquad u(0) = (M' + 1)(\varepsilon_n + \varepsilon_m).$$

Because $(M' + 1)(\varepsilon_n + \varepsilon_m) \to 0$ as $n, m \to \infty$, $\xi_{nm}(t) \to \xi(t)$ uniformly for $t \in [0, b/(M' + 1)]$ and $\xi(t)$ is a solution of

$$u'(t) = g(u), \qquad u(0) = 0.$$

Therefore, by the assumption of the theorem, (1.56) has only the trivial solution 0. Then $\xi(t) = 0$ for all $t \in [0, b/(M' + 1)]$. Combine (1.60); the sequence $\{\sigma_n(t, u_0)\}$ is convergent uniformly in $[0, b/(M' + 1)]$ and its limit $\sigma(t, u_0) \in \mathbf{C}\left([0, b/(M' + 1)], B(u_0, b)\right)$. Evidently, $\sigma(t, u_0) \in \mathcal{M}$. To complete the proof of the theorem, we have only to show that

$$\sigma(t, u_0) = u_0 + \int_0^t W(\sigma(s, u_0))ds, \qquad t \in \left[0, \frac{b}{M' + 1}\right]. \tag{1.61}$$

For each $t \in [0, b/(M'+1)]$, we may assume that $t \in [t_i^n, t_{i+1}^n]$ for some i, n. Then we have that

$$\begin{aligned}
&\left\|\sigma_n(t,u_0) - u_0 - \int_0^t W(\sigma_n(s,u_0))ds\right\| \\
&\quad \le \left\|\sigma_n(t,u_0) - \sigma_n(t_i^n,u_0) - \int_{t_i^n}^t W(\sigma_n(s,u_0))ds\right\| \\
&\qquad + \sum_{j=1}^{i}\left\|\sigma_n(t_j^n,u_0) - \sigma_n(t_{j-1}^n,u_0) - \int_{t_{j-1}^n}^{t_j^n} W(\sigma_n(s,u_0))ds\right\|.
\end{aligned}$$

By (4)–(5) of Lemma 1.47, we get that

$$\begin{aligned}
&\left\|\sigma_n(t_j^n,u_0) - \sigma_n(t_{j-1}^n,u_0) - \int_{t_{j-1}^n}^{t_j^n} W(\sigma_n(s,u_0))ds\right\| \\
&\quad \le \|\sigma_n(t_j^n,u_0) - \sigma_n(t_{j-1}^n,u_0) - (t_j^n - t_{j-1}^n)W(\sigma_n(t_{j-1}^n,u_0))\| \\
&\qquad + \int_{t_{j-1}^n}^{t_j^n} \|W(\sigma_n(t_{j-1}^n,u_0)) - W(\sigma_n(s,u_0))\|ds \\
&\quad \le 2\varepsilon_n(t_j^n - t_{j-1}^n).
\end{aligned}$$

Similarly, we have that

$$\left\|\sigma_n(t,u_0) - \sigma_n(t_i^n,u_0) - \int_{t_i^n}^t W(\sigma_n(s,u_0))ds\right\| \le 2\varepsilon_n(t - t_i^n).$$

It follows that

$$\left\|\sigma_n(t,u_0) - u_0 - \int_0^t W(\sigma_n(s,u_0))ds\right\| \le 2\varepsilon_n t. \tag{1.62}$$

Recall (1.57) and note that $\sigma_n(t,u_0) \to \sigma(t,u_0)$ uniformly in $[0, b/(M'+1)]$ as $n \to \infty$; thus we readily have

$$\int_0^t W(\sigma_n(s,u_0))ds \to \int_0^t W(\sigma(s,u_0))ds$$

uniformly for t in $[0, b/(M'+1)]$. Combining this with (1.62), we get (1.61). That is, $\sigma(t,u_0)$ is the solution we want. Finally, we show the uniqueness of the solution. Assume that there is another solution $\tilde{\sigma}(t,u_0)$ in $[0, b/(M'+1)]$. Let $\tilde{\pi}(t) = \|\sigma(t,u_0) - \tilde{\sigma}(t,u_0)\|$, then $\tilde{\pi}(0) = 0$ and the Dini derivative

$$\begin{aligned} D^+\tilde{\pi}(t) &\le \|\sigma'(t,u_0) - \tilde{\sigma}'(t,u_0)\| \\ &\le \|W(\sigma(t,u_0)) - W(\tilde{\sigma}(t,u_0))\| \\ &\le g(\|\sigma(t,u_0) - \tilde{\sigma}(t,u_0)\|) \\ &= g(\tilde{\pi}(t)). \end{aligned}$$

It follows that $\tilde{\pi}(t) \equiv 0$ because the problem (1.56) has only a trivial solution. This completes the proof of the theorem. □

The following theorem is an immediate consequence of Theorem 1.48.

Theorem 1.49. *Assume E is a Banach space, $O \subset E$ is an open subset, $\mathcal{M} \subset O$ is a closed subset of E, $W : O \to E$ is locally Lipschitz continuous, and*

$$\lim_{\lambda \to 0^+} \frac{\operatorname{dist}(u + \lambda W(u), \mathcal{M})}{\lambda} = 0, \qquad \forall u \in \mathcal{M}.$$

Then for any given $u_0 \in \mathcal{M}$, there exists a $\delta > 0$ such that the initial value problem

$$\tag{1.63} \begin{cases} \dfrac{d\sigma(t,u_0)}{dt} = W(\sigma(t,u_0)) \\ \sigma(0,u_0) = u_0 \end{cases}$$

has a unique solution $\sigma(t,u_0)$ defined on $[0,\delta)$. Moreover, $\sigma(t,u_0) \in \mathcal{M}$ for all $t \in [0,\delta)$.

Notes and Comments. The ideas of Theorems 1.42–1.44 can be traced back to Berger [57], Deimling [133], and Martin [221, 222]. Their proofs can also be found in Guo [163]. Lemma 1.47 and Theorem 1.48 are taken from Guo and Sun [164] and Lakshmikantham and Leela [188]. Theorem 1.49 can be found in Brezis [65] (see also Deimling [133, 134] and Chang [94]). We also refer readers to Barbu [26], Deimling [133], and Martin [222] for other results on ODEs in abstract spaces.

1.8 The (PS) Condition

Let $(E, \|\cdot\|)$ be a Banach space and $(\cdot,\cdot)$ denote the pairing of E with its dual space. Our purpose is to find the critical points, that is, solve

$$\tag{1.64} J'(u) = 0,$$

where J is a $\mathbf{C}^1$ functional on a Banach space E. Equation (1.64) is called the Euler–Lagrange equation of the functional J. However, under many circumstances, we just can derive a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c, \qquad J'(u_n) \to 0.$$

Obviously, to get a solution of (1.64), some kinds of compactness conditions are necessary.

Definition 1.50. Any sequence $\{u_n\}$ satisfying

$$\sup_n |J(u_n)| < \infty, \qquad J'(u_n) \to 0, \tag{1.65}$$

is called a Palais–Smale sequence ((PS) sequence, for short). If any (PS) sequence of J possesses a convergent subsequence, we say that J satisfies the (PS) condition.

The original idea of the (PS) condition was introduced by Palais [239], Smale [305] and Palais and Smale [240] (see also H. Brézis et al. [67]). The following weak version of the (PS) condition was proposed in Cerami [84].

Definition 1.51. Any sequence $\{u_n\}$ satisfying

$$\sup_n |J(u_n)| < \infty, \qquad (1 + \|u_n\|)J'(u_n) \to 0, \tag{1.66}$$

is called a weak Palais–Smale sequence (in short, w-PS sequence). If any weak (PS) sequence of J possesses a convergent subsequence, we say that J satisfies the w-PS condition. If the supremum in (1.66) is replaced by: $J(u_n) \to c$ as $n \to \infty$, we say that J satisfies the w-PS at level c, written as (w-PS)$_c$.

Theorem 1.52. *Let E be a Banach space, $J \in \mathbf{C}^1(E, \mathbf{R})$. Assume*

$$J'(u) = Lu + I'(u), \qquad u \in E,$$

where $L : E \to E'$ is a bounded linear invertible operator and I' maps bounded sets to relatively compact sets in E'. Then any bounded (PS)-sequence or weak (PS)-sequence is relatively compact.

Proof. Let $\{u_n\}$ be a bounded (PS)-sequence or weak (PS)-sequence, then $J'(u_n) \to 0$. The conclusion follows from the relative compactness of I' and $u_n = L^{-1}J'(u_n) - L^{-1}I'(u_n)$. □

Assume that $J \in \mathbf{C}^1(E, \mathbf{R})$. Let $\mathcal{K} := \{u \in E : J'(u) = 0\}$ and $\tilde{E} := E \backslash \mathcal{K}$.

Definition 1.53. A locally Lipschitz continuous map $W : \tilde{E} \to E$ is called a pseudo-gradient vector fieldfor J if

- $(J'(u), W(u)) \geq \frac{1}{2}\|J'(u)\|^2$ for all $u \in \tilde{E}$,
- $\|W(u)\| \leq 2\|J'(u)\|$ for all $u \in \tilde{E}$.

Let W be a pseudo-gradient vector field of J. Consider the initial value problem:

$$
\begin{cases} \dfrac{d\eta}{dt} = -W(\eta), \\ \eta(0, u_0) = u_0 \in \tilde{E}. \end{cases} \tag{1.67}
$$

By Lemma 1.43, it has a unique solution (called flow or trajectory) $\eta : [0, T(u_0)) \to E$, where $T(u_0) \in (0, \infty]$ is the maximal time of the existence of the flow with initial data u_0.

We consider a simple application of the (PS) condition (see, for example, Guo and Sun [164]).

Lemma 1.54. *Assume $J \in \mathbf{C}^1(E, \mathbf{R})$ is bounded below and satisfies the (PS) condition. Then there is a sequence $\{\eta(t_n, u_0)\}$ such that*

$$\eta(t_n, u_0) \to u^*, \qquad J(u^*) \le J(u_0), \qquad J'(u^*) = 0.$$

Proof. By the definition of the pseudo-gradient vector field, it easy to check that

$$\frac{dJ(\eta(t, u_0))}{dt} \le 0.$$

Therefore,

$$\inf_E J \le J(\eta(t, u_0)) \le J(u_0), \qquad \forall\, t \in [0, T(u_0)).$$

Let $s \ge t, s, t \in [0, T(u_0))$; then by the definition of the pseudo-gradient vector field,

$$
\begin{aligned}
&\|\eta(s, u_0) - \eta(t, u_0)\| \\
&\quad \le \int_t^s \|W(\eta(r, u_0))\| dr \\
&\quad \le |s - t|^{1/2} \left(\int_t^s \|W(\eta(r, u_0))\|^2 dr \right) \\
&\quad \le 2|s - t|^{1/2} \left(\int_t^s 2(J'(\eta(r, u_0)), W(\eta(r, u_0))) dr \right)^{1/2} \\
&\quad \le 4|s - t|^{1/2} \left(J(u_0) - \inf_E J \right)^{1/2}.
\end{aligned}
$$

If $T(u_0) < \infty$, then $\|\eta(s, u_0) - \eta(t, u_0)\| \to 0$ as $s, t \to T(u_0)$. Hence, there exists a limit

$$\lim_{t \to T(u_0) - 0} \eta(t, u_0) := u^*.$$

By Lemma 1.43, $u^* \in \partial\tilde{E} \subset \mathcal{K}$. Hence, $J(u^*) \le J(u_0)$ and $J'(u^*) = 0$. If $T(u_0) = \infty$, note that

$$\begin{aligned}
&\int_0^t \frac{1}{2}\|J'(\eta(r,u_0))\|^2 dr \\
&\le \int_0^t (J'(\eta(r,u_0)), W(\eta(r,u_0)))dr \\
&= J(\eta(0,u_0)) - J(\eta(t,u_0)) \\
&\le J(u_0) - \inf_E J.
\end{aligned}$$

It follows that there is a sequence $t_n \to \infty$ such that $J'(\eta(t_n,u_0)) \to 0$. By the (PS) condition, up to a subsequence, $\eta(t_n,u_0) \to u^*$ with $J'(u^*) = 0, J(u^*) \le J(u_0)$. □

1.9 Lax–Milgram Theorem and Weak Solutions

We recall the Lax–Milgram theorem from linear functional analysis, which will provide in certain circumstances the existence and uniqueness of a weak solution to elliptic problems. Let, in this section, E be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Let $(\cdot,\cdot)$ denote the pairing of E with its dual space.

Theorem 1.55. *Assume that $B : E \times E \to \mathbf{R}$ is a bilinear mapping. There are two constants $a, b > 0$ such that $|B[u,v]| \le a\|u\|\|v\|$ for all $u, v \in E$ and $B[u,u] \ge b\|u\|^2$ for all $u \in E$. For any given bounded linear functional $g : E \to \mathbf{R}$, there exists a unique element $u \in E$ such that $B[u,v] = (g,v)$ for all $v \in E$.*

Proof. For each fixed $u \in E$, the mapping $B[u,\cdot]$ is a bounded linear functional on E. By Riesz's representation theorem, we may find a unique element u_0 such that

$$B[u,v] = \langle u_0, v\rangle \tag{1.68}$$

for all $v \in E$. We define $M_0 u := u_0$. Then $B[u,v] = \langle M_0 u, v\rangle$ for all $u, v \in E$. It is easy to see that $M_0 : E \to E$ is linear and $\|M_0 u\| \le a\|u\|$ for all $u \in E$. Note that

$$b\|u\|^2 \le B[u,u] = \langle M_0 u, u\rangle \le \|M_0 u\|\|u\|.$$

It follows that M_0 is a bounded one-to-one mapping with a closed range $R(M_0)$. We claim that $R(M_0) = E$. Otherwise, we have a $v \in R(M_0)^\perp, v \ne 0$. But this is impossible because $b\|v\|^2 \le [Bv,v] = \langle M_0 v, v\rangle = 0$. By Riesz's representation theorem again, we have a $w \in E$ such that $(g, v) = \langle w, v\rangle$ for all $v \in E$. Hence, we have a $u \in E$ such that $M_0 u = w$. Therefore,

$$B[u, v] = \langle M_0 u, v\rangle = \langle w, v\rangle = (g, v)$$

for all $v \in E$. The uniqueness of u is obvious. □

In practice, one of the main research projects by means of critical point theory is the existence of solutions to elliptic equations. For example, consider

$$-\Delta u = f(x, u), \qquad x \in \Omega \subset \mathbf{R}^N. \tag{1.69}$$

Then the corresponding functional is defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u)dx,$$

where $F(x, u) = \int_0^u f(x, s)ds$. Roughly speaking, if I is of $\mathbf{C}^1$ on a Sobolev space $(E, \|\cdot\|)$ and $I'(u) = 0$ (critical point), then

$$\int_\Omega \nabla u \cdot \nabla w dx - \int_\Omega f(x, u)w dx = 0, \qquad \forall w \in E. \tag{1.70}$$

The critical point u satisfying (1.70) is called a weak solution of (1.69) and obviously u is not necessarily a classical solution of (1.69). In general, more assumptions on the smoothness of $\partial\Omega$ and of f are needed if the weak solution wants to be a classical solution.

We just give a simple example and show how the regularity theory of an elliptic equation can be applied to obtain a classical solution from a weak solution. The following proposition is due to Gilbarg and Trudinger [160, Theorems 6.6] (see also Struwe [313]).

Proposition 1.56. *Suppose that $u \in H^{2,p}_{loc}(\Omega)$ such that $-\Delta u = f$ in Ω with $f \in L^p(\Omega), 1 < p < \infty$. Then for any $\Omega' \subset\subset \Omega$, there holds*

$$\|u\|_{H^{2,p}(\Omega')} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where C depends on Ω, Ω', N, p. Assume in addition that Ω is a $\mathbf{C}^{1,1}$ domain and that there exists a function $u_0 \in H^{2,p}(\Omega)$ such that $u - u_0 \in H^{1,p}_0(\Omega)$; then

$$\|u\|_{H^{2,p}(\Omega)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|u_0\|_{H^{2,p}(\Omega)}),$$

where C depends on Ω, N, p.

The following proposition is due to Gilbarg and Trudinger [160, Theorems 6.14 and 6.19].

Proposition 1.57. *Assume that Ω is a $\mathbf{C}^{k+2,\alpha}$ domain and $f \in \mathbf{C}^{k,\alpha}(\Omega)$. Then the Dirichlet problem*

$$-\Delta u = f \quad in\ \Omega, \qquad u = 0 \quad in\ \partial\Omega$$

has a unique classical solution $u \in \mathbf{C}^{k+2,\alpha}(\Omega)$.

The following proposition is due to Gilbarg and Trudinger [160, Theorem 9.15].

Proposition 1.58. *Assume that Ω is a $\mathbf{C}^{1,1}$ domain and $f \in L^p(\Omega), p > 1$. Then the Dirichlet problem*

$$-\Delta u = f \quad in\ \Omega, \qquad u = 0 \quad in\ \partial\Omega$$

has a unique classical solution $u \in {W_0}^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

The following result is due to Brézis and Kato [68] (see also Struwe [313]).

Lemma 1.59. *Assume that Ω is a domain of $\mathbf{R}^N$ $(N \geq 3)$ and that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that*

$$|f(x,t)| \leq a(x)(1 + |u|), \qquad a.e.\ x \in \Omega,$$

where $a \in L^{N/2}_{loc}(\Omega)$. If $u \in H^{1,2}_{loc}(\Omega)$ is a weak solution of

$$-\Delta u = f(x,u) \qquad in\ \Omega,$$

then $u \in L^q_{loc}(\Omega)$ for any $q < \infty$. If $u \in H^{1,2}_0(\Omega)$ and $a \in L^{N/2}(\Omega)$, then $u \in L^q(\Omega)$ for any $q < \infty$.

Next, we give an example to illustrate when a weak solution becomes a classical solution (see, e.g., Lu [218, Theorem 7.5.4]).

Theorem 1.60. *Assume that Ω is a bounded domain of $\mathbf{R}^N$ $(N \geq 2)$, Ω is of $\mathbf{C}^{k+2,\varrho}$ $(\varrho \in (0,1))$, and that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that*

(1) There exists $\tau \in (0,1]$ such that

$$f(x,t) \in \mathbf{C}^{k,\tau}(\bar{\Omega} \times [-M', M'], \mathbf{R}), \qquad for\ any\ M' > 0,$$

(2) There are $C > 0$ and $2 \leq p \leq 2^*$ such that

$$|f(x,t)| \leq C(1 + |u|^{p-1}), \quad a.e.\ x \in \Omega.$$

(3) There exists a function $f_0(x) \in L^\infty(\Omega)$ such that

$$\lim_{t \to 0} \frac{f(x,t)}{t} = f_0(x) \qquad uniformly\ for\ x \in \Omega.$$

Assume $u \in H_0^{1,2}(\Omega)$ *is a weak solution of*

$$-\Delta u = f(x,u) \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega. \tag{1.71}$$

Then u *must be a classical solution of (1.71). In particular,* $u \in \mathbf{C}^{k+2,\beta}(\bar{\Omega})$*, where* $\beta = \varrho\tau^{k+1}$.

Proof. Define

$$\phi(x,t) = \begin{cases} \dfrac{f(x,t)}{t}, & \text{if } t \neq 0, \\ f_0(x), & \text{if } t = 0. \end{cases}$$

Then there are two constants $a > 0, b > 0$ such that

$$|\phi(x,t)| \le f_0(x) + a + b|t|^{p-2} \le f_0(x) + a + b|t|^{2^*-2} \tag{1.72}$$

for all $x \in \bar{\Omega}, t \in \mathbf{R}$. By the assumption, $u \in H_0^{1,2}(\Omega)$ is a weak solution of

$$-\Delta u = \phi(x, u(x))u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.73}$$

If $N \ge 3$, by Proposition 1.13, $u \in L^{2^*}(\Omega)$. By (1.72) and Theorem 1.25, $\phi(x,u(x)) \in L^{N/2}(\Omega)$. Then, Lemma 1.59 implies that $u \in L^s(\Omega)$ for all $s \ge 2$. This is naturally true if $N \le 2$. Note the conditions (2)–(3) and use Theorem 1.25 again; we see that $f(x,u(x)) \in L^{s/(p-1)}(\Omega), \quad \forall s \ge 2$. Choose $s \ge 2, s \ge p-1$. By Proposition 1.58, the following problem

$$-\Delta w = f(x, u(x)) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on} \quad \partial\Omega \tag{1.74}$$

has a unique solution

$$w \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega), \qquad \forall q = \frac{s}{(p-1)} > 1, s \ge 2.$$

Because u is a weak solution of (1.71), combining (1.74), we have $u = w$. If we choose $q = N/(1-\varrho)$, then $q \ge 2/(p-1)$ if $N \ge 2$. By Proposition 1.13, $u \in W_0^{1,q}(\Omega)$ implies that $u \in \mathbf{C}^{0,\varrho}(\bar{\Omega})$; here $1 - N/q = \varrho$. Then we may find $M' > 0$ such that

$$|u(x)| \le M', \qquad \forall x \in \bar{\Omega},$$

$$|u(x) - u(y)| \le M'|x-y|^{\varrho}, \qquad x, y \in \bar{\Omega}.$$

Note that Ω is of $\mathbf{C}^{k+2,\varrho}(\varrho \in (0,1))$; then f satisfies. Condition (1) with k replaced by $0, 1, 2, \dots, k-1$ (see Gilbarg and Trudinger [160, Lemma 6.35] and Lu [218, Theorem 7.5.4]). Hence, there exists a $C > 0$ such that

$$|f(x,u) - f(y,v)| \le C(|x-y|^{\tau} + |u-v|^{\tau})$$

for all $(x,u), (y,v) \in \bar{\Omega} \times [-M', M']$. Therefore,

$$
\begin{aligned}
&|f(x,u(x)) - f(y,v(y))| \\
&\quad \le C(|x-y|^\tau + |u(x)-v(y)|^\tau) \\
&\quad \le C(|x-y|^\tau + (M')^\varrho |x-y|^{\varrho\tau}) \\
&\quad \le C(d^{\tau(1-\varrho)} + (M')^\varrho |x-y|^{\varrho\tau})
\end{aligned}
$$

for $x, y \in \bar{\Omega}$, where d is the diameter of Ω. This shows that $f(x,u(x)) \in \mathbf{C}^{0,\varrho r}(\bar{\Omega})$. By Proposition 1.57, (1.74) has a unique classical solution $w = u$ in $\mathbf{C}^{2,\,\varrho\tau}(\bar{\Omega})$. By induction, if we assume that $u \in \mathbf{C}^{k+1,\,\varrho\tau^k}(\bar{\Omega})$, similarly, we may prove that $f(x,u(x)) \in \mathbf{C}^{k,\,\varrho\tau^{k+1}}(\bar{\Omega})$ and $u = w$. By Proposition 1.57 once again, we have that $u \in \mathbf{C}^{k+2,\,\varrho\tau^{1+k}}(\bar{\Omega})$. □

Finally, we want to review the unique continuation theorem.

Definition 1.61. Let Ω be a connected open subset of $\mathbf{R}^N$. We say that a function W on Ω has the unique continuation property if and only if every function u satisfying

$$
|\Delta u(x)| \le W(x)|u(x)| \tag{1.75}
$$

which is equal to zero on some open set is identically zero on Ω.

The following unique continuation theorem is due to Schechter and Simon [277].

Theorem 1.62. *Let u obey (1.75) and $W \in L^r_{loc}(\mathbf{R}^N)$ with $r = N-2$ if $N > 5$; $r > (2N-1)/3$ if $N \le 5$. Then if $u = 0$ in a small ball, then $u = 0$ everywhere.*

Notes and Comments. The Lax–Milgram theorem can be found in Lax [189] and Lax and Milgram [190]. We refer readers to Gilbarg and Trudinger [160], Evans [141], Struwe [313], and Taylor [325] for the regularity theory of elliptic equations. Results on unique continuation theorems also can be seen in Reed and Simon [262].

Note. In this book, we are only devoted to and satisfied with finding critical points (weak solutions) of differentiable functionals. All weak solutions are indiscriminately called solutions.

Chapter 2
Schechter–Tintarev Linking

The relationship between the classical linking theorem and the sign-changing critical point is established. The abstract theory is applied to elliptic equations with miscellaneous resonance.

2.1 Schechter–Tintarev Linking

Let E be a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Define a class of contractions of E as follows.

$$(2.1) \quad \Phi := \left\{ \Gamma : \begin{array}{l} \Gamma(\cdot,\cdot) \in \mathbf{C}([0,1] \times E, E), \quad \Gamma(0,\cdot) = \mathbf{id}, \\ \text{for each } t \in [0,1), \Gamma(t,\cdot) \text{ is a homeomorphism of } E \\ \text{onto itself and } \Gamma^{-1}(t,\cdot) \text{ is continuous on } [0,1) \times E; \\ \text{there exists a } x_0 \in E \text{ such that } \Gamma(1,x) = x_0 \\ \text{for each } x \in E \text{ and that } \Gamma(t,x) \to x_0 \text{ as } t \to 1 \\ \text{uniformly on bounded subsets of } E \end{array} \right\}.$$

Obviously, $\Gamma(t,u) = (1-t)u \in \Phi$.

Definition 2.1. A subset A of E is linked to a subset B of E if $A \cap B = \emptyset$ and, for every $\Gamma \in \Phi$, there is a $t \in [0,1]$ such that $\Gamma(t,A) \cap B \neq \emptyset$.

Proposition 2.2. *Let $H \in \mathbf{C}(E, \mathbf{R}^N)$ and $Q \subset E$ be such that $H_0 = H|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\mathbf{R}^N$. Let $p \in \Omega$; then $H_0^{-1}(\partial\Omega)$ links $H^{-1}(p)$.*

Proof. Assume by negation that $H_0^{-1}(\partial\Omega)$ does not link $H^{-1}(p)$. Then there is a $\Gamma \in \Phi$ such that

$$(2.2) \qquad \Gamma(t, H_0^{-1}(\partial\Omega)) \cap H^{-1}(p) = \emptyset, \qquad t \in [0,1].$$

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That is,

$$H(\Gamma(t, H_0^{-1}(\partial\Omega))) \cap \{p\} = \emptyset, \qquad t \in [0,1]. \tag{2.3}$$

Set

$$\theta(t) := H \circ \Gamma(t,\cdot) \circ H_0^{-1}. \tag{2.4}$$

Then we see that $\theta(t) \in \mathbf{C}(\bar{\Omega}, \mathbf{R}^N)$ for each $t \in [0,1]$ and $\theta(0) = \mathbf{id}$ on Ω. If $\Gamma(1, E) = x_0$, then $\theta(1)x = Hx_0 \neq p$ for all $x \in \bar{\Omega}$ because by (2.2)–(2.4),

$$H(\Gamma(1, H_0^{-1}(\partial\Omega))) \cap \{p\} = \emptyset.$$

By Brouwer's degree,

$$\deg(\theta(t), \Omega, p) = \deg(\theta(1), \Omega, p) = 1, \qquad \forall\ t\ \in [0,1].$$

This is a contradiction. □

Proposition 2.3. *Let A, B be two closed bounded subsets of E such that $E\backslash A$ is path connected. If A links B, then B links A as well.*

Proof. Assume by negation that B does not link A. Then we may find a $\Gamma \in \Phi$ such that

$$\Gamma(t, B) \cap A = \emptyset, \qquad \forall\, t \in [0,1]. \tag{2.5}$$

By the definition of Φ, we assume that $\Gamma(1, E) = x_0$, hence $x_0 \notin A$. Let Ω be a closed ball such that $A \subset \Omega$. Note that $E\backslash A$ is path connected; then there is a path γ connecting $x_0 \notin A$ to a point $x_1 \notin \Omega$. Let $t_0 \in [0,1)$ be such that the diameter of $\Gamma(t_0, B) - x_0$ is less than $\min\{\mathrm{dist}(\gamma, A), \mathrm{dist}(x_1, \Omega)\}$. Parameterize γ in such a way that it is given by $\gamma(t), t_0 \le t \le 1, \gamma(t_0) = x_0$, $\gamma(1) = x_1$. Then

$$(\Gamma(t_0, B) + \gamma(t) - x_0) \cap A = \emptyset, \qquad t_0 \le t \le 1 \tag{2.6}$$

and

$$(\Gamma(t_0, B) + x_1 - x_0) \cap \Omega = \emptyset. \tag{2.7}$$

Define

$$\Gamma_1(t,x) = \begin{cases} \Gamma(t,x) & \text{for } t \in [0, t_0], \\ \Gamma(t_0, x) - u_0 + \gamma(t) & \text{for } t \in [t_0, 1]. \end{cases}$$

By (2.5), we see that $\Gamma_1(t, B) \cap A = \emptyset$ for all $t \in [0,1]$. Hence,

$$B \cap \Gamma_1^{-1}(t, A) = \emptyset, \qquad \forall\, t \in [0,1]. \tag{2.8}$$

By (2.7),

$$\Gamma_1(1, B) \cap \Omega = \emptyset. \tag{2.9}$$

Let Γ_2 be any map in Φ such that $\Gamma_2(t, \Omega) \subset \Omega$ for all t. Take

$$\Gamma_3(t, \cdot) = \begin{cases} \Gamma_1(2t, \cdot)^{-1} & \text{for all } t \in [0, 1/2], \\ \Gamma_1(1, \cdot)^{-1} \circ \Gamma_2(2t - 1, \cdot) & \text{for all } (1/2, 1]. \end{cases}$$

It is easy to check that $\Gamma_3 \in \Phi$. But Equations (2.8)–(2.9) imply that

$$B \cap \Gamma_3(t, A) = \emptyset, \qquad t \in [0, 1].$$

It contradicts the fact that A links B. □

Proposition 2.4. *Let $E = M \oplus Y$, where M, Y are closed subspaces with one of them finite-dimensional. If $y_0 \in M \backslash \{0\}$ and $0 < \rho < R$, then the sets*

$$A := \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$
$$B := M \cap \partial B_\rho$$

link each other in the sense of Definition 2.1, where $B_r := \{u \in E : \|u\| < r\}$.

Proof. We first consider the case of $\dim Y < \infty$ and identify Y with some $\mathbf{R}^N$. We may assume that $\|y_0\| = 1$. Let

$$Q = \{sy_0 + v : v \in Y, s \geq 0, \|sy_0 + v\| \leq R\}.$$

Then $A = \partial Q$ in $\mathbf{R}^{N+1}$. Let $u = v + w$ with $v \in Y, w \in M$; we define $Fu = v + \|w\|y_0$. Then $F|_Q = \mathbf{id}$ and $B = F^{-1}(\rho y_0)$. We can apply Proposition 2.2 to conclude that A links B. Because A and B are bounded and $E \backslash A$ is path connected, B links A as well. □

Proposition 2.5. *Let $E = M \oplus Y$, where M, Y are closed subspaces with* $\dim Y < \infty$. *Let $B_R = \{u \in E : \|u\| < R\}$ and let $A = \partial B_R \cap Y, B = M$. Then A links B.*

Proof. We identify Y with some $\mathbf{R}^N$ and take $\Omega = B_R \cap Y, Q = \bar{\Omega}$. For $u = v + w, v \in Y, w \in M$, define the projection $Fu = v$. Because $F|_Q = \mathbf{id}$ and $M = F^{-1}(0)$, we observe by Proposition 2.2 that A links B. □

Proposition 2.6. *Let B be an open set in E and $A = \{a, b\}$ such that $a \in B$, $b \notin \bar{B}$. Then A links ∂B.*

Proof. Let $\Gamma \in \Phi$. If $\Gamma(1, E) = u_0$, then $\Gamma(t, a)$ $(\Gamma(t, b))$ is a curve in E connecting a (b, respectively) with u_0. If $u_0 \notin \bar{B}$, then $\Gamma(t, a)$ intersects ∂B. If $u_0 \in B$, then $\Gamma(t, b)$ intersects ∂B. Hence A links ∂B. □

Proposition 2.7. *Let $E = M \oplus Y$, where M, Y are closed subspaces with $\dim Y < \infty$. Let $B_R := \{u \in E : \|u\| < R\}$ and take $A = \partial B_R \cap Y$. Choose $z_0 \neq 0, z_0 \in Y$ and let*

$$B = \big\{u \in M : \|u\| \geq \delta\big\} \cup \big\{u = sz_0 + v : v \in M, s \geq 0, \|sz_0 + v\| = \delta\big\}.$$

Then A links B with respect to Φ for any $R > \delta > 0$.

Proof. Let $Q = \bar{B}_R \cap Y$. For simplicity, we may assume that $\|z_0\| = 1$ and that

$$E = \tilde{Y} \oplus \mathbf{R}z_0 \oplus M,$$

for each $u = \tilde{u} + sz_0 + v$ with $\tilde{u} \in \tilde{Y}, v \in M$. Define

$$H(u) = \begin{cases} \tilde{u} + (s + \delta - (\delta^2 - \|v\|^2)^{1/2})z_0, & \text{for } \|v\| \leq \delta, \\ \tilde{u} + (s + \delta)z_0, & \text{for } \|v\| > \delta. \end{cases}$$

We observe that $H|_Q = \mathbf{id}$ and that $H^{-1}(\delta z_0)$ is precisely the set B. We hence conclude by Proposition 2.2 that A links B. □

Proposition 2.8. *Let $E = M \oplus Y$, where M, Y are closed subspaces with $\dim N < \infty$. Let $B_R := \{u \in E : \|u\| < R\}$ and take $A = \partial B_R \cap Y$. Choose $z_0 \neq 0, z_0 \in Y, R_1 > \delta$ and let*

$$\begin{aligned} B = \{sz_0 + v &: s \leq 0; v \in M, R_1 \geq \|v\| \geq \delta\} \\ &\cup \{u = sz_0 + v : v \in M, s \geq 0, \|sz_0 + v\| = \delta\}. \end{aligned}$$

Then A links B with respect to Φ for any $R > \delta > 0$.

Proof. Let $Q = \bar{B}_R \cap Y$. For simplicity, we may assume that $\|z_0\| = 1$ and that

$$E = \tilde{Y} \oplus \mathbf{R}z_0 \oplus M.$$

For each $u = \tilde{u} + sz_0 + v$ with $\tilde{u} \in \tilde{Y}, v \in M$. Define

$$H(u) = \begin{cases} \tilde{u} + (s + \delta - (\delta^2 - \|v\|^2)^{1/2})z_0, & \text{for } \|v\| \leq \delta, \\ \tilde{u} + (s + \delta)z_0 + \dfrac{\|v\| - \delta}{R_1 - \delta}(s^2 + 1)z_0, & \text{for } R_1 \geq \|v\| > \delta, \\ \tilde{u} + (s^2 + s + 1 + \delta)z_0, & \text{for } \|v\| \geq R_1. \end{cases}$$

We observe that $H|_Q = \mathbf{id}$ and that $H^{-1}(\delta z_0)$ is precisely the set B. We hence conclude by Proposition 2.2 that A links B. □

Proposition 2.9. *If B is any subset of a bounded open set $\Omega \subset E$, then $\partial\Omega$ links B.*

Proof. Assume that $\partial\Omega$ does not link B, and we seek a contradiction. In this case, there is a $\Gamma \in \Phi$ such that

$$\Gamma(s, \partial\Omega) \cap B = \emptyset, \qquad \forall\, s \in [0,1]. \tag{2.10}$$

Hence,

$$\Gamma(1, \partial\Omega) := u \notin B. \tag{2.11}$$

Let v be any point in B. We now show that

$$\|\Gamma^{-1}(s, v)\| \to \infty, \quad \text{as } s \to 1. \tag{2.12}$$

If this were not true, we would have by the definition of Φ that

$$v = \Gamma(s, v)[\Gamma^{-1}(s, v)] \to u,$$

which contradicts (2.11). Therefore, (2.12) is correct and thus

$$\Gamma^{-1}(s, v) \notin \bar{\Omega} \tag{2.13}$$

for s close to 1 because Ω is bounded. Note that $v = \Gamma^{-1}(0, v) \in B \subset \Omega$; we may have an $s_0 \in (0, 1)$ such that

$$\Gamma^{-1}(s_0, v) \in \partial\Omega. \tag{2.14}$$

To see this, let

$$s_0 = \sup\{s \in [0,1] : \Gamma^{-1}(t, v) \in \Omega, \forall t \in [0, s)\}.$$

Then $\Gamma^{-1}(s_0, v) \notin \Omega$. Otherwise, there would be an interval containing s_0 in which $\Gamma^{-1}(s, v) \in \Omega$; this contradicts the definition of s_0. But

$$\Omega \ni \Gamma^{-1}(s, v) \to \Gamma^{-1}(s_0, v), \quad \text{as } s \to s_0.$$

Hence, (2.14) holds and this means that $v \in \Gamma(s_0, \partial\Omega)$ which contradicts (2.10). Thus, $\partial\Omega$ links B. $\square$

Proposition 2.10. *Let E be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. We assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \le C_\star\|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = M \oplus Y$, where M, Y are closed subspaces with $\dim Y < \infty$. If $y_0 \in M\backslash\{0\}$ with $\|y_0\| = 1$ and $0 < \rho < R$ with*

$$R^{p-2}\|y_0\|_\star^p + \frac{R\|y_0\|_\star}{1 + D_\star\|y_0\|_\star} > \rho, \qquad D_\star > 0, p > 2 \text{ are constants.}$$

Let

$$A := \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$

$$B := \left\{u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star} = \rho\right\},$$

where $\bar{B}_R$ *denotes the closed ball of* E *centered at zero with radius* R. *Then* A *links* B *in the sense of Definition 2.1.*

Proof. It is easy to check that $A \cap B = \emptyset$. We identify Y with some $\mathbf{R}^N$. Let

$$Q = \{sy_0 + v : v \in Y, s \geq 0, \|sy_0 + v\| \leq R\}.$$

Then $A = \partial Q$ in $\mathbf{R}^{N+1}$. Let $u = v + w$ with $v \in Y, w \in M$; we define

$$\xi_0(u) := \begin{cases} \dfrac{\|u\|_\star^p}{\|u\|^2} + \dfrac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases} \tag{2.15}$$

Then $\xi_0 : E \to E$ is continuous. We define

$$Fu = v + \xi_0(w)y_0.$$

Then for any $u = v + sy_0 \in Q$, we see that $s \in [0, R]$ and $F(u) = v + \xi_0(sy_0)y_0$. Here $\xi_0(sy_0) = 0$ iff $s = 0$. Otherwise

$$\xi_0(sy_0) = s^{p-2}\frac{\|y_0\|_\star^p}{\|y_0\|^2} + s\frac{\|y_0\|_\star\|y_0\|}{D_\star\|y_0\|_\star + \|y_0\|} := as^{p-2} + sb,$$

where $a, b > 0$ are two constants depending on y_0 only. Therefore, $F_0 = F|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\mathbf{R}^{N+1}$. Let $\rho y_0 \in \Omega$; then by Proposition 2.2, $F_0^{-1}(\partial\Omega) = \partial Q$ links $F^{-1}(\rho y_0) = B$. That is,

$$A \text{ links } B.$$

□

Proposition 2.11. *In Proposition 2.10, if we choose* $y_0 \in M$ *with* $\|y_0\|_\star = 1$ *and* $R > \rho\|y_0\|$, *then* A *links* B *where* B *is replaced by* $B := \{u \in M : \|u\|_\star = \rho\}$.

Proof. It is obvious. □

Notes and Comments. Definition 2.1 was introduced by Schechter and Tintarev [278] and Propositions 2.2–2.4 were proved there (see also Schechter [275]). Propositions 2.5–2.7 and 2.9 can be found in Schechter [275]. Proposition 2.8 is a new one. More examples can be seen in Schechter [273, 275] and Schechter and Zou [280, 283]. The original approach to linking required A to be of a special nature (e.g., the boundary of a manifold) in order to link a set B. This severely restricted the kind of sets that could be used. The linking in the current book seems more general and realistic. It only means that A cannot be continuously shrunk to a point without intersecting B. We refer readers to Benci and Rabinowitz's linking (infinite-dimensional and compact maps) in [55], Silva's linking of deformation type in [295], Corvellec's linking on metric spaces in [109], Tintarev's isotopic linking in [328], Li an Liu's local linking at zero in [197] (see also Chang [93], Brézis and Nirenberg [70], and Li and Willem [200]), Benci's homological linking in [51] (see also Perera [241] and Liu [207]), and Ramos and Sanchez's homotopical linking in [261].

2.2 Sign-Changing Critical Points via Linking

Let E be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. Consider the following type of functional $G \in \mathbf{C}^1(E, \mathbf{R})$. Its gradient G' is of the form

$$G'(u) = \kappa(u)u - \Theta_G u, \tag{2.16}$$

where $\kappa(u) : E \to [1/2, 1]$ is a locally Lipschitz continuous function; $\Theta_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E \backslash \mathcal{K}$.

Let $V : \tilde{E} \to E$ be a pseudo-gradient vector field for G(cf. Definition 1.53); that is,

(1) $\langle G'(u), V(u) \rangle \geq \frac{1}{2}\|G'(u)\|^2$ for all $u \in \tilde{E}$.
(2) $\|V(u)\| \leq 2\|G'(u)\|$ for all $u \in \tilde{E}$.

By Theorem 1.43, the following initial value problem

$$\begin{cases} \dfrac{d\sigma(t,u)}{dt} = -V(\sigma(t,u)), \\ \sigma(0,u) = u \in \tilde{E}, \end{cases}$$

has a unique solution (called flow or trajectory) $\sigma : [0, T(u)) \to E$, where $T(u) \in (0, \infty]$ is the maximal time of the existence of the flow with initial value u.

Let $\mathcal{P}$ be a closed convex and weakly closed subset of E such that $(\mathcal{P}) \backslash \{0\} \neq \emptyset$.

For any $\delta > 0$, define

$$\pm\mathcal{D}(\delta) := \{u \in E : \operatorname{dist}(u, \pm\mathcal{P}) < \delta\}, \tag{2.17}$$

$$\mathcal{D}^* := \mathcal{D}(\delta) \cup (-\mathcal{D}(\delta)), \qquad \mathcal{S} = E \backslash \mathcal{D}^*. \tag{2.18}$$

Then $\pm\mathcal{D}(\delta)$ are open convex, $\mathcal{D}^*$ is open, $\pm\mathcal{P} \subset \pm\mathcal{D}(\delta/2) \subset \pm\mathcal{D}(\delta)$, and $\mathcal{S}$ is closed. We make the following assumption.

($\mathbf{A_1}$) There exists a $\delta > 0$ (small enough) such that $\Theta_G(\pm\mathcal{D}(\delta)) \subset \pm\mathcal{D}(\delta/2)$.

Lemma 2.11. *Assume (A_1). Then there exists a locally Lipschitz continuous map $L_0 : \tilde{E} \to E$ such that $L_0(\pm\mathcal{D}(\delta) \cap \tilde{E}) \subset \pm\mathcal{D}(\delta/2)$ and that $V(u) := \kappa(u)u - L_0(u)$ is a pseudo-gradient vector field of G. Furthermore, V and L_0 can be chosen to be odd if G, κ are even.*

Proof. Note that $\|G'(v)\| \neq 0$ for any $v \in \tilde{E}$. Define

$$\Omega(v) := \left\{ u \in \tilde{E} : \|G'(u)\| > \frac{1}{2}\|G'(v)\|, \;\; \|\Theta_G u - \Theta_G v\| < \frac{1}{8}\|G'(v)\| \right\}.$$

Then $\{\Omega(v) : v \in \tilde{E}\}$ is an open covering of $\tilde{E}$ in the topology of E, and we can find a locally finite refinement open covering $\{\tilde{\Omega}(\lambda) : \lambda \in \Lambda\}$ of $\tilde{E}$, where Λ is the index set. For any $\lambda \in \Lambda$, only one of the following cases occurs.

(1) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) = \emptyset, \quad \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) = \emptyset$;
(2) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \neq \emptyset, \quad \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) = \emptyset$;
(3) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) = \emptyset, \quad \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) \neq \emptyset$;
(4) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)) \neq \emptyset$;
(5) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \neq \emptyset,\ \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) \neq \emptyset,\ \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)) = \emptyset$.

If the last case happens, we remove $\tilde{\Omega}(\lambda)$ from the covering and replace it with $\tilde{\Omega}(\lambda)\backslash\bar{\mathcal{D}}(\delta)$ and $\tilde{\Omega}(\lambda)\backslash(-\bar{\mathcal{D}}(\delta))$. In this way, we rearrange them so that the new covering has only the properties (1)–(4). In particular, the new covering is still a covering of $\tilde{E}$. To see this, we take any $w \in \tilde{E}$; then we have a $\tilde{\Omega}(\lambda)$ in the "old" covering $\{\tilde{\Omega}(\lambda) : \lambda \in \Lambda\}$ of $\tilde{E}$ such that $w \in \tilde{\Omega}(\lambda)$. If $\tilde{\Omega}(\lambda)$ is one of the cases (1)–(4), then w is covered by $\tilde{\Omega}(\lambda)$ which is also in the new covering. If $\tilde{\Omega}(\lambda)$ is of case (5), then we may distinguish the following cases.

- If $w \notin \bar{\mathcal{D}}(\delta)$, then $w \in \tilde{\Omega}(\lambda)\backslash\bar{\mathcal{D}}(\delta)$; hence, w is covered by the new covering.
- If $w \notin -\bar{\mathcal{D}}(\delta)$, then $w \in \tilde{\Omega}(\lambda)\backslash\big(-\bar{\mathcal{D}}(\delta)\big)$; hence, w is covered by the new covering.
- The remaining case is $w \in \bar{\mathcal{D}}(\delta) \cap (-\bar{\mathcal{D}}(\delta))$; we now show that this will not happen. Firstly, we observe that $w \notin \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$. Otherwise, it contradicts the fact that $\tilde{\Omega}(\lambda)$ is of Case (5). So, we must have the following cases.

  (a) $w \notin \mathcal{D}(\delta), w \in -\mathcal{D}(\delta)$; hence, $w \in \partial\mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$.
  (b) $w \notin \mathcal{D}(\delta), w \notin -\mathcal{D}(\delta)$; hence, $w \in \partial\mathcal{D}(\delta) \cap (-\partial\mathcal{D}(\delta))$.
  (c) $w \in \mathcal{D}(\delta), w \notin -\mathcal{D}(\delta)$; hence, $w \in \mathcal{D}(\delta) \cap (-\partial\mathcal{D}(\delta))$.

Because $0 \in \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$, which is open convex, then for these cases (a)–(c),

$$tw = tw + (1-t)0 \in \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)), \qquad t \in (0,1).$$

However, $tw \in \tilde{\Omega}(\lambda)$ for $t \to 1^-$ because $\tilde{\Omega}(\lambda)$ is open and $w \in \tilde{\Omega}(\lambda)$. This also contradicts the fact that $\tilde{\Omega}(\lambda)$ is of Case (5).

Therefore, we indeed get a new covering which has only the properties (1)–(4). For each $\lambda \in \Lambda$, define

$$\alpha_\lambda(u) := \mathrm{dist}(u, \tilde{E} \backslash \tilde{\Omega}_\lambda), \qquad \phi_\lambda(u) := \frac{\alpha_\lambda(u)}{\sum\limits_{\lambda \in \Lambda} \alpha_\lambda(u)}, \qquad u \in \tilde{E};$$

then $0 \le \phi_\lambda(u) \le 1$ and $\phi_\lambda : \tilde{E} \to E$ is locally Lipschitz continuous. For each $\lambda \in \Lambda$, choose $a_\lambda \in \tilde{\Omega}(\lambda)$ such that a_λ is arbitrary in Case (1); $a_\lambda \in \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta)$ in Case (2); $a_\lambda \in \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta))$ in Case (3); and $a_\lambda \in \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$ in Case (4). Define

$$L_0(u) = \sum_{\lambda \in \Lambda} \phi_\lambda(u) \Theta_G a_\lambda, \qquad u \in \tilde{E}.$$

Then $L_0 : \tilde{E} \to E$ is locally Lipschitz coninuous. Let

$$V(u) := \kappa(u)u - L_0 u.$$

We prove that L_0 and V are what we want.

For any $u \in \tilde{E}$, there are only finitely many numbers $\lambda_1, \ldots, \lambda_s \in \Lambda$ such that $u \in \tilde{\Omega}(\lambda_1) \cap \cdots \cap \tilde{\Omega}(\lambda_s)$. Moreover, there are $w_1, \ldots, w_s \in \tilde{E}$ such that $\tilde{\Omega}(\lambda_i) \subset \Omega(w_i)$ for $i = 1, \ldots, s$. Then

$$L_0(u) = \sum_{i=1}^{s} \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i},$$

where $a_{\lambda_i} \in \tilde{\Omega}(\lambda_i)$ for $i = 1, \ldots, s$. Note that

$$\begin{aligned}
&\|\Theta_G u - \Theta_G a_{\lambda_i}\| \\
&\quad \le \|\Theta_G u - \Theta_G w_i\| + \|\Theta_G w_i - \Theta_G a_{\lambda_i}\| \\
&\quad \le \frac{1}{4}\|G'(w_i)\| \\
&\quad \le \frac{1}{2}\|G'(u)\|
\end{aligned}$$

for $i = 1, \ldots, s$; it follows that

$$\begin{aligned} &\|\Theta_G u - L_0 u\| \\ &= \left\| \Theta_G u - \sum_{i=1}^{s} \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i} \right\| \\ &= \left\| \sum_{i=1}^{s} \phi_{\lambda_i}(u)(\Theta_G u - \Theta_G a_{\lambda_i}) \right\| \\ &\le \frac{1}{2}\|G'(u)\|. \end{aligned}$$

Hence

$$\begin{aligned} &\|V(u)\| \\ &= \|\kappa(u)u - L_0(u)\| \\ &\le \|\kappa(u)u - \Theta_G u\| + \|\Theta_G u - L_0(u)\| \\ &\le \|G'(u)\| + \left\| \sum_{i=1}^{s} \phi_{\lambda_i}(u)\Theta_G u - \sum_{i=1}^{s} \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i} \right\| \\ &\le \frac{3}{2}\|G'(u)\|. \end{aligned} \tag{2.19, 2.20}$$

and $|\langle G'(u), \Theta_G u - L_0 u\rangle| \le \frac{1}{2}\|G'(u)\|^2$. Furthermore,

$$\langle G'(u), V(u)\rangle = \|G'(u)\|^2 + \langle G'(u), \Theta_G u - L_0 u\rangle \ge \frac{1}{2}\|G'(u)\|^2. \tag{2.21}$$

Inequalities (2.20) and (2.21) imply that $V(u) := \kappa(u)u - L_0 u$ is a pseudo-gradient vector field for G. Next, we show that $L_0(\pm\mathcal{D}(\delta) \cap \tilde{E}) \subset \pm\mathcal{D}(\delta/2)$. In fact, for any $u \in \mathcal{D}(\delta) \cap \tilde{E}$, there are finitely many $\phi_\lambda(u)$, say $\phi_{\lambda_i}(u)$ $(i = 1, \ldots, s)$, which are nonzero. Then

$$L_0 u = \sum_{i=1}^{s} \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i}$$

and $u \in \tilde{\Omega}(\lambda_i) \cap \mathcal{D}(\delta)$ for $i = 1, \ldots, s$. Hence, $a_{\lambda_i} \in \tilde{\Omega}(\lambda_i) \cap \mathcal{D}(\delta)$ by the definition of a_λ. It follows that $\Theta_G a_{\lambda_i} \in \mathcal{D}(\delta/2)$ by recalling the condition (A_1). It implies that $L_0(u) \in \mathcal{D}(\delta/2)$, because $\mathcal{D}(\delta/2)$ is also convex. This proves that $L_0(\mathcal{D}(\delta) \cap \tilde{E}) \subset \mathcal{D}(\delta/2)$. Similarly, we have that $L_0(-\mathcal{D}(\delta) \cap \tilde{E}) \subset -\mathcal{D}(\delta/2)$.

Finally, we show that V and L_0 can be chosen to be odd if G and κ are even. Let $\bar{L}_0(u) = \frac{1}{2}(L_0 u - L_0(-u))$; then $\bar{L}_0 : \tilde{E} \to E$ is odd and locally

Lipschitz continuous. Define $\bar{V} := \kappa(\cdot) - \bar{L}_0$; then $\bar{V} : \tilde{E} \to E$ is also odd and locally Lipschitz continuous. We may show that $\bar{L}_0(\pm\mathcal{D}(\delta) \cap \tilde{E}) \subset \pm\mathcal{D}(\delta/2)$. Indeed, for any $u \in \pm\mathcal{D}(\delta) \cap \tilde{E}$, then $-u \in \pm\mathcal{D}(\delta) \cap \tilde{E}$ and hence

$$L_0(\pm u) \in \pm\mathcal{D}(\delta/2), \qquad -L_0(-u) \in \pm\mathcal{D}(\delta/2).$$

Therefore,

$$\bar{L}_0 u = \frac{1}{2} L_0 u + \frac{1}{2}(-L_0(-u)) \in \pm\mathcal{D}(\delta/2),$$

because $\pm\mathcal{D}(\delta/2)$ is convex. □

By checking the proof of Lemma 2.11, we readily have the following variant of Lemma 2.11.

Lemma 2.12. *Consider the functional $G \in \mathbf{C}^1(E, \mathbf{R})$. Its gradient G' is of the form*

$$G'(u) = u - \Theta_G u.$$

Let M_1, M_2 be two closed convex (or open convex) subsets of E. Suppose that

($\mathbf{A_2}$) $\Theta_G(M_i) \subset M_i,\ i = 1, 2.$

Then there exists a locally Lipschitz continuous map $L_0 : \tilde{E} \to E$ such that $L_0(M_i \cap \tilde{E}) \subset M_i,\ i = 1, 2$ and that $V(u) := u - L_0(u)$ is a pseudo-gradient vector field of G. Furthermore, V and L_0 can be chosen to be odd if G is even and $M_2 = -M_1$.

From now on, let $\mathcal{P}$ denote a positive cone of E; that is, $\mathcal{P}$ is a closed convex subset of E such that $t\mathcal{P} \subset \mathcal{P}$ for all $t \geq 0$ and $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. We always assume implicitly that $\mathcal{P} \neq \{0\}$. We call $-\mathcal{P}$ a negative cone. Consider the following vector field,

$$W(u) := \frac{(1 + \|u\|)^2 V(u)}{(1 + \|u\|)^2 \|V(u)\|^2 + 1}.$$

Then W is a locally Lipschitz continuous vector field on $\tilde{E}$. Obviously,

$$\|W(u)\| \leq \|u\| + 1$$

for all $u \in \tilde{E}$. We denote

$$\mathcal{K}[a, b] := \{u \in E : G'(u) = 0, a \leq G(u) \leq b\},$$

$$G^c := \{u \in E : G(u) \leq c\}, \qquad B_R(0) := \{u \in E : \|u\| \leq R\}.$$

Define

$$\Phi^* := \{\Gamma \in \Phi : \Gamma(t, \mathcal{D}^*) \subset \mathcal{D}^*\}. \tag{2.22}$$

Then $\Gamma(t, u) = (1 - t)u \in \Phi^*$.

Theorem 2.13. *Suppose that (2.16) and* (A_1) *hold. Assume that a compact subset* A *of* E *links a closed subset* B *of* $\mathcal{S}$ *and*

$$a_0 := \sup_A G \leq b_0 := \inf_B G.$$

Define

$$d^* := \inf_{\Gamma \in \Phi^*} \sup_{\Gamma([0,1],A)\cap \mathcal{S}} G(u);$$

then

$$d^* \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)\Big].$$

Furthermore, if G *satisfies the (w-PS)*$_c$ *condition for any* c *with*

$$c \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)\Big],$$

then

$$\mathcal{K}[d^* - \varepsilon, d^* + \varepsilon] \cap (E\backslash(-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ *small. Moreover, if* $d^* = b_0$*, then* $\mathcal{K}[d^*, d^*] \subset B$*.*

Proof. Obviously, d^* is well defined because A links B and $B \subset \mathcal{S}$. Moreover,

$$d^* \in \Big[b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)\Big].$$

We first consider the case of $d^* > b_0$. By contradiction, we assume that

$$\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \cap (E\backslash(-\mathcal{P} \cup \mathcal{P})) = \emptyset \tag{2.23}$$

for some $\varepsilon_0 > 0$ small enough. Then

$$\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \subset (-\mathcal{P} \cup \mathcal{P}). \tag{2.24}$$

Case 1. Assume $\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \neq \emptyset$.

$\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]$ is compact, thus we may assume that $\mathrm{dist}(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0], \mathcal{S}) := \delta_0 > 0$.

By the (w-PS) condition, there is an $\bar{\varepsilon} > 0$ such that

$$\frac{(1 + \|u\|)^2 \|G'(u)\|^2}{(1 + \|u\|)^2 \|G'(u)\|^2 + 1} \geq \bar{\varepsilon} \tag{2.25}$$

for

$$u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}]\backslash(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2},$$

where $(T)_c := \{u \in E : \mathrm{dist}(u, T) \leq c\}$. By decreasing $\bar{\varepsilon}$, we may assume that $\bar{\varepsilon} < d^* - b_0, \bar{\varepsilon} < \varepsilon_0/3$; then

$$\langle G'(u), W(u)\rangle \geq \bar{\varepsilon}/8$$

for any

$$u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}]\backslash(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2}.$$

Let

$$\Omega_1 = \{u \in E : |G(u) - d^*| \geq 3\bar{\varepsilon}\}, \qquad \Omega_2 = \{u \in E : |G(u) - d^*| \leq 2\bar{\varepsilon}\}$$

and

$$\vartheta(u) = \frac{\mathrm{dist}(u, \Omega_1)}{\mathrm{dist}(u, \Omega_1) + \mathrm{dist}(u, \Omega_2)}.$$

Let $\beta(u) : E \to [0, 1]$ be locally Lipschitz continuous such that

$$\beta(u) = \begin{cases} 1 & \text{for all } u \in E\backslash(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2}, \\ 0 & \text{for all } u \in (\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/3}. \end{cases} \tag{2.26}$$

Let $\bar{W}(u) = \vartheta(u)\beta(u)W(u)$ for $u \in \tilde{E}$; $\bar{W}(u) = 0$ otherwise. Then $\bar{W}$ is a locally Lipschitz vector field on E. We consider the following Cauchy initial value problem,

$$\begin{cases} \dfrac{d\varphi(t, u)}{dt} = -\bar{W}(\varphi(t, u)), \\ \varphi(0, u) = u, \end{cases} \tag{2.27}$$

which has a unique continuous solution $\varphi(t, u)$ in E. Evidently,

$$\frac{dG(\varphi(t, u))}{dt} \leq 0.$$

By the definition of d^*, there exists a $\Gamma \in \Phi^*$ such that

$$\Gamma([0, 1], A) \cap \mathcal{S} \subset G^{d^* + \bar{\varepsilon}}.$$

Therefore, $\Gamma([0, 1], A)$ is a subset of $G^{d^* + \bar{\varepsilon}} \cup \mathcal{D}^*$. Denote

$$A_1 := \Gamma([0, 1], A).$$

We claim that there exists a $T_1 > 0$ such that $\varphi(T_1, A_1) \subset G^{d^* - \bar{\varepsilon}/4} \cup \mathcal{D}^*$.

First, if $u \in \mathcal{D}^*$, we show that $\varphi(t, u) \in \mathcal{D}^*$ for all $t \geq 0$. Without loss of generality, we may assume that $u \in \mathcal{D}(\delta)$. Suppose there exists a $t_0 > 0$ such that $\varphi(t_0, u) \notin \mathcal{D}(\delta)$. We may choose a neighborhood $\mathcal{N}_u$ of u such that $\mathcal{N}_u \subset \mathcal{D}(\delta)$ because $\mathcal{D}(\delta)$ is open. By the theory of ordinary differential equations in Banach space, we can find a neighborhood $\mathcal{N}_{t_0}$ of $\varphi(t_0, u)$ such that $\varphi(t_0, \cdot) : \mathcal{N}_u \to \mathcal{N}_{t_0}$ is a homeomorphism. Because $\varphi(t_0, u) \notin \mathcal{D}(\delta)$, we can take a $w \in \mathcal{N}_{t_0}\backslash\bar{\mathcal{D}}(\delta)$. Correspondingly, we find a $v \in \mathcal{N}_u$ such that

$\varphi(t_0, v) = w$. Hence, we may find a $t_1 \in (0, t_0)$ such that $\varphi(t_1, v) \in \partial\mathcal{D}(\delta)$ and $\varphi(t, v) \notin \bar{\mathcal{D}}(\delta)$ for all $t \in (t_1, t_0]$.

On the other hand, for any $z \in \bar{\mathcal{D}}(\delta) \cap \mathcal{K}$, $\bar{W}(z) = 0$, hence

$$\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta)) = 0, \quad \text{for all } \lambda > 0. \tag{2.28}$$

For any $z \in \bar{\mathcal{D}}(\delta) \cap \tilde{E}$, we have $L_0(z) \in \bar{\mathcal{D}}(\delta/2)$ because $L_0(\mathcal{D}(\delta) \cap \tilde{E}) \subset \mathcal{D}(\delta/2)$ in view of Lemma 2.11. Therefore, by a property of the cone $\mathcal{P}$: $x\mathcal{P} + y\mathcal{P} \subset \mathcal{P}$ for all $x, y \geq 0$, we have

$$\begin{aligned}
&\operatorname{dist}(z + \lambda(-\bar{W}(z)), \mathcal{P}) \\
&\quad = \operatorname{dist}(z - \lambda\vartheta(z)\beta(z)W(z), \mathcal{P}) \\
&\quad = \operatorname{dist}\left(\left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right) z \right. \\
&\qquad \left. + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1} L_0(z), \mathcal{P}\right) \\
&\quad \leq \operatorname{dist}\left(\left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right) z + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1} L_0(z), \right. \\
&\qquad \left. \left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right) \mathcal{P} + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1} \mathcal{P}\right) \\
&\quad \leq \left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right) \operatorname{dist}(z, \mathcal{P}) \\
&\qquad + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1} \operatorname{dist}(L_0(z), \mathcal{P}) \\
&\quad \leq \left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right) \mu_0 + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1} \frac{\mu_0}{2} \\
&\quad \leq \mu_0
\end{aligned}$$

for $\lambda > 0$ small enough because $\kappa(z) \geq \frac{1}{2}$. That is, $z + \lambda(-\bar{W}(z)) \in \bar{\mathcal{D}}(\delta)$ for λ small. Once again, we get

$$\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta)) = 0, \quad \text{for all } \lambda > 0 \text{ small enough.} \tag{2.29}$$

Combining (2.28) and (2.29), we thus obtain

$$\lim_{\lambda \to 0+} \frac{\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta))}{\lambda} = 0, \qquad \forall z \in \bar{\mathcal{D}}(\delta).$$

Consider the following initial value problem

$$\begin{cases} \dfrac{d\varphi(t,\varphi(t_1,v))}{dt} = -\bar{W}(\varphi(t,\varphi(t_1,v))), \\ \varphi(0,\varphi(t_1,v)) = \varphi(t_1,v) \in \bar{\mathcal{D}}(\delta). \end{cases}$$

It has a unique solution $\varphi(t,\varphi(t_1,v))$. By Theorem 1.49, there is a $\tilde{\delta} > 0$ such that

$$\varphi(t,\varphi(t_1,v)) \in \bar{\mathcal{D}}(\delta) \quad \text{for all } t \in [0,\tilde{\delta}).$$

Hence, by the semigroup property, $\varphi(t,v) \in \bar{\mathcal{D}}(\delta)$ for all $t \in [t_1, t_1+\tilde{\delta})$, which contradicts the definition of t_1. Therefore, $\varphi(t,u) \in \mathcal{D}^*$ for all $t \geq 0$.

If $u \in A_1, u \notin \mathcal{D}^*$, then we observe that $G(u) \leq d^* + \bar{\varepsilon}$. If $G(u) \leq d^* - \bar{\varepsilon}$, then

$$G(\varphi(t,u)) \leq G(u) \leq d^* - \bar{\varepsilon}$$

for all $t \geq 0$. Assume $G(u) > d^* - \bar{\varepsilon}$. Then $u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}]$. If

$$\text{dist}(\varphi([0,\infty),u), \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]) \leq \delta_0/2, \tag{2.30}$$

then there exists a t_m such that $\text{dist}(\varphi(t_m,u),\mathcal{S}) \geq \delta_0/4$; that is, $\varphi(t_m,u) \in \mathcal{D}$. Assume that

$$\text{dist}(\varphi([0,\infty),u), \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]) > \delta_0/2 > 0. \tag{2.31}$$

Similarly, we assume that $G(\varphi(t,u)) > d^* - \bar{\varepsilon}$ for all $t \geq 0$ (otherwise, we are done). Then, by (2.26)–(2.31), we have that

$$\frac{(1+\|\varphi(t,u)\|)^2\|G'(\varphi(t,u))\|^2}{(1+\|\varphi(t,u)\|)^2\|G'(\varphi(t,u))\|^2+1} \geq \bar{\varepsilon}, \qquad \vartheta(\varphi(t,u)) = \beta(\varphi(t,u)) = 1 \tag{2.32}$$

for all $t \geq 0$. Therefore, by (2.32),

$$G(\varphi(24,u)) = G(u) + \int_0^{24} dG(\varphi(s,u)) \leq d^* - 2\bar{\varepsilon}. \tag{2.33}$$

By combining the above arguments, we see that for any $u \in A_1 \backslash \mathcal{D}^*$, there exists a $T_u > 0$ such that $\varphi(T_u,u) \in G^{d^*-\bar{\varepsilon}/2} \cup \mathcal{D}^*$. By continuity, there exists a neighborhood U_u such that $\varphi(T_u,U_u) \subset G^{d^*-\bar{\varepsilon}/3} \cup \mathcal{D}^*$. Because $A_1 \backslash \mathcal{D}^*$ is compact, we get a $T_1 > 0$ such that $\varphi(T_1, A_1 \backslash \mathcal{D}^*) \subset G^{d^*-\bar{\varepsilon}/4} \cup \mathcal{D}^*$. Then

$$\varphi(T_1, A_1) \subset G^{d^*-\bar{\varepsilon}/4} \cup \mathcal{D}^*. \tag{2.34}$$

Case 2. If $\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] = \emptyset$, then (2.26) holds with $(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2} = \emptyset$ and $\beta(u) \equiv 1$. Then, trivially, (2.32)–(2.34) are still true.

Now we define

$$\Gamma^*(s,u) = \begin{cases} \varphi(2T_1 s, u), & s \in [0,1/2], \\ \varphi(T_1, \Gamma(2s-1,u)), & s \in [1/2,1]. \end{cases}$$

Then, $\Gamma^* \in \Phi^*$. If $s \in [0,1/2]$, we have that

$$\Gamma^*(s,A) \cap \mathcal{S} \subset \varphi(2T_1 s, A) \cap \mathcal{S} \subset G^{a_0} \cap \mathcal{S} \subset G^{d^*-\bar{\varepsilon}/4}.$$

If $s \in [1/2,1]$, then

$$\begin{aligned} \Gamma^*(s,A) \cap \mathcal{S} &\subset \varphi(T_1, \Gamma(2s-1,A)) \cap \mathcal{S} \\ &\subset \varphi(T_1, A_1) \cap \mathcal{S} \\ &\subset (G^{d^*-\bar{\varepsilon}/4} \cup \mathcal{D}^*) \cap \mathcal{S} \\ &\subset G^{d^*-\bar{\varepsilon}/4} \cap \mathcal{S} \\ &\subset G^{d^*-\bar{\varepsilon}/4}. \end{aligned}$$

It follows that $G(\Gamma^*([0,1],A) \cap \mathcal{S}) \le d^* - \bar{\varepsilon}/4$, a contradiction.

Next we consider the case of $d^* = b_0$. Here we have to construct a different vector field and need a careful analysis of the flow. We prove that $\mathcal{K}[d^*,\ d^*] \cap B \ne \emptyset$. If it were not true, there would exist numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

$$\frac{(1+\|u\|)^2 \|G'(u)\|^2}{1+(1+\|u\|)^2 \|G'(u)\|^2} \ge \varepsilon_1 \tag{2.35}$$

for $|G(u) - d^*| < \varepsilon_2$ and $\mathrm{dist}(u,B) \le \varepsilon_3$. By decreasing ε_2, we may assume that $\varepsilon_2 < \varepsilon_1 \varepsilon_3/16$. Let

$$\begin{aligned} \Omega_3 &:= \{u \in E : \mathrm{dist}(u,B) \le \varepsilon_3/2, |G(u) - d^*| \le \varepsilon_2/2\}, \\ \Omega_4 &:= \{u \in E : \mathrm{dist}(u,B) \le \varepsilon_3/3, |G(u) - d^*| \le \varepsilon_2/3\}. \end{aligned}$$

Then $\mathcal{K} \subset E \backslash \Omega_3$. Choose $\Gamma \in \Phi^*$ such that

$$\sup_{\Gamma([0,1],A)\cap S} G(u) \le d^* + \varepsilon_2/3. \tag{2.36}$$

We can find a $u_0 \in \Gamma([0,1],A) \cap B \cap \mathcal{S} \ne \emptyset$ because A links B and $B \subset \mathcal{S}$. This implies that

$$b_0 \le G(u_0) \le \sup_{\Gamma([0,1],A)\cap S} G(u) \le d^* + \varepsilon_2/3; \tag{2.37}$$

that is, $u_0 \in \Omega_4 \subset \Omega_3$. Let

$$\vartheta_1(u) = \frac{\mathrm{dist}(u, E\backslash\Omega_3)}{\mathrm{dist}(u, E\backslash\Omega_3) + \mathrm{dist}(u, \Omega_4)},$$

and consider the following Cauchy initial value problem,

$$\begin{cases} \dfrac{d\varphi_1(t,u)}{dt} = -\vartheta_1(\varphi_1(t,u))W(\varphi_1(t,u)), \\ \varphi_1(0,u) = u \in E, \end{cases}$$

which has a unique continuous solution $\varphi_1(t,u)$ in E. Obviously, by (2.35),

$$\frac{dG(\varphi_1(t,u))}{dt} \le -\frac{\varepsilon_1}{8}\vartheta_1(\varphi_1(t,u)). \tag{2.38}$$

If $u \in G^{d^*+\varepsilon_2/3}$, then

$$G(\varphi_1(t,u)) \le G(u) \le d^* + \varepsilon_2/3$$

for all $t \ge 0$. If there is a $t_1 \le \varepsilon_3/4$ such that $\varphi_1(t_1,u) \notin \Omega_4$, then either $G(\varphi_1(t_1,u)) < d^* - \varepsilon_2/3$ or $\mathrm{dist}(\varphi_1(t_1,u),B) > \varepsilon_3/3$. For the latter case, we observe that $\mathrm{dist}(\varphi_1(t,u),B) \ge \varepsilon_3/12$, and hence, $\varphi_1(t,u) \notin B$ for all $t \in [0,\varepsilon_3/4]$. If $\varphi_1(t,u) \in \Omega_4$ for all $t \in [0,\varepsilon_3/4]$, then

$$\begin{aligned} G\left(\varphi_1\left(\frac{\varepsilon_3}{4},u\right)\right) &= G(u) + \int_0^{\varepsilon_3/4} dG(\varphi_1(t,u)) \\ &\le d^* + \frac{\varepsilon_2}{3} - \frac{\varepsilon_3\varepsilon_1}{32} \\ &\le d^* - \frac{\varepsilon_2}{6}. \end{aligned}$$

That is, either

$$G(\varphi_1(\varepsilon_3/4,u)) < d^* - \varepsilon_2/6 = b_0 - \varepsilon_2/6$$

or $\varphi_1(t,u) \notin B$ for all $t \in [0,\varepsilon_3/4]$ and each $u \in G^{d^*+\varepsilon_2/3}$. It follows that $\varphi_1(\varepsilon_3/4,u) \notin B$ for any $u \in G^{d^*+\varepsilon_2/3}$. Next we prove that for all $u \in A, t \in [0,\varepsilon_3/4]$, we must have $\varphi_1(t,u) \notin B$. Note that if $u \in A, u \notin \mathcal{S}$, then $u \in \mathcal{D}^*$. Following an argument similar to that of the proof of the first case, we see that $\varphi_1(t,u) \in \mathcal{D}^*$. Hence $\varphi_1(t,u) \notin B \subset \mathcal{S}$ for all $t \ge 0$. Therefore, we may only consider the case $u \in A \cap \mathcal{S}$. Evidently, $\varphi_1(\varepsilon_3/4,u) \notin B$. Furthermore, by (2.38), we see that

$$\begin{aligned} G(\varphi_1(t,u)) &\le G(u) - \frac{\varepsilon_1}{8}\int_0^t \vartheta_1(\varphi_1(t,u))dt \\ &\le d^* - \frac{\varepsilon_1}{8}\int_0^t \vartheta_1(\varphi_1(s,u))ds. \end{aligned}$$

If $\varphi_1(t,u) \in B$, then $G(\varphi_1(t,u)) \geq b_0 = d^*$, and we must have $\vartheta_1(\varphi_1(s,u)) \equiv 0$ for $s \in [0,t]$. This implies that $\varphi_1(s,u) \notin \Omega_4$ and either $G(\varphi_1(s,u)) < d^* - \varepsilon_2/3$ or $\operatorname{dist}(\varphi_1(s,u), B) > \varepsilon_3/3$ for all $s \in [0,t]$. Both cases imply $\varphi_1(t,u) \notin B$. This proves that $\varphi_1([0,\varepsilon_3/4], A) \cap B = \emptyset$. Let

$$\Gamma_1(t,u) = \begin{cases} \varphi_1(2t\varepsilon_3/4, u), & 0 \leq t \leq 1/2, \\ \varphi_1(\varepsilon_3/4, \Gamma(2t-1,u)), & 1/2 \leq t \leq 1. \end{cases}$$

Then it is easy to check that $\Gamma_1 \in \Phi^*$. But by the above arguments,

$$\Gamma_1([0,1], A) \cap B = \emptyset,$$

which contradicts the fact that A links B. □

Theorem 2.14. *Suppose that (2.16) and* (A_1) *hold and that* $\Theta_G : E \to E$ *is a compact operator. Assume that* $E = Y \oplus M, 1 < \dim Y < \infty$, *and that*

(1) $G(v) \leq \alpha$ *for all* $v \in Y$, *where* α *is a positive constant,*
(2) $G(w) \geq \alpha$ *for all* $w \in B := \{w : w \in M, \|w\| = \rho\} \subset \mathcal{S}$, *where* ρ *is a positive constant,*
(3) $G(sw_0 + v) \leq T_0$ *for all* $s \geq 0, v \in Y$; $w_0 \in M\backslash\{0\}$ *is a fixed element, and* T_0 *is a constant.*

If G *satisfies the (w-PS)*$_c$ *condition for all* $c > 0$, *then there exists a sequence* $\{u_n\} \subset E\backslash(-\mathcal{P} \cup \mathcal{P})$ *such that*

$$G'(u_n) \to 0, \qquad G'(u_n) = \frac{T_n}{n} u_n, \qquad G(u_n) \to c,$$

where $\{T_n\}$ *is a bounded sequence and* $c \in [\alpha/2,\ 2T_0]$.

Proof. Define $\psi \in \mathbf{C}^\infty(\mathbf{R})$ such that $\psi = 0$ in $(-\infty, 1/2)$ and $\psi = 1$ in $(1,\infty), 0 \leq \psi \leq 1$. We may assume that $\|w_0\| = 1$. Write $u \in E$ as $u = v + w, v \in Y, w \in M$. Let

$$G_n(u) = G(u) - \left(T_0 + \frac{1}{n}\right) \psi\left(\frac{\|u\|^2}{n}\right), \qquad n = 1, 2, \ldots.$$

Then

$$G'(u) - G_n'(u) = 2\left(T_0 + \frac{1}{n}\right) \psi'\left(\frac{\|u\|^2}{n}\right) \frac{u}{n},$$

$$\|G'(u) - G_n'(u)\| \leq T_1 n^{-1/2}.$$

We claim that G_n satisfies (w-PS) for each n sufficiently large if G does. In fact, assume that $\{u_k\}$ is a (w-PS) sequence: $G_n(u_k) \to c$ and $(1 + \|u_k\|)G_n'(u_k) \to 0$ as $k \to \infty$. If, for a renamed subsequence, $\|u_k\|^2/n > 1$, then $\psi'(\|u_k\|^2/n) = 0$ and

$$(1 + \|u_k\|)G_n'(u_k) = (1 + \|u_k\|)G'(u_k) \to 0.$$

Then $\{u_k\}$ has a convergent subsequence. If $\|u_k\|^2/n \le 1$, then $\{u_k\}$ is bounded and (w-PS) follows immediately. To see this, note that

$$G'(u_k) - 2\left(T_0 + \frac{1}{n}\right)\psi'\left(\frac{\|u_k\|^2}{n}\right)\frac{u_k}{n} \to 0.$$

Take n so large that

$$b(u) = \kappa(u) - \left[2\left(T_0 + \frac{1}{n}\right)\psi'\left(\frac{\|u\|^2}{n}\right)/n\right]$$

is bounded and bounded away from 0. Then

$$b(u_k)u_k - \Theta_G u_k \to 0 \quad \text{as} \quad k \to \infty.$$

Because the $\{u_k\}$ is bounded, there is a renamed subsequence such that both $b(u_k)$ and $\Theta_G u_k$ converge. Hence, this subsequence converges as well. Thus, in both cases, the (w-PS) condition is satisfied. Moreover, $G_n(v) \le \alpha$ for all $v \in Y$. For any $w \in M$, if $\|w\| = \rho$, then $\psi(\|w\|^2/n) = 0$ for $n > 2\rho^2$ and consequently $G_n(w) = G(w) \ge \alpha$. Choose $\|sw_0 + v\| := n^{1/2} := R_n$. Then $R_n > \rho$ if n large enough, and

$$G_n(sw_0 + v) = G(sw_0 + v) - (T_0 + 1/n)\psi\left(\frac{\|sw_0 + v\|^2}{n}\right) \le -\frac{1}{n}.$$

Let

$$B := \{w \in M : \|w\| = \rho\},$$

and

$$A_n := \{v \in Y : \|v\| \le R_n\} \cup \{sw_0 + v : s \ge 0, v \in Y, \|sw_0 + v\| = R_n\}.$$

Then A_n links B, and G_n satisfies all the conditions of Theorem 2.13. Hence, there exists a $u_n \in E\backslash(-\mathcal{P} \cup \mathcal{P})$ such that

$$G_n'(u_n) = 0, \qquad G_n(u_n) \in \left[\alpha/2, \sup_{(t,u)\in[0,1]\times A_n} G_n((1-t)u)\right].$$

Evidently,

$$\|G'(u_n) - G_n'(u_n)\| = \|G'(u_n)\| \le T_1 n^{-1/2} \to 0,$$

$$\alpha/2 \le G_n(u_n) \le G(u_n) \le G_n(u_n) + T_0 + 1/n,$$

$$\sup_{(t,u)\in[0,1]\times A_n} G_n((1-t)u) \le T_0.$$

Therefore, $G(u_n) \to c \in [\alpha/2, 2T_0]$. Finally,

$$G'(u_n) = G'(u_n) - G'_n(u_n) = 2\left(T_0 + \frac{1}{n}\right)\psi'\left(\frac{\|u_n\|}{n}\right)\frac{u_n}{n} = \frac{T_n}{n}u_n,$$

where $\{T_n\}$ is a bounded sequence. $\square$

The statement $G'(u_n) = (T_n/n)u_n$ in Theorem 2.14 is quite helpful for getting a sign-changing limit of the sequence $\{u_n\}$.

We now assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \le C_\star\|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. In the sequel, all properties are with respect to the norm $\|\cdot\|$ if without specific indication. Write $E = M \oplus Y$, where $Y, M := Y^\perp$ are closed subspaces with $\dim Y < \infty$ and $(M\backslash\{0\}) \cap (-\mathcal{P} \cup \mathcal{P}) = \emptyset$; that is, the nontrivial elements of M are sign-changing. Let $y_0 \in M\backslash\{0\}$ with $\|y_0\| = 1$ and $0 < \rho < R$ with

$$R^{p-2}\|y_0\|_\star^p + \frac{R\|y_0\|_\star}{1 + D_\star\|y_0\|_\star} > \rho, \qquad D_\star > 0, p > 2 \text{ are constants.}$$

Let

$$A := \{u = v + sy_0 : v \in Y, s \ge 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$

$$B := \left\{u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star} = \rho\right\}.$$

Then by Proposition 2.10, A links B in the sense of Definition 2.1. Choose

$$a_\star > \sup_{[0,1]\times A} G((1-t)u) + 2. \tag{2.39}$$

Define

$$B^\star := B \cap G^{a_\star}. \tag{2.40}$$

Choose $\Gamma(t,u) = (1-t)u \in \Phi^*$; then $\Gamma(t,a) \in B$ for some $(t,a) \in [0,1] \times A$. Moreover, $\Gamma(t,a) \in G^{a_\star}$, hence, $B^\star := B \cap G^{a_\star} \ne \emptyset$. Set

$$\Phi^{**} := \{\Gamma \in \Phi^* : \Gamma([0,1], A) \subset G^{a_\star}\}. \tag{2.41}$$

Then $\Gamma(t,u) = (1-t)u \in \Phi^* \cap \Phi^{**}$.

Lemma 2.15. $\|u\|_\star \le c_1, \quad \forall u \in B$; *here c_1 is a constant.*

($\mathbf{A_3}$) Assume that for any $a, b > 0$, there is a $c = c(a,b) > 0$ such that

$$G(u) \le a \quad \text{and} \quad \|u\|_\star \le b \;\Rightarrow\; \|u\| \le c.$$

Lemma 2.16. *Assume* (A_3). *Then we have that*

$$\operatorname{dist}(B^\star := B \cap G^{a_\star}, \mathcal{P}) := \delta_1 > 0.$$

Proof. By negation, we assume that $\operatorname{dist}\big(B^\star, \mathcal{P}\big) = 0$. Then we find $\{u_n\} \subset B^\star, \{p_n\} \subset \mathcal{P}$ such that $\|u_n - p_n\| \to 0$. Then $\{u_n\}$, hence $\{p_n\}$, is bounded in both $(E, \|\cdot\|)$ and $(E, \|\cdot\|_\star)$. We assume that $u_n \rightharpoonup u^* \in E$; $p_n \rightharpoonup p^* \in \mathcal{P}$ weakly in $(E, \|\cdot\|)$; $u_n \to u^*$ strongly in $(E, \|\cdot\|_\star)$. Then we observe that $u^* \in M$. Because

$$\frac{\|u_n\|_\star^p}{\|u_n\|^2} + \frac{\|u_n\|\|u_n\|_\star}{\|u_n\| + D_\star\|u_n\|_\star} = \rho$$

and $\|u_n - u^*\|_\star \to 0$, then $u^* \neq 0$. However, because $u^* = p^*$, we get a contradiction in as much as all nonzero elements of M are sign-changing. $\square$

In view of Lemma 2.16, we may assume that $B^\star \subset \mathcal{S}$ as long as the δ of Condition (A_1) is small enough; this is indeed true in our applications.

Theorem 2.17. *Suppose that (2.16),* (A_1)*, and* (A_3) *hold. Assume*

$$a_0 := \sup_A G \le b_0^\star := \inf_{B^\star} G.$$

Define

$$d^* := \inf_{\Gamma \in \Phi^{**}} \sup_{\Gamma([0,1],A)\cap\mathcal{S}} G(u);$$

then

$$d^* \in \big[b_0^\star, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)\big].$$

Furthermore, if G *satisfies the (w-PS)*$_c$ *condition for any* c *with*

$$c \in \big[b_0^\star, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)\big],$$

then

$$\mathcal{K}[d^* - \varepsilon, d^* + \varepsilon] \cap \big(E\backslash(-\mathcal{P}\cup\mathcal{P})\big) \neq \emptyset$$

for all $\varepsilon > 0$ *small. Moreover, if* $d^* = b_0^\star$*, then* $\mathcal{K}[d^*, d^*] \subset B^\star$.

Proof. It suffices to note that any flow ϕ considered in the proof of Theorem 2.13 is nonincreasing in the sense that $G(\phi(t,u))$ is nonincreasing in t. Then the proof of this theorem is the same as that of Theorem 2.13 where B is replaced by $B^\star$. $\square$

Theorem 2.18. *Suppose that* (A_3) *holds. Theorem 2.14 is still true if we replace* B *by* $B^\star$ *and* $a_\star := T_0 + 2$.

Notes and Comments. The ideas of the proofs for Lemmas 2.11 and 2.12 first come from Sun [316] (see also Guo [163] and a paper by Liu and Sun [211]). In [316], $\mathcal{D}^*$ itself is a convex set. In [211], it is assumed that $\Theta_G\big(\partial\mathcal{D}(\delta)\big) \subset \mathcal{D}(\delta)$

and that $G' = \mathbf{id} - \Theta_G$. Lemma 2.11 and Theorems 2.13 and 2.14 are proved in Schechter and Zou [288].

Condition (A_1) is applied in Conti et al. [107]. In particular, [107] is the pioneering paper where the neighborhood of a cone is introduced which satisfies the type of condition (A_1). By using the invariant sets of flows and lower (upper) solutions, Conti et al. obtained the existence of multiple solutions with ordering relations. Similar ideas are also used in Conti et al. [108] for the existence of many solutions for superlinear elliptic systems. Later, this idea of the neighborhood of a cone is used by Bartsch et al., Liu and Wang, Schechter and Zou, and Zou, among others.

In Theorem 2.14, T_0 is an arbitrary constant that is not necessarily equal to α. This novelty makes it powerful in applications, especially in dealing with asymptotically linear equations. Note that T_0 must be equal to α in classical linking (cf. Benci and Rabinowitz [55], Brézis and Nirenberg [70], Li and Liu [197], Li and Willem [200], Silva [295], and Tintarev [328]).

2.3 Jumping Dirichlet Equations

Consider the sign-changing solutions to the following Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.42}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$ and finite measure $\operatorname{meas}\Omega := |\Omega|$.

Let $E := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v\rangle := \int_\Omega (\nabla u \nabla v)dx$$

for $u, v \in E$ and the norm $\|u\| := \langle u,\ u\rangle^{1/2}$. Let

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

denote the distinct Dirichlet eigenvalues of $-\Delta$ on Ω with zero boundary value. Then each λ_k has finite multiplicity. The principal eigenvalue λ_1 is simple with a positive eigenfunction φ_1, and the eigenfunctions φ_k corresponding to λ_k $(k \geq 2)$ are sign-changing. Let N_k denote the eigenspace of λ_k. Then $\dim N_k < \infty$. We fix k and let $E_k := N_1 \oplus \cdots \oplus N_k$. In this section, we consider the case of

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = \beta_+(x), \qquad \lim_{t\to-\infty} \frac{f(x,t)}{t} = \beta_-(x) \tag{2.43}$$

uniformly for $x \in \Omega$. In particular,

$$\lambda_k < \beta_\pm(x) < c, \quad \text{where } c > 0 \text{ is a fixed constant.}$$

Throughout this section, we assume

(B_1) f is a Carathéodory function and $f(x,t)t \geq 0$ for $(x,t) \in \Omega \times \mathbf{R}$; $\lim_{t \to 0}(f(x,t))/t = 0$ uniformly for $x \in \Omega$.

(B_2) $2F(x,t) \geq \lambda_{k-1}t^2 - c_0$ for all $x \in \Omega, t \in \mathbf{R}$, where $F(x,t) = \int_0^t f(x, s)ds$; $c_0 > 0$ is a constant.

By the above assumptions, we may find a $C_F > 0$ such that

$$F(x,t) \leq \frac{1}{4}\lambda_1|t|^2 + C_F|t|^p, \qquad \forall x \in \Omega, \qquad t \in \mathbf{R}; \tag{2.44}$$

here $2 < p < 2^*$. Also, we can get another constant $\Lambda_0 > 0$ such that

$$2F(x,t) \leq \Lambda_0 t^2 \quad \text{for all } x \in \Omega, \qquad t \in \mathbf{R}. \tag{2.45}$$

Recall the Gagliardo–Nirenberg inequality,

$$\|u\|_p \leq c_p\|u\|^\alpha\|u\|_2^{(1-\alpha)}, \qquad u \in E, \tag{2.46}$$

where $\alpha \in (0,1)$ is defined by

$$\frac{1}{p} = \alpha\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\alpha)\frac{1}{2}. \tag{2.47}$$

On the other hand, we have a constant $\Lambda_p > 0$ such that

$$\|u\|_p \leq \Lambda_p\|u\|, \qquad u \in E. \tag{2.48}$$

Without loss of generality, we assume that $\Lambda_p > 1$ and $c_p > 1$. Set

$$\Lambda_p^* := \min\left\{\frac{1}{4\Lambda_p^2 c_p^{(p-2)}}, (4\Lambda_p^2 c_p^{(p-2)})^{-(1/(p-2))}\right\}, \tag{2.49}$$

$$T_1 := \min\{\lambda_k^{(1-\alpha)(p-2)}, \lambda_k^{(1-\alpha)}\}, \tag{2.50}$$

$$T_2 := \min\left\{\frac{1}{64C_F^2}, (8C_F)^{-(1/(p-2))}\right\}. \tag{2.51}$$

(B_3) Assume that

$$c_0 \leq \frac{1}{4|\Omega|}\left(\Lambda_p^*\right)^2 T_1 T_2.$$

The first result deals with the case of a jump not crossing eigenvalues: $\lambda_k < \beta_\pm(x) \leq \lambda_{k+1}$. Resonance may occur at λ_{k+1}.

Theorem 2.19. *Assume that* (B_1)–(B_3) *and (2.43) hold with* $\lambda_k < \beta_\pm(x) \leq \lambda_{k+1}$. *If either* $\beta_+(x) < \lambda_{k+1}$ *for* $x \in \Omega$ *or* $\beta_-(x) < \lambda_{k+1}$ *for* $x \in \Omega$, *then Equation (2.42) has a sign-changing solution.*

If we strengthen the condition on f, we have the following theorem where the jump is allowed to cross an arbitrarily finite number of eigen-values.

Theorem 2.20. *Assume that* (B_1)–(B_3) *and (2.43) hold. If* $\lambda_k < \beta_\pm(x)$ *for* $x \in \Omega$, *and*

$(\mathbf{B_4})$ *there exists a* $C_0(x) \in L^1(\Omega)$ *such that*

$$\text{(i) } f(x,t)t - 2F(x,t) \geq C_0(x), \quad \text{for } (x,t) \in \Omega \times \mathbf{R},$$
$$\text{(ii) } \lim_{|t|\to\infty} (f(x,t)t - 2F(x,t)) = \infty \quad \text{for } x \in \Omega.$$

then Equation (2.42) has a sign-changing solution.

Theorem 2.20 permits $\beta_\pm(x)$ to be arbitrary bounded functions greater than λ_k and to cross an arbitrarily finite number of eigenvalues of $-\Delta$ with zero boundary value condition. Therefore, the jump has much more freedom.

For the Dirichlet boundary value problem (2.42), it is usually called jumping nonlinearity at $\pm\infty$ if

$$\begin{cases} f(x,t)/t \to a & \text{a.e. } x \in \Omega \text{ as } t \to -\infty, \\ f(x,t)/t \to b & \text{a.e. } x \in \Omega \text{ as } t \to \infty. \end{cases}$$

The existence of solutions of (2.42) is closely related to the equation

$$-\Delta u = bu^+ - au^-, \quad \text{where } u^\pm = \max\{\pm u, 0\}.$$

Conventionally, the set

$$\Sigma := \{(a,b) \in \mathbf{R}^2 : -\Delta u = bu^+ - au^- \text{ has nontrivial solutions}\}$$

is called the Fučík spectrum of $-\Delta$ (see Dancer [127], Fučík [151], and Schechter [269]). It plays a key role in most results of this aspect. However, so far no complete description of Σ has been found. If

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

are the distinct Dirichlet eigenvalues of $-\Delta$ on Ω with zero boundary value, it was shown in Schechter [269] that in the square $(\lambda_{l-1}, \lambda_{l+1})^2$ there are decreasing curves C_{l1}, C_{l2} (which may or may not coincide) passing through the point (λ_l, λ_l) such that all points above or below both curves in the square (the so-called type (I) region) are not in Σ, whereas points on the curves are in Σ. Usually, the status of points between the curves (referred to as the type (II) region if the curves do not coincide) is unknown. However, it was shown in Gallouët and Kavian [154] that when λ is a simple eigenvalue, then points of the type (II) region are not in Σ. On the other hand, Margulies and Margulies [223] have shown that there are boundary value problems for which many curves in Σ emanate from a point (λ_l, λ_l) when λ_l is a multiple eigenvalue. Certainly, these curves are contained in region (II).

Next we proceed to prove the above theorems. Define

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx, \qquad u \in E.$$

Then $G \in \mathbf{C}^1(E, \mathbf{R})$ and $G'(u) = u - \Theta_G(u), u \in E$, where $\Theta_G : E \to E$ is a compact operator. Actually, $\Theta_G(u) = (-\Delta)^{-1}\big(f(x,u)\big)$.

Lemma 2.21. *Under the assumptions of Theorems 2.19–2.20, $G(u) \to -\infty$ for $u \in E_k$ as $\|u\| \to \infty$.*

Proof. Rewrite G as

$$G(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} \left(\frac{1}{2}\beta_+(x)(u^+)^2 + \frac{1}{2}\beta_-(x)(u^-)^2 + H(x,u)\right) dx, \quad u \in E,$$

where $H(x,u) := \int_0^u h(x,t)dt$;

$$h(x,t) = f(x,t) - (\beta_+(x)t^+ - \beta_-(x)t^-); \qquad t^\pm = \max\{\pm t, 0\}.$$

Therefore,

$$\begin{aligned} G(u) \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)dx \\ &\quad -\frac{1}{2}\left(\int_{\beta_-(x)\geq\beta_+(x)} + \int_{\beta_-(x)<\beta_+(x)}\right)(\beta_+(x)(u^+)^2 + \beta_-(x)(u^-)^2)dx \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\beta_-(x)\geq\beta_+(x)} \beta_+(x)u^2dx \\ &\quad -\frac{1}{2}\int_{\beta_-(x)\geq\beta_+(x)} (\beta_-(x) - \beta_+(x))(u^-)^2dx \\ &\quad -\frac{1}{2}\int_{\beta_-(x)<\beta_+(x)} \beta_-(x)u^2dx \\ &\quad -\frac{1}{2}\int_{\beta_-(x)<\beta_+(x)} (\beta_+(x) - \beta_-(x))(u^+)^2dx - \int_\Omega H(x,u)dx. \end{aligned}$$

Note $\min\{\beta_+(x), \beta_-(x)\} > \lambda_k$ and recall the variational characterization of eigenvalues $\{\lambda_k\}$; we have the following estimates for any $u \in E_k$.

$$
\begin{aligned}
&G(u)\\
&\quad\le \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\beta_-(x)\ge\beta_+(x)} \beta_+(x)u^2dx\\
&\qquad -\frac{1}{2}\int_{\beta_-(x)<\beta_+(x)} \beta_-(x)u^2dx - \int_\Omega H(x,u)dx\\
&\quad\le \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega \min\{\beta_+(x),\beta_-(x)\}u^2dx - \int_\Omega H(x,u)dx\\
&\quad\le -\delta\|u\|^2 - \int_\Omega H(x,u)dx,
\end{aligned}
$$

where $\delta > 0$ is a constant. The last inequality is due to the finite dimension of E_k and the Schechter–Simon Theorem 1.62. Therefore,

$$\lim_{\|u\|\to\infty, u\in E_k} \frac{G(u)}{\|u\|^2} \le -\delta$$

because

$$\lim_{t\to\infty} \frac{h(x,t)}{t} = 0$$

and $\dim E_k < \infty$. □

Lemma 2.22. *Assume* (B_2)*. Then*

$$G(u) \le \frac{c_0|\Omega|}{2}, \qquad \forall u \in E_{k-1}.$$

Proof. For $u \in E_{k-1}$,

$$
\begin{aligned}
G(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx\\
&\le \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega \lambda_{k-1}u^2dx + \frac{1}{2}\int_\Omega c_0dx\\
&\le \frac{c_0|\Omega|}{2}.
\end{aligned}
$$

□

For $p > 2$ given in (2.44), we let

$$\xi_0(u) := \begin{cases} \dfrac{\|u\|_p^p}{\|u\|^2} + \dfrac{\|u\|\|u\|_p}{\|u\| + \lambda_k^\beta\|u\|_p}, & \text{if } u \ne 0,\\ 0, & \text{if } u = 0, \end{cases} \tag{2.52}$$

where $\beta = (1-\alpha)(p-2)$. Then $\xi_0 : E \to E$ is continuous. Set

$$S_0 := \{u \in E_{k-1}^{\perp} : \xi_0(u) = \rho\}, \qquad \rho := \frac{1}{8C_F} > 0. \tag{2.53}$$

For $u \in S_0$, by (2.48) we have

$$\begin{aligned}
\rho &= \frac{\|u\|_p^p}{\|u\|^2} + \frac{\|u\|\|u\|_p}{\|u\| + \lambda_k^{\beta}\|u\|_p} \\
&\leq \frac{\|u\|\|u\|_p}{2(\|u\|\lambda_k^{\beta}\|u\|_p)^{1/2}} + \frac{\|u\|_p^2}{\|u\|^2}\|u\|_p^{p-2} \\
&\leq \frac{(\|u\|\|u\|_p)^{1/2}}{2(\lambda_k^{\beta})^{1/2}} + \varLambda_p^2\|u\|_p^{p-2} \\
&\leq \frac{(\varLambda_p)^{1/2}\|u\|}{2(\lambda_k^{\beta})^{1/2}} + \varLambda_p^2\|u\|_p^{p-2}.
\end{aligned}$$

By the Gagliardo–Nirenberg inequality in (2.46) and (2.47),

$$\|u\|_p^{p-2} \leq c_p^{p-2}\|u\|^{\alpha(p-2)}\|u\|_2^{(1-\alpha)(p-2)}. \tag{2.54}$$

But $u \in E_{k-1}^{\perp}$; we see that

$$\lambda_k\|u\|_2^2 \leq \|u\|^2 \quad \text{and} \quad \|u\|_2 \leq \frac{1}{\lambda_k^{1/2}}\|u\|.$$

Hence,

$$\|u\|_p^{p-2} \leq c_p^{p-2}\|u\|^{p-2}\lambda_k^{-(((1-\alpha)(p-2))/2)}. \tag{2.55}$$

Therefore,

$$\begin{aligned}
\rho &\leq \frac{(\varLambda_p)^{1/2}\|u\|}{2(\lambda_k^{\beta})^{1/2}} + (\varLambda_p)^2 c_p^{p-2}\|u\|^{p-2}\lambda_k^{-(((1-\alpha)(p-2))/2)} \\
&\leq \left(\frac{1}{(\lambda_k^{\beta})^{1/2}} + \frac{1}{\lambda_k^{((1-\alpha)(p-2)/2)}}\right)(2\varLambda_p^2 c_p^{p-2})\max\{\|u\|, \|u\|^{p-2}\}.
\end{aligned}$$

Then we have that

$$\frac{\lambda_k^{((1-\alpha)(p-2))/2}}{(4\varLambda_p^2 c_p^{p-2})}\rho \leq \max\{\|u\|, \|u\|^{p-2}\}. \tag{2.56}$$

Lemma 2.23. *For all* $u \in S_0$,

$$\|u\| \geq \Lambda_p^* \min\{\lambda_k^{((1-\alpha)(p-2))/2}, \lambda_k^{(1-\alpha)/2}\} \min\{\rho, \rho^{1/(p-2)}\}.$$

Lemma 2.24. $\|u\|_p^p/\|u\|^2 \leq \rho, \quad \forall u \in S_0.$

Lemma 2.25. $\|u\|_p \leq c_1, \quad \forall u \in S_0.$

Proof. If $\|u\|_p \to \infty$, then so does $\|u\| \to \infty$; hence

$$\frac{\|u\|\|u\|_p}{\|u\| + \lambda_k^\beta \|u\|_p} \to \infty,$$

a contradiction. □

By (2.44), we know that $F(x,u) \leq \lambda_1/4|u|^2 + C_F|u|^p, \quad \forall x \in \Omega,\ u \in \mathbf{R}$. Consider the functional

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx, \qquad u \in H_0^1(\Omega).$$

Then

$$\begin{aligned} G(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{4}\|u\|_2^2 - C_F\|u\|_p^p \\ &\geq \frac{1}{4}\|u\|^2 - C_F\|u\|_p^p \\ &\geq \|u\|^2\left(\frac{1}{4} - C_F\frac{\|u\|_p^p}{\|u\|^2}\right). \end{aligned}$$

Combining Lemma 2.23 and Lemma 2.24, we have the following.

Lemma 2.26. *For any* $u \in S_0$, *we have that*

$$G(u) \geq \frac{1}{8}(\Lambda_p^*)^2 T_1 T_2 \geq \frac{1}{2}|\Omega|c_0.$$

Lemma 2.27. *Under the assumptions of Theorem 2.19, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence:

$$G(u_n) \to c, \qquad (1 + \|u_n\|)G'(u_n) \to 0.$$

By negation, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $w_n = u_n/\|u_n\|$. Then $\|w_n\| = 1$ and there is a renamed subsequence such that $w_n \to w$ weakly in E, strongly in $L^2(\Omega)$, and a.e. in Ω. Moreover,

$$\langle G'(u_n), v\rangle = \langle u_n, v\rangle - \int_\Omega f(x, u_n) v dx \to 0$$

and

$$\langle w_n, v\rangle - \int_\Omega \frac{f(x, u_n)v}{\|u_n\|} dx \to 0.$$

By (2.43), we see that

$$-\Delta w = \beta_+ w^+ - \beta_- w^-.$$

Because

$$G(u_n)/\|u_n\|^2 = 1/2 - \int_\Omega F(x, u_n)dx/\|u_n\|^2 \to 0,$$

we see that $\int_\Omega (\beta_+(w^+)^2 + \beta_-(w^-)^2)dx = 1$. It implies that $w \neq 0$. Let $w := w_- + w_+$ with $w_- \in E_k$, $w_+ \in E_k^\perp$, $\tilde{w} := w_+ - w_-$. Let $q(x) = \beta_+(x)$ when $w(x) \geq 0$; $q(x) = \beta_-(x)$ when $w(x) < 0$. Then we have that $-\Delta w = q(x)w$ and hence

$$\|w_+\|^2 - \|w_-\|^2 = \int_\Omega q(x)(w_+)^2 dx - \int_\Omega q(x)(w_-)^2 dx.$$

It follows that

$$0 \leq \|w_+\|^2 - \lambda_{k+1}\|w_+\|_2^2 \leq \|w_+\|^2 - \int_\Omega q(x)(w_+)^2 dx$$

$$= \|w_-\|^2 - \int_\Omega q(x)(w_-)^2 dx \leq \|w_-\|^2 - \lambda_k \int_\Omega (w_-)^2 dx \leq 0.$$

That is, $\|w_\pm\|^2 = \int_\Omega q(x)(w_\pm)^2$. The only way this can happen is $q(x) = \lambda_k$ when $w_-(x) \neq 0$ and $q(x) = \lambda_{k+1}$ when $w_+(x) \neq 0$ and therefore, either w_- is an eigenfunction of λ_k or w_+ is an eigenfunction of λ_{k+1}. But the first case cannot occur because $\beta_\pm > \lambda_k$. If w_+ is an eigenfunction of λ_{k+1}, then w_+ is sign-changing. Because $-\Delta w_+ = \beta_+(x)w_+^+ - \beta_-(x)w_+^-$, we have $\beta_- = \lambda_{k+1}$ on a subset of Ω of positive measure and $\beta_+ = \lambda_{k+1}$ on another subset of Ω of positive measure. This contradicts the assumption of the theorem. □

Lemma 2.28. *Under the assumptions of Theorem 2.20, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence: $(1 + \|u_n\|)\|G'(u_n)\| \to 0$ and $\{G(u_n)\}$ is bounded. Then

$$G(u_n) - \frac{1}{2}\langle G'(u_n), u_n\rangle = \int_\Omega \left(\frac{1}{2} f(x, u_n)u_n - F(x, u_n)\right) dx < c \tag{2.57}$$

and

$$\frac{1}{2}\|u_n\|^2 \le c + \int_\Omega F(x,u_n)dx \le c + \int_\Omega \Lambda_0 u_n^2 dx.$$

If $\{\|u_n\|\}$ is unbounded, then, for a renamed subsequence,

$$1 \le 2\Lambda_0 \lim_{n\to\infty} \int_\Omega \frac{u_n^2}{\|u_n\|^2} dx.$$

It follows that $\lim_{n\to\infty} |u_n|^2 = \infty$ on a subset of Ω with a positive measure. Combining this with (B_4), we have $\int_\Omega \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right)dx \to \infty$, which contradicts (2.57). $\square$

To prove Theorems 2.19 and 2.20, we apply Theorem 2.17. First, we let

$$\mathcal{P} := \{u \in E : u(x) \ge 0 \text{ for a.e. } x \in \Omega\}.$$

Then $\mathcal{P}$ $(-\mathcal{P})$ is the positive (negative) cone of E and $\pm\mathcal{P}$ has an empty interior. Let

$$A := \{u = v + sy_0 : v \in E_{k-1}, s \ge 0, \|u\| = R\} \cup (E_{k-1} \cap B_R(0)),$$

where $y_0 \in E_{k-1}^\perp$, $\|y_0\| = 1$ and R large enough. Let ρ be defined in (2.53). By Proposition 2.10, A links S_0. Choose

$$a_\star > \sup_{[0,1]\times A} G((1-t)u) + 2. \tag{2.58}$$

Define

$$B^\star := S_0 \cap G^{a_\star}. \tag{2.59}$$

In Lemmas 2.15 and 2.16, we chose $\|\cdot\|_\star = \|\cdot\|_p$. Then by Lemma 2.16,

$$\operatorname{dist}(B^\star := S_0 \cap G^{a_\star}, \mathcal{P}) := \delta_1 > 0.$$

We define

$$\mathcal{D}(\mu_0) := \{u \in E : \operatorname{dist}(u,\mathcal{P}) < \mu_0\}.$$

Lemma 2.29. *Under the assumptions of* (B_1), *there exists a* $\mu_0 \in (0,\delta_1)$ *such that* $\Theta_G(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu_0/2)$.

Proof. Write $u^\pm = \max\{\pm u, 0\}$. For any $u \in E$,

$$\|u^+\|_2 = \min_{w\in(-\mathcal{P})} \|u - w\|_2$$

$$\leq \frac{1}{\lambda_1^{1/2}} \min_{w \in (-\mathcal{P})} \|u - w\|$$

$$= \frac{1}{\lambda_1^{1/2}} \operatorname{dist}(u, -\mathcal{P}) \tag{2.60}$$

and, for each $s \in (2, 2^*]$, there exists a $C_s > 0$ such that

$$\|u^{\pm}\|_s \leq C_s \operatorname{dist}(u, \mp\mathcal{P}). \tag{2.61}$$

By assumption (B_1), for each $\varepsilon' > 0$ small enough, there exists a $C_{\varepsilon'} > 0$ such that

$$f(x,t)t \leq \varepsilon' t^2 + C_{\varepsilon'}|t|^p, \qquad x \in \Omega, t \in \mathbf{R}, \tag{2.62}$$

where $p > 2$ is a constant. Let $v = \Theta_G(u)$. Then by (2.60)–(2.62),

$$\begin{aligned}
&\operatorname{dist}(v, -\mathcal{P})\|v^+\| \\
&\quad \leq \|v^+\|^2 \\
&\quad = \langle v, v^+ \rangle \\
&\quad = \int_\Omega f(x, u^+) v^+ dx \\
&\quad \leq \int_\Omega (\varepsilon'|u^+| + C_{\varepsilon'}|u^+|^{p-1})|v^+| dx \\
&\quad \leq \left(\frac{2}{5} \operatorname{dist}(u, -\mathcal{P}) + C(\operatorname{dist}(u, -\mathcal{P}))^{p-1} \right) \|v^+\|.
\end{aligned}$$

That is,

$$\operatorname{dist}(\Theta_G(u), -\mathcal{P}) \leq \left(\frac{2}{5}\right) \operatorname{dist}(u, -\mathcal{P}) + C(\operatorname{dist}(u, -\mathcal{P}))^{p-1}.$$

So, there exists a $\mu_0 < \delta_1$ such that $\operatorname{dist}(\Theta_G(u), -\mathcal{P}) < \frac{1}{2}\mu_0$ for every $u \in -\mathcal{D}(\mu_0)$. Similarly, $\operatorname{dist}(\Theta_G(u), \mathcal{P}) < \frac{1}{2}\mu_0$ for every $u \in \mathcal{D}(\mu_0)$. The conclusion follows. □

Proofs of Theorems 2.19–2.20. By Theorem 2.17, there exists a $u \in E\backslash(-\mathcal{P} \cup \mathcal{P})$ (sign-changing critical point) such that

$$G'(u) = 0, \;\; G(u) \in \left[b_0^\star - \bar{\varepsilon}, \sup_{(t,u) \in [0,1] \times A} G((1-t)u) + \bar{\varepsilon} \right],$$

where $b_0^* = c_0|\Omega|/2$; $\bar{\varepsilon}$ is small enough. □

Notes and Comments. The study of the Fučík spectrum began with Ambrosetti and Prodi [17], Dancer [125], and Fučík [151]. They first realized that the set Σ is an important factor in the study of semilinear elliptic boundary value problems with jumping nonlinearities. There are many papers on the existence of solutions to Dirichlet elliptic boundary value problems with jumping nonlinearities; see Các [75], Dancer [127, 128], Giannoni and Micheletti [159], Hirano and Nishimura [169], Lazer and Mckenna [191, 192], Liu and Wu [209], Margulies and Margulies [223], Marino and Saccon [220], Perera and Schechter [244–246], Schechter [269, 272, 273, 275, 276], and their references cited therein. They were mainly concerned with the existence results without sign-changingness of the solution. Dancer and Du [130] (jumping at zero), Dancer and Zhang [132], Li and Zhang [201], and Schechter et al. [279] got sign-changing solutions for Dirichlet zero-boundary value problems where the Fučík spectrum of Dirichlet boundary value problems is essential to their arguments. Note that in Schechter et al. [279], the authors first proved that the sign-changing solutions of Dirichlet boundary value problems are independent of the Fučík spectrum. The Fučík spectrum for Schrödinger equations is an open question. Lemma 2.29 is due to Bartsch, Liu and Weth [37], whose earlier ideas can be found in Conti, Merizzi and Terracini [107, 108].

2.4 Oscillating Dirichlet Equations

In this section, we consider the following case,

$$\liminf_{t\to\pm\infty} \frac{f(x,t)}{t} := \theta_\pm(x); \qquad \limsup_{t\to\pm\infty} \frac{f(x,t)}{t} := \vartheta_\pm(x), \tag{2.63}$$

where $\theta_\pm, \vartheta_\pm \in L^\infty(\Omega)$. Assumption (2.63) implies that the nonlinearities are jumping and oscillating. Assume

($\mathbf{B_5}$) $2F(x,t) \geq \max\{\lambda_{k-1}t^2, \theta_+(x)(t^+)^2 + \theta_-(x)(t^-)^2\} - c_0$ for $x \in \Omega, t \in \mathbf{R}$; $c_0 > 0$ is a constant.

Theorem 2.30. *Assume $(B_1), (B_3)$, and (B_5). For each pair of numbers α_+, β_- in the interval $(\lambda_k, \lambda_{k+1})$ there are numbers $\alpha_- < \lambda_k$ and $\beta_+ > \lambda_{k+1}$ such that*

$$\alpha_\pm \leq \theta_\pm(x) \leq \vartheta_\pm(x) \leq \beta_\pm, \quad x \in \Omega.$$

Then Equation (2.42) has a sign-changing solution.

Theorem 2.31. *Assume $(B_1), (B_3)$, and (B_5). For each pair of numbers α_-, β_+ in the interval $(\lambda_k, \lambda_{k+1})$ there are numbers $\alpha_+ < \lambda_k$ and $\beta_- > \lambda_{k+1}$ such that*

$$\alpha_\pm \leq \theta_\pm(x) \leq \vartheta_\pm(x) \leq \beta_\pm, \quad x \in \Omega.$$

Then Equation (2.42) has a sign-changing solution.

Theorem 2.32. *Assume* $(B_1), (B_3)$, *and* (B_5). *Suppose that*

$$\|v\|^2 \leq \int_{\mathbf{R}^N} (\theta_+(v^+)^2 + \theta_-(v^-)^2)dx, \quad \forall v \in E_k; \quad \vartheta_\pm(x) \leq \lambda_{k+1}, \quad x \in \Omega$$

and that no eigenfunction corresponding to λ_{k+1} *satisfies*

$$-\Delta u + V(x)u = \vartheta_+ u^+ - \vartheta_- u^-,$$

and no function in $E_k \backslash \{0\}$ *satisfies* $-\Delta u + V(x)u = \theta_+ u^+ - \theta_- u^-$. *Then Equation (2.42) has a sign-changing solution.*

Theorem 2.33. *Assume* $(B_1), (B_3)$, *and* (B_5). *Suppose that*

$$\lambda_k \leq \theta_\pm(x) \leq \vartheta_\pm(x) \leq \lambda_{k+1}, \qquad x \in \Omega$$

and that no eigenfunction corresponding to λ_k *satisfies*

$$-\Delta u = \theta_+ u^+ - \theta_- u^-$$

and that no eigenfunction corresponding to λ_{k+1} *satisfies* $-\Delta u = \vartheta_+ u^+ - \vartheta_- u^-$. *Then Equation (2.42) has a sign-changing solution.*

Some lemmas are necessary for proving the above theorems.

Lemma 2.34. *For each pair of numbers* $\alpha_+, \beta_- \in (\lambda_k, \lambda_{k+1})$, *there are numbers* $\alpha_- < \lambda_k, \beta_+ > \lambda_{k+1}$ *such that*

$$\|u\|^2 < \int_\Omega (\alpha_+(u^+)^2 + \alpha_-(u^-)^2)dx, \qquad \forall\, u \in E_k \backslash \{0\}; \tag{2.64}$$

$$\|u\|^2 > \int_{\mathbf{R}^N} (\beta_+(u^+)^2 + \beta_-(u^-)^2)dx, \qquad \forall\, u \in E_k^\perp \backslash \{0\}. \tag{2.65}$$

Proof. To prove (2.64), we define

$$\bar{c}_0 := \max_{u \in E_k, \|u\|_2 = 1} \left(\|u\|^2 - \int_\Omega \alpha_+(u^+)^2 dx - \int_\Omega \lambda_k (u^-)^2 dx \right).$$

Because $\dim E_k < \infty$, $\bar{c}_0$ exists and is attained at a point $u_0 \in E_k$ with $\|u_0\|_2 = 1$. Then,

$$\bar{c}_0 = \left(\|u_0\|^2 - \int_\Omega \lambda_k u_0^2 dx \right) + \int_\Omega (\lambda_k - \alpha_+)(u_0^+)^2 dx \leq 0.$$

Note that both terms in the middle above are less than or equal to zero. If $\|u_0\|^2 - \int_\Omega \lambda_k u_0^2 dx = 0$, then $u_0 \in N_k$ is an eigenfunction of λ_k. Hence, $u_0^+ \not\equiv 0$ because the eigenfunction is sign-changing. Thus, the second term, hence $\bar{c}_0$, is less than zero. Therefore, for all $u \in E_k \backslash \{0\}$,

$$\|u\|^2 - \int_\Omega (\alpha_+(u^+)^2 + \alpha_-(u^-)^2)dx \le (\bar{c}_0 + \lambda_k - \alpha_-)\int_\Omega u^2 dx < 0$$

for an appropriate $\alpha_- < \lambda_k$. To prove (2.65), we define

$$d_0 = \inf_{u \in E_k^\perp, \|u\|_2 = 1} \left(\|u\|^2 - \int_\Omega \lambda_{k+1}(u^+)^2 dx - \int_\Omega \beta_-(u^-)^2 dx \right).$$

Note that $\int_\Omega \lambda_{k+1}(u)^2 dx \le \|u\|^2$ for all $u \in E_k^\perp$; we see that

$$\begin{aligned} &\|u\|^2 - \int_\Omega \lambda_{k+1}(u^+)^2 dx - \int_\Omega \beta_-(u^-)^2 dx \\ &\ge \int_\Omega (\lambda_{k+1} - \beta_-)(u^-)^2 dx \\ &\ge 0. \end{aligned} \tag{2.66}$$

It follows that $d_0 \ge 0$. It suffices to show that $d_0 > 0$. But, if this were not true, we would have a sequence $\{u_n\} \subset E_k^\perp$, $\|u_n\|_2 = 1$ such that

$$d_n := \|u_n\|^2 - \int_\Omega \lambda_{k+1}(u_n)^2 dx - \int_\Omega (\lambda_{k+1} - \beta_-)(u_n^-)^2 dx \to 0$$

as $n \to \infty$. It follows that

$$\|u_n\|^2 \le \lambda_{k+1} + d_n.$$

We may assume that $u_n \to u_*$ weakly in E and strongly in $L^2(\Omega)$, hence, $\|u_*\|_2 = 1$. Therefore,

$$\|u_*\|^2 - \int_{\mathbf{R}^N} \lambda_{k+1}(u_*)^2 dx - \int_\Omega (\lambda_{k+1} - \beta_-)(u_*^-)^2 dx \le \lim_{n\to\infty} d_n = 0.$$

This implies that $u_*^- = 0$ and

$$\|u_*\|^2 = \int_\Omega \lambda_{k+1}(u_*)^2 dx.$$

All these mean that u_* is a positive eigenfunction of λ_{k+1}. This contradiction completes the proof of the lemma. □

Lemma 2.35. *Under the assumptions of Theorem 2.30, G satisfies the (PS) condition.*

Proof. Lemma 2.34 and the conditions of Theorem 2.30 imply that

$$\|u\|^2 < \int_{\mathbf{R}^N} (\theta_+(u^+)^2 + \theta_-(u^-)^2)dx, \qquad \forall u \in E_k \backslash \{0\}; \tag{2.67}$$

$$(2.68) \qquad \|u\|^2 > \int_\Omega (\vartheta_+(u^+)^2 + \vartheta_-(u^-)^2)dx, \qquad \forall u \in E_k^\perp \backslash \{0\}.$$

Now let $\{u_n\}$ be a (PS) sequence: $\|G'(u_n)\| \to 0$ and $\{G(u_n)\}$ is bounded. We just have to show that $\{u_n\}$ is bounded. To show this, assume that $\|u_n\| \to \infty$. Let $\bar{u}_n = u_n/\|u_n\|$. Then $\bar{u}_n \to \bar{u}$ weakly in E, strongly in $L^2(\Omega)$, and a.e. in Ω. Because $|f(x,u_n)|/\|u_n\| \le F_0|\bar{u}_n|$, we may assume that $(f(x,u_n))/\|u_n\|$ converges strongly in $L^2(\Omega)$ to a function $h(x)$. Observe that

$$\liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \ge \bar{u}(x) \liminf_{t\to\infty} \frac{f(x,t)}{t} = \bar{u}(x)\theta_+(x), \quad \text{if } \bar{u}(x) > 0.$$

In a similar way, we can show that

$$\bar{u}(x)\theta_+(x) \le \liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \limsup_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \bar{u}(x)\vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$\bar{u}(x)\vartheta_-(x) \le \liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \limsup_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \bar{u}(x)\theta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

This gives

$$\bar{u}(x)\theta_+(x) \le h(x) \le \bar{u}(x)\vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$\bar{u}(x)\vartheta_-(x) \le h(x) \le \bar{u}(x)\theta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

Let $q(x) = h(x)/\bar{u}(x)$ if $\bar{u}(x) \ne 0$; otherwise, $q(x) = 0$. Then

$$(2.69) \qquad \theta_+(x) \le q(x) \le \vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$(2.70) \qquad \theta_-(x) \le q(x) \le \vartheta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

On the other hand, $G'(u_n) \to 0$ implies that

$$(2.71) \qquad \langle \bar{u}(x), v\rangle - \int_\Omega h(x)v dx = \langle \bar{u}(x), v\rangle - \int_\Omega q(x)\bar{u}v dx = 0.$$

Let $\bar{u} = \bar{v} + \bar{w}$ with $\bar{v} \in E_k$, $\bar{w} \in E_k^\perp$, and $\tilde{u} = \bar{w} - \bar{v}$. Therefore, by (2.71),

$$(2.72) \qquad \|\bar{w}\|^2 - \|\bar{v}\|^2 = \int_\Omega q(x)(\bar{w})^2 dx - \int_\Omega q(x)(\bar{v})^2 dx.$$

Recalling (2.67)–(2.69) and (2.72), we have

$$\begin{aligned} 0 &\le \int_\Omega (\theta_+(\bar{v}^+)^2 + \theta_-(\bar{v}^-)^2)dx - \|\bar{v}\|^2 \\ &\le \int_\Omega q(x)(\bar{v})^2 dx - \|\bar{v}\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega q(x)(\bar{w})^2 dx - \|\bar{w}\|^2 \\
&\leq \int_\Omega (\vartheta_+(\bar{w}^+)^2 + \vartheta_-(\bar{w}^-)^2) dx - \|\bar{w}\|^2 \\
&\leq 0.
\end{aligned}
$$

It follows that

$$\int_\Omega (\theta_+(\bar{v}^+)^2 + \theta_-(\bar{v}^-)^2) dx = \|\bar{v}\|^2; \tag{2.73}$$

$$\int_\Omega (\vartheta_+(\bar{w}^+)^2 + \vartheta_-(\bar{w}^-)^2) dx = \|\bar{w}\|^2. \tag{2.74}$$

Using (2.67) and (2.68) once again, we see that $\bar{v} = \bar{w} = \bar{u} = 0$. Hence,

$$\left\langle G'(u_n), \frac{u_n}{\|u_n\|^2} \right\rangle = 1 - \int_{\mathbf{R}^N} \frac{f(x, u_n)}{\|u_n\|} \bar{u}_n(x) dx \to 1,$$

providing a contradiction. □

By a similar argument, we can prove the following.

Lemma 2.36. *Under the assumptions of Theorem 2.31, G satisfies the (PS) condition.*

Lemma 2.37. *Under the assumptions of Theorem 2.32, G satisfies the (PS) condition.*

Proof. By the assumptions of the Theorem 2.32, we have

$$\|u\|^2 \leq \int_\Omega (\theta_+(u^+)^2 + \theta_-(u^-)^2) dx, \quad u \in E_k;$$

$$\|u\|^2 \geq \lambda_{k+1} \|u\|_2^2 \geq \int_\Omega (\vartheta_+(u^+)^2 + \vartheta_-(u^-)^2) dx, \quad u \in E_k^\perp.$$

Then (2.74) still holds. Hence

$$\int_\Omega (\lambda_{k+1} - \vartheta_+)(\bar{w}^+)^2 dx + \int_\Omega (\lambda_{k+1} - \vartheta_-)(\bar{w}^-)^2 dx = 0.$$

It follows that $\vartheta_+ = \lambda_{k+1}$ if $\bar{w} > 0$, and $\vartheta_- = \lambda_{k+1}$ if $\bar{w} < 0$ and $\bar{w}$ is an eigenfunction of λ_{k+1}. Therefore,

$$-\Delta \bar{w} + V(x) \bar{w} = \lambda_{k+1} \bar{w} = \vartheta_+ \bar{w}^+ - \vartheta_- \bar{w}^-,$$

which implies that $\bar{w}=0$. Furthermore,

$$\int_{\Omega}(\theta_+(\bar{v}^+)^2+\theta_-(\bar{v}^-)^2)dx=\int_{\Omega}q(x)(\bar{v})^2dx.$$

Thus, $q(x)=\theta_+(x)$ if $\bar{u}>0$; $q(x)=\theta_-(x)$ if $\bar{u}<0$ and

$$-\Delta\bar{u}+V(x)\bar{u}=\theta_+\bar{u}^+-\theta_-\bar{u}^-.$$

It follows that $\bar{u}=\bar{v}=0$. Using an argument similar to that used in proving Lemma 2.35, we get a contradiction if the (PS) sequence is unbounded. □

Similarly, we have

Lemma 2.38. *Under the assumptions of Theorem 2.33, G satisfies the (PS) condition.*

Proofs of Theorems 2.30–2.33. By Lemmas 2.22–2.26, we see that $G(u)\le c_0|\Omega|/2$ for all $u\in E_{k-1}:=Y$ and $G(u)\ge c_0|\Omega|/2$ for all $u\in S_0$. By (B_5), (2.63), and (2.67), similar to the proof of Lemma 2.22, we have $G(u)\le T_0$ for all $u\in E_k$; here T_0 is a constant. Then, G satisfies all the conditions of Theorem 2.18. Therefore, there exists a sequence $\{u_k\}\in E\backslash(-\mathcal{P}\cup\mathcal{P})$ such that

$$G'(u_k)\to 0,\qquad G'(u_k)=C_ku_k/k,G(u_k)\in\left[\frac{1}{4}c_0|\Omega|,2T_0\right],$$

as $k\to\infty$, where the sequence $\{C_k\}$ is bounded. By Lemmas 2.35–2.38, $u_k\to u$, where u satisfies

$$G'(u)=0,\qquad G(u)\in\left[\frac{1}{4}c_0|\Omega|,2T_0\right].$$

We now show that u is sign-changing. In fact, because $G'(u_k)-C_ku_k/k=0$, we have

$$\|u_k^{\pm}\|^2-\frac{C_k}{k}\|u_k^{\pm}\|^2=\int_{\mathbf{R}^N}f(x,u_k^{\pm})u_k^{\pm}dx\le\frac{1}{3}\|u_k^{\pm}\|^2+C\|u_k^{\pm}\|_p^p.$$

It follows that $\|u_k^{\pm}\|\ge s_0>0$, where s_0 is a constant independent of k. This implies that the limit u is sign-changing and $G'(u)=0,G(u)\in\left[\frac{1}{4}c_0|\Omega|,2T_0\right]$. □

Notes and Comments. The existence results of Theorems 2.30–2.33 are essentially known (cf. Các [75], Berestycki and de Figueiredo [58], Furtado et al. [152, 153], Habets et al. [165], and Schechter [269]). But in those papers the signs of the solutions cannot be decided. Theorems 2.30–2.33 are neither consequences of the usual linking theorems nor straightforward results of the methods developed in Bartsch [30], Li and Wang [199], and Bartsch et al. [37]. A similar result to that of Lemma 2.34 can be found in Các [75], Lazer and Mckenna [191], and Schechter [269].

2.5 Double Resonant Cases

Consider the following case,

$$\lambda_k \le \Psi_1(x) := \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \le \limsup_{|t|\to\infty} \frac{f(x,t)}{t} := \Psi_2(x) \le \lambda_{k+1} \tag{2.75}$$

uniformly for $x \in \Omega$. We have the following.

Theorem 2.39. *Assume that* (B_1)–(B_4) *and (2.75) hold with* $\Psi_1(x) \not\equiv \lambda_k$. *Then Equation (2.42) has a sign-changing solution.*

Lemma 2.40. *Under the assumptions of Theorem 2.39,* $G(u) \to -\infty$ *for* $u \in E_k$ *and* $\|u\| \to \infty$.

Proof. Because $\Psi_1(x) \ge \lambda_k, \Psi_1(x) \not\equiv \lambda_k$, and $\dim E_k < \infty$, by the variational characterization of the eigenvalues $\{\lambda_k\}$, there is a $\rho > 0$ such that

$$\|u\|^2 - \int_\Omega \Psi_1(x)u^2 dx \le -\rho\|u\|^2 \quad \text{for all } u \in E_k. \tag{2.76}$$

In fact, this is an immediate consequence of the Schechter–Simon Theorem 1.62. Furthermore, by (2.75), for $\varepsilon > 0$ small enough, there exists a $C_\varepsilon > 0$ such that

$$\frac{1}{2}\Psi_1(x)t^2 - F(x,t) \le \frac{1}{2}\varepsilon t^2 + C_\varepsilon$$

for all $x \in \Omega, t \in \mathbf{R}$. Therefore, combining (2.76),

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega \Psi_1(x)u^2 dx + \int_\Omega \left(\frac{1}{2}\Psi_1(x)u^2 - F(x,u)\right) dx \\ &\le -\frac{3\rho}{8}\|u\|^2 + \int_\Omega \left(\frac{1}{2}\varepsilon u^2 + C_\varepsilon\right) dx \\ &\le -\frac{\rho}{4}\|u\|^2 + \int_\Omega C_\varepsilon dx. \end{aligned}$$

The lemma follows immediately. □

Proof of Theorem 2.39. Similar to Lemma 2.28, G satisfies the (w-PS) condition. The remainder is analogous to the proof of Theorem 2.20. We leave the details to the readers. □

Notes and Comments. To study the sign-changing solutions, several authors developed some methods. In Bartsch [30], the author established an abstract critical theory in partially ordered Hilbert spaces by virtue of critical groups and studied superlinear problems. In Li and Wang [199], a Ljusternik–Schnirelman theory was established for studying the sign-changing solutions of an even functional. Some linking-type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of Bartsch [30], Bartsch and Weth [45], and Li and Wang [199, 198] involved the dense Banach space $\mathbf{C}(\Omega)$ of continuous functions in the Hilbert space $H_0^1(\Omega)$, where the cone has a nonempty interior. This plays a crucial role. To fit that framework, much stronger hypotheses (e.g., boundedness of the domain and stronger smoothness of the nonlinearities) were imposed. In [37], the method of dealing with superlinear non-odd f was based on Liu and Sun [211] by using arguments of invariant sets. In [37], this idea of the neighborhood of a cone due to Conti et al. [107] was applied and modified by the authors to construct the invariant set. Their idea also can be traced back to Bartsch [29].

Under the Ambrosetti–Rabinowitz's super-quadratic (ARS, for short), Wang [331] obtained the existence result of three solutions (one is positive, another one is negative) on a superlinear Dirichlet elliptic equation and later in Bartsch and Wang [40], the authors proved for semilinear Dirichlet problems that the third solution is sign-changing. This result was generalized to nonlinear Schrödinger equations in Bartsch and Wang [41] where the (ARS) condition plays an important role. Recall the papers of Coti Zelati and Rabinowitz [121, 122], where $V(x)$ and $f(x,t)$ were periodic for each x variable and infinitely many sign-changing solutions were obtained by a totally different theory.

In Bartsch and Wang [44], the existence of sign-changing entire solutions defined on $\mathbf{R}^N$ was studied. They constructed a series of Dirichlet problems on the ball and then expanded the ball to whole space.

Other papers on sign-changing solutions include Bartsch et al. [31], Castro et al. [80, 81], Castro and Finan [82], Dancer and Du [129], Dancer and Yan [131], and Schechter et al. [279]. Other variants of the linking theorem can be found in Schechter [273, 267, 270, 274, 271]. Finally, we mention other papers on resonant problems. In Arcoya and Costa [18], Bartolo et al. [27], Bartsch and Li [36], Hirano et al. [171], and Hirano and Nishimura [169], the strong resonant elliptic equation was studied. In Schechter [276] and Zou and Liu [348], general resonant problems were considered.

Chapter 3
Sign-Changing Saddle Point

3.1 Rabinowitz's Saddle Points

Let E be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Assume that E has an orthogonal decomposition $E = Y \oplus M$ with $\dim Y < \infty$. Consider a $\mathbf{C}^1$-functional G defined on E.

Theorem 3.1. *Suppose that $G \in \mathbf{C}^1(E, \mathbf{R})$ satisfies the Palais–Smale condition. If there is a constant α and a bounded neighborhood D of 0 in Y such that*

$$G|_{\partial D} \leq \alpha, \qquad \inf_M G \geq \beta > \alpha,$$

then G has a critical value $\geq \beta$.

This is the saddle point theorem. It can be found in the well-known brochure of Rabinowitz (cf. Theorem 4.6 of Rabinowitz [255]). The saddle point theorem is an elementary but very useful result that has been applied in various variational problems (see, e.g., Rabinowitz [255] and Struwe [313]). Some variants were obtained (cf., e.g., Benci and Rabinowitz [55] and Lazer and Solimini [193]). The following generalization was given by Silva [299] (see also Furtado et al. [152, 153]).

Theorem 3.2. *Assume that $G \in \mathbf{C}^1(E, \mathbf{R})$ satisfies a weak Palais–Smale condition. If*

$$a_0 := \sup_Y G \neq \infty, \qquad b_0 := \inf_M G \neq -\infty, \tag{3.1}$$

then G has a critical point.

Unfortunately, no more information on this critical point produced in the above theorems was obtained. Theorem 3.2 does not get an estimate of the critical value. Particularly, both theorems cannot exclude the trivial point 0 if zero is a critical point because, in practice, $\inf_M G \leq 0$. Therefore, additional

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conditions must be assumed in order to get a nontrivial critical point. For example, in Furtado et al. [152] (see also Lazer and Solimini [193] for an earlier version), under the hypotheses of Theorem 3.2, the authors assumed furthermore that

$$(3.2)\quad \begin{cases} G \in \mathbf{C^2}(E, \mathbf{R}), G'(0) = 0, & G''(0) \text{ is a Fredholm operator and} \\ \text{either} \quad \dim Y < m(G, 0) \quad \text{or} & \bar{m}(G, 0) < \dim Y, \end{cases}$$

where $m(G, 0)(\bar{m}(G, 0))$ is the Morse index (augmented Morse index) of G at 0. In this case, G has a nonzero critical point. Condition (3.2) was introduced in Lazer and Solimini [193] which was related to Amann and Zehnder's theorem in [10]. Under the assumptions of (3.2) and of Theorem 3.1, the authors of [193] also got a nontrivial solution including an estimate of the Morse index. In all those papers, no further property on this nonzero critical point was obtained even though (3.2) was imposed. Sometimes, in application, Condition (3.2) is somewhat hard to verify and many more requirements are needed.

The questions are twofold. If $G'(0) = 0$, when will the saddle point be nontrivial if (3.2) is cancelled? But on the other hand, can we get a further property for this point, say, sign-changingness or nodal structure of the saddle point? We devote this chapter to these open questions. More precisely, we generalize Theorem 3.2 by showing that there is another critical point in addition to zero which is sign-changing with respect to a positive cone of E. We do not need the assumptions as in (3.2). We apply the new abstract result to study the existence of sign-changing solutions to the semilinear elliptic boundary value problem of the form

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and the Schrödinger equation

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x, u), & x \in \mathbf{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $f(x, t)$ is a Carathéodory function. We establish the existence results on sign-changing solutions.

3.2 Sign-Changing Saddle Points

Let $G \in \mathbf{C}^1(E, \mathbf{R})$ have the gradient G' of the form:

$$G'(u) = u - \Theta_G(u),$$

where $\Theta_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E \backslash \mathcal{K}$. The locally Lipschitz continuous map $V : \tilde{E} \to E$ is a pseudo-gradient vector field of G (cf. Definition 1.53). Let $\mathcal{P}$ denote a closed convex positive cone of E and $\mathcal{D}_0^{(i)}$ be an open convex subset of $E, i = 1, 2$. In applications, we may choose $\mathcal{D}_0^{(i)}$ appropriately so that either $\mathcal{D}_0^{(1)}$ contains all possible positive critical points or $\mathcal{D}_0^{(2)}$ includes all possible negative critical points. Let

$$\mathcal{S} := E \backslash \mathcal{W}, \qquad \mathcal{W} := \mathcal{D}_0^{(1)} \cup \mathcal{D}_0^{(2)}. \tag{3.3}$$

We make the following assumptions.

(A₁) $\Theta_G(\mathcal{D}_0^{(i)}) \subset \mathcal{D}_0^{(i)}$, $i = 1, 2$.

Let

$$W(u) := \frac{(1 + \|u\|)^2 V(u)}{(1 + \|u\|)^2 \|V(u)\|^2 + 1}. \tag{3.4}$$

Then W is a locally Lipschitz continuous vector field on $\tilde{E}$. Let Φ be the set of contractions defined in (2.1) of Section 2.1. Obviously, for a fixed $e_0 \in E$, $\Gamma(t, u) := (1 - t)u + te_0 \in \Phi$.

In this chapter, we always use the following weaker version of the (PS) condition. It is a variant of Definition 1.51 due to Cerami [84].

Definition 3.3. The functional G is said to satisfy the (w^*-PS) condition if for any sequence $\{u_n\}$ such that $\{G(u_n)\}$ is bounded and $G'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|G'(u_n)\| \|u_n\| \to \infty$. If in particular, $\{G(u_n)\} \to c$, we say that (w^*-PS)$_c$ is satisfied.

(A₂) There exists a $\delta > 0$ and $z_0 \in Y$ with $\|z_0\| = 1$ such that

$$B := \{u \in M : \|u\| \geq \delta\} \cup \{sz_0 + v : v \in M, s \geq 0, \|sz_0 + v\| = \delta\} \subset \mathcal{S}.$$

In applications, usually the first eigenfunction is positive, and the orthogonal complement of the first eigenspace ($\subset Y$) contains sign-changing elements and zero. Therefore, (A_2) can be verified readily.

Theorem 3.4. *Assume (A_1) and (A_2). Let G be a $\mathbf{C}^1$-functional on E that maps bounded sets to bounded sets and satisfies (w^*-PS) and*

$$b_0 := \inf_M G \neq -\infty, \qquad a_0 := \sup_Y G \neq \infty.$$

Then G has a critical point in $\mathcal{S}$ with critical value $\geq \inf_B G$.

In comparison with Theorems 3.1 and 3.2, we observe the following novelties of Theorem 3.4. If zero is a critical point of G, we still obtain another nonzero saddle point, no matter what $\inf_B G$ is (possibly $\inf_B G \leq 0$).

The nonzero saddle point is sign-changing, the critical value of which is not necessarily nonzero. We do not need the assumptions as in (3.2) and G is only of $\mathbf{C}^1$. To contrast Theorem 3.2, we get a lower bound of the critical value. After finishing the proof of Theorem 3.4, we give an estimate of the upper bound of the critical value.

In applications, by choosing different $\mathcal{W}$, hence $\mathcal{S}$, we may obtain different locations of critical points. In particular, we can get nontrivial sign-changing critical points even though $\inf_B G < 0$.

Proof of Theorem 3.4. First of all, we define

$$d_0^* := \inf_{B \cup M} G; \tag{3.5}$$

then $d_0^* > -\infty$. We divide the proof into five steps.

Step 1. We show that there exists a flow $\vartheta \in \mathbf{C}([0,\infty) \times E, E)$ such that $\vartheta(t,u) = u$ for any $u \in M \cup B$ and that $G(\vartheta(t,u))$ is nonincreasing with respect to variable $t \in (0,\infty)$ for every $u \in E$. More important, ϑ has the properties stated in Steps 2–4 below. By analyzing the flow carefully, we may find a critical point in $\mathcal{S}$.

To prove these, we first choose

$$c_0 := 64(a_0 - d_0^* + 1)\left(\ln \frac{5}{4}\right)^{-1} + 1.$$

Then by the (w^*-PS) condition, there exist $\varepsilon_1 \in (0,1), R_1 > \delta > 0$ such that

$$\|G'(u)\|(1 + \|u\|) \geq c_0 \tag{3.6}$$

for all $u \in G^{-1}[d_0^* - \varepsilon_1, a_0 + \varepsilon_1]$ with $\|u\| \geq R_1$, where δ comes from the set B in (A_2). Let $\varepsilon_0 \in (0, \varepsilon_1)$ and

$$\begin{cases} \Omega_1 := \{u : G(u) \geq a_0 + \varepsilon_1 \ \ or \ \ G(u) \leq d_0^* - \varepsilon_1\}, \\ \Omega_2 := \{u : d_0^* - \varepsilon_0 \leq G(u) \leq a_0 + \varepsilon_0\}, \\ \Omega_3 := \{u = u^- + u^+ : u^- \in Y, u^+ \in M, \|u^-\| \leq R_1\}, \\ \Omega_4 := \{u = u^- + u^+ : u^- \in Y, u^+ \in M, \|u^-\| \geq R_1 + 1\}. \end{cases} \tag{3.7}$$

Hence, $B \subset \Omega_3$. Define

$$g(u) = \frac{\mathrm{dist}(u, \Omega_1)}{\mathrm{dist}(u, \Omega_1) + \mathrm{dist}(u, \Omega_2)}, \tag{3.8}$$

$$l(u) = \frac{\mathrm{dist}(u, \Omega_3)}{\mathrm{dist}(u, \Omega_3) + \mathrm{dist}(u, \Omega_4)}. \tag{3.9}$$

Let

$$W^*(u) := \begin{cases} g(u)l(u)W(u) = \dfrac{g(u)l(u)(1+\|u\|)^2 V(u)}{(1+\|u\|)^2\|V(u)\|^2+1}, & u \in \tilde{E}, \\ 0, & u \in \mathcal{K}. \end{cases} \tag{3.10}$$

For any $u \in \partial\mathcal{K}$, note that $\partial\mathcal{K} \subset \mathcal{K} \subset \Omega_1 \cup \Omega_3$; we distinguish the cases $u \in \Omega_1$ and $u \in \Omega_3 \backslash \Omega_1$. First, we assume that $u \in \Omega_1$. Then either $G(u) \geq a_0 + \varepsilon_1$ or $G(u) \leq d_0^* - \varepsilon_1$. We consider $G(u) \geq a_0 + \varepsilon_1$ first. If $G(u) = a_0 + \varepsilon_1$ and $\|u^-\| \geq R_1$, then $\|G'(u)\|(1+\|u\|) \geq c_0$, which contradicts the fact that $u \in \partial\mathcal{K}$. So we must have either

$$G(u) = a_0 + \varepsilon_1 \quad \text{with } \|u^-\| < R_1 \quad \text{or} \quad G(u) > a_0 + \varepsilon_1.$$

Both cases imply that there is an open neighborhood U_u of u such that $U_u \subset \Omega_1 \cup \Omega_3$. If $G(u) \leq d_0^* - \varepsilon_1$, in a similar way, we find a neighborhood U_u of u such that $U_u \subset \Omega_1 \cup \Omega_3$. Second, if $u \in \Omega_3 \backslash \Omega_1$, we may also find this kind of neighborhood U_u of u. These arguments show that W^* is locally Lipschitz continuous on whole E. Moreover, $\|W^*(u)\| \leq 1 + \|u\|$ on E. Now we can consider the following Cauchy problem

$$\begin{cases} \dfrac{d\vartheta(t,u)}{dt} = -W^*(\vartheta), \\ \vartheta(0,u) = u \in E. \end{cases} \tag{3.11}$$

It has a unique solution $\vartheta(t,u) : [0,\infty) \times E \to E$ satisfying the following properties.

(1) $\vartheta(t,u)$ is a homeomorphism of E onto E for each $t \geq 0$.
(2) $\vartheta(t,u) = u$ for all $u \in M \cup B$.
(3) $G(\vartheta(t,u))$ is nonincreasing with respect to $t \geq 0$.

Step 2. We show that

$$\vartheta([0,+\infty), \mathcal{W}) \subset \mathcal{W}. \tag{3.12}$$

We first show

$$\vartheta([0,+\infty), \bar{\mathcal{W}}) \subset \bar{\mathcal{W}}. \tag{3.13}$$

By Lemma 2.11, there exists a locally Lipschitz continuous map O such that

$$O(\mathcal{D}_0^{(i)} \cap \tilde{E}) \subset \mathcal{D}_0^{(i)},$$

hence,

$$O(\bar{\mathcal{D}}_0^{(i)} \cap \tilde{E}) \subset \bar{\mathcal{D}}_0^{(i)}, \qquad i = 1, 2.$$

Because $\mathcal{K} \subset \Omega_1 \cup \Omega_3$, then $\vartheta(t,u) = u$ for all $t \ge 0$ and $u \in \bar{\mathcal{W}} \cap \mathcal{K}$. Next, we assume that $u \in \bar{\mathcal{D}}_0^{(1)} \cap \tilde{E}$. We show that $\vartheta(t,u) \in \bar{\mathcal{D}}_0^{(1)}$ for all $t > 0$. By negation, assume that there is a $T_0 > 0$ such that $\vartheta(T_0,u) \notin \bar{\mathcal{D}}_0^{(1)}$; we may find a number $s_0 \in [0,T_0)$ such that $\vartheta(s_0,u) \in \partial\bar{\mathcal{D}}_0^{(1)}$ and $\vartheta(t,u) \notin \bar{\mathcal{D}}_0^{(1)}$ for $t \in (s_0,T_0]$. Consider the following initial value problem

$$\begin{cases} \dfrac{d\vartheta(t,\vartheta(s_0,u))}{dt} = -W^*(\vartheta(t,\vartheta(s_0,u))), \\ \vartheta(0,\vartheta(s_0,u)) = \vartheta(s_0,u) \in E. \end{cases}$$

It has a unique solution $\vartheta(t,\vartheta(s_0,u))$. For any $v \in \bar{\mathcal{D}}_0^{(1)}$, if $v \in \mathcal{K}$, then $W^*(v) = 0$. Hence,

$$v + \beta(-W^*(v)) = v \in \bar{\mathcal{D}}_0^{(1)}.$$

Assume that $v \in \tilde{E} \cap \bar{\mathcal{D}}_0^{(1)}$; then $O(v) \in \bar{\mathcal{D}}_0^{(1)}$. By Lemma 2.11 and noting that $\bar{\mathcal{D}}_0^{(1)}$ is convex, we have

$$\begin{aligned} &v + \beta(-W^*(v)) \\ &\quad = v - \beta\frac{g(v)l(v)(1+\|v\|)^2 V(v)}{(1+\|v\|)^2\|V(v)\|^2+1} \\ &\quad = v - \beta\frac{g(v)l(v)(1+\|v\|)^2}{(1+\|v\|)^2\|V(v)\|^2+1}(v - O(v)) \\ &\quad = \left(1 - \beta\frac{g(v)l(v)(1+\|v\|)^2}{(1+\|v\|)^2\|V(v)\|^2+1}\right)v \\ &\qquad + \beta\frac{g(v)l(v)(1+\|v\|)^2}{(1+\|v\|)^2\|V(v)\|^2+1}O(v) \in \bar{\mathcal{D}}_0^{(1)} \end{aligned}$$

for β small enough. Summing up, we have

$$\lim_{\beta\to 0^+}\frac{\operatorname{dist}(v+\beta(-W^*(v)),\bar{\mathcal{D}}_0^{(1)})}{\beta} = 0, \qquad \forall v \in \bar{\mathcal{D}}_0^{(1)}.$$

By Lemma 1.49, there exists an $\varepsilon > 0$ such that $\vartheta(t,\vartheta(s_0,u)) \in \bar{\mathcal{D}}_0^{(1)}$ for all $t \in [0,\varepsilon)$. By the semigroup property, we see that $\vartheta(t,u) \in \bar{\mathcal{D}}_0^{(1)}$ for all $t \in [s_0, s_0+\varepsilon)$, which contradicts the definition of s_0. Therefore,

$$\vartheta([0,+\infty),\bar{\mathcal{D}}_0^{(1)}) \subset \bar{\mathcal{D}}_0^{(1)}.$$

Similarly, $\vartheta([0,+\infty),\bar{\mathcal{D}}_0^{(2)}) \subset \bar{\mathcal{D}}_0^{(2)}$. That is, $\vartheta([0,+\infty),\bar{\mathcal{W}}) \subset \bar{\mathcal{W}}$. Thus (3.13) is true. To prove $\vartheta([0,+\infty),\mathcal{W}) \subset \mathcal{W}$, we just show that $\vartheta([0,+\infty),\mathcal{D}_0^{(1)}) \subset$

$\mathcal{D}_0^{(1)}$. By a contradiction, assume that there exists a $u^* \in \mathcal{D}_0^{(1)}, T_0 > 0$ such that $\vartheta(T_0, u^*) \notin \mathcal{D}_0^{(1)}$. Choose a neighborhood U_{u^*} of u^* such that $U_{u^*} \subset \bar{\mathcal{D}}_0^{(1)}$. Then by the theory of ordinary differential equations in Banach spaces, we may find a neighborhood U_{T_0} of $\vartheta(T_0, u^*)$ such that $\vartheta(T_0, \cdot) : U_{u^*} \to U_{T_0}$ is a homeomorphism. Because $\vartheta(T_0, u^*) \notin \mathcal{D}_0^{(1)}$, we take a $w \in U_{T_0} \backslash \bar{\mathcal{D}}_0^{(1)}$. Correspondingly, we find a $v \in U_{u^*}$ such that $\vartheta(T_0, v) = w$; this contradicts the fact that $\vartheta([0, \infty), \bar{\mathcal{D}}_0^{(1)}) \subset \bar{\mathcal{D}}_0^{(1)}$. This completes the proof of (3.12).

Step 3. For any $R > 0$, let

$$A_R := \{u \in Y : \|u\| = R\}.$$

We show that there exist $R_0 > 0$, $T_0 > 0$ such that

$$\vartheta(T_0, A_{R_0}) \subset G^{d_0^* - \varepsilon_0} := \{u \in E : G(u) \leq d_0^* - \varepsilon_0\}. \tag{3.14}$$

Choose $R_0 = 2(R_1 + 1)$, where R_1 comes from (3.6) of Step 1. Let $\phi_0(t) = \|\vartheta(t, u)\|$, where ϑ comes from (3.11). If $g(u)l(u) \neq 0$ for some $u \in E$, then by (3.6)–(3.8), we must have that

$$\|G'(u)\|(1 + \|u\|) \geq c_0 > 1.$$

Hence,

$$\begin{aligned} \|g(u)l(u)W(u)\| &\leq \frac{(1+\|u\|)^2\|V(u)\|}{(1+\|u\|)^2\|V(u)\|^2+1} \\ &\leq \frac{8(1+\|u\|)^2\|G'(u)\|}{(1+\|u\|)^2\|G'(u)\|^2+1} \\ &\leq \frac{8c_0}{1+c_0^2}(1+\|u\|). \end{aligned}$$

Therefore,

$$\|g(u)l(u)W(u)\| \leq \frac{8c_0}{1+c_0^2}(1+\|u\|) \quad \text{for all } u \in E. \tag{3.15}$$

Furthermore,

$$\left|\frac{d\phi_0(t)}{dt}\right| \leq \frac{8c_0}{1+c_0^2}(1+\phi_0(t)).$$

It follows that

$$\phi_0(t) = \|\vartheta(t,u)\| \leq e^{(8c_0/(1+c_0^2))t}(1+\|u\|) - 1 \quad \text{for all } u \in E, t \geq 0. \tag{3.16}$$

Choose

$$T_0 := \frac{1+c_0^2}{8c_0} \ln \frac{5}{4}.$$

For any $u \in Y$ with $\|u\| = R_0 = 2(R_1 + 1)$, write

$$\vartheta(t,u) = \vartheta_1(t,u) \oplus \vartheta_2(t,u) \quad \text{with } \vartheta_1(t,u) \in Y, \vartheta_2(t,u) \in M.$$

By (3.15) and (3.16),

$$\begin{aligned} \big|\|\vartheta_1(t,u)\| - \|u\|\big| &\le \|\vartheta(t,u) - u\| \\ &= \int_0^t d\vartheta(t,u) \\ &\le \frac{8c_0}{1+c_0^2}\int_0^t (1 + \|\vartheta(t,u)\|)dt \\ &\le \frac{8c_0}{1+c_0^2}\int_0^t (1+\|u\|)e^{(c_0/(1+c_0^2))t}dt \\ &= (1+\|u\|)(e^{(8c_0/(1+c_0^2))t} - 1). \end{aligned} \tag{3.17}$$

It implies that

$$\|\vartheta(t,u)\| \ge \|\vartheta_1(t,u)\| \ge \|u\| - (1+\|u\|)(e^{(8c_0/(1+c_0^2))t} - 1) \ge R_1 + 1 \tag{3.18}$$

for all $t \in [0, T_0]$. Then, by (3.7) and (3.8), $l(\vartheta(t,u)) = 1$ for all $t \in [0, T_0]$.

If there exists a $t_1 \in [0, T_0]$ such that $G(\vartheta(t_1, u)) \le d_0^* - \varepsilon_0$, then

$$G(\vartheta(T_0, u)) \le d_0^* - \varepsilon_0. \tag{3.19}$$

Otherwise,

$$d_0^* - \varepsilon_0 < G(\vartheta(t,u)) \le G(u) \le a_0 \le a_0 + \varepsilon_0$$

for all $t \in [0, T_0]$. By (3.6), (3.8), and (3.18), we have that

$$\|G'(\vartheta(t,u))\|(1 + \|\vartheta(t,u)\|) \ge c_0$$

and $g(\vartheta(t,u)) = 1$ for all $t \in [0, T_0]$. Therefore, if we keep in mind the choice of c_0, ε_0, and T_0, we then have

$$\begin{aligned} &G(\vartheta(T_0,u)) \\ &\quad = G(u) + \int_0^{T_0} dG(\vartheta(t,u)) \\ &\quad \le G(u) - \int_0^{T_0} \left\langle G'(\vartheta(t,u)), \frac{(1+\|\vartheta(t,u)\|)^2 V(\vartheta(t,u))}{(1+\|\vartheta(t,u)\|)^2\|V(\vartheta(t,u))\|^2 + 1} \right\rangle dt \\ &\quad \le G(u) - \frac{1}{8}\int_0^{T_0} \frac{(1+\|\vartheta(t,u)\|)^2\|G'(u)\|^2}{(1+\|\vartheta(t,u)\|)^2\|G(u)\|^2+1} dt \end{aligned}$$

$$\leq a_0 - \frac{1}{8}\int_0^{T_0} \frac{c_0^2}{1+c_0^2}dt$$

$$\leq d_0^* - \varepsilon_0.$$

Combining (3.19), we observe that

$$\vartheta(T_0, A_{R_0}) \subset G^{d_0^*-\varepsilon_0}, \quad \text{hence } \vartheta(T_0, A_{R_0}) \cap B = \emptyset. \tag{3.20}$$

Step 4. In this step, we show that $\vartheta(T_0, A_{R_0})$ links B with respect to Φ and then we can define a critical value of minimax type. First, we note that $\vartheta(t,u) = u$ for all $u \in B; A_{R_0} \cap B = \emptyset$ and that $\vartheta(t,\cdot) : E \to E$ is a homeomorphism for any $t \geq 0$, we see that

$$\vartheta(t, A_{R_0}) \cap B = \emptyset \quad \text{for all } t \geq 0. \tag{3.21}$$

Let Γ be any map in Φ. Define

$$\Gamma_1(t,u) = \begin{cases} \vartheta(2tT_0, u) & \text{if } t \in [0,1/2]; \\ \Gamma((2t-1), \vartheta(T_0,u)) & \text{if } t \in [1/2,1]. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Because A_{R_0} links B by Proposition 2.7, there is a $t_1 \in [0,1]$ such that $\Gamma_1(t_1, A_{R_0}) \cap B \neq \emptyset$. Because $\vartheta(2tT_0, A_{R_0}) \cap B = \emptyset$ for all $t \in \left[0, \frac{1}{2}\right]$, we must have $t_1 > \frac{1}{2}$. Hence,

$$\Gamma((2t_1 - 1), \vartheta(T_0, A_{R_0})) \cap B \neq \emptyset.$$

Invoking (3.21), this shows that $\vartheta(T_0, A_{R_0})$ links B with respect to Φ.

Now, take any $\Gamma \in \Phi$, then $\Gamma([0,1], \vartheta(T_0, A_{R_0})) \cap B \neq \emptyset$. Because $B \subset \mathcal{S}$, we see that

$$\Gamma([0,1], \vartheta(T_0, A_{R_0})) \cap \mathcal{S} \neq \emptyset.$$

Define

$$d_0 := \inf_{\Gamma \in \Phi} \sup_{\Gamma([0,1],\ \vartheta(T_0, A_{R_0})) \cap \mathcal{S}} G. \tag{3.22}$$

Evidently, by (3.20),

$$\bar{a}_0 := \sup_{\vartheta(T_0, A_{R_0})} G \leq d_0^* - \varepsilon_0 < d_0^* \leq \inf_B G \leq d_0. \tag{3.23}$$

Step 5. We prove that $\mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}] \cap \mathcal{S} \neq \emptyset$ for all $\bar{\varepsilon} > 0$; here and in the sequel, $\mathcal{K}[e,f] := \{u \in E : G'(u) = 0, e \leq G(u) \leq f\}$. Once this is done, note that $\mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]$ is compact due to the (w^*-PS) condition and that $\mathcal{S}$ is closed; we may find a critical point in $\mathcal{S}$ with critical value $d_0 \geq \inf_B G$.

We assume by negation that $\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}]\cap\mathcal{S}=\emptyset$ for some $\bar{\varepsilon}>0$ and try to get a contradiction. In this case, $\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}]\subset\mathcal{W}$. By the ($w^*$-PS) condition, $\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}]$ is compact. We may assume that $\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}]\neq\emptyset$ (otherwise, it is simpler).

It follows that

$$\operatorname{dist}(\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}],\ \mathcal{S}) := \delta_0 > 0. \tag{3.24}$$

Again, by the (w^*-PS) condition, there is an $\varepsilon_2>0$ such that

$$\frac{(1+\|u\|)^2\|G'(u)\|^2}{1+(1+\|u\|)^2\|G'(u)\|^2}\geq\varepsilon_2 \tag{3.25}$$

for all

$$u\in G^{-1}[d_0-\varepsilon_2, d_0+\varepsilon_2]\backslash(\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}])_{\delta_0/2},$$

where $(A)_{\delta_0}:=\{u\in E:\operatorname{dist}(u,A)\leq\delta_0\}$. By decreasing ε_2 and invoking (3.23), we may assume that

$$\varepsilon_2<\bar{\varepsilon}/3, \qquad \varepsilon_2<d_0-\bar{a}_0. \tag{3.26}$$

Particularly, we still have that $\mathcal{K}[d_0-\varepsilon_2, d_0+\varepsilon_2]\cap\mathcal{S}=\emptyset$. Furthermore by (3.25),

$$\langle G'(u), W(u)\rangle\geq\varepsilon_2/8 \tag{3.27}$$

for all $u\in G^{-1}[d_0-\varepsilon_2, d_0+\varepsilon_2]\backslash(\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}])_{\delta_0/2}$. Let

$$\Omega_5 := \{u\in E: |G(u)-d_0|\geq 3\varepsilon_2\}, \tag{3.28}$$

$$\Omega_6 := \{u\in E: |G(u)-d_0|\leq 2\varepsilon_2\}, \tag{3.29}$$

$$\xi(u)=\frac{\operatorname{dist}(u,\Omega_5)}{\operatorname{dist}(u,\Omega_6)+\operatorname{dist}(u,\Omega_5)}, \tag{3.30}$$

$$\kappa(u):=\begin{cases}1, & u\in E\backslash(\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}])_{\delta_0/2},\\ 0, & u\in(\mathcal{K}[d_0-\bar{\varepsilon}, d_0+\bar{\varepsilon}])_{\delta_0/3}.\end{cases} \tag{3.31}$$

Take any $v\in\partial\mathcal{K}$.

If $|G(v)-d_0|>3\varepsilon_2$, then there is an open neighborhood U_v of v such that

$$\xi|_{U_v}=0. \tag{3.32}$$

If $|G(v) - d_0| \leq 3\varepsilon_2$, note $3\varepsilon_2 < \bar{\varepsilon}$; we may find a $\delta^* << \delta_0/8$ such that

$$|G(w) - d_0| \leq \bar{\varepsilon} \quad \text{for all } w \in U_v := \{w \in E : \|w - v\| < \delta^*\}. \tag{3.33}$$

Because $v \in \partial\mathcal{K}$, there exists a $v_1 \in U_v \cap \mathcal{K}$. Hence, $v_1 \in \mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]$. It follows that

$$\operatorname{dist}(v, \mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]) \leq \operatorname{dist}(v, v_1) < \delta^* < \delta_0/8.$$

Therefore, we may find an open neighborhood $\tilde{U}_v$ of v such that

$$\operatorname{dist}(x, \mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]) < \delta_0/5, \quad \text{for all } x \in \tilde{U}_v.$$

This implies that

$$\kappa|_{\tilde{U}_v} = 0. \tag{3.34}$$

Combining (3.28)–(3.34), the vector field defined by

$$Y^*(u) := \begin{cases} \xi(u)\kappa(u)W(u) = \dfrac{\xi(u)\kappa(u)(1 + \|u\|)^2 V(u)}{(1 + \|u\|)^2\|V(u)\|^2 + 1}, & u \in \tilde{E}, \\ 0, & u \in \mathcal{K}, \end{cases} \tag{3.35}$$

is locally Lipschitz on whole E. We now consider the following Cauchy initial value problem,

$$\begin{cases} \dfrac{d\pi(t,u)}{dt} = -Y^*(\pi(t,u)), \\ \pi(0,u) = u \in E, \end{cases}$$

which has a unique continuous solution $\pi(t,u)$ in E. Evidently,

$$\frac{dG(\pi(t,u))}{dt} \leq 0. \tag{3.36}$$

Similarly to Step 2, we can prove that

$$\pi([0,\infty), \mathcal{W}) \subset \mathcal{W}. \tag{3.37}$$

By the definition of d_0 in (3.22), there exists a $\Gamma \in \Phi$ such that

$$\Gamma([0,1], \vartheta(T_0, A_{R_0})) \cap \mathcal{S} \subset G^{d_0+\varepsilon_2}. \tag{3.38}$$

Therefore,

$$\Gamma([0,1], \vartheta(T_0, A_{R_0})) \subset G^{d_0+\varepsilon_2} \cup \mathcal{W}. \tag{3.39}$$

Denote $A^* := \Gamma([0,1], \vartheta T_0, A_{R_0}))$. Next, we show that there exists a $T_1 > 0$ such that $\pi(T_1, A^*) \subset G^{d_0 - \varepsilon_2/4} \cup \mathcal{W}$.

In fact, if $u \in A^* \cap \mathcal{W}$, then $\pi(t,u) \in \mathcal{W}$ for all $t > 0$ by (3.37).

If $u \in A^*, u \notin \mathcal{W}$, then we see that $G(u) \le d_0 + \varepsilon_2$. If it happens that $G(u) \le d_0 - \varepsilon_2$, then by (3.36),

$$G(\pi(t,u)) \le G(u) \le d_0 - \varepsilon_2$$

for all $t \ge 0$.

If $G(u) > d_0 - \varepsilon_2$, then $u \in G^{-1}[d_0 - \varepsilon_2, d_0 + \varepsilon_2]$. If

$$\mathrm{dist}(\pi([0,\infty), u), \mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]) \le \delta_0/2,$$

then there exists a t_N such that $\pi(t_N, u) \in \mathcal{W}$. Moreover,

$$\mathrm{dist}(\pi(t_N,u), \mathcal{S}) \ge \delta_0/4, \qquad \pi(t,u) \in \mathcal{W} \quad \text{for all } t \ge t_N \text{ (by (3.37))}. \tag{3.40}$$

Assume

$$\mathrm{dist}(\pi([0,\infty), u), \mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}]) > \delta_0/2.$$

Similarly, we assume that $G(\pi(t,u)) > d_0 - \varepsilon_2$ for all t; then

$$\pi(t,u) \in G^{-1}[d_0 - \varepsilon_2, d_0 + \varepsilon_2] \backslash (\mathcal{K}[d_0 - \bar{\varepsilon}, d_0 + \bar{\varepsilon}])_{\delta_0/2}.$$

Therefore, by (3.29)–(3.31),

$$\xi(\pi(t,u)) = \kappa(\pi(t,u)) = 1 \quad \text{for all } t \ge 0. \tag{3.41}$$

Moreover, by (3.27) and (3.41),

$$\begin{aligned} G(\vartheta(24,u)) &= G(u) + \int_0^{24} dG(\pi(s,u)) \\ &\le G(u) - \int_0^{24} \langle G'(\pi(s,u)), W(\pi(s,u))\rangle ds \\ &\le d_0 - 2\varepsilon_2. \end{aligned} \tag{3.42}$$

By combining the above arguments (cf. (3.40)–(3.42)), for any $u \in A^* \backslash \mathcal{W}$, there exists a $T_u > 0$ such that

$$\begin{cases} \text{either } \pi(T_u, u) \in G^{d_0 - \varepsilon_2/2} \quad \text{or} \\ \mathrm{dist}(\pi(T_u,u), \mathcal{S}) \ge \delta_0/4 \quad \text{and} \quad \pi(t,u) \in \mathcal{W} \quad \text{for all } t \ge T_u. \end{cases} \tag{3.43}$$

By continuity, (3.43) implies that there exists a neighborhood U_u of $u \in A^* \backslash \mathcal{W}$ such that

$$\text{(3.44)} \qquad \begin{cases} \text{either } \pi(T_u, U_u) \subset G^{d_0-\varepsilon_2/4} \quad \text{or} \quad \operatorname{dist}(\pi(T_u, U_u), \mathcal{S}) \geq \delta_0/5 \\ \text{and } \pi(T_u, U_u) \subset \mathcal{W}, \quad \text{hence, } \pi(t, U_u) \subset \mathcal{W} \text{ for all } t \geq T_u. \end{cases}$$

Because $A^*\backslash\mathcal{W}$ is compact, by (3.43) and (3.44) we get a $T_1 > 0$ such that

$$\text{(3.45)} \quad \pi(T_1, A^*\backslash\mathcal{W}) \subset G^{d_0-\varepsilon_2/4} \cup \mathcal{W} \quad \text{hence, } \pi(T_1, A^*) \subset G^{d_0-\varepsilon_2/4} \cup \mathcal{W}.$$

Now we define

$$\Gamma^*(s,u) = \begin{cases} \pi(2T_1 s, u), & s \in \left[0, \frac{1}{2}\right], \\ \pi(T_1, \Gamma(2s-1, u)), & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, $\Gamma^* \in \Phi$. We consider two cases.

If $s \in \left[0, \frac{1}{2}\right]$, we have that

$$\begin{aligned} &\Gamma^*(s, \vartheta(T_0, A_{R_0})) \cap \mathcal{S} \\ &\quad \subset \pi(2T_1 s, \vartheta(T_0, A_{R_0})) \cap \mathcal{S} \\ &\quad \subset G^{\bar{a}_0} \cap \mathcal{S} \quad \text{(by (3.23) and (3.36))} \\ \text{(3.46)} \qquad &\quad \subset G^{d_0-\varepsilon_2/4} \quad \text{(by (3.26))}. \end{aligned}$$

If $s \in \left[\frac{1}{2}, 1\right]$, we have

$$\begin{aligned} &\Gamma^*(s, \vartheta(T_0, A_{R_0})) \cap \mathcal{S} \\ &\quad \subset \pi(T_1, \Gamma(2s-1, \vartheta(T_0, A_{R_0}))) \cap \mathcal{S} \\ &\quad \subset \pi(T_1, A^*) \cap \mathcal{S} \\ &\quad \subset (G^{d_0-\varepsilon_2/4} \cup \mathcal{W}) \cap \mathcal{S} \quad \text{(by (3.45))} \\ &\quad \subset G^{d_0-\varepsilon_2/4} \cap \mathcal{S} \\ \text{(3.47)} \qquad &\quad \subset G^{d_0-\varepsilon_2/4}. \end{aligned}$$

It follows from (3.46) and (3.47) that

$$G(\Gamma^*([0,1], \vartheta(T_0, A_{R_0})) \cap \mathcal{S}) \leq d_0 - \varepsilon_2/4,$$

which contradicts the definition of d_0 in (3.22). □

Remark 3.5. From the proof of Theorem 3.4, we may estimate the upper bound of the critical value, which is helpful in applications. In fact, by (3.17), for all $u \in A_{R_0}$,

$$\text{(3.48)} \qquad \|\vartheta(T_0, u)\| \leq (1 + \|R_0\|)(e^{(8c_0/(1+c_0^2))T_0} - 1) + R_0 := R_2.$$

Because $\Gamma(t,u) = (1-t)u + te_0 \in \Phi$ for a fixed $e_0 \in E$ with $\|e_0\| = 1$, by (3.22) and (3.48),

$$\begin{aligned} d_0 &:= \inf_{\Gamma\in\Phi} \sup_{\Gamma([0,1],\, \vartheta(T_0,A_{R_0}))\cap\mathcal{S}} G \\ &\le \sup_{t\in[0,1],\, w\in\vartheta(T_0,A_{R_0})} G((1-t)w + te_0) \\ &\le \sup_{u\in E,\, \|u\|\le R_2+\|e_0\|} G. \end{aligned}$$

Therefore,

$$d_0 \in \Big[\inf_B G, \quad \sup_{u\in E,\, \|u\|\le R_2+1} G\Big],$$

where R_2 given in (3.48) depends on R_0, c_0, T_0, hence, on a_0, d_0^*, R_1 in (3.6).

Notes and Comments. Various versions of the weak (PS) condition were used in Costa [112], Costa and Magalhães [117, 111, 116, 113–115], Silva [293–298, 300], and the references cited therein. Other variants of the saddle point (linking) theorem and its applications can be found in Amann [9], Ambrosotti and Rabinowitz [15], Liu [208] (on product spaces with a compact manifold), and Schechter [273, 267, 270, 274, 271]. The estimates of the Morse index for the saddle point were given in Lazer and Solimini [193], Perera and Schechter [243], Ramos and Sanchez [261], and Solimini [307]. In particular, by way of the critical groups, the Morse indices of sign-changing solutions for nonlinear elliptic problems can be determined in the paper by Bartsch et al. [31] where the functionals are of $\mathbf{C}^2$ and the cones have nonempty interiors. Another version of the saddle point theorem, called the sandwich theorem, was obtained by Schechter (see Schechter [275, Theorem 2.9.1]). Theorem 3.4 was originally established in Zou [347] where the following stronger condition was imposed: if $\mathcal{D}_0^{(1)} \cap \mathcal{D}_0^{(2)} = \emptyset$, then either $\mathcal{D}_0^{(1)} = \emptyset$ or $\mathcal{D}_0^{(2)} = \emptyset$. Actually, this is unnecessary.

3.3 Schrödinger Equations with Potential Well

Consider the Schrödinger equation:

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x,u), & x \in \mathbf{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{3.49}$$

where $f \in \mathbf{C}(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$. In this section, we study the basic properties of the spectrum of $-\Delta + V_\lambda$. About the potential, we make the following assumptions.

(D₁) $V_\lambda(x) := \lambda g_0(x) + 1$, $g_0 \in \mathbf{C}(\mathbf{R}^N, \mathbf{R})$; $g_0 \geq\not\equiv 0$ and $\Omega := int(g_0^{-1}(0)) \neq \emptyset$;

(D₂) There exist $M_0 > 0$ and $r_0 > 0$ such that

$$\text{meas}\big(\{x \in B_{r_0}(y) : g_0(x) \leq M_0\}\big) \to 0 \quad \text{as } |y| \to \infty,$$

where $B_{r_0}(y)$ denotes the ball centered at y with radius r_0;

(D₃) $\bar{\Omega} := g_0^{-1}(0)$ and $\partial\Omega$ is locally Lipschitz.

Condition (D_1) means that V_λ has a steep potential well whose steepness is controlled by λ. Let

$$E = \left\{u \in H^{1,2}(\mathbf{R}^N) : \int_{\mathbf{R}^N} g_0(x)u^2dx < \infty\right\}$$

endowed with the norm

$$\|u\|_E = \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + (1 + g_0(x))u^2)dx\right)^{1/2}.$$

Equivalently, let E_λ be the Hilbert space

$$E_\lambda := \left\{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + V_\lambda(x)u^2)dx < \infty\right\}$$

endowed with the inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbf{R}^N} (\bigtriangledown u \bigtriangledown v + V_\lambda(x)uv)dx$$

for $u, v \in E_\lambda$ and norm $\|u\|_\lambda := \langle u, u\rangle_\lambda^{1/2}$.

The operator $S_\lambda := -\Delta + V_\lambda$ is a self-adjoint operator on $L^2(\mathbf{R}^N)$, bounded below by 1. We write $\langle\cdot,\cdot\rangle_{L^2}$ and $\|\cdot\|_2$ for the usual inner product and the associated norm in $L^2(\mathbf{R}^N)$. We denote

$$\langle S_\lambda u, u\rangle_{L^2} = \int_{\mathbf{R}^N} (|\nabla u|^2 + V_\lambda u^2)dx, \qquad u \in E.$$

For given elements $\phi_1, \phi_2, \ldots, \phi_m$ of E, set

$$\begin{aligned} &Q_\lambda(\phi_1, \ldots, \phi_k) \\ &\quad = \inf\{\langle S_\lambda\phi, \phi\rangle_{L^2} : \phi \in E,\ \|\phi\|_2 = 1,\ \langle\phi,\ \phi_i\rangle_{L^2} = 0,\ \ i = 1, \ldots, k\}. \end{aligned}$$

For each $k \in \mathbf{N}$ we define spectral values of S_λ by the kth Rayleigh quotient

$$\mu_k(S_\lambda) := \sup_{\phi_1,\dots,\phi_{k-1}\in E} Q_\lambda(\phi_1,\dots,\phi_{k-1}).$$

Then $\mu_k(S_\lambda)$ is nondecreasing with respect to k and λ. By Reed and Simon's theorems [262, Theorems XIII.1 and XIII.2], either $\mu_k(S_\lambda)$ is an eigenvalue of S_λ or $\mu_k(S_\lambda) = \mu_{k+1}(S_\lambda) = \cdots = \inf \sigma_{ess}(S_\lambda)$, the infimum of the essential spectrum.

Choose a domain $\Omega_0 \subset \Omega$ with meas $\Omega_0 < \infty$. Consider $L := S_\lambda|_{L^2(\Omega_0)} = -\Delta+1$. This operator is self-adjoint and positive with the domain $W_0^{1,2}(\Omega_0)\cap W^{2,2}(\Omega_0)$ and the form domain $W_0^{1,2}(\Omega_0)$. Then the spectrum $\sigma(L)$ is discrete and consists of eigenvalues $\mu_k(L)$ with finite multiplicity and

$$0 < \mu_1(L) < \mu_2(L) \le \mu_3(L) \le \cdots \to \infty.$$

We may consider $W_0^{1,2}(\Omega_0)$ as a subspace of E. Then as a simple consequence of the Courant minimax description of the eigenvalues (cf., e.g., Reed and Simon's theorems [262, Section XIII.1]), we observe that

$$\mu_k(S_\lambda) \le \mu_k(L)$$

for all k. We may assume that $\lim_{\lambda\to\infty} \mu_k(S_\lambda) := \mu_k$. Then

$$\mu_1 \le \mu_2 \le \cdots$$

is a nondecreasing sequence because $\mu_k(S_\lambda)$ is nondecreasing in k for each λ. In particular, $\mu_k \le \mu_k(L)$. Given an open set D in $\mathbf{R}^N$, define

$$\mu^*(-\Delta + V_\lambda, D) = \inf_{u\in H^1(D),u\ne 0} \frac{\int_D(|\nabla u|^2 + V_\lambda u^2)dx}{\|u\|^2_{L^2(D)}}.$$

Lemma 3.6. *Assume (D_1) and (D_2); then there exists a sequence $r_i \to \infty$ such that*

$$\lim_{\lambda\to\infty}\lim_{i\to\infty} \mu^*(-\Delta + V_\lambda, \mathbf{R}^N\backslash \bar{B}_{r_i}(0)) = \infty.$$

Proof. We first show that (D_2) implies that

$$\lim_{\lambda\to\infty}\lim_{y\to\infty} \mu^*(-\Delta + V_\lambda, B_{r_0}(y)) = \infty. \tag{3.50}$$

Let

$$O(y) = \{x \in B_{r_0}(y) : g_0(x) > M_0\},$$

$$P(y) = \{x \in B_{r_0}(y) : g_0(x) \le M_0\}.$$

Then

$$\int_{O(y)} (|\nabla u|^2 + V_\lambda u^2)dx \geq (\lambda M_0 + 1) \int_{O(y)} u^2 dx. \tag{3.51}$$

Now we fix $p \in (1, N/(N-2))$ and let $q = p/(p-1)$ be the dual exponent. By Sobolev's inequality,

$$\|u\|_{L^{2p}(B_{r_0}(y))} \leq c\|u\|_{W^{1,2}(B_{r_0}(y))}, \qquad u \in W^{1,2}(B_{r_0}(y)). \tag{3.52}$$

By (3.52) and the Hölder inequality

$$\begin{aligned} \int_{P(y)} u^2 dx &= (\text{meas}\,(P(y)))^{1/q} \left(\int_{P(y)} u^{2p} dx \right)^{1/p} \\ &\leq (\text{meas}\,(P(y)))^{1/q} \|u\|^2_{L^{2p}(B_{r_0}(y))} \\ &\leq c(\text{meas}\,(P(y)))^{1/q} \|u\|^2_{W^{1,2}(B_{r_0}(y))}. \end{aligned} \tag{3.53}$$

Note that meas $(P(y)) \to 0$ as $y \to \infty$. Then by (3.53),

$$\begin{aligned} &\mu^*(-\Delta + V_\lambda, B_{r_0}(y)) \\ &\quad = \inf_{u \in W^{1,2}(B_{r_0}(y)),\, u \neq 0} \frac{\int_{B_{r_0}(y)} (|\nabla u|^2 + V_\lambda u^2)dx}{\int_{B_{r_0}(y)} u^2 dx} \\ &\quad \to \infty \end{aligned}$$

as $y \to \infty$ and $\lambda \to \infty$. Thus, (3.50) is true. It follows that

$$\lim_{\lambda \to \infty} \lim_{y \to \infty} \mu^*(-\Delta + V_\lambda, B_r(y)) = \infty \tag{3.54}$$

holds for all $r > 0$.

Finally, we show that (3.54) implies the conclusion of the current lemma. Choose $r_i = ir_0$ $(i > 1)$ and decompose $\mathbf{R}^N$ into a countable family of pairwise disjoint balls $B_{r_0}(x_m)$ such that $\overline{\mathbf{R}^N \backslash B_{r_i}(0)}$ is the union of all $\bar{B}_{r_0}(x_m)$ with $m \in \text{Index}(i) := \{m : x_m \in \mathbf{R}^N \backslash B_{r_i}(0)\}$. Let

$$\gamma_i(\lambda) = \inf_{m \in \text{Index}(i)} \mu^*(S_\lambda, B_{r_0}(x_m)),$$

then $\lim_{\lambda \to \infty} \lim_{i \to \infty} \gamma_i(\lambda) = \infty$. For each $u \in H^1(\mathbf{R}^N \backslash B_{r_i}(0))$,

$$\int_{\mathbf{R}^N \backslash B_{r_i}(0)} (|\nabla u|^2 + V_\lambda u^2)dx$$

$$
\begin{aligned}
&= \sum_{m \in \mathrm{Index}(i)} \int_{B_{r_0}(x_m)} (|\nabla u|^2 + V_\lambda u^2) dx \\
&\geq \gamma_i(\lambda) \sum_{m \in \mathrm{Index}(i)} \|u\|^2_{L^2(B_{r_0}(x_m))} \\
&= \gamma_i(\lambda) \|u\|^2_{L^2(\mathbf{R}^N \setminus B_{r_i}(0))};
\end{aligned}
$$

this implies the conclusion of the lemma. □

Under (D_1)–(D_3), the Schrödinger operator $-\Delta + V_\lambda$ has a finite number of eigenvalues below the infimum of the essential spectrum. More precisely, we have the following.

Proposition 3.7. *Assume* (D_1) *and* (D_2)*. Then*

(1) $\inf \sigma_{ess}(-\Delta + V_\lambda) \to \infty$ *as* $\lambda \to \infty$; *where* σ_{ess} *denotes the essential spectrum.*
(2) For any $k > 0$*, there exists a* Λ_k *such that* $-\Delta + V_\lambda$ *has at least* k *eigenvalues below the essential spectrum provided* $\lambda \geq \Lambda_k$*.*

Proof. It suffices to show that

$$
\lim_{k \to \infty} \mu_k = \infty. \tag{3.55}
$$

Assume by negation that $\lim_{k\to\infty} \mu_k = \sup_k \mu_k < c_0 < \infty$. By Lemma 3.6, there exist $\lambda_0 > 0$ and $r_i > 0$ such that

$$
\mu^*(S_{\lambda_0}, \mathbf{R}^N \setminus B_{r_i}(0)) \geq 2c_0. \tag{3.56}
$$

Let $\mu_k^*(S_0, B_{r_i}(0))$ denote the kth Rayleigh quotient of the operator S_0 on the domain $B_{r_i}(0)$. Then

$$
\lim_{k\to\infty} \mu_k^*(S_0, B_{r_i}(0)) = \infty.
$$

Hence, there exists a $k_0 \in \mathbf{N}$ with

$$
\mu_k^*(S_0, B_{r_i}(0)) \geq 2c_0.
$$

Using the notation (S_λ^*, D) for the operator S_λ on $L^2(D)$ with Neumann boundary conditions, by Reed and Simon's propositions ([262, Propositions 3–4, Section XIII.15]) we have for $\lambda \geq \lambda_0$ that

$$
\begin{aligned}
(S_\lambda, \mathbf{R}^N) &\geq (S_{\lambda_0}, \mathbf{R}^N) \\
&\geq (S^*_{\lambda_0}, B_{r_i}(0) \cup (\mathbf{R}^N \setminus B_{r_i}(0))
\end{aligned}
$$

$$= (S^*_{\lambda_0}, B_{r_i}(0)) \oplus (S^*_{\lambda_0}, \mathbf{R}^N \backslash B_{r_i}(0))$$
$$\geq (S^*_0, B_{r_i}(0)) \oplus (S^*_0, \mathbf{R}^N \backslash B_{r_i}(0)).$$

By (3.56),

$$\begin{aligned}\mu_{k_0}(S_\lambda, \mathbf{R}^N) &\geq \mu_{k_0}(S^*_0, B_{r_i}(0)) \oplus (S^*_{\lambda_0}, \mathbf{R}^N \backslash B_{r_i}(0)) \\ &= \mu^*_{k_0}(S_0, B_{r_i}(0)) \\ &\geq 2c_0.\end{aligned}$$

But, $\mu_{k_0}(S_\lambda, \mathbf{R}^N) \leq \mu_{k_0} \leq c_0$; this is a contradiction. □

Lemma 3.8. *Assume* (D_1) *and* (D_2)*. Then* $H^1_0(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$*, where* Ω *comes from* (D_1)*.*

Proof. Let $C(R) = \mathbf{R}^N \backslash B_R$, where B_R is the open ball centered at 0 with radius R. We first show that for any $\varepsilon > 0$, there exists an $R(\varepsilon) > 0$ such that

$$\|u\|^2_{L^2(\Omega \cap C(R(\varepsilon)))} \leq \varepsilon \|u\|^2_{H^1_0(\Omega)}, \tag{3.57}$$

where

$$\|u\|_{H^1_0(\Omega)} := \left(\int_\Omega (|\nabla u|^2 + u^2) dx \right)^{1/2}.$$

For this, we choose a countable set of coordinates $x_i \in \mathbf{R}^N$ such that each x of $\mathbf{R}^N$ belongs to at most κ balls centered at x_i with radius r_0 (cf. (D_1)). Then for any $R > r_0$ and $u \in H^1_0(\Omega)$,

$$\int_{C(R)\cap\Omega} u^2 dx \leq \sum_{|x_i|>R-r_0} \int_{B_{r_0}(x_i)\cap\Omega} u^2 dx.$$

For any fixed $p \in (2, N/(N-2))$, we have

$$\int_{C(R)\cap\Omega} u^2 dx \leq \sum_{|x_i|>R-r_0} \left(\int_{B_{r_0}(x_i)\cap\Omega} u^{2p} dx \right)^{1/p} (\mathrm{meas}(B_{r_0}(x_i) \cap \Omega))^{1/p'},$$

where $p' = p/(p-1)$. Note that $H^1(\mathbf{R}^N) \hookrightarrow L^{2q}(\mathbf{R}^N)$; we may find a constant $c > 0$ that is independent of x_i such that

$$\left(\int_{B_{r_0}(x_i)\cap\Omega} u^{2p} dx \right)^{1/p} \leq c \int_{B_{r_0}(x_i)\cap\Omega} (|\nabla u|^2 + u^2) dx.$$

For any $\varepsilon > 0$, by (D_2), we have a $R(\varepsilon) > 0$ such that

$$(\mathrm{meas}(B_{r_0}(x_i) \cap \Omega))^{1/p'} \le \frac{\varepsilon}{c\kappa}, \qquad \text{for } |x_i| > R(\varepsilon) - r_0.$$

Thus,

$$\begin{aligned}
&\int_{C(R)\cap\Omega} u^2 dx \\
&\quad \le \frac{\varepsilon}{\kappa} \sum_{|x_i|>R-r_0} \int_{B_{r_0}(x_i)\cap\Omega} (|\nabla u|^2 + u^2) dx \\
&\quad \le \varepsilon \int_{C(R)\cap\Omega} (|\nabla u|^2 + u^2) dx \\
&\quad \le \varepsilon \|u\|_{H_0^1(\Omega)}.
\end{aligned}$$

By this, it is easy to see that $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$. $\square$

Lemma 3.9. *Assume (D_1) and (D_2). For any $\varepsilon > 0$, there exist $\Lambda_\varepsilon > 0$ and $R_\varepsilon > 0$ such that*

$$\|u\|^2_{L^2(O_\varepsilon)} \le \varepsilon \|u\|^2_\lambda$$

for all $u \in E_\lambda$ and $\lambda \ge \Lambda_\varepsilon$, where $O_\varepsilon := \{x \in \mathbf{R}^N : |x| \ge R_\varepsilon\}$.

Proof. For any $\varepsilon > 0$, similarly to the proof of (3.57), we may find an $R_\varepsilon > 0$ such that

$$\int_{\Omega_\varepsilon} u^2 dx \le \frac{\varepsilon}{2} \|u\|^2_\lambda, \tag{3.58}$$

where $\Omega_\varepsilon := \{x \in \mathbf{R}^N : |x| > R_\varepsilon, g_0(x) < M_0\}$ (see (D_2)). Let

$$\tilde{\Omega}_\varepsilon := \{x \in \mathbf{R}^N : |x| \ge R_\varepsilon, g_0(x) \ge M_0\}.$$

Then

$$\begin{aligned}
\int_{\tilde{\Omega}_\varepsilon} u^2 dx &\le \frac{1}{\lambda M_0 + 1} \int_{\tilde{\Omega}_\varepsilon} (\lambda g_0(x) + 1) u^2 dx \\
&\le \frac{1}{\lambda M_0 + 1} \|u\|^2_\lambda \\
&\le \frac{\varepsilon}{2} \|u\|^2_\lambda
\end{aligned} \tag{3.59}$$

as λ large enough. Combining (3.58) and (3.59), we get the conclusion of the lemma. $\square$

Consider the following eigenvalue problem.

$$-\Delta u + u = \nu u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.60}$$

Proposition 3.10. *Under (D_1)–(D_3), (3.60) has positive isolated eigenvalues with finite multiplicity:*

$$0 < \nu_1 < \nu_2 < \cdots < \nu_m < \nu_{m+1} < \cdots . \tag{3.61}$$

Proof. For each $g \in L^2(\Omega)$, consider the Dirichlet problem on Ω:

$$-\Delta u + u = g \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Define the following functional

$$I(u) = \int_\Omega (|\nabla u|^2 + u^2)dx - \int_\Omega gudx.$$

By Lemma 3.8, there is a unique $u \in H_0^1(\Omega)$ such that $I'(u) = 0$. Define the linear operator $H : L^2(\Omega) \to H_0^1(\Omega)$ by $Hg = u$. Then

$$\|Hg\|^2 = \|u\|^2 = I'(u)u + \int_\Omega gudx \le \|g\|_{L^2(\Omega)}\|u\|_{H_0^1(\Omega)}.$$

Combining this and Lemma 3.8, $H : L^2(\Omega) \to L^2(\Omega)$ is compact. Therefore, by Hislop and Sigal [172, Theorem 9.10] (and Kato [183]), Equation (3.60) has a discrete spectrum (eigenvalues), and each eigenvalue has finite multiplicity. Obviously, the first eigenvalue is greater than zero. □

Let $\dim(\nu_i)$ denote the dimension of the eigenspace corresponding to the eigenvalue ν_i. Let $d_k := \sum_{i=1}^k \dim(\nu_i)$. Let ψ_1 be the first eigenfunction of (3.60) corresponding to ν_1. We may assume that $\psi_1 > 0$.

We may rewrite (3.61) with their multiplicity taken into consideration:

$$0 < \mu_1(L^*) < \mu_2(L^*) \le \mu_3(L^*) \cdots \le \mu_k(L^*) \cdots , \tag{3.62}$$

where $L^* := -\Delta + 1$ on Ω. Consider $W_0^{1,2}(\Omega)$ as a subspace of $W^{1,2}(\mathbf{R}^N)$ and note that $\langle S_\lambda u, u\rangle_\lambda = \langle L^* u, u\rangle, \forall u \in W_0^{1,2}(\Omega)$. We see that

$$\mu_k(S_\lambda) \le \mu_k(L^*) \tag{3.63}$$

for all $k \in \mathbf{N}$ and $\lambda > 0$. Because

$$\mu_k = \lim_{\lambda\to\infty} \mu_k(S_\lambda), \tag{3.64}$$

then

$$\mu_k \le \mu_k(L^*) \tag{3.65}$$

for all $k \in \mathbf{N}$. By (3.55), $\lim_{k\to\infty} \mu_k = \infty$. Furthermore, we have the following.

Proposition 3.11. *$\mu_k = \mu_k(L^*)$ for all $k \in \mathbf{N}$.*

Proof. Let $\varphi_{k,\lambda} \in E_\lambda$ be a normalized eigenfunction of S_λ corresponding to $\mu_k(S_\lambda)$. Hence, $\|\varphi_{k,\lambda}\|_{L^2(\mathbf{R}^N)} = 1$ and

$$(3.66)\qquad \int_{\mathbf{R}^N} (\nabla\varphi_{k,\lambda}\nabla v + V_\lambda \varphi_{k,\lambda} v)dx = \mu_k(S_\lambda)\int_{\mathbf{R}^N} \varphi_{k,\lambda} v dx,$$

for all $v \in \mathbf{C}_0^\infty(\mathbf{R}^N)$. Combining this with (3.62)–(3.65), we may assume that

$$(3.67)\qquad \varphi_{k,\lambda} \to \varphi_k \text{ weakly in } W^{1,2}(\mathbf{R}^N), \qquad \varphi_{k,\lambda} \to \varphi_k \text{ strongly in } L^2(\mathbf{R}^N)$$

as $\lambda \to \infty$. We claim that $\varphi_k \in W_0^{1,2}(\Omega)$. In fact, by (D_3) it suffices to show that $\varphi_k = 0$ a.e. in $\mathbf{R}^N\backslash\Omega$. However, if this were not true, there would exist a compact $\Omega_1 \subset \mathbf{R}^N\backslash\Omega$ and a $c_1 > 0$ such that

$$(3.68)\qquad \lim_{\lambda\to\infty}\int_{\Omega_0} \varphi_{k,\lambda}^2 dx = \int_{\Omega_0} \varphi_k^2 dx \geq c_1.$$

By (D_1), we get a $c_2 > 0$ such that $g_0(x) \geq c_2$ for all $x \in \Omega_0$. Therefore, by (3.68),

$$\mu_k(L^*) \geq \|\varphi_{k,\lambda}\|_\lambda^2 \geq \lambda\int_{\Omega_0} g_0(x)\varphi_{k,\lambda}^2 dx \geq \lambda c_2 \int_{\Omega_0} \varphi_{k,\lambda}^2 dx \to \infty,$$

as $\lambda \to \infty$. This is a contradiction. Next, we prove that

$$(3.69)\qquad \|\varphi_{k,\lambda}\|_{L^2(\mathbf{R}^N)} \to \|\varphi_k\|_{L^2(\mathbf{R}^N)}, \qquad \lambda \to \infty.$$

First, by Lemma 3.9 and Equations (3.63) and (3.66), for any $\varepsilon > 0$ we find an $R_\varepsilon > 0$ such that

$$(3.70)\qquad \lim_{\lambda\to\infty} \|\varphi_{k,\lambda}\|_{L^2(O_\varepsilon)}^2 \leq \varepsilon/2,$$

where $O_\varepsilon := \{x \in \mathbf{R}^N : |x| \geq R_\varepsilon\}$. Choose R_ε so large that

$$(3.71)\qquad \|\varphi_k\|_{L^2(O_\varepsilon)}^2 \leq \varepsilon/2.$$

Combine (3.70) and (3.71); we get (3.69), which implies that $\|\varphi_k\|_{L^2(\Omega)} = 1$. In (3.66), we take any $v \in \mathbf{C}_0^\infty(\Omega)$ and let $\lambda \to \infty$. We see that

$$\int_\Omega (\nabla\varphi_k\nabla v + \varphi_k v)dx = \mu_k \int_\Omega \varphi_k v dx.$$

This completes the proof of the lemma. $\square$

Notes and Comments. Lemma 3.6 and Proposition 3.7 were established in Bartsch et al. [38]. Lemma 3.8 and Proposition 3.10 were obtained in van Heerden and Wang [168]. Lemma 3.9 was also due to van Heerden and Wang [168]. Proposition 3.11 was given in Liu et al. [210], which was essentially based on van Heerden [166]. Equation (3.49) with a potential well was also considered recently in Stuart and Zhou [309] where the existence results had been obtained.

3.4 Flow-Invariant Sets

In this section, we construct the invariant set of the gradient flow. Let ψ_1 be the first eigenfunction of (3.60) corresponding to $\nu_1 = \mu_1(L^*)$, and ϕ_1 be the first eigenfunction corresponding to $\mu_1(S_\lambda)$. We may assume that $\psi_1 > 0, \phi_1 > 0$.

Lemma 3.12. *Assume that*

$$\liminf_{t\to 0} \frac{f(x,t)}{t} > \nu_1 \quad \textit{uniformly for } x \in \mathbf{R}^N. \tag{3.72}$$

If $U^+(x)$ is a positive solution of Equation (3.49), then there exists a constant $t_0 > 0$ such that

$$t_0\psi_1(x) < U^+(x), \qquad \forall\ x \in \mathbf{R}^N$$

and

$$t_0\phi_1(x) < U^+(x), \qquad \forall\ x \in \mathbf{R}^N.$$

Proof. By (3.72), we have two positive constants $\varepsilon_1, \varepsilon_2$ such that

$$\frac{f(x,t)}{t} > \nu_1 + \varepsilon_1, \qquad \forall\, |t| \in (0, \varepsilon_2), \qquad x \in \mathbf{R}^N. \tag{3.73}$$

Choose $R > 0$ such that $U^+(x) < \varepsilon_2$ for $|x| \geq R$. Choose $t_0 > 0$ so small that

$$t_0\psi_1(x) < U^+(x), \qquad \text{for } |x| \leq R. \tag{3.74}$$

We just have to show that (3.74) is true for all x. By negation, we assume that $\Theta := \{x \in \mathbf{R}^N : t_0\psi_1(x) > U^+(x)\} \neq \emptyset$; then $\Theta \subset \{x \in \Omega : |x| \geq R\}$. Moreover,

$$-\Delta(t_0\psi_1) + t_0\psi_1 = \nu_1 t_0\psi_1 \quad \text{in } \Theta, \tag{3.75}$$

$$-\Delta U^+ + V_\lambda U^+ = f(x, U^+) \quad \text{in } \Theta, \tag{3.76}$$

$$t_0\psi_1(x) = U^+(x) \quad \text{on } \partial\Theta, \tag{3.77}$$

$$t_0\frac{\partial\psi_1}{\partial\nu} \leq \frac{\partial U^+}{\partial\nu} \quad \text{on } \partial\Theta, \tag{3.78}$$

where ν denotes the outer unit normal on $\partial\Theta$. By (3.73),

$$\int_{\Theta} t_0\psi_1(\nu_1 U^+ - f(x,U^+))dx < -\int_{\Theta}(t_0\psi_1)(\varepsilon_1)U^+dx < 0. \tag{3.79}$$

But by (3.75)–(3.78) and the divergence theorem,

$$\int_{\Theta} t_0\psi_1(\nu_1 U^+ - f(x,U^+))dx \tag{3.80}$$

$$= \int_{\Theta}(t_0\psi_1 \Delta U^+ - U^+\Delta(t_0\psi_1))dx \tag{3.81}$$

$$= \int_{\partial\Theta} t_0\psi_1 \left(\frac{\partial U^+}{\partial\nu} - t_0\frac{\partial\psi_1}{\partial\nu}\right) ds \tag{3.82}$$

$$\geq 0; \tag{3.83}$$

it contradicts (3.79). The second conclusion can be proved analogously. □

Given any $u_1, u_2 \in E_\lambda$ such that $u_1(x) \geq u_2(x)$ for all $x \in \mathbf{R}^N$. Define

$$g(x,t) = \begin{cases} f(x,u_1(x)), & \text{for } t > u_1(x), \\ f(x,t), & \text{for } u_2(x) \leq t \leq u_1(x), \\ f(x,u_2(x)), & \text{for } t < u_2(x). \end{cases}$$

Set

$$J(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbf{R}^N} G(x,u)dx, \qquad u \in E_\lambda,$$

where $G(x,u) = \int_0^u g(x,t)dt$.

Lemma 3.13. *Assume that there exists an $F_0 > 0$ such that*

$$|f(x,t)| \leq F_0|t| \quad \textit{for all } (x,t) \in \mathbf{R}^N \times \mathbf{R}; \tag{3.84}$$

then J satisfies the (PS) condition and $J(u) \to \infty$ as $\|u\| \to \infty$.

Proof. By the assumption, we observe that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - c\int_{\mathbf{R}^N}(|u_1| + |u_2|)|u|dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - c(\|u_1\|_2 + \|u_2\|_2)\|u\|_\lambda. \end{aligned} \tag{3.85}$$

This implies the second part of the lemma. Now let $\{v_n\}$ be a (PS) sequence; that is,

$$\sup_n |J(v_n)| < \infty, \quad J'(v_n) \to 0.$$

By (3.85), we see that $\{\|v_n\|_\lambda\}$ is bounded. Then, up to a subsequence, $v_n \to v$ weakly in E_λ and strongly in $L^2_{loc}(\mathbf{R}^N)$, with v a solution of

$$-\Delta u + V_\lambda u = g(x,u).$$

Furthermore,

$$\begin{aligned}
\|v_n\|_\lambda^2 - \|v\|_\lambda^2 &= \langle J'(v_n), v_n\rangle - \int_{\mathbf{R}^N} (g(x,v_n)v_n - g(x,v)v)dx \\
&\le o(1) + \left| \int_{|x|\ge R} (g(x,v_n)v_n - g(x,v)v)dx \right| \\
&\le o(1) + c\int_{|x|\ge R} (|u_1| + |u_2|)(|v_n| + |v|)dx \\
&= o(1).
\end{aligned}$$

This means $\|v_n\|_\lambda \to \|v\|_\lambda$. The (PS) condition is satisfied. □

Lemma 3.14. *Assume that (D_1)–(D_3), (3.72), and (3.84) hold. Moreover, there exists an $L \ge 0$ such that $f(x,t) + Lt$ is increasing in t. If there exists a positive (negative) solution u_1 (u_2, resp.) to Equation (3.49), then there exists a minimal positive (maximal negative) solution U^+ (U^-, resp.) to Equation (3.49).*

Proof. We assume the existence of a positive solution u_1. Define

$$g(x,t) = \begin{cases} f(x,u_1(x)), & \text{for } t > u_1(x), \\ f(x,t), & \text{for } 0 \le t \le u_1(x), \\ 0, & \text{for } t < 0. \end{cases}$$

Consider the solution $u \in E_\lambda$ of the equation

$$\begin{cases} -\Delta u + V_\lambda(x)u = g(x,u), & x \in \mathbf{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{3.86}$$

Let

$$J(u) := \frac{1}{2}\|u\|_\lambda - \int_{\mathbf{R}^N} G(x,u)dx, \tag{3.87}$$

which is of $\mathbf{C}^1$. We claim that any solution u of (3.86) belongs to the interval $[0, u_1]$; that is, $0 \le u(x) \le u_1(x)$ for $x \in \mathbf{R}^N$. Otherwise, the open set $\Theta := \{x \in \mathbf{R}^N : u(x) > u_1(x)\} \ne \emptyset$ and

$$-\Delta u + V_\lambda(x)u = f(x,u_1) = -\Delta u_1 + V_\lambda(x)u_1.$$

Hence,

$$-\Delta(u-u_1)+V_\lambda(x)(u-u_1)=0.$$

Because $u(x), u_1(x) \to 0$ as $|x| \to \infty$, the maximum principle implies that $u(x) = u_1(x)$ for $x \in \Theta$, a contradiction. Similarly, we have $u(x) \geq 0$. By Lemma 3.12, there exist constants $t_0 > 0$ such that

$$0 < t_0\psi_1(x) < u_1(x), \qquad \forall\ x \in \mathbf{R}^N. \tag{3.88}$$

By (3.72), we may choose t_0 so small that

$$F(x, t\psi_1) > \frac{1}{2}(\nu_1+\delta)t^2\psi_1^2, \qquad t \in (0, t_0], s \in \mathbf{R}^N$$

and that $t\psi_1$ is not a critical point (solution) of J (of (3.86)) for all $t \in (0, t_0]$. Therefore,

$$\begin{aligned} J(t\psi_1) &= \frac{1}{2}\|t\psi_1\|_\lambda - \int_{\mathbf{R}^N} G(x, t\psi_1)dx \\ &= \frac{1}{2}\|t\psi_1\|_\lambda - \int_{\mathbf{R}^N} F(x, t\psi_1)dx \\ &= \frac{\nu_1 t^2}{2}\int_\Omega \psi_1^2 dx - \int_\Omega \frac{\nu_1+\varepsilon_1}{2}t^2\psi_1^2 dx \\ &< 0. \end{aligned}$$

for all $t \in (0, t_0]$. Hence $\inf_E J < 0$. Because $f(x,t)+Lt$ is increasing in t, we may assume that $L = 0$. Otherwise, we may replace the norm $\|u\|_\lambda$ by the equivalent norm $\|u\|_* := \int_{\mathbf{R}^N} |(\nabla u|^2 + V_\lambda u^2 + Lu^2)dx$. Note that $0, u_1$ are solutions to (3.49) and $f(x,t)$ is increasing in t; we have that

$$0 \leq (-\Delta + V_\lambda)^{-1} f(x,u) \leq u_1 \quad \text{ if } 0 \leq u \leq u_1.$$

By (3.87),

$$J'(u) = u - (-\Delta + V_\lambda)^{-1} g(x,u).$$

Because

$$0 \leq (-\Delta + V_\lambda) g(x,u) = (-\Delta + V_\lambda) f(x,u) \leq u_1$$

if $0 \leq u \leq u_1$, then by Lemma 2.12, there exists an operator L_0 such that

$$L_0([0, u_1]) \subset [0, u_1]$$

and that $V = \mathbf{id} - L_0$ is the pseudo-gradient vector field of J, where $[0, u_1] := \{u \in E_\lambda : 0 \leq u \leq u_1\}$. For each $t \in (0, t_0]$, note that $t\psi_1$ is not a critical point of J. We consider the initial value problem:

$$\begin{cases} \dfrac{\sigma(s,t\psi_1)}{ds} &= -V(\sigma(s,t\psi_1)) \\ \sigma(0,t\psi_1) &= t\psi_1. \end{cases}$$

By Lemmas 3.13 and 1.54, there exist

(3.89)
$$\sigma(s_n^t, t\psi_1) \to u_t^* \qquad (n \to \infty), \qquad J(u_t^*) \le J(t\psi_1) < 0, \qquad J'(u_t^*) = 0$$

for each $t \in (0, t_0]$. Recalling Lemma 1.49 and noting $L_0([0,u_1]) \subset [0,u_1]$, we may assume that

$$0 \le \sigma(s,t\psi_1) \le u_1, \qquad s \ge 0.$$

Therefore, we may assume that

$$u_t^* \in [0,u_1], \qquad t \in (0,t_0]$$

for each $t \in (0,t_0]$. By (3.89), we get a critical point u_t^* of J such that

$$0 \le u_t^* \le u_1.$$

Then u_t^* is also a solution of (3.49). Because $J(u_t^*) < 0$, we may assume that

$$0 < u_t^* \le u_1.$$

That is, u_t^* is a positive solution of (3.49) for all $t \in (0,t_0]$. Obviously,

$$\inf_{E_\lambda} J \le J(u_t^*) < 0, \qquad J'(u_t^*) = 0.$$

By the (PS) condition, there is a U^+ such that $u_t^* \to U^+$ $(t \to 0)$ in E_λ and $u_t^*(x) \to U^+(x) \ge 0$ for $x \in \mathbf{R}^N$. Evidently, U^+ is a solution of (3.49). Next, we show that $U^+(x) > 0$. If $U^+ = 0$, then by L^p_{loc} estimates on any ball $B_R := \{x \in \mathbf{R}^N : |x| \le R\}$ $(R > 0)$, we have $\|u_t^*\|_{L^\infty(B_R)} \le c\|u_t^*\|_\lambda$, where c is independent of t. For any $\varepsilon > 0$, there is a T_0 such that $((f(x,t))/t) \ge (\nu_1 + \varepsilon)$ for $|s| \in (0,T_0)$. For this T_0, we find a $R > 0$ such that $u_t^* \le u_{t_0}^* < T_0$ for $|x| \ge R$ and all $t \in (0,t_0]$. For this R, we find a $\bar{t} \in (0,t_0)$ such that

$$\|u_t^*\|_{L^\infty(B_R)} \le c\|u_t^*\|_\lambda < T_0, \qquad \forall\, t \in (0,\bar{t}).$$

Because u_t^* is a solution of (3.49), that is,

$$-\Delta u_t^* + V_\lambda u_t^* = f(x,u_t^*),$$

hence,

(3.90)
$$-\Delta u_t^* + V_\lambda u_t^* \ge (\nu_1 + \varepsilon)u_t^*.$$

Let $\mu_1(S_\lambda)$ be the first eigenvalue of S_λ with eigenfunction $\phi_1(\lambda) > 0$. Then $\nu_1 \geq \mu_1(S_\lambda)$. Multiplying (3.90) by $\phi_1(\lambda)$ and integrating, we have

$$\begin{aligned}\mu_1(S_\lambda) \int_{\mathbf{R}^N} u_t^* \phi_1(\lambda) dx \\ = \int_{\mathbf{R}^N} f(x, u_t^*) \mu_1(S_\lambda) dx \\ \geq (\mu_1(S_\lambda) + \varepsilon) \int_{\mathbf{R}^N} u_t^* \phi_1(\lambda) dx,\end{aligned}$$

a contradiction. Finally, we show that U^+ is indeed minimal. Assume that U_1 is another positive solution to (3.49); then we find a $t_1 < t_0$ such that

$$t\psi_1(x) \leq U_1(x), \quad \text{for all } x \in \mathbf{R}^N, \qquad t \in (0, t_1].$$

Then, for each $t \in (0, t_1]$ we may find a flow $\sigma(s, t\psi_1)$ such that

$$0 \leq \sigma(s, t\psi_1) \leq U_1, \qquad s \geq 0.$$

Therefore,

$$U_1 \geq \sigma(s_n^t, t\psi_1) \to u_t^*, \qquad n \to \infty; \qquad \forall\, t \in (0, t_1].$$

Let $t \to 0$; we have

$$U_1 \geq U^+.$$

In the same way, we may find a maximal negative solution. □

By (3.72), for $\varepsilon_0 > 0$ small enough, we find a $t_0 > 0$ such that

$$\frac{f(x,t)}{t} > \nu_1 + \varepsilon_0 \geq \mu_1(S_\lambda) + \varepsilon_0, \qquad x \in \mathbf{R}^N, \qquad t \in (0, t_0]. \tag{3.91}$$

Let

$$P_\lambda^* := \{u \in E_\lambda : u \geq \phi_1(\lambda)\}, \tag{3.92}$$

where $\phi_1(\lambda)$ is the positive eigenfunction of $\mu_1(S_\lambda)$. Then P_λ^* is closed and convex. By Lemmas 3.12 and 3.14, we may assume, up to multiplying $\phi_1(\lambda)$ by a small coefficient, that P_λ^* $(-P_\lambda^*)$ includes all positive (negative) solutions of (3.49) if they exist. Moreover, we may choose

$$\phi_1(\lambda) \leq t_0. \tag{3.93}$$

Let

$$\mathcal{D}_0(\varepsilon, \lambda) = \{u \in E_\lambda : \operatorname{dist}(u, P_\lambda^*) < \varepsilon\} \tag{3.94}$$

and

$$J_0 := (-\Delta + V_\lambda)^{-1} f. \tag{3.95}$$

Theorem 3.15. *Under the assumptions of Lemma 3.14, there exists an* $\varepsilon^* > 0$ *and* $\Lambda^* > 0$ *such that*

$$J_0(\pm\mathcal{D}_0(\varepsilon, \lambda)) \subset \pm\mathcal{D}_0\left(\frac{1}{2}\varepsilon, \lambda\right) \qquad \textit{for all } \varepsilon \in (0, \varepsilon^*), \lambda \geq \Lambda^*.$$

Proof. For any $u \in E_\lambda$, let $w = \max\{\phi_1(\lambda), J_0 u\} \in P_\lambda^*$. Therefore,

$$\operatorname{dist}(J_0 u, P_\lambda^*) \leq \|J_0 u - w\|_\lambda. \tag{3.96}$$

Because w is either $J_0 u$ or $\phi_1(\lambda)$, we have

$$\begin{aligned} \|J_0 u - w\|_\lambda^2 \quad & \qquad (3.97) \\ &= \langle J_0 u - \phi_1(\lambda), J_0 u - w\rangle_\lambda \\ &= \int_{\mathbf{R}^N} (-\Delta(J_0 u - \phi_1(\lambda)) + V_\lambda(J_0 u - \phi_1(\lambda)))(J_0 u - w) dx \\ &= \int_{\mathbf{R}^N} (f(x,u) - \mu_1(S_\lambda)\phi_1(\lambda))(J_0 u - w) dx \\ &= \int_{\mathbf{R}^N} (\mu_1(S_\lambda)\phi_1(\lambda) - f(x,u))(w - J_0 u) dx. \qquad (3.98) \end{aligned}$$

Keeping (3.93) in mind and noting that $f(x,t)$ is increasing in t, then

$$f(x,u) \geq f(x,t_0) > (\mu_1(\lambda) + \varepsilon_0)t_0 \geq (\mu_1(S_\lambda) + \varepsilon_0)\phi_1(\lambda), \tag{3.99}$$

for $u \geq t_0$. Hence,

$$\begin{aligned} &\int_{\mathbf{R}^N} (f(x,u) - \mu_1(S_\lambda)\phi_1(\lambda))(J_0 u - w) dx. \\ &\leq \int_{u(x) \leq t_0} (\mu_1(S_\lambda)\phi_1(\lambda) - f(x,u))(w - J_0 u) dx \qquad (3.100) \end{aligned}$$

Because $f(x,u) \geq F_0 u$ for $u \leq 0$ (see (3.84)), combining (3.93), we have that

$$\int_{u(x) \leq t_0} (\mu_1(S_\lambda)\phi_1(\lambda) - f(x,u))(w - J_0 u) dx \tag{3.101}$$

$$\le \int_{t_0 \ge u \ge 0} (\mu_1(S_\lambda)\phi_1(\lambda) - (\mu_1(S_\lambda) + \varepsilon_0)u)(w - J_0 u)dx$$

$$+ \int_{u<0} (\mu_1(S_\lambda)\phi_1(\lambda) - F_0 u)(w - J_0 u)dx$$

$$\le (\mu_1(S_\lambda) + \varepsilon_0) \int_{\xi_\lambda \ge u \ge 0} (\phi_1(\lambda) - u)(w - J_0 u)dx$$

$$+ (\mu_1(S_\lambda) + F_0) \int_{u<0} (\phi_1(\lambda) - u)(w - J_0 u)dx$$

$$\le (\mu_1(S_\lambda) + \varepsilon_0 + F_0) \int_\Gamma (\phi_1(\lambda) - u)(w - J_0 u)dx$$

$$\le (\nu_1 + \varepsilon_0 + F_0) \int_\Gamma (\phi_1(\lambda) - u)(w - J_0 u)dx, \tag{3.102}$$

where

$$\xi_\lambda := \frac{\mu_1(S_\lambda)\phi_1(\lambda)}{(\mu_1(S_\lambda) + \varepsilon_0)},$$

$$\Gamma := \{x \in \mathbf{R}^N : u(x) \le \xi_\lambda(x)\}.$$

Given $R > 0$, we may find an $\varepsilon_R > 0$ such that

$$(\phi_1(\lambda) - u) \ge \frac{\varepsilon_0}{\mu_1(S_\lambda) + \varepsilon_0}\phi_1(\lambda) \ge \varepsilon_R, \qquad \text{for } x \in \Gamma, |x| \le R. \tag{3.103}$$

On the other hand, $u(x) \le \phi_1(\lambda)$ on Γ; combining the definition of P_λ^*, we observe, for any $\Gamma' \subset \Gamma$, that

$$\|\phi_1(\lambda) - u\|_{L^p(\Gamma')} = \inf_{v \in P_\lambda^*} \|v - u\|_{L^p(\Gamma')}, \qquad \forall\, p \in [2, 2^*]. \tag{3.104}$$

Therefore, by the Sobolev inequality and (3.103) and (3.104),

$$\int_{\Gamma \cap \{|x| \le R\}} (\phi_1(\lambda) - u)(w - J_0 u)dx \tag{3.105}$$

$$\le c(R) \int_{\Gamma \cap \{|x| \le R\}} |\phi_1(\lambda) - u|^{2^*-1}(w - J_0 u)dx$$

$$\le c(R)\|\phi_1(\lambda) - u\|_{L^{2^*}(\Gamma)}^{2^*-1}\|w - J_0 u\|_\lambda$$

$$\le c(R)(\mathrm{dist}(u, P_\lambda^*))^{2^*-1}\|w - J_0 u\|_\lambda,$$

where $c(R)$s are constants depending on R, whose values are irrelevant to each other. On the other hand, by (3.104)

$$\begin{aligned}
(3.106)\qquad & \int_{\Gamma\cap\{|x|>R\}} (\phi_1(\lambda)-u)(w-J_0u)dx \\
& \le \|\phi_1(\lambda)-u\|_{L^2(\Gamma\cap\{|x|>R\})}\|w-J_0u\|_2 \\
& \le \|\phi_1(\lambda)-u\|_{L^2(\Gamma\cap\{|x|>R\})}\|w-J_0u\|_2 \\
(3.107)\qquad & \le \inf_{v\in P_\lambda^*} \|u-v\|_{L^2(\Gamma\cap\{|x|>R\})}\|w-J_0u\|_\lambda .
\end{aligned}$$

For the constant

$$\varepsilon' := \frac{1}{4(\nu_1+\varepsilon_0+F_0)},$$

by Lemma 3.9, we have $R_{\varepsilon'}>0, \Lambda_{\varepsilon'}>0$ such that

$$\begin{aligned}
(3.108)\qquad & \inf_{v\in P_\lambda^*} \|u-v\|_{L^2(\Gamma\cap\{|x|>R\})} \\
& \le \inf_{v\in P_\lambda^*} \|u-v\|_{L^2(\{|x|>R\})} \\
& \le \inf_{v\in P_\lambda^*} \varepsilon'\|u-v\|_\lambda \\
& = \varepsilon'\mathrm{dist}(u,P_\lambda^*).
\end{aligned}$$

Combining (3.96)–(3.108), we have

$$\begin{aligned}
& \mathrm{dist}(J_0u,P_\lambda^*) \\
& \le (\nu_1+\varepsilon_0+F_0)(\varepsilon'\mathrm{dist}(u,P_\lambda^*)+c(R)(\mathrm{dist}(u,P_\lambda^*))^{2^*-1}) \\
& \le \left(\frac{1}{4}\mathrm{dist}(u,P_\lambda^*)+\frac{c(R)}{4\varepsilon'}(\mathrm{dist}(u,P_\lambda^*))^{2^*-1}\right).
\end{aligned}$$

Therefore, if $\mathrm{dist}(u,P_\lambda^*)<\varepsilon<\varepsilon^*$, where ε^* is small enough, then

$$\mathrm{dist}(J_0u,P_\lambda^*) \le \left(\frac{1}{4}\varepsilon+\frac{c(R)}{4\varepsilon'}\varepsilon^{2^*-1}\right) < \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon^*;$$

that is,

$$J_0(\mathcal{D}_0(\varepsilon,\lambda)) \subset \mathcal{D}_0\left(\frac{1}{2}\varepsilon,\lambda\right), \qquad \varepsilon\in(0,\varepsilon^*).$$

This completes the proof for the case of "+" of the theorem. The proof for the case of "−" is similar; we omit the details. □

Notes and Comments. Theorem 3.15 and other lemmas of this section were established in Liu et al. [210]. The ideas of decreasing flow-invariant sets can also be found in Sun [315], Sun and Hu [317], and Sun and Xu [318]. More references have been mentioned in previous notes and comments.

3.5 Sign-Changing Homoclinic-Type Solutions

We form the following hypotheses on the nonlinearity f.

($\mathbf{E}_1$) $f \in \mathbf{C}(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$; there exist $H_0 > 0, L \geq 0$ such that $|f(x,t)| \leq H_0|t|$ for all $(x,t) \in \mathbf{R}^N \times \mathbf{R}$ and that $f(x,t) + Lt$ is increasing in t.
($\mathbf{E}_2$) $\nu_1 < \liminf_{t\to 0}((f(x,t))/t) \leq \limsup_{t\to 0}((f(x,t))/t) < \nu_k$ uniformly for $x \in \mathbf{R}^N$.
($\mathbf{E}_3$) $2F(x,t) \leq \kappa t^2, \quad x \in \mathbf{R}^N, t \in \mathbf{R}$, where $\kappa < \nu_{k+1}$.
($\mathbf{E}_4$) $\lim_{|t|\to\infty}(2F(x,t))/t^2 = \theta(x) \geq \nu_k$ uniformly for $x \in \mathbf{R}^N$, where $\theta(x) \not\equiv \nu_k$.

We need the following alternatives to guarantee the (w^*-PS) condition. That is, either

($\mathbf{E}_5$) $f(x,t)t - 2F(x,t) \geq H(x) \in L^1(\mathbf{R}^N)$ for $x \in \mathbf{R}^N, t \in \mathbf{R}$ and

$$\lim_{|t|\to\infty} (f(x,t)t - 2F(x,t)) = \infty \quad \text{for each } x \in \mathbf{R}^N,$$

or

($\mathbf{E}_6$) $f(x,t)t - 2F(x,t) \leq H(x) \in L^1(\mathbf{R}^N)$ for $x \in \mathbf{R}^N, t \in \mathbf{R}$ and

$$\lim_{|t|\to\infty} (f(x,t)t - 2F(x,t)) = -\infty \quad \text{for each } x \in \mathbf{R}^N.$$

Theorem 3.16. *Assume* (D_1)–(D_3)*,* (E_1)–(E_4)*, and* (E_5) *(or* (E_6)*). Then there exists a* $\Lambda > 0$ *such that Equation (3.49) has a (nontrivial) sign-changing solution for each* $\lambda \geq \Lambda$*.*

By Condition (E_2), f is allowed to be jumping (or oscillating) between ν_1 and ν_k around zero in the sense that

$$f(x,t)/t \to a \quad \text{as } t \to 0^+, \qquad f(x,t)/t \to b \quad \text{as } t \to 0^-,$$

where $a, b \in (\nu_1, \nu_k)$. By assumption (E_4), the resonance might be happening at ν_k. Due to our assumptions (E_2) and (E_4), the energy functional does not have mountain pass geometry.

Then by (D_1), $E_\lambda \hookrightarrow H^1(\mathbf{R}^N)$ is continuous. Consider

$$G_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbf{R}^N} F(x,u)dx. \tag{3.109}$$

Then $G_\lambda \in \mathbf{C}^1(E_\lambda, \mathbf{R})$ and $G'_\lambda = \mathbf{id} - J_0$, where $J_0 := (-\Delta + V_\lambda)^{-1} f$. The weak solution of (3.49) corresponds to the critical point of G_λ.

By Proposition 3.7, for λ large enough, the operator $-\Delta + V_\lambda$ has at least d_k eigenvalues:

$$\mu_1(S_\lambda),\ \mu_2(S_\lambda),\ \ldots,\ \mu_{d_k}(S_\lambda)$$

with corresponding eigenfunctions $\phi_1(\lambda), \phi_2(\lambda), \ldots, \phi_{d_k}(\lambda)$ and $\|\phi_i(\lambda)\|_2 = 1$ for all $i = 1, \ldots, d_k$. Set

$$E_{d_k}(\lambda) := \operatorname{span}\{\phi_1(\lambda), \phi_2(\lambda), \ldots, \phi_{d_k}(\lambda)\}.$$

Lemma 3.17. *Assume (E_4). Then there exist $\Lambda_1 > 0, C_1 > 0$ such that*

$$G_\lambda(u) \le C_1 \quad \text{for all } u \in E_{d_k}(\lambda) := Y \quad \text{and} \quad \lambda > \Lambda_1.$$

Proof. It suffices to show that $G(u) \le 0$ for $u \in E_{d_k}(\lambda)$ and $\|u\|$ large enough. By a contradiction, we assume that there is a sequence $\{u_n\} \subset E_{d_k}(\lambda)$ with $\|u_n\|_\lambda \to \infty$ such that $G(u_n) > 0$. By (E_4), we write

$$F(x, u) = \frac{\theta(x)}{2}|u|^2 + P(x, u),$$

where $P(x, u) = o(|u|^2)$ uniformly for $x \in \mathbf{R}^N$ as $|u| \to \infty$. Furthermore, we observe that $\|u\|_\lambda^2 \le \mu_{d_k}(E_\lambda)|u|_2^2$ for all $u \in E_{d_k}(\lambda)$ and $\mu_{d_k}(E_\lambda) \le \nu_k$ for λ large enough. Moreover, by the standard elliptic theory (here we need $\dim E_{d_k}(\lambda) < \infty$ and the Schechter–Simon theorem (cf. Theorem 1.62) on the unique continuation property for Schrödinger operators), we may prove that there exists an $\varepsilon_0 > 0$ such that

$$\|u\|_\lambda^2 - \int_{\mathbf{R}^N} \theta(x)u^2 dx \le -\varepsilon_0 \|u\|_\lambda^2, \quad \text{for any } u \in E_{d_k}(\lambda).$$

Note $\dim E_{d_k}(\lambda) < \infty$; we assume that $u_n/\|u_n\|_\lambda \to w_0$ in $E_{d_k}(\lambda)$ with $\|w_0\|_\lambda = 1$. Then by (E_4),

$$\begin{aligned}
0 &< \frac{G(u_n)}{\|u_n\|_\lambda^2} \\
&= \frac{1}{2} - \frac{1}{2}\int_{\mathbf{R}^N} \theta(x)\frac{|u_n|^2}{\|u\|_\lambda^2} dx + \int_{\mathbf{R}^N} \frac{P(x, u_n)}{\|u\|_\lambda^2} dx \\
&\to \frac{1}{2} - \frac{1}{2}\int_{\mathbf{R}^N} \theta(x) w_0^2 dx + o(1) \\
&< -\frac{\varepsilon_0}{2} + o(1),
\end{aligned}$$

this is impossible. □

Lemma 3.18. *Assume (E_2). Then there exists a $\Lambda_2 > 0$ and a $\delta > 0$ independent of λ such that $G_\lambda(u) \ge c > 0$ for all $u \in E_{d_{k-1}}^{\perp}(\lambda)$ with $\|u\|_\lambda = \delta$ and all $\lambda \ge \Lambda_2$.*

Proof. By (E_2), there are $t_0 > 0, \nu^* < \nu_k$ such that $f(x,t)t \le \nu^* t^2$ for all $x \in \mathbf{R}^N$ and $|t| \le t_0$. Therefore,

$$2F(x,t) \le \nu^* t^2 \quad \text{for all } x \in \mathbf{R}^N, |t| \le t_0. \tag{3.110}$$

Furthermore, by (E_1),

$$2F(x,t) \le 2H_0 t^2 - H_0 t_0^2 \quad \text{for } |t| \ge t_0, x \in \mathbf{R}^N. \tag{3.111}$$

By Proposition 3.7 and (3.55),

$$\lim_{\lambda\to\infty} \mu_n(E_\lambda) = \mu_n; \qquad \lim_{n\to\infty} \mu_n \to \infty.$$

We first choose $\lambda > \Lambda^*$ such that $\mu_{d_k}(E_\lambda)$ approaches $\mu_{d_k} = \nu_k$; hence $\mu_{d_k}(E_\lambda) > \nu^*$ because $\nu_k > \nu^*$. Next, we choose λ large enough (say $\lambda > \Lambda^{**}$) such that the Schrödinger operator $-\Delta + V_\lambda$ has d_m eigenvalues $\mu_1(E_\lambda), \dots,\ \mu_{d_m}(E_\lambda)$. In particular, we may want d_m large enough so that

$$(2H_0 + \mu_{d_m}(E_\lambda) - 2\nu^*)(\mu_{d_k}(E_\lambda) - \nu^*) \ge 4\nu^*, \tag{3.112}$$

$$(\mu_{d_m}(E_\lambda) - 2H_0)(\mu_{d_k}(E_\lambda) - \nu^*) \ge 32\nu^*, \tag{3.113}$$

$$\mu_{d_m}(E_\lambda) - 2H_0 - 4|2H_0 - \nu^*| \ge 0, \tag{3.114}$$

$$\nu^* \mu_{d_m}(E_\lambda) > 2H_0 \mu_{d_k}(E_\lambda). \tag{3.115}$$

For any $u \in E^{\perp}_{d_{k-1}}(\lambda)$, we write $u = v + w$ with

$$v \in X_{d_k}(\lambda) \oplus X_{d_{k+1}}(\lambda) \oplus \cdots \oplus X_{d_m}(\lambda)$$

and $w \in E^{\perp}_{d_m}(\lambda)$, where d_m is given in (3.112)–(3.115) and $X_{d_i}(\lambda)$ $(i = k, \dots, m)$ is the eigenspace associated with $\mu_{d_i}(E_\lambda)$. Let

$$\iota_1 := \frac{(2H_0 + \mu_{d_m}(E_\lambda))}{4} w^2 + \frac{(\mu_{d_k}(E_\lambda) + \nu^*)}{4} v^2 - F(x,u). \tag{3.116}$$

If $|v + w| \le t_0$, then by (3.110) and (3.112) and the choice of $\mu_{d_m}(E_\lambda)$, we see that

$$\begin{aligned}
\iota_1 &\ge \frac{2H_0 + \mu_{d_m}(E_\lambda)}{4} w^2 + \frac{\mu_{d_k}(E_\lambda) + \nu^*}{4} v^2 - \frac{1}{2}\nu^*(v+w)^2 \\
&\ge \frac{2H_0 + \mu_{d_m}(E_\lambda) - 2\nu^*}{4} w^2 + \frac{\mu_{d_k}(E_\lambda) - \nu^*}{4} v^2 - \nu^*|vw| \\
&\ge \left(\frac{((2H_0 + \mu_{d_m}(E_\lambda) - 2\nu^*)(\mu_{d_k}(E_\lambda) - \nu^*))^{1/2}}{2} - \nu^* \right) |vw| \\
&\ge 0.
\end{aligned} \tag{3.117}$$

If $|v+w| > t_0$, then by (3.111), we conclude that

$$\begin{aligned}\iota_1 &\geq \Bigg(\frac{\mu_{d_m}(E_\lambda) - 2H_0}{4} w^2 + \frac{(\mu_{d_k}(E_\lambda) + \nu^*) - 4H_0}{4} v^2 \\ &\quad -2H_0 vw + \frac{H_0 t_0^2}{2}\Bigg) \\ &:= \iota_2 + \iota_3, \end{aligned} \tag{3.118}$$

where

$$\iota_2 := \frac{\mu_{d_m}(E_\lambda) - 2H_0}{8} w^2 + \frac{(\mu_{d_k}(E_\lambda) - \nu^*)}{4} v^2 - \nu^* vw, \tag{3.119}$$

$$\iota_3 := \frac{\mu_{d_m}(E_\lambda) - 2H_0}{8} w^2 - \frac{2H_0 - \nu^*}{2} v^2 - (2H_0 - \nu^*)vw + \frac{H_0 t_0^2}{2}. \tag{3.120}$$

Next, we estimate ι_2 and ι_3. If

$$\frac{(\mu_{d_k}(E_\lambda) - \nu^*)}{4}|v| - \nu^*|w| \geq 0,$$

then by (3.113),

$$\iota_2 \geq \frac{\mu_{d_m}(E_\lambda) - 2H_0}{8} w^2 + \left(\frac{\mu_{d_k}(E_\lambda) - \nu^*}{4}|v| - \nu^*|w|\right)|v| \geq 0. \tag{3.121}$$

Otherwise,

$$\frac{(\mu_k(E_\lambda) - \nu^*)}{4}|v| - \nu^*|w| \leq 0,$$

by the choice of $\mu_{d_m}(E_\lambda)$ in (3.113); we deduce that

$$\iota_2 \geq \left(\frac{\mu_{d_m}(E_\lambda) - 2H_0}{8} - \frac{4(\nu^*)^2}{\mu_{d_k}(E_\lambda) - \nu^*}\right) w^2 + \frac{\mu_{d_k}(E_\lambda) - \nu^*}{4} v^2 \geq 0. \tag{3.122}$$

On the other hand, by (3.114),

$$\begin{aligned}\iota_3 &\geq \frac{\mu_{d_m}(E_\lambda) - 2H_0}{8} w^2 - \frac{(2H_0 - \nu^*)}{2} v^2 - (2H_0 - \nu^*)vw + \frac{H_0 r_0^2}{2} \\ &\geq \frac{\mu_{d_m}(E_\lambda) - 2H_0 - 4|2H_0 - \nu^*|\nu^*}{8} w^2 \\ &\quad - \frac{2H_0 - \nu^* + |2H_0 - \nu^*|}{2} v^2 + \frac{H_0 t_0^2}{2} \\ &\geq -(1 + |2H_0 - \nu^*|)v^2 + \frac{H_0 t_0^2}{2}. \end{aligned} \tag{3.123}$$

Choose

$$\delta := \left(\frac{H_0 t_0^2}{2(1 + |2H_0 - \nu^*|)C_m^2}\right)^{1/2},$$

where C_m is a constant such that

$$\|v\|_\infty \le C_m \|v\|_\lambda$$

for all

$$v \in X_{d_k}(\lambda) \oplus X_{d_{k+1}}(\lambda) \oplus \cdots \oplus X_{d_m}(\lambda),$$

which is finite-dimensional. Now, $\|u\|_\lambda = \delta$; then

$$\|v\|_\infty \le C_m \|v\|_\lambda \le C_m \|u\|_\lambda = C_m \delta.$$

Hence, $\iota_3 \ge 0$. Therefore, by (3.118)–(3.123), $\iota_1 \ge 0$. Finally,

$$\begin{aligned}
G_\lambda(u) \\
&= G_\lambda(v+w) \\
&= \frac{1}{2}(\|v\|_\lambda^2 + \|w\|_\lambda^2) - \int_{\mathbf{R}^N} F(x,u)dx \\
&\ge \frac{1}{4}\|v\|_\lambda^2 + \frac{1}{4}\|w\|_\lambda^2 + \frac{1}{4}\mu_{d_k}(E_\lambda)|v|_2^2 + \frac{1}{4}\mu_{d_m}(E_\lambda)|w|_2^2 - \int_{\mathbf{R}^N} F(x,u)dx \\
&\ge \frac{1}{4}\left(1 - \frac{\nu^*}{\mu_{d_k}(E_\lambda)}\right)\|v\|_\lambda^2 + \frac{1}{4}\left(1 - \frac{2H_0}{\mu_{d_m}(E_\lambda)}\right)\|w\|_\lambda^2 + \int_{\mathbf{R}^N} \iota_1 dx \\
&\ge \frac{1}{4}\min\left\{\left(1 - \frac{\nu^*}{\mu_{d_k}(E_\lambda)}\right), \left(1 - \frac{2H_0}{\mu_{d_m}(E_\lambda)}\right)\right\}\|u\|_\lambda^2 \\
&\ge \frac{1}{4}\left(1 - \frac{\nu^*}{\mu_{d_k}(E_\lambda)}\right)\delta^2 \ \ \text{(by (3.115))} \\
&> 0.
\end{aligned}$$

□

Now, we choose

$$M := E_{d_k}^\perp, \qquad z_0 \in E_{d_{k-1}}^\perp \backslash E_{d_k}^\perp \quad \text{with } \|z_0\|_\lambda = 1,$$

$$B := \{u \in M : \|u\|_\lambda \ge \delta\} \cup \{u = sz_0 + v : s \ge 0, v \in M, \|u\|_\lambda = \delta\}, \tag{3.124}$$

where δ comes from Lemma 3.18.

Lemma 3.19. *There exists a* $\Lambda_3 > \max\{\Lambda_1, \Lambda_2\}$ *such that* $\inf_M G_\lambda \ge 0$ *and* $\inf_B G_\lambda > c > 0$ *for all* $\lambda > \Lambda_3$.

Proof. For any $u \in M$ with $\|u\|_\lambda \ge \delta$, by (E_3),

$$G_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbf{R}^N} F(x,u)dx$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\kappa}{2}\int_{\mathbf{R}^N} u^2 dx \\
&\geq \frac{1}{2}\left(1 - \frac{\kappa}{\mu_{d_{k+1}}(E_\lambda)}\right)\|u\|_\lambda^2 \\
&\geq \frac{1}{3}\left(1 - \frac{\kappa}{\nu_{k+1}}\right)\|u\|_\lambda^2 \\
&\geq \frac{1}{3}\left(1 - \frac{\kappa}{\nu_{k+1}}\right)\delta^2 \\
&> 0.
\end{aligned}
$$

Combining Lemma 3.18, we have $\inf_B G_\lambda > c > 0$. The proof also implies that $\inf_M G_\lambda \geq 0$. □

Lemma 3.20. *Under the assumptions of Theorem 3.16, there exists a $\Lambda_4 > 0$ such that G_λ satisfies the (w^*-PS) condition for each $\lambda \geq \Lambda_4$.*

Proof. Let $\{u_n\}$ be a (w^*-PS) sequence:

$$G'_\lambda(u_n) \to 0, \qquad G_\lambda(u_n) \to c.$$

We assume that $\{\|u_n\|_\lambda \|G'_\lambda(u_n)\|_\lambda\}$ is bounded (otherwise, we are done). We are going to show that $\{\|u_n\|_\lambda\}$ is bounded and has a convergent subsequence. Note that

$$
\begin{aligned}
&\left|G_\lambda(u_n) - \frac{1}{2}\langle G'_\lambda(u_n), u_n\rangle_\lambda\right| \\
&= \left|\int_{\mathbf{R}^N}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx\right| \\
&< c
\end{aligned}
\tag{3.125}
$$

and

$$
\begin{aligned}
\frac{1}{2}\|u_n\|_\lambda^2 &\leq c + \int_{\mathbf{R}^N} F(x,u_n)dx \\
&\leq c + c\int_{\mathbf{R}^N} |u_n|^2 dx.
\end{aligned}
$$

If $\{\|u_n\|_\lambda\}$ is unbounded, then, for a renamed subsequence,

$$1 \leq c \lim_{n\to\infty}\int_{\mathbf{R}^N}\frac{u_n^2}{\|u_n\|_\lambda^2}dx. \tag{3.126}$$

By Lemma 3.9, for any $\varepsilon > 0$ there exists an $R > 0$ and a $\Lambda_4 > 0$ such that

$$\|v\|_{L^2(B_R^c)}^2 \leq \varepsilon\|v\|_\lambda^2, \qquad \forall\, v \in E_\lambda, \qquad \lambda \geq \Lambda_4, \tag{3.127}$$

where $B_R^c := \{x \in \mathbf{R}^N : |x| > R\}$. Applying (3.126) to (3.127), we may find $R > 0$ such that

$$\lim_{n\to\infty} \int_{|x|\le R} \frac{u_n^2}{\|u_n\|_\lambda^2} dx > c > 0.$$

It follows that $\lim_{n\to\infty} |u_n|^2 = \infty$ on a subset Ω with positive measure. Combining this with (E_5) or (E_6), we have

$$\left| \int_{\mathbf{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n) \right) dx \right| \to \infty,$$

which contradicts (3.125). Thus, we see that $\{\|u_n\|_\lambda\}$ is bounded. Next, we show that $\{u_n\}$ has a convergent subsequence. Suppose that $u_n \to u$ weakly in E_λ and $u_n \to u$ strongly in $L^2_{\text{loc}}(\mathbf{R}^N)$ for some $u \in E_\lambda$. Then $G'_\lambda(u) = 0$. Recall that $\|G'_\lambda(u_n)\| \to 0$; then

$$\begin{aligned}
\|u_n - u\|_\lambda^2 &= o(1) + \int_{\mathbf{R}^N} (f(x,u_n) - f(x,u))(u_n - u) dx \\
&\le o(1) + H_0 \int_{|x|\ge R} (|u_n| + |u|)|u_n - u| dx \\
&\quad + \int_{|x|\le R} (f(x,u_n) - f(x,u))(u_n - u) dx \\
&\le o(1) + H_0 \int_{|x|\ge R} |u||u_n - u|^2 dx + 2H_0 \int_{|x|\ge R} |u_n - u|^2 dx \\
&\quad + \int_{|x|\le R} (f(x,u_n) - f(x,u))(u_n - u) dx \\
&\le o(1) + \frac{1}{2}\|u_n - u\|_\lambda^2 + 2H_0 \|u_n - u\|_\lambda \left(\int_{|x|\ge R} |u|^2 dx \right)^{1/2} \\
&\quad + \int_{|x|\le R} (f(x,u_n) - f(x,u))(u_n - u) dx.
\end{aligned}$$

It implies that we may make $\|u_n - u\|_\lambda$ small enough by choosing R, n large enough; that is, $\|u_n - u\|_\lambda \to 0$. □

Let P_λ^* and $\mathcal{D}_0(\varepsilon, \lambda)$ be as in (3.92) and (3.94).

Lemma 3.21. *Under the assumptions of (D_1)–(D_3) and (E_1), there exists a $\Lambda_5 > 0$ such that*

$$\operatorname{dist}((E_1(\lambda))^\perp, \pm P_\lambda^*) > 0, \quad \textit{for all } \lambda \ge \Lambda_5. \tag{3.128}$$

Proof. We just prove the case of "+"; the other case is analogous. If

$$\operatorname{dist}((E_1(\lambda))^{\perp}, P_{\lambda}^*) = 0,$$

then there would exist $\{w_n\} \subset (E_1(\lambda))^{\perp}$, $\{e_n\} \subset P_{\lambda}^*$ such that $\operatorname{dist}(w_n, e_n) \to 0$ as $n \to \infty$. Then

$$\langle e_n, \phi_1(\lambda)\rangle_{\lambda} = \langle e_n - w_n, \phi_1(\lambda)\rangle_{\lambda} + \langle w_n, \phi_1(\lambda)\rangle_{\lambda} \to 0 \quad \text{as } n \to \infty.$$

However, $e_n \geq \phi_1(\lambda)$ implies that

$$\langle e_n, \phi_1(\lambda)\rangle_{\lambda} = \mu_1(E_{\lambda})\langle e_n, \phi_1(\lambda)\rangle_{L^2} \geq \mu_1(E_{\lambda}) \int_{\mathbf{R^N}} (\phi_1(\lambda))^2 dx > 0,$$

a contradiction. □

Proof of Theorem 3.16. By Lemmas 3.17 and 3.18, we have

$$\inf_{M} G_{\lambda} \geq 0, \qquad \sup_{Y} G_{\lambda} < \infty$$

for λ large enough. Choose $\varepsilon \in \big(0, \ \operatorname{dist}((E_1(\lambda))^{\perp}, P_{\lambda}^*)\big)$ small enough. Let

$$\mathcal{D}_0^{(1)} := \mathcal{D}_0(\varepsilon), \qquad \mathcal{D}_0^{(2)} := -\mathcal{D}_0(\varepsilon), \qquad \mathcal{S} = E\backslash\mathcal{W}, \qquad \mathcal{W} := \mathcal{D}_0^{(1)} \cup \mathcal{D}_0^{(2)};$$

then Lemma 3.21 implies that (A_1) and (A_2) (cf. (3.124)) of Theorem 3.4 hold. Therefore, we have a critical point $u_1 \in \mathcal{S}$ with $G_{\lambda}(u_1) \geq \inf_B G_{\lambda} > 0$; then u_1 is sign-changing. □

Notes and Comments. After the paper by Bartsch et al. [38], there are some papers on (3.49). In van Heerden and Wang [168] and van Heerden [166], the authors studied using the mountain pass theorem, the existence of one positive solution to (3.49) with asymptotically linear nonlinearities and of multiple solutions if $f(x,t)$ is odd in t. In van Heerden [166], under the assumptions that $f(x,t)$ is odd in t and that

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} := \xi(0) \in (\nu_m, \nu_{m+1}),$$

$$\lim_{|t|\to \infty} \frac{f(x,t)}{t} := \xi(\infty) \in (\nu_k, \nu_{k+1})$$

uniformly for $x \in \mathbf{R}^N$, where $k \neq m$, the authors obtained multiple solutions. In Liu et al. [210] it was proved by the genus-method of the even functional (see also Li and Wang [199]) those solutions obtained in van Heerden [166] are sign-changing. If $\min\{k, m\} = 0$ ($\nu_0 := -\infty$), under which the mountain pass theorem can be applied readily, a positive and a negative solution are also obtained. The existence of a solution to asymptotically linear scalar field equations was considered in Stuart and Zhou [310] and Li and Zhou [194].

Theorem 3.16 was originally obtained in Zou [347] where an alternative result was given.

The first paper where modern global variational methods were employed in order to find homoclinic type solutions (for a Hamiltonian system) seems to be Coti Zelati et al. [120]. The Hamiltonian considered there was strictly convex and superlinear. Subsequently multibump type solutions for this system have been found in Séré [290, 291]. Existence of multibump type solutions has been shown in Coti Zelati and Rabinowitz [121] (see also [123]) for the second-order Hamiltonian systems and [122] for a semilinear elliptic PDE on $\mathbf{R}^N$ with periodic potentials and nonlinearities under the assumption of the superlinearity condition. In these papers the multibumps have been obtained starting from a mountain pass point at a level c, under the assumption that there are only finitely many geometrically distinct homoclines below a somewhat higher level $c + \varepsilon$. In the very recent paper of Arioli et al. [20], it is shown that a multibump construction can be carried out from any isolated homocline having a nontrivial critical group.

If the potential V and the nonlinearity f are periodic in variable x, we also refer readers to van Heerden [167], Kryszewski and Szulkin [185], Liu and Wang [212], Schechter and Zou [281, 284], Troestler and Willem [329], Willem [335], and Willem and Zou [336] for the existence results. In particular, in [122, 212] the authors obtained infinitely many sign-changing solutions. Interested readers may also consult the following papers for homoclinic orbit problems for Hamiltonian systems and Schrödinger equations; they are Ackermann [4] (multibump solutions by using nontrivial local degree), Ackermann and Weith [5] (multibump solutions for periodic Schrödinger equations in a degenerate setting), Bartsch and Ding [33, 34] (no multibump type solutions), Rabinowitz [258] (handbook), Rabinowitz and Tanaka [259], Alama and Li [7] (on multibump bound states), Li and Wang [206] and Szulkin and Zou [320], among others. We also mention the paper by Arioli et al. [19] where multibump solutions have been found for an infinite lattice of particles (a Fermi–Pasta–Ulam type problem). The paper of Berti and Bolle [60] considers the homoclines and chaotic behavior for perturbed second-order ODE systems and PDEs. For the Schrödinger equation

$$-\Delta u + V(x)u = |u|^{2^*-2}u, \qquad x \in \mathbf{R}^N, \qquad u \in \mathbf{R}, \tag{3.129}$$

where 2^* is the critical Sobolev exponent, V is 1-periodic in $x_1, \ldots, x_N$ and the spectrum $\sigma(-\Delta + V) \subset (0, \infty)$, the first result is due to Arioli et al. [20]. They show that if V changes sign and $N \geq 4$, then (3.129) has a solution $u \neq 0$ which is a minimizer for the associated functional on the Nehari manifold. Moreover, there exist multibumps whenever this solution is isolated. Hence, it implies that (3.129) always has infinitely many solutions that are geometrically distinct. The nonlinear term can be much more general and not odd there. Some computations on nontrivial critical groups are given in [20].

Chapter 4
On a Brezis–Nirenberg Theorem

4.1 Introduction

Let E be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Assume that E has an orthogonal decomposition $E = Y \oplus M$ with $\dim Y < \infty$. In Brezis and Nirenberg [70], it is assumed that G is a $\mathbf{C}^1$-functional on E satisfying the Palais–Smale condition. Suppose that there is a continuous map p^* of the boundary of the half ball:

$$K := \{u = sv_0 + w : w \in Y, s \geq 0, \|u\| \leq R\}, \qquad R > 0,$$

into E, where v_0 is a fixed unit vector in M, with the following properties,

$$p^*(u) = u, \qquad \forall\, u \in Y, \qquad \|u\| \leq R;$$
$$\|p^*(u)\| \geq r_0 > 0, \qquad \forall\, u \in K, \qquad \|u\| = R$$

and $G(p^*(u)) \leq 0$ for all $u \in \partial K$. Assume furthermore that for some positive $\rho < r_0$,

$$G(u) \geq 0 \quad \text{for } u \in M, \|u\| = \rho.$$

Then G has a nonzero critical point u_0 where $G(u_0) \geq 0$.

The question is when will this critical point be sign-changing? In this chapter we are concerned with this problem on the location and nodal structure of the critical point. More precisely, we generalize Brezis and Nirenberg's result (cf. [70]) by giving a sufficient condition on the existence of sign-changing critical points.

4.2 Generalized Brezis–Nirenberg Theorems

Let $G \in \mathbf{C}^1(E, \mathbf{R})$ and the gradient G' be of the form

$$G'(u) = u - \Theta_G(u), \tag{4.1}$$

W. Zou, *Sign-Changing Critical Point Theory*, doi: 10.1007/978-0-387-76658-4,

where $\Theta_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E\backslash\mathcal{K}$. Let $\mathcal{D}_0$ be an open convex subset of E. Denote $\mathcal{S} := E\backslash\mathcal{D}$, $\mathcal{D} := -\mathcal{D}_0 \cup \mathcal{D}_0$. Assume

($\mathbf{A_1}$) $\Theta_G(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_0$.

Lemma 4.1. *Assume that* (A_1) *holds. Let* $G \in \mathbf{C}^1(E, R)$ *and let* B, $M^\star$ *be two closed and disjoint subsets of* E. *Assume that* $M^\star$ *is compact and* $\|G'(u)\| \geq \delta > 0$ *for all* $u \in M^\star$. *Then there exists a deformation* $\psi \in \mathbf{C}([0, +\infty) \times E,\ E)$ *satisfying*

(1) $\psi(t, u) = u$ *for all* $u \in B$ *and* $t \geq 0$; $\psi(0, u) = u$ *for all* $u \in E$.
(2) $\|\psi(t, u) - u\| \leq t$ *for all* $u \in E$ *and* $t \geq 0$.
(3) There exists a $t_0 > 0$ *and an open neighborhood* $\mathcal{U}_{M^\star}$ *of* $M^\star$ *such that*

$$G(\psi(t, u)) - G(u) \leq -\frac{\delta^2}{8 + 2\delta^2} t$$

for all $u \in \mathcal{U}_{M^\star}$ *and* $t \in [0, t_0]$.
(4) $\psi([0, +\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$; $\quad \psi([0, +\infty), \mathcal{D}) \subset \mathcal{D}$.

Proof. Let $\delta_1 > 0$ and

$$M_1 := \{u \in E : \mathrm{dist}(u, M^\star) < \delta_1\},$$

$$M_2 := \{u \in E : \mathrm{dist}(u, M^\star) < \delta_1/2\}.$$

Because $M^\star$ is compact, we may take a $\delta_1 > 0$ small enough such that

$$\|G'(u)\| \geq \delta/2 \quad \text{for all } u \in \bar{M}_1; \qquad \bar{M}_1 \cap B = \emptyset; \tag{4.2}$$

here $\bar{M}_1$ is the closure of M_1. Let

$$\theta(u) = \frac{\mathrm{dist}(u, E\backslash M_1)}{\mathrm{dist}(u, E\backslash M_1) + \mathrm{dist}(u, M_2)}.$$

Define

$$W(u) := \begin{cases} \theta(u)\dfrac{V(u)}{1 + \|V(u)\|^2}, & \text{for } u \in \tilde{E}, \\ 0, & \text{for } u \in \mathcal{K}, \end{cases} \tag{4.3}$$

where $V(u)$ is the pseudo-gradient vector field provided by Lemma 2.12. Note that if $u \in \partial\mathcal{K}$, then $u \notin \bar{M}_1$ by (4.2) because $\mathcal{K}$ is closed. Hence we may find a neighborhood $\mathcal{U}_u$ of u such that $\mathcal{U}_u \subset E\backslash\bar{M}_1 \subset E\backslash M_1$ and $\theta(\mathcal{U}_u) = 0$. Then W is a locally Lipschitz continuous vector field from E to E. Moreover, $\|W(u)\| \leq 1$ on E for all $u \in E$.

Consider the following Cauchy initial value problem

$$\frac{d\psi(t,u)}{dt} = -W(\psi(t,u)), \qquad \psi(0,u) = u \in E. \tag{4.4}$$

By Theorem 1.41, (4.4) has a unique continuous solution $\psi\colon [0,\infty) \times E \to E$. Evidently, $\psi(t,u) = u$ for all $u \in B$ and $t \geq 0$ and $\|\psi(t,u) - u\| \leq t$ for all $u \in E$ and $t \geq 0$. Choose

$$\mathcal{U}_{M^\star} := \{u \in E : \operatorname{dist}(u, M^\star) < \delta_1/10\},$$

which is an open neighborhood of $M^\star$. For any $u \in \mathcal{U}_{M^\star}$ and $0 \leq t \leq t_0 := \delta_1/3$, choose $w \in M^\star$ such that $\|u - w\| \leq \delta_1/8$. Then

$$\operatorname{dist}(\psi(t,u), M^\star) \leq \|\psi(t,u) - w\| \leq 11\delta_1/24 < \delta_1/2.$$

Therefore, $\psi((0,\delta_1/3], \mathcal{U}_{M^\star}) \subset M_2$, and hence, $\mathcal{U}_{M^\star} \subset M_2$, $\theta(\psi(t,u)) = 1$, and $\|G'(\psi(t,u))\| \geq \delta/2$ for all $t \in [0,t_0]$ and $u \in \mathcal{U}_{M^\star}$. Hence,

$$\begin{aligned}
&G(\psi(t,u)) - G(u) \\
&\quad = \int_0^t \frac{dG(\psi(s,u))}{ds} ds \\
&\quad \leq -\int_0^t \theta(\psi(s,u)) \langle G'(\psi(s,u)), \frac{V(\psi(s,u))}{1 + \|V(\psi(s,u))\|^2} \rangle ds \\
&\quad = -\frac{1}{2} \int_0^t \frac{\|G'(\psi(s,u))\|^2}{1 + \|G'(\psi(s,u))\|^2} ds \\
&\quad \leq -\frac{\delta^2 t}{8 + 2\delta^2}
\end{aligned}$$

for all $t \in [0,t_0]$ and $u \in \mathcal{U}_{M^\star}$. Finally, we show that

$$\psi([0,+\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \psi([0,+\infty), \mathcal{D}) \subset \mathcal{D}.$$

The idea is similar to that in Theorem 2.13. By Lemma 2.12, we first observe that

$$O(\pm\mathcal{D}_0 \cap \tilde{E}) \subset (\pm\mathcal{D}_0) \Rightarrow O(\pm\bar{\mathcal{D}}_0 \cap \tilde{E}) \subset (\pm\bar{\mathcal{D}}_0).$$

Because $\mathcal{K} \subset E \backslash M_1, \psi(t,u) = u$ for all $t \geq 0$ and $u \in \bar{\mathcal{D}} \cap \mathcal{K}$. Assume that $u \in \bar{\mathcal{D}}_0 \cap \tilde{E}$. If there is a $T_0 > 0$ such that $\psi(T_0,u) \notin \bar{\mathcal{D}}_0$, then we may find a number $s_0 \in [0,T_0)$ such that $\psi(s_0,u) \in \partial\bar{\mathcal{D}}_0$ and $\psi(t,u) \notin \bar{\mathcal{D}}_0$ for $t \in (s_0,T_0]$. Consider the following initial value problem

$$\begin{cases} \dfrac{d\psi(t,\psi(s_0,u))}{dt} = -W(\psi(t,\psi(s_0,u))), \\ \psi(0,\psi(s_0,u)) = \psi(s_0,u) \in E. \end{cases}$$

It has a unique solution $\psi(t, \psi(s_0, u))$. For any $v \in \bar{\mathcal{D}}_0$, if $v \in \mathcal{K}$, then $W(v) = 0$. Hence, $v + \lambda(-W(v)) = v \in \bar{\mathcal{D}}_0$. Assume that $v \in \tilde{E} \cap \bar{\mathcal{D}}_0$. By Lemma 2.12 and noting that $\bar{\mathcal{D}}_0$ is convex, we have

$$\begin{aligned}
&v + \rho(-W(v)) \\
&\quad = v - \rho\theta(v)\frac{V(v)}{1 + \|V(u)\|^2} \\
&\quad = v + \rho\left(-\frac{\theta(v)}{1 + \|V(v)\|^2}\right)(v - O(v)) \\
&\quad = \left(1 - \frac{\rho\theta(v)}{1 + \|V(v)\|^2}\right)v + \frac{\rho\theta(v)}{1 + \|V(v)\|^2}O(v)
\end{aligned}$$

for ρ small enough. It implies that $v + \rho(-W(v)) \in \bar{\mathcal{D}}_0$ for $\rho > 0$ small enough. Summing up, we have

$$\lim_{\rho \to 0^+} \frac{\operatorname{dist}(v + \rho(-W(v)), \bar{\mathcal{D}}_0)}{\rho} = 0, \qquad \forall v \in \bar{\mathcal{D}}_0.$$

By Lemma 1.49, there exists an $\varepsilon > 0$ such that $\psi(t, \psi(s_0, u)) \in \bar{\mathcal{D}}_0$ for all $t \in [0, \varepsilon)$. By the semigroup property, we see that $\psi(t, u) \in \bar{\mathcal{D}}_0$ for all $t \in [s_0, s_0 + \varepsilon)$, which contradicts the definition of s_0. Therefore, $\psi([0, +\infty), \bar{\mathcal{D}}_0) \subset \bar{\mathcal{D}}_0$. Similarly, $\psi([0, +\infty), -\bar{\mathcal{D}}_0) \subset -\bar{\mathcal{D}}_0$. Consequently, $\psi([0, +\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$. To prove $\psi([0, +\infty), \mathcal{D}) \subset \mathcal{D}$, we just show that $\psi([0, +\infty), \mathcal{D}_0) \subset \mathcal{D}_0$ by a contradiction. Assume that there exists a $u^* \in \mathcal{D}_0, T_0 > 0$ such that $\psi(T_0, u^*) \notin \mathcal{D}_0$. Choose a neighborhood $\mathcal{U}_{u^*}$ of u^* such that $\mathcal{U}_{u^*} \subset \bar{\mathcal{D}}_0$. Then by the theory of ordinary equations in Banach space, we may find a neighborhood $\mathcal{U}_{T_0}$ of $\psi(T_0, u^*)$ such that $\psi(T_0, \cdot) : \mathcal{U}_{u^*} \to \mathcal{U}_{T_0}$ is a homeomorphism. Because $\psi(T_0, u^*) \notin \mathcal{D}_0$, we take a $w \in \mathcal{U}_{T_0} \backslash \bar{\mathcal{D}}_0$. Correspondingly, we find a $v \in \mathcal{U}_{u^*}$ such that $\psi(T_0, v) = w$; this contradicts the fact that $\psi([0, \infty), \bar{\mathcal{D}}_0) \subset \bar{\mathcal{D}}_0$. □

Definition 4.2. Let B be a closed subset of E. Define a class $\mathcal{F}$ of compact subsets of E satisfying

(1) $A \cap \mathcal{S} \neq \emptyset$ for all $A \in \mathcal{F}$.
(2) For any $\sigma \in \mathbf{C}([0, 1] \times E, E)$ satisfying $\sigma(t, x) = x$ for all (t, x) in $(\{0\} \times E) \cup ([0, 1] \times B)$, there holds $\sigma(1, A) \in \mathcal{F}$ for any $A \in \mathcal{F}$.

Class $\mathcal{F}$ is called a homotopy-stable family with extended boundary B.

Lemma 4.3. *Assume (A_1). Let B be a closed subset of E and assume that it has a homotopy-stable family with extended boundary B. Define*

$$c = \inf_{A \in \mathcal{F}} \sup_{A \cap \mathcal{S}} G$$

and assume that there is a closed subset Π of E with

$$A \cap \Pi \cap \mathcal{S} \backslash B \neq \emptyset, \qquad \forall A \in \mathcal{F}$$

and

$$\sup_B G \leq c \leq \inf_\Pi G.$$

Then there exists a sign-changing (PS) sequence $\{u_m\}$ such that

$$G'(u_m) \to 0, \qquad G(u_m) \to c, \qquad u_m \in \mathcal{S}, \qquad \mathrm{dist}(u_m, \Pi) \to \infty.$$

Proof. For any $\varepsilon \in (0, 100^{-10})$, there is an $A \in \mathcal{F}$ such that

$$c \leq \sup_{A \cap \mathcal{S}} G \leq c + \varepsilon. \tag{4.5}$$

Let

$$\Pi_\varepsilon := \{u \in E : \mathrm{dist}(u, \Pi) < \varepsilon^{1/2}\}.$$

Let $\mathcal{L} \subset C([0,1] \times E, E)$ be the set of mappings ψ satisfying

$$\psi(t,u) = u, \qquad \forall\ (t,u) \in (\{0\} \times E) \cup ([0,1] \times ((A \backslash \Pi_\varepsilon) \cup B))$$

and

$$\sup_{(t,u) \in [0,1] \times E} \|\psi(t,u) - u\| < \infty.$$

Then $\mathcal{L}$ is a complete metric space equipped with the metric

$$\mathbf{d}(\psi, \psi') = \sup\{\|\psi(t,u) - \psi'(t,u)\| : (t,u) \in [0,1] \times E\}.$$

Consequently, $\psi(1, A) \subset \mathcal{F}$ for any $\psi \in \mathcal{L}$. Define

$$\phi_1(u) = \max\{0, \varepsilon - \varepsilon^{1/2} \mathrm{dist}(u, \Pi)\}, \tag{4.6}$$

$$\phi_2(u) = \varepsilon \min\{1, \mathrm{dist}(u, (A \backslash \Pi_\varepsilon) \cup B)\}, \tag{4.7}$$

$$G_0 = G + \phi_1 + \phi_2.$$

Also, define $\Theta : \mathcal{L} \to \mathbf{R}$ by

$$\Theta(\psi) = \sup G_0(\psi(1, A) \cap \mathcal{S}).$$

Then Θ is a lower semicontinuous function on $\mathcal{L}$. Note that

$$\Theta(\psi) = \sup_{\psi(1,A) \cap \mathcal{S}} (G + \phi_1 + \phi_2) \tag{4.8}$$

$$\geq \sup_{\psi(1,A) \cap \mathcal{S} \cap \Pi} (G + \phi_1) \tag{4.9}$$

$$\geq c + \varepsilon.$$

Therefore, $d := \inf_{\psi\in\mathcal{L}} \Theta \geq c+\varepsilon$. Let $\bar{\psi} = \mathbf{id}$ (i.e., $\bar{\psi}(t,u) = u, \forall(t,u)$); then $\bar{\psi} \in \mathcal{L}$. By (4.5)–(4.7),

$$d \leq \Theta(\bar{\psi}) = \sup_{A\cap\mathcal{S}}(G+\phi_1+\phi_2) \leq c+3\varepsilon.$$

Hence,

$$\Theta(\bar{\psi}) \leq \inf_{\psi\in\mathcal{L}} \Theta + 2\varepsilon.$$

By Ekeland's variational principle (cf. Lemma 1.4), we have a $\psi_0 \in \mathcal{L}$ such that $\Theta(\psi_0) \leq \Theta(\bar{\psi}), \mathbf{d}(\psi_0,\bar{\psi}) \leq 4\varepsilon^{1/2}$ and

$$\Theta(\psi) \geq \Theta(\psi_0) - \varepsilon^{1/2}\mathbf{d}(\psi,\psi_0)/2, \qquad \forall\psi\in\mathcal{L}. \tag{4.10}$$

Let

$$M^\star := \{u \in \psi_0(1,A)\cap\mathcal{S} : G_0(u) = \Theta(\psi_0)\}.$$

Because A is compact and $\psi_0(1,A)\cap\mathcal{S} \neq \emptyset$, $M^\star \neq \emptyset$. Next we show that

$$M^\star \cap ((A\backslash\Pi_\varepsilon)\cup B) = \emptyset.$$

In as much as

$$\psi_0(1,A)\cap\Pi\cap\mathcal{S}\backslash B \neq \emptyset,$$

we may find a u_0 in $\Pi \subset \Pi_\varepsilon$ satisfying $\phi_2(u_0) > 0$. For any $u \in M^\star$, we have

$$\begin{aligned} G_0(u) &= \max_{\psi_0(1,A)\cap\mathcal{S}} G_0 \\ &\geq \max_{\psi_0(1,A)\cap\Pi\cap\mathcal{S}} G_0 \\ &= \max_{\psi_0(1,A)\cap\Pi\cap\mathcal{S}} (G+\phi_1+\phi_2) \\ &\geq c+\varepsilon+\phi_2(u_0). \end{aligned} \tag{4.11}$$

On the other hand, for any $u \in (A\backslash\Pi_\varepsilon)\cup B$, we may assume that $u \in \mathcal{S}$ (otherwise, $u \notin M^\star$).

If $u \in (A\backslash\Pi_\varepsilon)\cap\mathcal{S}$, we see that $\phi_1(u) = \phi_2(u) = 0$. Then by (4.5),

$$G_0(u) = G(u) \leq \sup_{A\cap\mathcal{S}} G \leq c+\varepsilon. \tag{4.12}$$

If $u \in B\cap\mathcal{S}$, then by (4.6) and (4.7), $G_0(u) \leq c+\varepsilon$. Both cases imply that

$$\sup_{((A\backslash\Pi_\varepsilon)\cup B)\cap\mathcal{S}} G_0 \leq c+\varepsilon.$$

By (4.11) and (4.12),

$$M^\star \cap ((A\backslash\Pi_\varepsilon)\cup B) = \emptyset. \tag{4.13}$$

Next, we show that there exists a $u_\varepsilon \in M^\star$ such that $\|G'(u_\varepsilon)\| \leq \varepsilon^{1/5}$ and $c - \varepsilon \leq G(u_\varepsilon) \leq c + 3\varepsilon$. In particular, $u_\varepsilon \in \mathcal{S}$ and $\operatorname{dist}(u_\varepsilon, \varPi) \leq 5\varepsilon^{1/2}$.

First we note that any $u_\varepsilon \in M^\star$ implies that $u_\varepsilon \in \psi_0(1, A), u_\varepsilon \in \mathcal{S}$, and

$$G_0(u_\varepsilon) = \max G_0(\psi_0(1, A) \cap \mathcal{S}) = \varTheta(\psi_0) \geq d$$

and $\varTheta(\psi_0) \leq \varTheta(\bar{\psi}) \leq c + 3\varepsilon$. Then

$$c + \varepsilon \leq G_0(u_\varepsilon) = G(u_\varepsilon) + \phi_1(u_\varepsilon) + \phi_2(u_\varepsilon) \leq c + 3\varepsilon;$$

it implies that

$$c - \varepsilon \leq G(u_\varepsilon) \leq c + 3\varepsilon.$$

Choose $w \in A$ such that $u_\varepsilon = \psi_0(1, w)$. By the definition of ψ_0 and (4.13), we see that $w \in \varPi_\varepsilon$. Hence, $\operatorname{dist}(w, \varPi) < \varepsilon^{1/2}$, because $d(\psi_0, \bar{\psi}) < 4\varepsilon^{1/2}$. Hence, $\operatorname{dist}(u_\varepsilon, \varPi) < 5\varepsilon^{1/2}$.

By way of contradiction, assume that $\|G'(u)\| > \varepsilon^{1/5}$ for all $u \in M^\star$. We apply Lemma 4.1 to the set $M^\star$ and $B \cup (A \backslash \varPi_\varepsilon)$; we have a $\sigma \in C([0,1] \times E, E)$, $t_0 > 0$ (assume $t_0 < 1$) and an open neighborhood $\mathcal{U}_{M^\star}$ of $M^\star$ such that

(1) $\sigma(0, u) = u$ for all $u \in E$.
(2) $\sigma(t, u) = u$ for all $u \in B \cup (A \backslash \varPi_\varepsilon)$ for all $t \geq 0$.
(3) $\|\sigma(t, u) - u\| \leq t$ for all $u \in E$ for all $t \geq 0$.
(4) $G(\sigma(t, u)) - G(u) \leq -(\varepsilon^{2/5}/(8 + 2\varepsilon^{2/5}))t$ for all $u \in \mathcal{U}_{M^\star}$ and $t \in [0, t_0]$.

In addition, we have

$$\sigma(t, \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \sigma(t, \mathcal{D}) \subset \mathcal{D}, \qquad \forall t \in [0, \infty). \tag{4.14}$$

Define

$$\psi_\lambda(t, u) = \sigma(t\lambda, \psi_0(t, u)), \qquad \lambda \in [0, 1].$$

Then $\psi_\lambda(t, u) \in \mathcal{L}$ and

$$\mathbf{d}(\psi_\lambda, \psi_0) = \sup\{\|\psi_\lambda(t, u) - \psi_0(t, u)\| : u \in E, t \in [0, 1]\} \leq \lambda.$$

Hence, by (4.10),

$$\varTheta(\psi_\lambda) \geq \varTheta(\psi_0) - \varepsilon^{1/2}\lambda/2.$$

Because

$$\varTheta(\psi_\lambda) = \sup G_0(\psi_\lambda(1, A) \cap \mathcal{S}) \tag{4.15}$$

and A is compact, there is a $u_\lambda \in A$ such that $\varTheta(\psi_\lambda) = G_0(\psi_\lambda(1, u_\lambda))$ and $\psi_\lambda(1, u_\lambda) = \sigma(\lambda, \psi_0(1, u_\lambda)) \in \mathcal{S}$. Furthermore, by (4.14), $\psi_0(1, u_\lambda) \in \mathcal{S}$. Consequently,

$$G_0(\psi_\lambda(1, u_\lambda)) \geq \varTheta(\psi_0) - \varepsilon^{1/2}\lambda/2 \tag{4.16}$$

$$
\begin{aligned}
&= \sup_{\psi_0(1,A)\cap\mathcal{S}} G_0 - \varepsilon^{1/2}\lambda/2 \\
&\geq G_0(\psi_0(1,u)) - \varepsilon^{1/2}\lambda/2
\end{aligned}
$$

for all $u \in A$ with $\psi_0(1,u) \in \mathcal{S}$. Because A is compact, we may assume that $u_\lambda \to u_0 \in A$ as $\lambda \to 0$. Then $\psi_0(1,u_0) \in \mathcal{S}$. Moreover, by (4.16),

$$G_0(\psi_0(1,u_0)) = \sup_{\psi_0(1,A)\cap\mathcal{S}} G_0$$

and therefore, $\psi_0(1,u_0) \in M^\star$. It follows that $\psi_0(1,u_\lambda) \in \mathcal{U}_{M^\star}$ for λ small enough. Hence,

$$
\begin{aligned}
(4.17)\qquad & G(\psi_\lambda(1,u_\lambda)) - G(\psi_0(1,u_\lambda)) \\
&= G(\sigma(\lambda,\psi_0(1,u_\lambda))) - G(\psi_0(1,u_\lambda)) \\
&\leq -\frac{\varepsilon^{2/5}\lambda}{8+2\varepsilon^{2/5}}
\end{aligned}
$$

for λ small enough. Note that

$$
\begin{aligned}
(4.18)\qquad & |\phi_1(\psi_0(1,u_\lambda)) - \phi_1(\psi_\lambda(1,u_\lambda))| \\
&\leq \varepsilon^{1/2}\|\psi_\lambda(1,u_\lambda) - \psi_0(1,u_\lambda)\| \\
&\leq \lambda\varepsilon^{1/2}
\end{aligned}
$$

and that

$$
\begin{aligned}
(4.19)\qquad & |\phi_2(\psi_0(1,u_\lambda)) - \phi_2(\psi_\lambda(1,u_\lambda))| \\
&\leq \varepsilon\|\psi_\lambda(1,u_\lambda) - \psi_0(1,u_\lambda)\| \\
&\leq \lambda\varepsilon.
\end{aligned}
$$

By combining (4.13)–(4.19), we get

$$\frac{\varepsilon^{2/5}}{8+2\varepsilon^{2/5}}\lambda \leq \varepsilon^{1/2}\lambda/2 + 2\varepsilon\lambda + \varepsilon^{1/2}\lambda.$$

This implies $\varepsilon \geq 96^{-10}$, a contradiction. □

Lemma 4.4. *Assume (A_1). Let K be a compact subset of E and γ_0 be a given continuous function from a closed subset K_0 of K into E and consider the family*

$$\Gamma := \{\gamma \in \mathbf{C}(K,E) : \gamma = \gamma_0 \ \text{ on } \ K_0\}.$$

Let Π be a closed subset of E. Assume that $\gamma(K)\cap\mathcal{S}\cap\Pi\backslash\gamma_0(K_0) \neq \emptyset$ for any $\gamma \in \Gamma$. Define

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma(K) \cap \mathcal{S}} G.$$

If

$$\sup_{\gamma_0(K_0)} G \leq \inf_{\Pi} G,$$

then there exists a sequence $\{u_m\}$ *such that*

$$G'(u_m) \to 0, \qquad G(u_m) \to c, \qquad u_m \in \mathcal{S}.$$

Moreover, $\mathrm{dist}(u_m, \Pi) \to 0$ *if* $c = \inf_{\Pi} G$.

Proof. Let

$$\mathcal{F} := \{A : A = \gamma(K), \ \ \gamma \in \Gamma\}, \qquad B = \gamma_0(K_0).$$

Then one checks that $\mathcal{F}$ is a homotopy-stable family with extended boundary B,

$$A \cap \Pi \cap \mathcal{S} \backslash B \neq \emptyset, \qquad \forall A \in \mathcal{F},$$

and

$$\sup_B G \leq \inf_{\Pi} G \leq c.$$

If $\inf_{\Pi} G < c$, then for any $A \in \mathcal{F}$, there is a $u_0 \in A \cap \mathcal{S}$ such that

$$c \leq \sup_{A \cap \mathcal{S}} G = G(u_0).$$

Hence

$$\{u \in E : G(u) \geq c\} \cap \mathcal{S} \cap A \backslash B \neq \emptyset$$

and

$$\sup_B G < c \leq \inf_{\Pi'} G, \quad \text{where } \Pi' = \{u \in E : G(u) \geq c\}.$$

Apply Lemma 4.3 to $\mathcal{F}, B, \Pi'$, and $\mathcal{S}$; we get a sign-changing sequence $\{u_m\}$ such that

$$G'(u_m) \to 0, \qquad G(u_m) \to c, \qquad u_m \in \mathcal{S}, \qquad \mathrm{dist}(u_m, \Pi') \to \infty.$$

If $\inf_{\Pi} G = c$, then $\sup_B G \leq c = \inf_{\Pi} G$. Apply Lemma 4.3 to $\mathcal{F}, B, \Pi$, and $\mathcal{S}$. Then we get a sign-changing sequence $\{u_m\}$ such that

$$G'(u_m) \to 0, \qquad G(u_m) \to c, \qquad u_m \in \mathcal{S}, \qquad \mathrm{dist}(u_m, \Pi) \to \infty.$$

$\square$

Lemma 4.5. *Let* $E = Y \oplus M$ *with* $\dim Y < \infty$. *Assume that* $z \in M$ *with* $\|z\| = 1$ *and that there is a continuous map* $\gamma_0 : \partial K \to E$, *where*

$$K := \{y + sz : y \in Y, s \geq 0, \|u\| \leq R\},$$

satisfying

(1) $\gamma_0(y) = y$ *for any* $y \in Y, \|y\| \leq R$.
(2) $\|\gamma_0(u)\| \geq r > 0$ *for* $u \in K, \|u\| = R$.

Let $K_0 = \partial K$;

$$\Gamma := \{\gamma \in \mathbf{C}(K, E) : \gamma = \gamma_0 \ \ on \ \ K_0\}.$$

Let $F = M \cap S_\rho,\ \rho < r,$ *where* $S_\rho := \{u \in E : \|u\| = \rho\}$. *Then* $\gamma(K) \cap F \neq \emptyset$ *for all* $\gamma \in \Gamma$.

Proof. Let $P : E \to Y$ be the projection onto Y along M. Let $Y_0 = Y \oplus \mathbf{R}z$. Define $H : K \to Y_0$ as the following.

$$H(u) = P\gamma(u) + \|(\mathbf{id} - P)\gamma(u)\|z.$$

Let

$$\Sigma_1 = \{(y, 0) : y \in Y, \|y\| \leq R\}$$

and $\Sigma_2 = \{u \in K : \|u\| = R\}$. Then $\partial K = \Sigma_1 \cup \Sigma_2$. By the assumption, $H(u) \neq \rho z$ for any $u \in \partial K$. Therefore, $\deg(H, K, \rho z)$ is well defined. Let

$$H(t, u) = tH(u) + (1 - t)H^*(u), \qquad t \in [0, 1], \qquad u \in \partial K,$$

where

$$H^*(u) = \begin{cases} u & \text{if } u \in \Sigma_1 \\ \dfrac{H(u)}{\|H(u)\|}R & \text{if } u \in \Sigma_2. \end{cases}$$

Note that $H(u) = u$ for $u \in \Sigma_1$ and $\|H(u)\| \geq r > 0$ for $u \in \Sigma_2$. Then it is easy to check that $H(t, u) \neq \rho z$ for all $u \in \partial K, t \in [0, 1]$. Because the degree depends only on the boundary values of $H(t, \cdot)$, we have that

$$\deg(H, K, \rho z) = \deg(H(t, \cdot), K, \rho z) = \deg(H^*, K, \rho z).$$

To compute the degree on the right-hand side of the above identities, we note that $u \in \Sigma_1$; then $H^*(u) = u \neq \rho z$. And $H^*(\Sigma_2) \subset \Sigma_2$, $H^* = \mathbf{id}$ on $\partial\Sigma_2$. Because Σ_2 is homeomorphic to a ball, there is a continuous deformation $H^*(t, u)$ connecting $H^*(u)$ to the identity in Σ_2 with $H^*(t, u) = \mathbf{id}$ on $\partial\Sigma_2$ for all $t \in [0, 1]$. Therefore,

$$\deg(H^*, K, \rho z) = \deg(\mathbf{id}, K, \rho z) = 1.$$

It follows that $\deg(H, K, \rho z) = 1$; this completes the proof of the lemma. □

Theorem 4.6 (Generalized Brezis–Nirenberg Theorem). *Assume* (A_1). *Let* $E = Y \oplus M$ *with* $\dim Y < \infty$. *Assume that* $z \in M$ *with* $\|z\| = 1$ *and that there is a continuous map* $\gamma_0 : \partial K \to E$, *where*

$$K := \{y + sz : y \in Y, s \geq 0, \|u\| \leq R\},$$

satisfying

(1) $\gamma_0(y) = y$ *for any* $y \in Y, \|y\| \leq R$ *and* $\|\gamma_0(u)\| \geq r > 0$ *for* $u \in K$, $\|u\| = R$.
(2) $G(\gamma_0(u)) \leq a$ *for all* $u \in \partial K$, *where* a *is a constant may be* ≤ 0 *or* > 0.
(3) For some positive $\rho < r$, $\inf_{M \cap S_\rho} G \geq a$, *where* $S_\rho := \{u \in E : \|u\| = \rho\}$.
(4) $M \cap S_\rho \subset \mathcal{S}$.

Then there exists a sequence $\{u_m\}$ *such that*

$$G'(u_m) \to 0, \qquad G(u_m) \to c \in \mathbf{R}, \qquad u_m \in \mathcal{S}.$$

If G *satisfies the (PS) condition at level* c, *then* G *has a sign-changing critical point in* $\mathcal{S}$.

Proof. Let $K_0 = \partial K$,

$$\Gamma := \{\gamma \in \mathbf{C}(K, E) : \gamma = \gamma_0 \ \ on \ \ K_0\}.$$

Let $F = M \cap S_\rho$. By Lemma 4.5, $\gamma(K) \cap F \neq \emptyset$ for all $\gamma \in \Gamma$. Hence, by (1) and (3), $\gamma(K) \cap \mathcal{S} \cap F \backslash \gamma_0(K_0) \neq \emptyset$ for any $\gamma \in \Gamma$. Define

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma(K) \cap \mathcal{S}} G.$$

By (2) and (3),

$$\sup_{\gamma_0(K_0)} G \leq \inf_F G.$$

By Lemma 4.4, there exists a sign-changing sequence $\{u_m\}$ such that

$$G'(u_m) \to 0, \qquad G(u_m) \to c, \qquad u_m \in \mathcal{S}.$$

Moreover, $\mathrm{dist}(u_m, F) \to 0$ if $c = \inf_F G$. By the (PS) condition, we get a sign-changing critical point in $\mathcal{S}$. □

Notes and Comments. Lemma 4.5 is due to Brezis and Nirenberg [71]. An earlier version of Definition 4.2 was given by Ghoussoub [156, 157]. Theorem 4.6 reveals the relationship between the classical linking theorem and sign-changing solution. We refer readers to Rabinowitz [255], Schechter [275], Struwe [313], Willem [335], and Zou and Schechter [351] for some earlier linking theorems without the nodal structure information.

4.3 Schrödinger Equations

Consider the existence of a sign-changing homoclinic orbit to the Schrödinger equation:

$$(4.20) \qquad -\Delta u + V_\lambda(x)u = f(x,u), \qquad x \in \mathbf{R}^N,$$

that is, a solution satisfying $u(x) \to 0$ as $|x| \to \infty$. Suppose that V_λ satisfies all the conditions (D_1)–(D_3) of Chapter 3. By the theory of the previous chapter, the Schrödinger operator $-\Delta + V_\lambda$ has a finite number of eigenvalues below the infimum of the essential spectrum. The eigenvalue problem

$$-\Delta u + u = \nu u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega$$

has positive isolated eigenvalues with finite multiplicity:

$$0 < \nu_1 < \nu_2 < \cdots < \nu_m < \nu_{m+1} < \cdots .$$

Let $\dim(\nu_i)$ denote the dimension of the eigenspace corresponding to the eigenvalue ν_i. Let $d_k := \sum_{i=1}^k \dim(\nu_i)$. We make the following hypotheses on the nonlinearity f.

(G$_1$) $f \in \mathbf{C}(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$; there exist $H_0 > 0, L \geq 0$ such that

$$|f(x,t)| \leq H_0|t|, \qquad \forall (x,t) \in \mathbf{R}^N \times \mathbf{R}$$

and $f(x,t) + Lt$ is increasing in t.

(G$_2$) There is a $k > 2$ such that $2F(x,t) \geq \nu_{k-1}t^2$ for all $x \in \mathbf{R}^N, t \in \mathbf{R}$. In particular, $\liminf_{t\to 0}((f(x,t))/t) > \nu_1$ uniformly for $x \in \mathbf{R}^N$.

(G$_3$) $2F(x,t) \leq ((\nu_k + \nu_{k-1})/2)t^2$ for all $x \in \mathbf{R}^N$ and $|t| \leq T_0$, where $T_0 > 0$ is a constant.

(G$_4$) $\liminf_{|t|\to\infty}(2F(x,t))/t^2 = H_\infty > \nu_k$; $\lim_{|t|\to 0}(f(x,t)t)/(F(x,t)) = \pi_0 > 2$ uniformly for $x \in \mathbf{R}^N$, where π_0 is a constant.

(G$_5$) $\lim_{|t|\to\infty}(f(x,t)t - 2F(x,t))/|t|^\beta = c > 0$ uniformly for $x \in \mathbf{R}^N$, where $\beta \in (1,2)$ is a constant.

(G$_6$) $f(x,t)t - 2F(x,t) > 0$ for all $x \in \mathbf{R}^N, t \in \mathbf{R}\backslash\{0\}$.

The above assumptions include the following case.

$$f(x,t)/t \to a \quad \text{as } t \to -\infty, \qquad f(x,t)/t \to b \quad \text{as } t \to \infty.$$

Here a, b are allowed to be any (different) numbers greater than ν_k. In other words, the jump at infinity may cross an arbitrarily finite number of eigenvalues in the spectrum of $-\Delta + V_\lambda$. In particular, a and b may belong to the continuous spectrum of $-\Delta + V_\lambda$ and therefore the resonance may occur at the continuous spectrum.

Theorem 4.7. *Assume (D_1)–(D_3) and (G_1)–(G_6). Then there exists a $\Lambda > 0$ such that Equation (4.20) has a sign-changing solution for each $\lambda \geq \Lambda$.*

Let E_λ be the Hilbert space

$$E_\lambda := \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + V_\lambda(x)u^2)dx < \infty \right\}$$

endowed with the inner product $\langle u, v\rangle_\lambda := \int_{\mathbf{R}^N}(\triangledown u \triangledown v + V_\lambda(x)uv)dx$ for $u, v \in E_\lambda$ and norm $\|u\|_\lambda := \langle u, u\rangle_\lambda^{1/2}$. Then by (D_1), $E_\lambda \hookrightarrow H^1(\mathbf{R}^N)$ is continuous. Consider

$$G_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbf{R}^N} F(x, u)dx. \tag{4.21}$$

Then $G_\lambda \in \mathbf{C}^1(E_\lambda, \mathbf{R})$ and $G'_\lambda = \mathbf{id} - \Theta_G$, where $\Theta_G := (-\Delta + V_\lambda)^{-1} f$. The weak solutions of (4.20) correspond to the critical points of G_λ. By Proposition 3.7, for λ large enough, the operator $-\Delta + V_\lambda$ has at least d_k eigenvalues:

$$\mu_1(S_\lambda),\ \mu_2(S_\lambda),\ \ldots,\ \mu_{d_k}(S_\lambda)$$

with corresponding eigenfunctions $e_1(\lambda), e_2(\lambda), \ldots, e_{d_k}(\lambda)$ and $|e_i(\lambda)|_2 = 1$ for all $i = 1, \ldots, d_k$. Set

$$E_{d_k}(\lambda) := \operatorname{span}\{e_1(\lambda), e_2(\lambda), \ldots, e_{d_k}(\lambda)\}.$$

By (3.63) $\mu_n(S_\lambda) \le \mu_n(L^*)$ for all $n \in \mathbf{N}$. Note that $\mu_n(S_\lambda) \to \mu_n$ as $\lambda \to \infty$ for all $n > 0$. By (3.55), $\lim_{n\to\infty} \mu_n = \infty$. By Proposition 3.11, $\mu_n = \mu_n(L^*)$ for all $n \in \mathbf{N}$.

Lemma 4.8. *Assume (G_2). Then there exists a $\Lambda_1 > 0$ such that*

$$G_\lambda(u) \le 0 \quad \textit{for all } u \in E_{d_{k-1}}(\lambda) \quad \textit{and} \quad \lambda > \Lambda_1.$$

Proof. We first observe that $\|u\|_\lambda^2 \le \mu_{d_{k-1}}(S_\lambda)|u|_2^2$ for all $u \in E_{d_{k-1}}(\lambda)$ and $\mu_{d_{k-1}}(S_\lambda) \le \nu_{k-1}$. Then by (G_2),

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbf{R}^N} F(x, u)dx \\ &\le \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int_{\mathbf{R}^N} \nu_{k-1}u^2 dx \\ &\le \frac{1}{2}\left(1 - \frac{\nu_{k-1}}{\mu_{d_{k-1}}(S_\lambda)}\right)\|u\|_\lambda^2 \\ &\le 0. \end{aligned}$$

□

Lemma 4.9. *Under the assumption of (D_4), there exists a $\Lambda_2 > 0$ such that $G_\lambda(u) \to -\infty$ for $u \in E_{d_k}(\lambda)$ as $\|u\| \to \infty$ for each $\lambda > \Lambda_2$.*

Proof. First, we observe that

$$\|u\|_\lambda^2 - \int_{\mathbf{R}^N} H_\infty u^2 dx \le -\kappa\|u\|_\lambda^2, \qquad \forall u \in E_{d_k}(\lambda), \tag{4.22}$$

where $\kappa := (H_\infty - \mu_{d_k}(S_\lambda))/\mu_{d_k}(S_\lambda) > 0$ because $\mu_{d_k}(S_\lambda) \leq \nu_k < H_\infty$. Furthermore, note that $\dim E_{d_k}(\lambda) < \infty$. We may find an $R > 0$ such that

$$(4.23) \qquad \int_{\mathbf{R}^N \setminus B_R(0)} \frac{H_\infty u^2}{\|u\|_\lambda^2} dx \leq \frac{\kappa}{4}, \qquad \int_{\mathbf{R}^N \setminus B_R(0)} |F(x,u)| dx \leq \frac{\kappa}{8} \|u\|_\lambda^2$$

for all $u \in E_{d_k}(\lambda)$. Here and in the sequel, $B_R(0) := \{x \in \mathbf{R}^N : |x| \leq R\}$. It follows from (4.22) and (4.23) that

$$(4.24) \qquad \|u\|_\lambda^2 - \int_{B_R(0)} H_\infty u^2 dx \leq -\frac{3\kappa}{4} \|u\|_\lambda^2 \quad \text{for all } u \in E_{d_k}(\lambda).$$

Furthermore, by (G_4), for $\varepsilon > 0$ small enough, there exists a $C_\varepsilon > 0$ such that

$$\frac{1}{2} H_\infty t^2 - F(x,t) \leq \frac{1}{2} \varepsilon t^2 + C_\varepsilon, \qquad \forall\, x \in B_R(0), \qquad t \in \mathbf{R}.$$

Therefore, combining (4.22)–(4.24),

$$\begin{aligned} G_\lambda(u) & \\ &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2} \int_{B_R(0)} H_\infty u^2 dx + \int_{B_R(0)} \left(\frac{1}{2} H_\infty u^2 - F(x,u) \right) dx \\ &\quad - \int_{\mathbf{R}^N \setminus B_R(0)} F(x,u) dx \\ &\leq -\frac{3\kappa}{8} \|u\|_\lambda^2 + \frac{\kappa}{8} \|u\|_\lambda^2 + \int_{B_R(0)} \left(\frac{1}{2} \varepsilon u^2 + C_\varepsilon \right) dx \\ &\leq -\frac{\kappa}{5} \|u\|_\lambda^2 + \int_{B_R(0)} C_\varepsilon dx. \end{aligned}$$

The lemma follows immediately. □

Lemma 4.10. *Assume (G_2). Then there exists a $\Lambda_3 > 0$ and a $\rho_0 > 0$ independent of λ such that*

$$G_\lambda(u) \geq c > 0$$

for all $u \in E_{d_{k-1}}^\perp(\lambda)$ with $\|u\| = \rho_0$ and all $\lambda \geq \Lambda_3$.

Proof. This is similar to the proof of Lemma 3.18. We give it here for completeness. Note that

$$(4.25) \qquad 2F(x,t) \leq 2H_0 t^2 - H_0 T_0^2 \quad \text{for } |t| \geq T_0, \qquad x \in \mathbf{R}^N,$$

where T_0 comes from (G_3). We first choose λ large enough (say $\lambda > \Lambda^*$) such that $\mu_{d_k}(S_\lambda)$ approaches $\mu_{d_k} = \nu_k$; hence $\mu_{d_k}(S_\lambda) > \bar{\Lambda}$ because $\nu_k > \bar{\Lambda}$, where $\bar{\Lambda} = (\nu_k + \nu_{k-1})/2$. Next, we choose λ large enough (say $\lambda > \Lambda^{**}$) such that the Schrödinger operator $-\Delta + V_\lambda$ has d_m eigenvalues

$$\mu_1(S_\lambda), \ldots, \ \mu_{d_m}(S_\lambda).$$

In particular, we may want d_m large enough so that

$$(2H_0 + \mu_{d_m}(S_\lambda) - 2\bar{\Lambda})(\mu_{d_k}(S_\lambda) - \bar{\Lambda}) \geq 4\bar{\Lambda}^2, \tag{4.26}$$

$$\mu_{d_m}(S_\lambda) \geq 2H_0, \tag{4.27}$$

$$(\mu_{d_m}(S_\lambda) - 2H_0)(\mu_{d_k}(S_\lambda) - \bar{\Lambda}) \geq 32\bar{\Lambda}^2, \tag{4.28}$$

$$\mu_{d_m}(S_\lambda) - 6H_0 + 2\bar{\Lambda} > 0, \tag{4.29}$$

$$\mu_{d_m}(S_\lambda) > \frac{2H_0}{\bar{\Lambda}}\mu_{d_k}(S_\lambda). \tag{4.30}$$

For any $u \in E^{\perp}_{d_{k-1}}(\lambda)$, we write $u = v + w$ with

$$v \in X_{d_k}(\lambda) \oplus X_{d_{k+1}}(\lambda) \oplus \cdots \oplus X_{d_m}(\lambda)$$

and $w \in E^{\perp}_{d_m}(\lambda)$, where d_m is given in (4.26)–(4.30) and $X_{d_i}(\lambda)$ $(i = k, \ldots, m)$ is the eigenspace associated with $\mu_{d_i}(S_\lambda)$. Let

$$\theta_1 := \frac{(2H_0 + \mu_{d_m}(S_\lambda))}{4}w^2 + \frac{(\mu_{d_k}(S_\lambda) + \bar{\Lambda})}{4}v^2 - F(x, v + w). \tag{4.31}$$

If $|v + w| \leq T_0$, then by condition (G_3) and the choice of $\mu_{d_m}(S_\lambda)$, we see that

$$\begin{aligned} \theta_1 &\geq \frac{(2H_0 + \mu_{d_m}(S_\lambda))}{4}w^2 + \frac{(\mu_{d_k}(S_\lambda) + \bar{\Lambda})}{4}v^2 - \frac{1}{2}\bar{\Lambda}(v + w)^2 \\ &\geq \frac{(2H_0 + \mu_{d_m}(S_\lambda)) - 2\bar{\Lambda}}{4}w^2 + \frac{(\mu_{d_k}(S_\lambda) + \bar{\Lambda}) - 2\bar{\Lambda}}{4}v^2 - \bar{\Lambda}|vw| \\ &\geq \left(\frac{((2H_0 + \mu_{d_m}(S_\lambda) - 2\bar{\Lambda})(\mu_{d_k}(S_\lambda) + \bar{\Lambda} - 2\bar{\Lambda}))^{1/2}}{2} - \bar{\Lambda}\right)|vw| \\ &\geq 0. \quad \text{(By (4.26))} \end{aligned} \tag{4.32}$$

If $|v + w| > T_0$, then by (4.25), we conclude that

$$\theta_1 \ \geq \ \Bigg(\frac{(\mu_{d_m}(S_\lambda) + 2H_0) - 4H_0}{4}w^2 \tag{4.33}$$

$$+\frac{(\mu_{d_k}(S_\lambda) + \bar{\Lambda}) - 4H_0}{4}v^2 - 2H_0vw + \frac{H_0T_0^2}{2}\Bigg) \tag{4.34}$$

$$:= \theta_2 + \theta_3,$$

where

(4.35)
$$\theta_2 := \frac{\mu_{d_m}(S_\lambda) + 2H_0 - 4H_0}{8} w^2 + \frac{(\mu_{d_k}(S_\lambda) - \bar{\Lambda})}{4} v^2 - \bar{\Lambda} v w,$$

(4.36)
$$\theta_3 := \frac{\mu_{d_m}(S_\lambda) + 2H_0 - 4H_0}{8} w^2 - \frac{2H_0 - \bar{\Lambda}}{2} v^2 - (2H_0 - \bar{\Lambda}) v w + \frac{H_0 T_0^2}{2}.$$

If
$$\frac{(\mu_{d_k}(S_\lambda) - \bar{\Lambda})}{4} |v| - \bar{\Lambda} |w| \geq 0,$$

then by (4.27),

$$\theta_2 \geq \frac{\mu_{d_m}(S_\lambda) + 2H_0 - 4H_0}{8} w^2 + \left(\frac{\mu_{d_k}(S_\lambda) - \bar{\Lambda}}{4} |v| - \bar{\Lambda} |w| \right) |v| \geq 0. \tag{4.37}$$

If
$$\frac{(\mu_k(S_\lambda) - \bar{\Lambda})}{4} |v| - \bar{\Lambda} |w| \leq 0,$$

by the choice of $\mu_{d_m}(S_\lambda)$ in (4.27) and (4.28), we deduce that

$$\begin{aligned} \theta_2 & \\ &\geq \left(\frac{\mu_{d_m}(S_\lambda) + 2H_0 - 4H_0}{8} - \frac{4\bar{\Lambda}^2}{\mu_{d_k}(S_\lambda) - \bar{\Lambda}} \right) w^2 + \frac{\mu_{d_k}(S_\lambda) - \bar{\Lambda}}{4} v^2 \\ &\geq 0. \end{aligned} \tag{4.38}$$

On the other hand, by (4.29),

$$\theta_3 \geq \frac{(\mu_{d_m}(S_\lambda) + 2H_0) - 4H_0}{8} w^2 - \frac{(2H_0 - \bar{\Lambda})}{2} v^2 - \frac{(2H_0 - \bar{\Lambda})}{2} v w + \frac{H_0 r_0^2}{2} \tag{4.39}$$

$$\geq \frac{(\mu_{d_m}(S_\lambda) + 2H_0) - 6H_0 + 2\bar{\Lambda}}{8} w^2 - \frac{6H_0 - 3\bar{\Lambda}}{4} v^2 + \frac{H_0 T_0^2}{2} \tag{4.40}$$

$$\geq -\frac{6H_0 - 3\bar{\Lambda}}{4} v^2 + \frac{H_0 T_0^2}{2}.$$

Choose
$$\rho_0 := \left(\frac{2H_0 T_0^2}{(6H_0 - 3\bar{\Lambda}) C_m^2} \right)^{1/2},$$

where C_m is a constant such that $\|v\|_\infty \le C_m\|v\|$ for all

$$v \in X_{d_k}(\lambda) \oplus X_{d_{k+1}}(\lambda) \oplus \cdots \oplus X_{d_m}(\lambda)$$

which is finite-dimensional. Then $\|u\| = \rho_0$ and

$$\|v\|_\infty \le C_m\|v\| \le C_m\|u\| = C_m\rho_0.$$

Hence, $\theta_3 \ge 0$. Therefore, by (4.33)–(4.39), $\theta_1 \ge 0$. Finally,

$$\begin{aligned}
&G_\lambda(u)\\
&\quad = G_\lambda(v+w)\\
&\quad = \frac{1}{2}(\|v\|_\lambda^2 + \|w\|_\lambda^2) - \int_{\mathbf{R}^N} F(x, v+w)dx\\
&\quad \ge \frac{1}{4}\|v\|_\lambda^2 + \frac{1}{4}\|w\|_\lambda^2 + \frac{1}{4}\mu_{d_k}(S_\lambda)|v|_2^2\\
&\qquad + \frac{1}{4}\mu_{d_m}(S_\lambda)|w|_2^2 - \int_{\mathbf{R}^N} F(x,u)dx\\
&\quad \ge \frac{1}{4}\left(1 - \frac{\bar{\Lambda}}{\mu_{d_k}(S_\lambda)}\right)\|v\|_\lambda^2 + \frac{1}{4}\left(1 - \frac{2H_0}{\mu_{d_m}(S_\lambda)}\right)\|w\|_\lambda^2 + \int_{\mathbf{R}^N} \theta_1 dx\\
&\quad \ge \frac{1}{4}\min\left\{\left(1 - \frac{\bar{\Lambda}}{\mu_{d_k}(S_\lambda)}\right), \left(1 - \frac{2H_0}{\mu_{d_m}(S_\lambda)}\right)\right\}\|u\|^2\\
&\quad \ge \frac{1}{4}\left(1 - \frac{\bar{\Lambda}}{\mu_{d_k}(S_\lambda)}\right)\rho_0^2\\
&\quad > 0.
\end{aligned}$$

□

Lemma 4.11. *Under the assumptions of Theorem 4.7, for each c, there exists a $\Lambda_4 > 0$ such that G_λ satisfies the (PS) condition at level c for each $\lambda \ge \Lambda_4$.*

Proof. Let $\{u_n\}$ be a (PS) sequence at level c:

$$G'_\lambda(u_n) \to \infty, \qquad G_\lambda(u_n) \to c.$$

We first show that $\{\|u_n\|_\lambda\}$ is bounded. By (G_4), let $\varepsilon_0 > 0$ be such that $\pi_0 - \varepsilon_0 > 2$. Hence, there exists an $R_0 > 0$ such that

$$f(x,u)u \ge (\pi_0 - \varepsilon_0)F(x,u) \tag{4.41}$$

for all $x \in \mathbf{R}^N$ and $|u| \le R_0$. On the other hand, by (G_5) and (G_6), we may choose $c > 0$ small enough such that

$$f(x,u)u - 2F(x,u) \ge c|u|^\beta \tag{4.42}$$

for all $x \in \mathbf{R}^N$ and $|u| \geq R_0$. Then

$$(4.43)\quad \left(\frac{1}{2} - \frac{1}{\pi_0 - \varepsilon_0}\right) \|u_n\|_\lambda^2 + \int_{\mathbf{R}^N} \left(\frac{1}{\pi_0 - \varepsilon_0} f(x, u_n)u_n - F(x, u_n)\right) dx \leq c + o(1)\|u_n\|_\lambda.$$

Hence, by (4.42) and (4.43) and (G_1) and (G_6), we get that

$$\begin{aligned}(4.44)\quad \|u_n\|_\lambda^2 &\leq c + o(1)\|u_n\|_\lambda \\ &\quad + c\left(\int_{|u_n| \leq R_0} + \int_{|u_n| \geq R_0}\right)\left(F(x, u_n) - \frac{1}{\pi_0 - \varepsilon_0} f(x, u_n)u_n\right) dx \\ &\leq c + o(1)\|u_n\|_\lambda + c\int_{|u_n| \geq R_0}\left(F(x, u_n) - \frac{1}{\pi_0 - \varepsilon_0} f(x, u_n)u_n\right) dx \\ &\leq c + o(1)\|u_n\|_\lambda + c\left(\frac{1}{2} - \frac{1}{\pi_0 - \varepsilon_0}\right)\int_{|u_n| \geq R_0} f(x, u_n)u_n dx \\ &\leq c + o(1)\|u_n\|_\lambda + c\int_{|u_n| \geq R_0} |u_n|^2 dx.\end{aligned}$$

Furthermore,

$$G_\lambda(u_n) - \frac{1}{2}\langle G'_\lambda(u_n), u_n\rangle_\lambda \leq c + o(1)\|u_n\|_\lambda$$

and (G_5) and (G_6) and (4.42) imply that

$$\begin{aligned} c + o(1)\|u_n\|_\lambda &\geq \int_{\mathbf{R}^N}\left(\frac{1}{2} f(x, u_n)u_n - G(x, u_n)\right) dx \\ (4.45)\qquad &\geq c\int_{|u_n| \geq R_0} |u_n|^\beta dx.\end{aligned}$$

Choose

$$\theta = \frac{(2 - \beta)(N + 2)}{(2N + 4 - N\beta)}.$$

Then $\theta \in (0, 1)$ and, by (4.45) and the Hölder inequality,

$$\begin{aligned}&\int_{|u_n| \geq R_0} |u_n|^2 \\ &\quad = \int_{|u_n| \geq R_0} |u_n|^{2(1-\theta)}|u_n|^{2\theta} dx\end{aligned}$$

$$\leq \left(\int_{|u_n| \geq R_0} |u_n|^\beta dx \right)^{(1-\theta)2/\beta} \left(\int_{|u_n| \geq R_0} |u_n|^{(2N+4)/N} dx \right)^{2\theta N/2N+4}$$

$$\leq (c + c\|u_n\|_\lambda)^{2(1-\theta)/\beta} \|u_n\|_\lambda^{2\theta}.$$

It follows by (4.44) that

$$\|u_n\|_\lambda^2 \leq o(1)\|u_n\|_\lambda + c + c(c + c\|u_n\|_\lambda)^{2(1-\theta)/\beta} \|u_n\|_\lambda^{2\theta}.$$

Note that $2(1-\theta)/\beta + 2\theta < 2$ because $\beta \in (1,2)$ and $\theta \in (0,1)$. Thus, we see that $\{\|u_n\|_\lambda\}$ is bounded. Suppose that $u_n \to u$ weakly in E_λ and $u_n \to u$ strongly in $L^2_{\mathrm{loc}}(\mathbf{R}^N)$ for some $u \in E_\lambda$. Then $G'_\lambda(u) = 0$. By Lemma 3.9, for any $\varepsilon > 0$ there exists an $R > 0$ and a $\Lambda > 0$ such that

$$\|v\|^2_{L^2(B_R^c)} \leq \varepsilon \|v\|_\lambda^2, \qquad \forall v \in E_\lambda, \qquad \lambda \geq \Lambda, \tag{4.46}$$

where $B_R^c := \{x \in \mathbf{R}^N : |x| > R\}$. Recall that $\|G'_\lambda(u_n)\| \to 0$ and $G'_\lambda(u) = 0$. Thus,

$$\begin{aligned}
&\|u_n - u\|_\lambda^2 \\
&\quad = o(1) + \int_{\mathbf{R}^N} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\quad \leq o(1) + \int_{|x| \geq R} (|u_n| + |u|)|u_n - u|dx \\
&\qquad + \int_{|x| \leq R} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\quad \leq o(1) + H_0 \int_{|x| \geq R} |u||u_n - u|^2 dx + 2H_0 \int_{|x| \geq R} |u_n - u|^2 dx \\
&\qquad + \int_{|x| \leq R} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\quad \leq o(1) + \frac{1}{2}\|u_n - u\|_\lambda^2 + 2H_0 \|u_n - u\|_\lambda \left(\int_{|x| \geq R} |u|^2 dx \right)^{1/2} \\
&\qquad + \int_{|x| \leq R} (f(x,u_n) - f(x,u))(u_n - u)dx.
\end{aligned}$$

Obviously, we may make $\|u_n - u\|_\lambda$ small enough by choosing R, n large enough; that is, $\|u_n - u\|_\lambda \to 0$. $\square$

Let $P_\lambda := \{u \in E_\lambda : u \geq \phi_1(\lambda)\}$, where $\phi_1(\lambda)$ is the positive eigenfunction of $\mu_1(S_\lambda)$. Then P_λ is closed and convex. By Lemma 3.21, all positive solutions belong to P_λ. By Lemma 3.21 and the proof of Theorem 3.15, we have the following.

Lemma 4.12. *Under the assumptions of* (D_1)–(D_3) *and* (G_1), *there exist* $\varepsilon_0 > 0$ *and* $\Lambda_5 > 0$ *such that*

$$\operatorname{dist}(S_\rho \cap (E_1(\lambda))^\perp, -P_\lambda \cup P_\lambda) > \varepsilon_0 > 0 \qquad \text{for all } \rho > 0$$

and

$$\Theta_G(\pm \mathcal{D}_0(\varepsilon)) \subset \pm \mathcal{D}_0(\varepsilon) \qquad \text{for all } \varepsilon \in (0, \varepsilon_0), \lambda \geq \Lambda_5,$$

where $S_\rho := \{u \in E_\lambda : \|u\|_\lambda = \rho\}$, $\mathcal{D}_0(\varepsilon) = \{u \in E_\lambda : \operatorname{dist}(u, P_\lambda) < \varepsilon\}$.

Proof of Theorem 4.7. Let $\mathcal{S} = E_\lambda \backslash (-\mathcal{D}_0(\varepsilon) \cup \mathcal{D}_0(\varepsilon))$. Then a critical point of G_λ in $\mathcal{S}$ is a sign-changing solution of (4.20). Let $Y = E_{d_{k-1}}(\lambda), M := E_{d_{k-1}}(\lambda)^\perp$; then $\dim Y < \infty$ and $E_\lambda = Y \oplus M$. Assume that $z \in M \backslash E_{d_k}(\lambda)^\perp$ with $\|z\| = 1$. Let

$$K := \{y + sz : y \in Y, s \geq 0, \|u\| \leq R\}.$$

Then there exists a $\Lambda > 0$ such that for each $\lambda \geq \Lambda$, $G_\lambda(u) \leq 0$ for all $u \in \partial K$ with R large enough (by Lemmas 4.8 and 4.9). For some positive ρ, by Lemma 4.10, $\inf_{M \cap S_\rho} G \geq a$. By Lemma 4.12, $M \cap S_\rho \subset \mathcal{S}$. Combining the (PS) condition of Lemma 4.11, the second conclusion of Lemma 4.12 and Theorem 4.6, G has a sign-changing critical point in $\mathcal{S}$. $\square$

Notes and Comments. Evidently, $P_\lambda \cap (-P_\lambda) = \emptyset$. Then, $-\mathcal{D}_0(\varepsilon) \cap \mathcal{D}_0(\varepsilon) = \emptyset$ if ε small enough. Moreover, $(1-t)(\pm P_\lambda) \not\subset \pm P_\lambda, (1-t)(\pm \mathcal{D}_0(\varepsilon)) \not\subset \pm \mathcal{D}_0(\varepsilon)$ for some $t \in [0,1]$. So, we cannot construct linking as in Chapter 2. The results of Chapter 2 cannot be directly applied to (4.20). The methods of Li and Wang [199] and Schechter et al. [279] need the functionals to be of $\mathbf{C}^2$.

Chapter 5
Even Functionals

5.1 Introduction

Let E be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We first recall a well-known result.

Suppose that $G \in \mathbf{C}^1(E, \mathbf{R})$ is even and satisfies the (PS) condition. Let $Y, M \subset E$ be two closed subspaces of E with $\dim Y < \infty, \dim Y - \mathrm{codim} M = 1$, $G(0) = 0$. Assume that there exists $\gamma > 0, \rho > 0$ such that $G(u) \geq \gamma$ for $u \in Q(\rho) := \{u \in M : \|u\| = \rho\}$ and that there exists an $R > 0$ such that $G(u) \leq 0$ for all $u \in Y$ with $\|u\| \geq R$. Let

$$\Gamma = \{\phi \in \mathbf{C}(E, E) : \phi \text{ is odd } ; \phi(u) = u \text{ if } u \in Y, \|u\| \geq R\}.$$

Then the number

$$b := \inf_{\phi \in \Gamma} \sup_{u \in Y} G(\phi(u)) \geq \gamma$$

is a critical value of G.

In applications on superlinear problems, one can prove that $\gamma \to \infty$ by choosing a sequence of subspaces Y. In this manner, one can obtain infinitely many critical points. This is a well-known version of the symmetric mountain pass theorem due to Ambrosetti and Rabinowitz (see Ambrosotti and Rabinowitz [15] and Rabinowitz [255]). This result has been applied to elliptic equations, Hamiltonian systems, and other variational problems to get infinitely many solutions.

In this chapter we are concerned with when the critical points of the symmetric mountain pass theorem will be sign-changing.

5.2 An Abstract Theorem

Let $G \in \mathbf{C}^1(E, \mathbf{R})$ and the gradient G' be of the form

$$G'(u) = u - K_G(u),$$

W. Zou, *Sign-Changing Critical Point Theory*, doi: 10.1007/978-0-387-76658-4,

where $K_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E\backslash\mathcal{K}$, $\mathcal{K}[a,b] := \{u \in \mathcal{K} : G(u) \in [a,b]\}$. Let $\mathcal{P}$ be a positive cone of E. For $\mu_0 > 0$, define

$$\mathcal{D}_0 := \{u \in E : \operatorname{dist}(u,\mathcal{P}) < \mu_0\}. \tag{5.1}$$

Then $\mathcal{D}_0$ is an open convex set containing the positive cone $\mathcal{P}$ in its interior. Set

$$\mathcal{D} := \mathcal{D}_0 \cup (-\mathcal{D}_0), \qquad \mathcal{S} = E\backslash\mathcal{D}.$$

In this chapter, we use the following assumption.

(A) $K_G(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_0$.

Let

$$\pi(u) := \frac{(1+\|u\|)^2}{(1+\|u\|)^2\|V(u)\|^2+1}, \tag{5.2}$$

where V is a pseudo-gradient vector field of G from Lemma 2.12. Then $\pi(u)$ is locally Lipschitz continuous. Write $W(u) = \pi(u)V(u)$. Then W is a locally Lipschitz continuous vector field over $\tilde{E}$. Obviously, $\|W(u)\| \le \|u\|+1$ for all $u \in \tilde{E}$. Moreover, $W(u)$ is odd if $V(u)$ is odd.

Lemma 5.1. *Assume that* $E = \tilde{E} + \hat{E}$ *with* $\dim \tilde{E} < \infty, \dim \tilde{E} - \operatorname{codim} \hat{E} \ge 1$. *Let* $\zeta : E \to E$ *be a continuous and odd mapping and*

$$\Theta := \{u \in \tilde{E} : \|u\| \le R\},$$

$$\Upsilon := \{u \in \hat{E} : \|u\| = \rho\},$$

where $R > \rho > 0$. *If* $\zeta(u) = u$ *for all* $u \in \partial\Theta$, *then*

$$\zeta(\Theta) \cap \Upsilon \neq \emptyset.$$

Proof. Let $U_\zeta := \{u \in \tilde{E} : \|\zeta(u)\| < \rho\} \cap \{u \in \tilde{E} : \|u\| < R\}$. Because ζ is odd, then U_ζ is an open bounded symmetric neighborhood of 0 in $\tilde{E}$. Write $E = E' \oplus \hat{E}$; then $E' \subset \tilde{E}$ and $\dim \tilde{E} > \dim E'$. Let $P : E \to E'$ be the projection onto E'. Then $P\zeta : \partial U_\zeta \to E'$ is a continuous odd map. By the Borsuk–Ulam theorem (cf. Theorem 1.33), there exists a $u \in \partial U_\zeta$ such that $P\zeta(u) = 0$; that is, $\zeta(u) \in \hat{E}$. Note that $\zeta(u) = u$ for $u \in \partial\Theta$ and $R \ge \rho$; we may check that $u \in \partial U_\zeta$ implies that $\|\zeta(u)\| = \rho$. This completes the proof. □

Theorem 5.2. *Assume (A). Let* Y, M *be two subspaces of* E *with* $\dim Y < \infty$; $\dim Y - \operatorname{codim} M = 1$. *Suppose that*

$$Q(\rho) := \{u \in M : \|u\| = \rho\} \subset \mathcal{S}.$$

Assume that $G \in \mathbf{C}^1(E, \mathbf{R})$ *is even and that there exist* $\gamma > 0, \beta \in \mathbf{R}$ *such that*

$$G(u) \geq \gamma \quad \textit{for all } u \in Q(\rho), \qquad G(u) \leq \beta, \quad \textit{for all } u \in Y.$$

If G *satisfies the* $(w^*$*-PS*$)_c$ *condition (see Definition 3.3) at level* c *for each* $c \in [\gamma, \beta]$, *then*

$$\mathcal{K}[\gamma - \varepsilon, \beta + \varepsilon] \cap (E \backslash (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon \in (0, \gamma)$.

Proof.
Step 1. We claim that there exists a $\sigma \in \mathbf{C}([0, \infty) \times E, E)$ such that $\sigma(t, u)$ is odd in u and that $\sigma(t, u) = u$ for any $u \in G^0 \cup Q(\rho)$ for all $t \geq 0$. Moreover, σ possesses some properties stated in the next steps.

To prove this, we choose $c_0 = 72(\beta - \gamma + 1)(\ln(5/4))^{-1} + 2$; then by the $(w^*\text{-PS})_c$ condition, there exist $\varepsilon_1 \in (0, \gamma), R_1 > 2\rho$ such that

$$\|G'(u)\|(1 + \|u\|) \geq c_0$$

for all $u \in G^{-1}[\gamma - \varepsilon_1, \beta + \varepsilon_1]$ with $\|u\| \geq R_1$. Let $\varepsilon_0 \in (0, \varepsilon_1), \varepsilon_0 < 1$ and

$$U_1 := \{u \in E : \text{either } \ G(u) \leq \gamma - \varepsilon_1 \text{ or } G(u) \geq \beta + \varepsilon_1\}, \tag{5.3}$$

$$U_2 := \{u \in E : \gamma - \varepsilon_0 \leq G(u) \leq \beta + \varepsilon_0\}, \tag{5.4}$$

$$U_3 := \{u \in E : \|u\| \leq R_1\}, \tag{5.5}$$

$$U_4 := \{u \in E : \|u\| \geq R_1 + 1\}. \tag{5.6}$$

Then $Q(\rho) \subset U_3$, $\mathcal{K} \subset U_1 \cup U_3$. Moreover, for any $u \in \mathcal{K}$, there exists a neighborhood U_u of u in E such that either $U_u \subset U_1$ or $U_u \subset U_3$. Define

$$q(u) = \frac{\text{dist}(u, U_1)}{\text{dist}(u, U_1) + \text{dist}(u, U_2)}, \tag{5.7}$$

$$j(u) = \frac{\text{dist}(u, U_3)}{\text{dist}(u, U_3) + \text{dist}(u, U_4)}; \tag{5.8}$$

both $q(u)$ and $j(u)$ are locally Lipschitz continuous functions on E. Set

$$W^*(u) = j(u)q(u)W(u)$$

for $u \in \tilde{E}$ and $W^*(u) = 0$ otherwise, where $W(u)$ is odd because G is even. Then by construction, W^* is locally Lipschitz continuous and odd on E. Consider the following Cauchy problem

$$\text{(5.9)} \qquad \begin{cases} \dfrac{d\sigma(t,u)}{dt} = -W^*(\sigma(t,u)), \\ \sigma(0,u) = u \in E. \end{cases}$$

Note that $\|W^*(u)\| \le (1+\|u\|)$; the unique solution $\sigma(t,\cdot) : E \to E$ is a homeomorphism and has the following properties.

(1) $\sigma(t,u)$ is odd in $u \in E$.
(2) $\sigma(t,u) = u$ for all $u \in G^0 \cup Q(\rho)$ for all $t \ge 0$.
(3) $G(\sigma(t,u))$ is nonincreasing with respect to $t \ge 0$ for any u in E.

Step 2. We show that

$$\text{(5.10)} \qquad \sigma([0,+\infty),\bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \sigma([0,+\infty),\mathcal{D}) \subset \mathcal{D}.$$

We first observe by Lemma 2.12 that $L_0(\pm\mathcal{D}_0 \cap \tilde{E}) \subset (\pm\mathcal{D}_0)$ implies that $L_0(\pm\bar{\mathcal{D}}_0 \cap \tilde{E}) \subset (\pm\bar{\mathcal{D}}_0)$. Obviously, $\sigma(t,u) = u$ for all $t \ge 0$ and $u \in \bar{\mathcal{D}} \cap \mathcal{K}$. Next, we assume that $u \in \bar{\mathcal{D}}_0 \cap \tilde{E}$. If there were a $t_0 > 0$ such that $\sigma(t_0,u) \notin \bar{\mathcal{D}}_0$, then there would be a number $s_0 \in [0,t_0)$ such that $\sigma(s_0,u) \in \partial\bar{\mathcal{D}}_0$ and $\sigma(t,u) \notin \bar{\mathcal{D}}_0$ for $t \in (s_0,t_0]$. Consider the following initial value problem

$$\text{(5.11)} \qquad \begin{cases} \dfrac{d\sigma(t,\sigma(s_0,u))}{dt} = -W^*(\sigma(t,\sigma(s_0,u))), \\ \sigma(0,\sigma(s_0,u)) = \sigma(s_0,u) \in E. \end{cases}$$

It has a unique solution $\sigma(t,\sigma(s_0,u))$. For any $v \in \bar{\mathcal{D}}_0$, if $v \in \mathcal{K}$, then $W^*(v) = 0$; hence $v + \lambda(-W^*(v)) = v \in \bar{\mathcal{D}}_0$. Therefore, we assume that $v \in \tilde{E}$. Noting $v \in \bar{\mathcal{D}}_0$ implies that $\operatorname{dist}(v,\mathcal{P}) \le \mu_0$. By Lemma 2.12, for any $p \in \mathcal{P}$, we have

$$\begin{aligned}
&\|v + \lambda(-W^*(v)) - p\| \\
&\quad = \|v + \lambda(-j(v)q(v)\pi(v)(v - L_0(v))) - p\| \\
&\quad = \|(1 - \lambda j(v)q(v)\pi(v))v + \lambda j(v)q(v)\pi(v)L_0(v) \\
&\qquad -\lambda j(v)q(v)\pi(v)p - (1 - \lambda j(v)q(v)\pi(v))p\| \\
&\quad = (1 - \lambda j(v)q(v)\pi(v))\|v - p\| + \lambda j(v)q(v)\pi(v)\|L_0(v) - p\| \\
&\quad \le (1 - \lambda j(v)q(v)\pi(v))\mu_0 + \lambda j(v)q(v)\pi(v)\mu_0 \\
&\quad = \mu_0
\end{aligned}$$

for λ small enough. It means that

$$\lim_{\lambda\to 0^+}\frac{\mathrm{dist}(v+\lambda(-W^*(v)),\bar{\mathcal{D}}_0)}{\lambda}=0,\qquad \forall v\in\bar{\mathcal{D}}_0.$$

By Lemma 1.49 and (5.1), there exists a $\delta>0$ such that $\sigma(t,\sigma(s_0,u))\in\bar{\mathcal{D}}_0$ for all $t\in[0,\delta)$. By the semigroup property, we see that $\sigma(t,u)\in\bar{\mathcal{D}}_0$ for all $t\in[s_0,s_0+\delta)$, which contradicts the definition of s_0. Therefore,

$$\sigma([0,+\infty),\bar{\mathcal{D}}_0)\subset\bar{\mathcal{D}}_0. \tag{5.12}$$

Similarly,

$$\sigma([0,+\infty),-\bar{\mathcal{D}}_0)\subset-\bar{\mathcal{D}}_0. \tag{5.13}$$

That is, $\sigma([0,+\infty),\bar{\mathcal{D}})\subset\bar{\mathcal{D}}$. To prove $\sigma([0,+\infty),\mathcal{D})\subset\mathcal{D}$, we just show that $\sigma([0,+\infty),\mathcal{D}_0)\subset\mathcal{D}_0$ by negation. Assume there exist $u^*\in\mathcal{D}_0, t_0>0$ such that $\sigma(t_0,u^*)\notin\mathcal{D}_0$. Choose a neighborhood U_{u^*} of u^* such that $U_{u^*}\subset\bar{\mathcal{D}}_0$. Then by the theory of ordinary differential equations in Banach space, we may find a neighborhood U_{t_0} of $\sigma(t_0,u^*)$ such that $\sigma(t_0,\cdot):U_{u^*}\to U_{t_0}$ is a homeomorphism. Because $\sigma(t_0,u^*)\notin\mathcal{D}_0$, we take a $w\in U_{t_0}\backslash\bar{\mathcal{D}}_0$. Correspondingly, we find a $v\in U_{u^*}$ such that $\sigma(t_0,v)=w$, which contradicts (5.13). We get (5.31).

Step 3. There exists a $T_0>0$ such that

$$\sigma(T_0,(G^\beta\backslash B_{R_0}(0))\cap Y)\subset G^{\gamma-\varepsilon_0}, \tag{5.14}$$

where $R_0=2(R_1+1)$. Note that

$$\|q(u)j(u)W(u)\|\le\frac{8c_0}{4+c_0^2}(1+\|u\|);$$

then by calculation, $\|\sigma(t,u)\|\le e^{(8c_0/(4+c_0^2))t}(1+\|u\|)-1$ for all $u\in E, t\ge 0$. We choose $T_0=9(\beta-\gamma+1)$. For any $u\in(G^\beta\backslash B_{R_0}(0))\cap Y$, then $\|u\|\ge R_0$ and $G(u)\le\beta$, and it follows that

$$\begin{aligned}\|\sigma(t,u)-u\| &= \left\|\int_0^t d\sigma(t,u)\right\|\\ &\le\frac{8c_0}{4+c_0^2}\int_0^t(1+\|\sigma(t,u)\|)dt\\ &\le\frac{8c_0}{4+c_0^2}\int_0^t(1+\|u\|)e^{(8c_0/(4+c_0^2))t}dt\\ &=(1+\|u\|)(e^{(8c_0/(4+c_0^2))t}-1).\end{aligned}$$

It implies that $\|\sigma(t,u)\| \ge \|u\| - (1+\|u\|)(e^{(8c_0/(4+c_0^2))t} - 1) \ge R_1 + 1$ for all $t \in [0, T_0]$. Hence, $j(\sigma(t,u)) = 1$ for all $t \in [0, T_0]$. If there exists a $t_1 \in [0, T_0]$ such that $G(\sigma(t_1, u)) \le \gamma - \varepsilon_0$, then $G(\sigma(T_0, u)) \le \gamma - \varepsilon_0$ and we are done. Otherwise,

$$\gamma - \varepsilon_0 \le G(\sigma(t,u)) \le G(u) \le \beta \le \beta + \varepsilon_0$$

for all $t \in [0, T_0]$. It implies that

$$\|G'(\sigma(t,u))\|(1 + \|\sigma(t,u)\|) \ge c_0$$

and $q(\sigma(t,u)) = 1$ for all $t \in [0, T_0]$. Therefore,

$$\begin{aligned}
&G(\sigma(T_0, u)) \\
&\quad = G(u) + \int_0^{T_0} dG(\sigma(t,u)) \\
&\quad = G(u) + \int_0^{T_0} \left(\frac{-(1+\|\sigma(t,u)\|)^2 \langle G'(\sigma(t,u), V(\sigma(t,u))\rangle}{(1+\|\sigma(t,u)\|)^2 \|V(\sigma(t,u))\|^2 + 1} \right) dt \\
&\quad \le G(u) - \int_0^{T_0} \left(\frac{(1+\|\sigma(t,u)\|)^2 \|G'(\sigma(t,u))\|^2}{8(1+\|\sigma(t,u)\|)^2 \|G'(\sigma(t,u))\|^2 + 2} \right) dt \\
&\quad \le G(u) - \frac{c_0^2}{2 + 8c_0^2} T_0 \\
&\quad \le \beta - \frac{c_0^2}{2 + 8c_0^2} T_0 \\
&\quad \le \gamma - \varepsilon_0;
\end{aligned}$$

we get (5.32).

Step 4. Let $D(R) := B_R(0) \cap Y, R > R_0, R > \rho$, and

$$\begin{aligned}
\Gamma := \{\Phi : \Phi \in \mathbf{C}([0,\infty) \times E, E), &\Phi(t,u) \text{ is odd in } u, \Phi(0,\cdot) = \mathbf{id}, \\
&\Phi(t, \sigma(T_0, u)) = \sigma(T_0, u), \forall u \in \partial D(R), \forall t \in [0,\infty); \Phi(t, \mathcal{D}) \subset \mathcal{D}\},
\end{aligned}$$

Then $\mathbf{id} \in \Gamma$. We claim that

$$\sigma^{-1}\Phi(1, \sigma(T_0, D(R))) \cap Q(\rho) \ne \emptyset,$$

where $\sigma^{-1}(\cdot) := \sigma^{-1}(T_0, \cdot)$. Set

$$\Phi^*(u) := \sigma^{-1}\Phi(1, \sigma(T_0, u)),$$

which is odd. Note $\Phi^*(u) = u$ for all $u \in \partial D(R)$; by Lemma 5.1,

$$\Phi^*(D(R)) \cap Q(\rho) \ne \emptyset.$$

Hence, by Steps (1) and (2),

$$\Phi(1, \sigma(T_0, D(R))) \cap Q(\rho) \neq \emptyset, \qquad \Phi(1, \sigma(T_0, D(R))) \cap \mathcal{S} \neq \emptyset.$$

Consider

$$b = \inf_{\Phi \in \Gamma} \sup_{u \in \Phi(1, \sigma(T_0, D(R))) \cap \mathcal{S}} G(u). \tag{5.15}$$

Obviously, b is well defined and $\beta \geq b \geq \gamma > 0$.

Step 5. We prove that

$$\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}] \cap (E \backslash (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\bar{\varepsilon} \in (0, \gamma)$. That is, there is a sign-changing critical point.

By negation, we assume that

$$\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}] \cap (E \backslash (-\mathcal{P} \cup \mathcal{P})) = \emptyset$$

for some $\bar{\varepsilon} \in (0, \gamma)$; then

$$\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}] \subset (-\mathcal{P} \cup \mathcal{P}). \tag{5.16}$$

Case (i). Assume $\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}] \neq \emptyset$. Because $\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}]$ is compact in E, by the definition of $\mathcal{S}$, we must have

$$\operatorname{dist}(\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}], \mathcal{S}) := \delta_0 > 0.$$

By the $(w^*\text{-PS})_c$ condition for $c \in [\gamma, \beta]$, there is an $\varepsilon_2 \in (0, \bar{\varepsilon}/3)$ such that

$$\frac{(1 + \|u\|)^2 \|G'(u)\|^2}{1 + (1 + \|u\|)^2 \|G'(u)\|^2} \geq \varepsilon_2 \tag{5.17}$$

for $u \in G^{-1}[b - \varepsilon_2, b + \varepsilon_2] \backslash (\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}])_{\delta_0/2}$; here and in the sequel, $(A)_c := \{u \in E : \operatorname{dist}(u, A) \leq c\}$. By decreasing ε_2, we may assume that $\varepsilon_2 < \varepsilon_0/3$, where ε_0 comes from Step 1. Then

$$\langle G'(u), W(u) \rangle \geq \varepsilon_2/8$$

for any $u \in G^{-1}[b - \varepsilon_2, b + \varepsilon_2] \backslash (\mathcal{K}[b - \bar{\varepsilon}, b + \bar{\varepsilon}])_{\delta_0/2}$. Let

$$U_5 = \{u \in E : |G(u) - b| \geq 3\varepsilon_2\},$$

$$U_6 = \{u \in E : |G(u) - b| \leq 2\varepsilon_2\}.$$

Let $y(u) : E \to [0,1]$ be locally Lipschitz continuous such that

$$y(u) := \begin{cases} 1 & \text{for all } u \in E\backslash(\mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}])_{\delta_0/2} \\ 0 & \text{for all } u \in \mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}])_{\delta_0/3}. \end{cases}$$

Consider

$$h(u) := \frac{\operatorname{dist}(u, U_5)}{\operatorname{dist}(u, U_6) + \operatorname{dist}(u, U_5)}.$$

Let $\bar{W}(u) := y(u)h(u)W(u)$ if $u \in \tilde{E}$ and $\bar{W}(u) = 0$ otherwise; then $\bar{W}$ is a locally Lipschitz vector field on E. We consider the following Cauchy initial value problem,

$$\begin{cases} \dfrac{d\eta(t,u)}{dt} = -\bar{W}(\eta(t,u)), \\ \eta(0,u) = u \in E, \end{cases}$$

which has a unique continuous odd solution $\eta(t,u)$ in E. Evidently,

$$\frac{dG(\eta(t,u))}{dt} \leq 0.$$

By the definition of b in (5.35), there exists a $\Phi \in \Gamma$ such that

$$\Phi(1, \sigma(T_0, D(R))) \cap \mathcal{S} \subset G^{b+\varepsilon_2}.$$

Therefore,

$$\Phi(1, \sigma(T_0, D(R))) \subset G^{b+\varepsilon_2} \cup (E\backslash\mathcal{S}).$$

Denote $A := \Phi(1, \sigma(T_0, D(R)))$. We claim that there exists a $T_1 > 0$ such that

$$\eta(T_1, A) \subset G^{b-\varepsilon_2/4} \cup \mathcal{D}. \tag{5.18}$$

In fact, if $u \in A \cap \mathcal{D}$, then similar to Step 2, $\eta(t,u) \in \mathcal{D}$ for all $t \geq 0$.

If $u \in A, u \notin \mathcal{D}$, then we see that $G(u) \leq b + \varepsilon_2$. If $G(u) \leq b - \varepsilon_2$, then

$$G(\eta(t,u)) \leq G(u) \leq b - \varepsilon_2$$

for all $t \geq 0$. If $G(u) > b - \varepsilon_2$, then $u \in G^{-1}[b-\varepsilon_2, b+\varepsilon_2]$. If

$$\operatorname{dist}(\eta([0,\infty), u), \mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}]) \leq \delta_0/2,$$

then there exists a t_m such that $\eta(t_m, u) \notin \mathcal{S}$. Moreover, we may choose m so that $\operatorname{dist}(\eta(t_m, u), \mathcal{S}) \geq \frac{1}{3}\delta_0 > 0$.

Assume

$$\operatorname{dist}(\eta([0,\infty), u), \mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}]) > \delta_0/2.$$

Similarly, we assume that $G(\eta(t,u)) > b - \varepsilon_2$ for all t (otherwise, we are done); then

$$\eta(t,u) \in G^{-1}[b-\varepsilon_2, b+\varepsilon_2] \backslash (\mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}])_{\delta_0/2}.$$

Hence,

$$h(\eta(t,u)) = 1, y(\eta(t,u)) = 1, \langle G'(\eta(t,u)), W(\eta(t,u))\rangle \geq \varepsilon_2/8$$

for all $t \geq 0$. Therefore,

$$G(\eta(24,u)) = G(u) + \int_0^{24} dG(\eta(s,u)) \leq b - 2\varepsilon_2. \tag{5.19}$$

By combining the above arguments, for any $u \in A\backslash\mathcal{D}$, there exists a $T_u > 0$ such that either $\eta(T_u,u) \in G^{b-\varepsilon_2/2}$ or $\operatorname{dist}(\eta(T_u,u),\mathcal{S}) \geq \frac{1}{3}\delta_0$. By continuity, there exists a neighborhood U_u of u such that either $\eta(T_u,U_u) \subset G^{b-\varepsilon_2/3}$ or $\operatorname{dist}(\eta(T_u,U_u),\mathcal{S}) \geq \frac{1}{4}\delta_0$. Both cases imply that

$$\eta(T_u,U_u) \subset G^{b-\varepsilon_2/3} \cup (E\backslash\mathcal{S}).$$

Because $A\backslash\mathcal{D}$ is compact in E, we get a $T_1 > 0$ such that

$$\eta(T_1,A) \subset G^{b-\varepsilon_2/4} \cup (E\backslash\mathcal{S}). \tag{5.20}$$

Case (ii). If $\mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}] = \emptyset$, then (5.17) holds with $(\mathcal{K}[b-\bar{\varepsilon}, b+\bar{\varepsilon}])_{\delta_0/2} = \emptyset$. Then, trivially, (5.19) and (5.20) are still true.

Now we define

$$\Phi^{**}(t,u) = \eta(tT_1, \Phi(t,u)).$$

Then $\Phi^{**}(t,u)$ is odd in u for every t and $\Phi^{**}(0,u) = u$. Moreover, if $u \in \partial D(R)$, then $G(u) \leq \beta$ and $G(\sigma(T_0,u)) \leq \gamma - \varepsilon_0$ (by Step 3) $\leq b - 3\varepsilon_2$; that is, $\sigma(T_0,u) \in U_5$. Therefore,

$$\begin{aligned} \Phi^{**}(t,\sigma(T_0,u)) &= \eta(tT_1, \Phi(t,\sigma(T_0,u))) \\ &= \eta(tT_1, \sigma(T_0,u)) \\ &= \sigma(T_0,u), \end{aligned}$$

for all $t \geq 0$. Evidently, by the construction of η, $\Phi^{**}(t,\mathcal{D}) \subset \mathcal{D}$. Then $\Phi^{**} \in \Gamma$. But

$$G(\Phi^{**}(1,\sigma(T_0,D(R))) \cap \mathcal{S}) \leq b - \varepsilon_2/4,$$

a contradiction. □

Again, let Y, M be two subspaces of E with $\dim Y < \infty, \dim Y - \operatorname{codim} M \geq 1$ and $(M\backslash\{0\}) \cap (-\mathcal{P} \cup \mathcal{P}) = \emptyset$; that is, the nontrivial elements of M are sign-changing. We assume that $\mathcal{P}$ is weakly closed; that is, if

$\mathcal{P} \ni u_k \rightharpoonup u$ weakly in $(E, \|\cdot\|)$, then $u \in \mathcal{P}$. In all applications of this book, this is satisfied automatically. Next we assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \le C_\star \|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = M_1 \oplus M$. Let

$$Q^\star(\rho) := \left\{ u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star \|u\|_\star} = \rho \right\},$$

where $\rho > 0, D_\star > 0, p > 2$ are fixed constants. Evidently, we have

Lemma 5.3. $\|u\|_\star \le c_1, \quad \forall u \in Q^\star(\rho)$; *where $c_1 > 0$ is a constant.*

We first make the following assumption.

($\mathbf{A_1^\star}$) Assume that for any $a, b > 0$, there is a $c_2 = c_2(a, b) > 0$ such that

$$G(u) \le a \text{ and } \|u\|_\star \le b \quad \Rightarrow \quad \|u\| \le c_2.$$

Lemma 5.4. *Assume ($A_1^\star$). For any $a > 0$, we have that*

$$\operatorname{dist}(Q^\star(\rho) \cap G^a, \mathcal{P}) := \delta(a) > 0.$$

Proof. By negation, we assume that

$$\operatorname{dist}(Q^\star(\rho) \cap G^a, \mathcal{P}) = 0.$$

Then we find $\{u_n\} \subset Q^\star(\rho) \cap G^a, \{p_n\} \subset \mathcal{P}$ such that $\|u_n - p_n\| \to 0$. Then by Lemmas 5.3 and ($A_1^\star$), $\{u_n\}$, hence $\{p_n\}$, is bounded in both $(E, \|\cdot\|)$ and $(E, \|\cdot\|_\star)$. We assume that

$$u_n \rightharpoonup u^* \in E, \qquad p_n \rightharpoonup p^* \in \mathcal{P} \quad \text{weakly in } (E, \|\cdot\|);$$

$$u_n \to u^* \quad \text{strongly in } (E, \|\cdot\|_\star).$$

Then we observe that $u^* \in M$. Because

$$\frac{\|u_n\|_\star^p}{\|u_n\|^2} + \frac{\|u_n\|\|u_n\|_\star}{\|u_n\| + D_\star \|u_n\|_\star} = \rho$$

and $\|u_n - u^*\|_\star \to 0$, then $u^* \ne 0$. However, because $u^* = p^*$, we get a contradiction because all nonzero elements of M are sign-changing. $\square$

Let

$$\Gamma_Y^* := \{h : h \in \mathbf{C}(\Theta_Y, E), \ \ h|_{\partial\Theta_Y} = \mathbf{id}, \ \ h \ \text{ is odd}\},$$

where

$$\Theta_Y := \{u \in Y : \ \|u\| \le R_Y\}, \qquad R_Y > 0.$$

Note that both $\|\cdot\|_\star$ and $\|\cdot\|$ are equivalent in Y; we have a constant ϱ_Y such that

$$\|u\| \le \varrho_Y \|u\|_\star, \qquad \text{for all } u \in Y.$$

We assume that $R_Y \ge \varrho_Y + 2$ and

$$\frac{\left(\frac{R_Y}{\varrho_Y}\right)^p}{R_Y^2} + \frac{R_Y\left(\frac{R_Y}{\varrho_Y}\right)}{R_Y + D_\star C_\star R_Y} > \rho. \tag{5.21}$$

Lemma 5.5. $h(\Theta_Y) \cap Q^\star(\rho) \ne \emptyset, \quad \forall\, h \in \Gamma_Y^*.$

Proof. Let

$$\beta^*(u) := \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star}$$

if $u \ne 0$ and $\beta^*(0) = 0$. Then $\beta^* : E \to E$ is continuous. Let

$$U := \{u \in Y : \beta^*(h(u)) < \rho\} \cap \{u \in Y : \|u\| < R_Y\};$$

then U is a neighborhood of zero in Y. Let $P : E \to M_1$ be the projection; then $P \circ h : \partial U \to M_1$ is odd and continuous. By the Borsuk–Ulam theorem, we have that $P \circ h(u) = 0$ for some $u \in \partial U$. Hence, $h(u) \in M$. We claim $u \notin \partial\{u \in Y : \|u\| < R_Y\}$. Otherwise, $\|u\| = R_Y$ and then $h(u) = u, P(u) = 0$. It follows that

$$\frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star} \le \rho.$$

Note that $\|u\|_\star \le C_\star\|u\| \le C_\star\varrho_Y\|u\|_\star$ in Y. Therefore,

$$\begin{aligned}
&\frac{\left(\frac{R_Y}{\varrho_Y}\right)^p}{R_Y^2} + \frac{R_Y\left(\frac{R_Y}{\varrho_Y}\right)}{R_Y + D_\star C_\star R_Y} \\
&\quad = \frac{\left(\frac{\|u\|}{\varrho_Y}\right)^p}{\|u\|^2} + \frac{\|u\|\left(\frac{\|u\|}{\varrho_Y}\right)}{\|u\| + D_\star C_\star\|u\|} \\
&\quad \le \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star} \\
&\quad \le \rho;
\end{aligned}$$

this is impossible in view of (5.21). So, our claim is true. It means

$$u \in \partial\{u \in Y : \beta^*(h(u)) < \rho\}, \qquad \|u\| \le R_Y, \qquad u \in Y.$$

Hence,

$$h(u) \in M,$$

$$\frac{\|h(u)\|_\star^p}{\|h(u)\|^2} + \frac{\|h(u)\|\|h(u)\|_\star}{\|h(u)\| + D_\star\|h(u)\|_\star} = \rho \quad \Rightarrow h(u) \in Q^\star(\rho).$$

□

We need the following assumption.

($\mathbf{A_2^\star}$) $\lim_{u\in Y, \|u\|\to\infty} G(u) = -\infty, \quad \sup_Y G := \beta.$

By Lemma 5.4 and $(A_1^\star)$ and $(A_2^\star)$, we may assume

$$Q^{\star\star} := Q^\star(\rho) \cap G^\beta \subset \mathcal{S}, \tag{5.22}$$

where $Q^{\star\star}$ is a bounded set by Lemma 5.3 and $(A_1^\star)$. Let

$$\inf_{Q^{\star\star}} G := \gamma. \tag{5.23}$$

It is easy to check that $Q^{\star\star} \cap Y \neq \emptyset$. Then $\beta \geq \gamma$.

Theorem 5.6. *Assume (A) and $(A_1^\star)$ and $(A_2^\star)$. If the even functional G satisfies the $(w^*$-$PS)_c$ condition (see Definition 3.3) at level c for each $c \in [\gamma, \beta]$, then*

$$\mathcal{K}[\gamma - \varepsilon, \beta + \varepsilon] \cap (E \backslash (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ small.

Proof. The proof is the same as that of Theorem 5.2. We just sketch the main points.

Step 1. We claim that there exists a $\sigma \in \mathbf{C}([0,\infty) \times E, E)$ such that $\sigma(t,u)$ is odd in u and that $\sigma(t,u) = u$ for any $u \in Q^{\star\star}$ for all $t \geq 0$. Moreover, σ possesses some properties stated in next steps.

To prove this, we choose $c_0 = 72(\beta - \gamma + 1)(\ln 5/4)^{-1} + 2$, then by the $(w^*$-PS$)_c$ condition, there exist $\varepsilon_1 > 0, R_1 > 2\rho$ such that

$$\|G'(u)\|(1 + \|u\|) \geq c_0$$

for all $u \in G^{-1}[\gamma - \varepsilon_1, \beta + \varepsilon_1]$ with $\|u\| \geq R_1$. Let $\varepsilon_0 \in (0, \varepsilon_1), \varepsilon_0 < 1$ and

$$U_1 := \{u \in E : \text{either } \ G(u) \leq \gamma - \varepsilon_1 \quad \text{or} \quad G(u) \geq \beta + \varepsilon_1\}, \tag{5.24}$$

$$U_2 := \{u \in E : \gamma - \varepsilon_0 \leq G(u) \leq \beta + \varepsilon_0\}, \tag{5.25}$$

$$U_3 := \{u \in E : \|u\| \leq R_1\}, \tag{5.26}$$

$$U_4 := \{u \in E : \|u\| \geq R_1 + 1\}. \tag{5.27}$$

If necessary, we may enlarge R_1 such that $Q^{\star\star} \subset U_3$, $\mathcal{K} \subset U_1 \cup U_3$. Moreover, for any $u \in \mathcal{K}$, there exists a neighborhood U_u of u in E such that either $U_u \subset U_1$ or $U_u \subset U_3$. Define

$$q(u) = \frac{\operatorname{dist}(u, U_1)}{\operatorname{dist}(u, U_1) + \operatorname{dist}(u, U_2)}, \tag{5.28}$$

$$j(u) = \frac{\operatorname{dist}(u, U_3)}{\operatorname{dist}(u, U_3) + \operatorname{dist}(u, U_4)}; \tag{5.29}$$

both $q(u)$ and $j(u)$ are locally Lipschitz continuous functions on E. Set

$$W^*(u) = j(u)q(u)W(u)$$

for $u \in \tilde{E}$ and $W^*(u) = 0$ otherwise, where $W(u)$ is odd because G is even. Then by construction, W^* is locally Lipschitz continuous and odd on E. Consider the following Cauchy problem

$$\begin{cases} \dfrac{d\sigma(t,u)}{dt} = -W^*(\sigma(t,u)), \\ \sigma(0,u) = u \in E. \end{cases} \tag{5.30}$$

Note that $\|W^*(u)\| \leq (1 + \|u\|)$, the unique solution $\sigma(t,\cdot) : E \to E$ is a homeomorphism and has the following properties.

(1) $\sigma(t,u)$ is odd in $u \in E$.
(2) $\sigma(t,u) = u$ for all $u \in Q^{\star\star}$ for all $t \geq 0$.
(3) $G(\sigma(t,u))$ is nonincreasing with respect to $t \geq 0$ for any u in E.

Step 2. We show that

$$\sigma([0,+\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \sigma([0,+\infty), \mathcal{D}) \subset \mathcal{D}. \tag{5.31}$$

It is the same as that of Theorem 5.2.

Step 3. There exists a $T_0 > 0$ such that

$$\sigma(T_0, (G^\beta \backslash B_{R_0}(0)) \cap Y) \subset G^{\gamma - \varepsilon_0}, \tag{5.32}$$

where $R_0 = 2(R_1 + 1)$. It is the same as that of Theorem 5.2.

Step 4. Let $D(R) := B_R(0) \cap Y, R > R_0, R > \rho$ and by Condition $(A_2^\star)$, we may choose R so large that

$$\sup_{u \in Y, \|u\| = R} G \leq \gamma - \varepsilon_1 - 1. \tag{5.33}$$

Thus, $\partial D(R) \subset U_1$ of step 1. Hence

$$\sigma(t,u) = u \quad \text{for all } \ u \in \partial D(R), t \geq 0. \tag{5.34}$$

Let

$$\Gamma := \left\{ \Phi : \begin{array}{l} \Phi \in \mathbf{C}([0,\infty) \times E, E), \quad \Phi(t,u) \text{ is odd in } u; \quad \Phi(0,\cdot) = \mathbf{id} \\ \Phi(t, \sigma(T_0, u)) = \sigma(T_0, u), \quad \forall u \in \partial D(R), \quad \forall t \in [0,\infty) \\ G(\Phi(t, \sigma(T_0, u))) \le G(u), \quad \forall u \in D(R), \quad \forall t \in [0,\infty) \\ \Phi(t, \mathcal{D}) \subset \mathcal{D} \end{array} \right\}.$$

Then $\mathbf{id} \in \Gamma$. Set

$$\Phi^*(u) := \Phi(1, \sigma(T_0, u)),$$

which is odd. Note $\Phi^*(u) = u$ for all $u \in \partial D(R)$ and

$$G(\Phi^*(u)) \le G(u) \le \beta, \qquad u \in D(R).$$

By Lemma 5.5 (we may enlarge $R = R_Y$),

$$\Phi^*(D(R)) \cap Q^{\star\star} \ne \emptyset.$$

Hence,

$$\Phi(1, \sigma(T_0, D(R))) \cap Q^{\star\star} \ne \emptyset, \qquad \Phi(1, \sigma(T_0, D(R))) \cap \mathcal{S} \ne \emptyset.$$

Consider

$$b = \inf_{\Phi \in \Gamma} \sup_{u \in \Phi(1, \sigma(T_0, D(R)) \cap \mathcal{S}} G(u). \tag{5.35}$$

Obviously, b is well defined and $\beta \ge b \ge \gamma$.

Step 5. Similarly to Step 5 of Theorem 5.2, we get that

$$\Phi^{**}(t,u) = \eta(tT_1, \Phi(t,u)).$$

Then $\Phi^{**}(t,u)$ is odd in u for every t and $\Phi^{**}(0,u) = u$. Moreover, if $u \in \partial D(R)$, then $G(u) \le \beta$ and $G(\sigma(T_0,u)) \le \gamma - \varepsilon_0$ (by Step 3) $\le b - 3\varepsilon_2$; that is, $\sigma(T_0, u) \in U_5$. Therefore,

$$\begin{aligned} \Phi^{**}(t, \sigma(T_0,u)) &= \eta(tT_1, \Phi(t, \sigma(T_0, u))) \\ &= \eta(tT_1, \sigma(T_0, u)) \\ &= \sigma(T_0, u), \end{aligned}$$

for all $t \ge 0$. Evidently, by the construction of η, $\Phi^{**}(t, \mathcal{D}) \subset \mathcal{D}$. Moreover,

$$G(\Phi^{**}(t, \sigma(T_0,u))) \le G(\Phi(t, \sigma(T_0, u))) \le G(u), \qquad \forall u \in D(R).$$

Therefore, $\Phi^{**} \in \Gamma$. But

$$G(\Phi^{**}(1, \sigma(T_0, D(R))) \cap \mathcal{S}) \leq b - \varepsilon_2/4,$$

a contradiction. □

Notes and Comments. Lemma 5.1 can be found in Rabinowitz [255] and Willem [335]. A recent paper by Bartsch et al. [37] studied the sign-changing critical points of even functionals by using genus. An estimate of the number of nodal domains was given there. In particular; Li and Wang [199], a Ljusternik–Schnirelmann theory was established for studying the sign-changing critical points of even functionals of $\mathbf{C}^2$.

5.3 A Classical Superlinear Problem

We consider the following superlinear elliptic equation.

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.36}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$ $(N \geq 3)$. Assume

(J₁) $f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with subcritical growth:

$$|f(x, u)| \leq c(1 + |u|^{s-1}) \quad \text{for all } u \in \mathbf{R} \quad \text{and} \quad x \in \bar{\Omega},$$

where $s \in (2, 2^*)$; $f(x, u)u \geq 0$ for all $x \in \bar{\Omega}, u \in \mathbf{R}$, and $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \bar{\Omega}$.

(J₂) There exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x, u) \leq f(x, u)u, \qquad x \in \Omega, \qquad |u| \geq R,$$

where $F(x, u) = \int_0^u f(x, v)dv$.

(J₃) $f(x, u)$ is odd in u.

Theorem 5.7. *Assume* (J_1)–(J_3). *Then (5.36) has infinitely many sign-changing solutions.*

Let $E := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v \rangle := \int_\Omega (\nabla u \nabla v) dx$$

for $u, v \in E$ and the norm $\|u\| := \langle u,\ u \rangle^{1/2}$. Let

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

denote the distinct Dirichlet eigenvalues of $-\Delta$ on Ω with zero boundary value. Then each λ_k has finite multiplicity. The principal eigenvalue λ_1 is simple with a positive eigenfunction φ_1, and the eigenfunctions φ_k corresponding to λ_k ($k \geq 2$) are sign-changing. Let N_k denote the eigenspace of λ_k. Then $\dim N_k < \infty$. We fix k and let $E_k := N_1 \oplus \cdots \oplus N_k$. Let

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx, \qquad u \in E.$$

Then G is of $\mathbf{C}^1(E, \mathbf{R})$ and

$$\langle G'(u), v\rangle = \langle u, v\rangle - \int_\Omega f(x,u)v dx, \qquad v \in E,$$

$$G' = \mathbf{id} - K_G.$$

Lemma 5.8. *Assume* (J_1) *and* (J_2) *hold; then* G *satisfies the* (PS) *condition.*

Lemma 5.9. $G(u) \to -\infty$ *as* $\|u\| \to \infty$, *for all* $u \in E_k$.

Proof. Because $\dim E_k < \infty$, then by (J_2),

$$\frac{G(u)}{\|u\|^2} \leq \frac{1}{2} - \int_\Omega \frac{F(x,u)}{\|u\|^2} dx \to -\infty$$

as $\|u\| \to \infty, u \in E_k$. The lemma follows immediately. □

Consider another norm $\|\cdot\|_\star := \|\cdot\|_s$ of $E, s \in (2, 2^*)$. Then $\|u\|_s \leq C_\star \|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant and $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = E_{k-1} \oplus E_{k-1}^\perp$. Let

$$Q^\star(\rho) := \left\{ u \in E_{k-1}^\perp : \frac{\|u\|_s^s}{\|u\|^2} + \frac{\|u\|\|u\|_s}{\|u\| + D_\star \|u\|_s} = \rho \right\},$$

where $\rho, D_\star$ are fixed constants. Evidently, we have the following.

Lemma 5.10. $\|u\|_s \leq c_1, \forall u \in Q^\star(\rho)$; *where* $c_1 > 0$ *is a constant.*

By the assumptions, we may find a $C_F > 0$ such that

$$F(x,t) \leq \frac{1}{4}\lambda_1 |t|^2 + C_F |t|^s, \qquad \forall x \in \Omega, \qquad t \in \mathbf{R}; \tag{5.37}$$

here $2 < s < 2^*$. For any $a, b > 0$, there is a $c_2 = c_2(a,b) > 0$ such that

$$G(u) \leq a \quad \text{and} \quad \|u\|_s \leq b \quad \Rightarrow \quad \|u\| \leq c_2.$$

By Lemma 5.9,

$$\lim_{u \in Y, \|u\| \to \infty} G(u) = -\infty,$$

where $Y = E_k$. Then $(A_1^\star)$ and $(A_2^\star)$ are satisfied. We define

$$\sup_Y G := \beta.$$

Let

$$Q^{\star\star} := Q^\star(\rho) \cap G^\beta, \qquad \inf_{Q^{\star\star}} G := \gamma.$$

Set $\mathcal{P} := \{u \in E : u(x) \geq 0 \text{ for a.e. } x \in \Omega\}$. Then $\mathcal{P}$ $(-\mathcal{P})$ is the positive (negative) cone of E and weakly closed. By Lemma 5.4, there is a $\delta := \delta(\beta)$ such that $\operatorname{dist}(Q^{\star\star}, \mathcal{P}) := \delta(\beta) > 0$. We define

$$\mathcal{D}_0(\mu_0) := \{u \in E : \operatorname{dist}(u, \mathcal{P}) < \mu_0\},$$

where μ_0 is determined by the following lemma.

Lemma 5.11. *Under the assumptions of* (J_1) *and* (J_2)*, there exists a* $\mu_0 \in (0, \delta)$ *such that* $K_G(\pm\mathcal{D}_0(\mu_0)) \subset \pm\mathcal{D}_0(\mu_0)$.

Proof. The proof is quite similar to that of Lemma 2.29. □

Let

$$\mathcal{D} := -\mathcal{D}_0(\mu_0) \cup \mathcal{D}_0(\mu_0), \qquad \mathcal{S} := E \backslash \mathcal{D}.$$

We may assume

$$Q^{\star\star} := Q^\star(\rho) \cap G^\beta \subset \mathcal{S}. \tag{5.38}$$

Proof of Theorem 5.7. By Theorem 5.6,

$$\mathcal{K}[\gamma - \varepsilon, \beta + \varepsilon] \cap (E \backslash (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ small. That is, there exists a $u_k \in E\backslash(-\mathcal{P} \cup \mathcal{P})$ (sign-changing critical point) such that

$$G'(u_k) = 0, G(u_k) \in [\gamma - 1, \beta - 1].$$

Next we estimate the $\gamma = \inf_{Q^{\star\star}} G$. Similarly to Lemma 2.23, by choosing the constants $D_\star$ and ρ, for all $u \in Q^\star(\rho)$, we may get

$$\|u\| \geq \Lambda_s^* \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\} \min\{\rho, \rho^{1/(p-2)}\}.$$

By Lemma 2.26, for any $u \in Q^\star(\rho)$, we have that

$$G(u) \geq \frac{1}{8}(\Lambda_s^*)^2 T_1 T_2,$$

where Λ_s^*, T_1, T_2 are defined in (2.49)–(2.51) with p replaced by s, $\alpha \in (0,1)$ is a constant, and Λ_s^*, T_2 are independent of k. In particular,

$$T_1 := \min\{\lambda_k^{(1-\alpha)(s-2)}, \lambda_k^{(1-\alpha)}\} \to \infty, \quad \text{as } k \to \infty.$$

Therefore, $\gamma \to \infty$ as $k \to \infty$; hence the proof of Theorem 5.7 is finished. □

5.4 Composition Convergence Lemmas

Sometimes we need strong convergence results for the composition of nonlinearities with a weakly convergent sequence. The following result is the well-known Brézis–Lieb lemma (cf. Brézis and Lieb [69]).

Lemma 5.12. *Let Ω be an open subset of $\mathbf{R}^N$ and let $\{u_n\} \subset L^p(\Omega), \infty > p \geq 1$. Assume that $\{u_n\}$ is bounded in $L^p(\Omega)$ and that $u_n(x) \to u(x)$ a.e. on Ω. Then*

$$\lim_{n\to\infty} \int_\Omega (|u_n|^p - |u_n - u|^p - |u|^p)dx = 0.$$

Proof. For any $\varepsilon > 0$, there exists a $c_\varepsilon > 0$ such that

$$\big||a+b|^p - |a|^p\big| \leq \varepsilon|a|^p + c_\varepsilon|b|^p.$$

Let $D_n = |u_n|^p - |u_n - u|^p - |u|^p$; then

$$|D_n| \leq \varepsilon|u_n - u|^p + (c_\varepsilon + 1)|u|.$$

Let $L_n = (|D_n| - \varepsilon|u_n - u|^p)^+$; then $0 \leq L_n \leq (c_\varepsilon + 1)|u|^p$ and $L_n(x) \to 0$ a.e. on Ω as $n \to \infty$. By Fatou's lemma,

$$\|u\|_p \leq \liminf_{n\to\infty} \|u_n\|_p \leq C_0,$$

where C_0 is a constant. Therefore, Lebesgue's theorem implies that $\int_\Omega L_n(x)\, dx \to 0$ as $n \to \infty$. Note that

$$||u_n|^p - |u_n - u|^p - |u|^p| \leq L_n + \varepsilon|u_n - u|^p;$$

we have that

$$\limsup_{n\to\infty} \int_\Omega ||u_n|^p - |u_n - u|^p - |u|^p|dx \leq 2\varepsilon C_0.$$

This implies the conclusion of the lemma. □

The following result is known as Strauss' lemma (cf. Strauss [311]).

Lemma 5.13. *Let $F, H : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ be Carathéodory functions satisfying*

$$\sup_{x\in\mathbf{R}^N, |t|\leq c} |F(x,t)| < \infty, \qquad \sup_{x\in\mathbf{R}^N, |t|\leq c} |H(x,t)| < \infty \tag{5.39}$$

for each $c > 0$ *and*

$$\lim_{|t|\to\infty} \frac{F(x,t)}{H(x,t)} = 0 \qquad \text{uniformly for } x \in \mathbf{R}^N. \tag{5.40}$$

Assume that $\{u_n\}$ *and* u^* *are measurable functions on* $\mathbf{R}^N$ *such that*

$$C := \sup_n \int_{\mathbf{R}^N} |H(x,u_n)|dx < \infty \tag{5.41}$$

and that

$$\lim_{n\to\infty} F(x,u_n(x)) = u^* \quad \text{a.e. on} \quad \mathbf{R}^N. \tag{5.42}$$

Then for each bounded Borel set $\Omega \subset \mathbf{R}^N$, *we have*

$$\lim_{n\to\infty} \int_\Omega |F(x,u_n(x)) - u^*(x)|dx = 0. \tag{5.43}$$

Furthermore, if

$$\lim_{|t|\to 0} \frac{F(x,t)}{H(x,t)} = 0 \qquad \text{uniformly for } x \in \mathbf{R}^N \tag{5.44}$$

and

$$\lim_{|x|\to\infty} u_n(x) = 0 \qquad \text{uniformly in } n, \tag{5.45}$$

then $F(x,u_n(x)) \to u^*$ *in* $L^1(\mathbf{R}^N)$.

Proof. To prove the first conclusion of the lemma, it suffices to show that $\{F(x,u_n(x))\}$ is uniformly integrable. By (5.39) and (5.40), for any $\varepsilon > 0$, there exists a $c_\varepsilon > 0$ such that

$$|F(x,u_n(x))| \le c_\varepsilon + \varepsilon|H(x,u_n(x))| \quad \text{on } \mathbf{R}^N \text{ for all } n. \tag{5.46}$$

For each $R > 0$, we have that

$$\begin{aligned} &\text{meas}(\Omega \cap \{x \in \mathbf{R}^N : |F(x,u_n(x))| \ge R\}) \\ &\quad \le \frac{1}{R}\int_{\Omega\cap\{x\in\mathbf{R}^N:|F(x,u_n(x))|\ge R\}} |F(x,u_n(x))|dx \\ &\quad \le \frac{1}{R}\left(c_\varepsilon \,\text{meas}\; \Omega + \varepsilon \sup_n \int_{\mathbf{R}^N} |H(x,u_n(x))|dx\right) \\ &\quad \to 0 \end{aligned} \tag{5.47}$$

as $R \to \infty$. Combining (5.46), (5.47), and (5.41), we have that

$$\lim_{R\to\infty} \int_{\Omega\cap\{x\in\mathbf{R}^N:|F(x,u_n(x))|\geq R\}} |F(x,u_n(x))|dx = 0.$$

This implies the uniform integrability of $\{F(x,u_n(x))\}$.

To prove the second part of the lemma, take any $\varepsilon > 0$; by (5.44) and (5.45) we may find a $K > 0$ such that

$$|F(x,u_n(x))| \leq \varepsilon|H(x,u_n(x))| \tag{5.48}$$

for all $|x| \geq K$ and all n. By (5.46)–(5.48), we have that

$$\begin{aligned}
&\int_{\mathbf{R}^N} |F(x,u_n)|dx \\
&\quad \leq \int_{|x|\leq K} (c_\varepsilon + \varepsilon|H(x,u_n)|)dx + \varepsilon \int_{\mathbf{R}^N} |H(x,u_n)|dx \\
&\quad < \infty
\end{aligned}$$

uniformly for all n. Invoking Fatou's lemma, we observe that $u^* \in L^1(\mathbf{R}^N)$. By (5.48),

$$\int_{|x|\geq K} |u^*|dx \leq \varepsilon C. \tag{5.49}$$

By the first part of the lemma, we find an n_0 such that

$$\int_{|x|\leq K} |F(x,u_n(x)) - u^*|dx \leq \varepsilon, \quad \text{if } n \geq n_0. \tag{5.50}$$

Inequalities (5.48)–(5.50) imply that

$$\int_{\mathbf{R}^N} |F(x,u_n) - u^*|dx \leq 2\varepsilon C + \varepsilon \quad \text{if } n \geq n_0.$$

This completes the proof of the lemma. □

Lemma 5.14. *Let $g : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function satisfying $|g(x,t)| \leq c(|t| + |t|^{2^*-1})$ for all $x \in \mathbf{R}^N$ and $t \in \mathbf{R}$. Furthermore,*

$$\lim_{|t|\to\infty} \frac{g(x,t)}{|t|^{2^*-1}} = 0, \quad \textit{uniformly for } x \in \mathbf{R}^N. \tag{5.51}$$

If $u_n \to u^*$ weakly in $H^1(\mathbf{R}^N)$ and $u_n \to u^*$ a.e. on $\mathbf{R}^N$, then

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (G(x,u_n) - G(x,u^*) - G(x,u_n - u^*))dx = 0,$$

where $G(x,u) = \int_0^u g(x,s)ds$.

Proof. For any fixed $R > 0$, by the mean value theorem we have a $\beta \in (0, 1)$ depending on x, R such that

$$
\begin{aligned}
(5.52) \qquad & \int_{\mathbf{R}^N} G(x, u_n)dx \\
&= \int_{|x|\le R} G(x, u_n)dx + \int_{|x|\ge R} G(x, u_n - u^* + u^*)dx \\
&= \int_{|x|\le R} G(x, u_n)dx \\
&\quad + \int_{|x|\ge R} (G(x, u_n - u^*) + g(x, (u_n - u^*) + \beta u^*)u^*)dx.
\end{aligned}
$$

By Lemma 5.13, we know that

$$
(5.53) \qquad \lim_{n\to\infty} \int_{|x|\le R} (G(x, u_n) - G(x, u^*))dx = 0
$$

and that

$$
(5.54) \qquad \lim_{n\to\infty} \int_{|x|\le R} G(x, u_n - u^*)dx = 0.
$$

By the assumption on the growth of g and the Hölder inequality, we have that

(5.55)

$$
\begin{aligned}
&\left| \int_{|x|\ge R} g(x, (u_n - u^*) + \beta u^*)u^*)dx \right| \\
&\quad \le c \int_{|x|\ge R} (|\beta u^* + (u_n + u^*)||u^*| + |\beta u^* + (u_n - u^*)|^{2^*-1}|u^*|)dx \\
&\quad \le c \left(\int_{|x|\ge R} |u^*|^2 dx \right)^{1/2} \left(\int_{|x|\ge R} |\beta u^* + (u_n + u^*)|^2 dx \right)^{1/2} \\
&\qquad + c \left(\int_{|x|\ge R} |u^*|^{2^*} dx \right)^{1/2^*} \left(\int_{|x|\ge R} |\beta u^* + (u_n + u^*)|^{2^*} dx \right)^{(N+2)/2N}.
\end{aligned}
$$

Because

$$
\left(\int_{|x|\ge R} |u^*|^2 dx \right)^{1/2} \to 0, \qquad \left(\int_{|x|\ge R} |u^*|^{2^*} dx \right)^{1/2^*} \to 0 \quad \text{as } R \to \infty
$$

and $\{u_n\}$ is bounded in $H^1(\mathbf{R}^N)$, we know that (5.55) can be made small enough as R is sufficiently large. Finally, we write

$$\begin{aligned}(5.56)\quad &\left|\int_{\mathbf{R}^N}(G(x,u_n)-G(x,u^*)-G(x,u_n-u^*))dx\right| \\ &\le \left|\int_{|x|\le R}(G(x,u_n)-G(x,u^*))dx\right| + \left|\int_{|x|\le R}G(x,u_n-u^*)dx\right| \\ &+ \left|\int_{|x|\ge R}G(x,u^*)dx\right| + \left|\int_{|x|\ge R}g(x,(u_n-u^*)+\beta u^*)u^*dx\right|.\end{aligned}$$

By (5.53)–(5.56), by choosing R large enough and then letting $n\to\infty$, we may make (5.56) as small as we like. This completes the proof. □

Notes and Comments. Lemma 5.12 can be found in Willem [335]. Lemmas 5.13 and 5.14 are adopted from Chabrowski [88] with more applications there.

5.5 Improved Hardy–Poincaré Inequalities

We first establish the following improved Hardy–Poincaré inequalities.

Theorem 5.15. *Let Ω be a bounded subset of $\mathbf{R}^N$ $(N\ge 2)$. Then there is a constant $C>0$ such that*

$$(5.57)\qquad \int_\Omega\left(|\nabla u|^2-\frac{(N-2)^2}{4}\frac{u^2}{|x|^2}\right)dx\ge C\|u\|_2^2$$

for all $u\in H_0^1(\Omega)$.

Proof. Obviously, it is true when $N=2$. We assume that $N\ge 3$. We make a symmetrization that replaces Ω by a ball $B_R:=\{w\in H_0^1(\Omega):\|w\|\le R\}$ with the same volume as Ω and the function u by its symmetric rearrangement. It is well known that the rearrangement does not change the L^2-norm. Therefore, it is enough to prove the results in the symmetric case. Moreover, by a simple scaling, we assume that $R=1$. We define the new variable

$$v(r)=u(r)r^{(N-2)/2},\qquad r=|x|.$$

Then

$$\begin{aligned}&\int_{B_1}|\nabla u|^2dx-\frac{(N-2)^2}{4}\int_{B_1}\frac{u^2}{r^2}dx\\ &=N\omega_N\left(\int_0^1(v')^2rdr-(N-2)\int_0^1 v(r)v'(r)dr\right),\end{aligned}$$

where ω_N is the volume of B_1. Taking for instance $u \in \mathbf{C}_0^1(B_1)$, the last integral is zero and then

$$\int_{B_1} |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_{B_1} \frac{u^2}{r^2} dx = N\omega_N \int_0^1 (v')^2 r dr.$$

By the Poincaré inequality in two dimensions, we have

$$\int_0^1 (v')^2 r dr \geq C \int_0^1 v^2 r dr.$$

Because

$$\int_{B_1} |u|^2 dx = N\omega_N \int_0^1 v^2 r dr,$$

the proof is done by density. □

Theorem 5.16. *Let Ω be a bounded subset of $\mathbf{R}^N$ $(N \geq 3)$. Then for any $p \in [1,2)$ there exists a constant $C(p,\Omega) > 0$ such that*

$$\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4} \frac{u^2}{|x|^2} \right) dx \geq C(p,\Omega) \|u\|^2_{W^{1,p}(\Omega)} \tag{5.58}$$

for all $u \in H_0^1(\Omega)$.

Proof. We divide the proof into several steps.

Step 1. Assume Ω is a ball centered at zero and u is radial. By scaling we may assume that Ω is the unit ball $B = B_1(0)$. Because $u = u(r), r = |x|$, we have to show that there exists a constant $C = C(p,B) > 0$ such that

$$C \left(\int_0^1 |u'|^p r^{N-1} dr \right)^{2/p} \leq \int_0^1 \left(|u'|^2 - \frac{(N-2)^2}{4} \frac{u^2}{r^2} \right) r^{N-1} dr \tag{5.59}$$

holds for every smooth function $u(r)$ defined for $r \in [0,1]$ and $u(1) = 0$. By density, it is true for radial functions in $H_0^1(B)$. By changing the variables, we have that

$$\int_0^1 \left(|u'|^2 - \frac{(N-2)^2}{4} \frac{u^2}{r^2} \right) r^{N-1} dr = \int_0^1 |v'|^2 r dr, \tag{5.60}$$

where $v(r) = r^{(N-2)/2} u(r)$. On the other hand,

$$\int_0^1 |u'|^p r^{N-1} dr$$

$$= \int_0^1 \left| r^{-(N-2)/2} v'(r) - \frac{N-2}{2} r^{-N/2} v(r) \right|^p r^{N-1} dr$$

$$\le C(p)\int_0^1 |v'|^p r^{N-1-(N-2)p/2}dr + C(p,N)\int_0^1 |v|^p r^{N-1-Np/2}dr$$
$$:= A + Q.$$

Furthermore,

$$A \le \left(\int_0^1 |v'|^2 r dr\right)^{p/2} \left(\int_0^1 r^{N-1}dr\right)^{(2-p)/2} \le C\left(\int_0^1 |v'|^2 r dr\right)^{p/2}. \tag{5.61}$$

Choose $q > \max\{p, 4p/(N(2-p))\}$ and let

$$s = \left(N - 1 - \frac{Np}{2} - \frac{p}{q}\right)\frac{q}{q-p} > -1;$$

then

$$Q \le \left(\int_0^1 |v|^q r dr\right)^{p/q} \left(\int_0^1 r^s dr\right)^{(q-p)/q}. \tag{5.62}$$

By the standard embedding of $H_0^1(B_2)$ into $L^q(B_2)$ in the two-dimensional ball, we see that

$$\int_0^1 |v|^p r dr \le C(p)\left(\int_0^1 |v'|^2 r dr\right)^{q/2}.$$

Combining this with (5.60)–(5.62), we get (5.59). The result is proved for radial functions in a ball.

Step 2. Assume Ω is a ball centered at zero and u is nonradial. Once again, we assume that $\Omega = B$. By using spherical coordinates $x = (r, \sigma)$ in B, we decompose u into spherical harmonics to write

$$u = \sum_{k=0}^{\infty} u_k(r)e_k(\sigma),$$

where $\{e_k\}$ constitute an orthogonal basis of $L^2(S^{N-1})$ consisting of eigenfunctions of the Laplace–Beltrami operator, which has eigenvalues $c_k = k(N+k-2), \quad k \ge 0$. In particular, $e_0(\sigma) = 1$ and $u_0(r)$ is the projection of $u \in H_0^1(B)$ onto the space of radially symmetric functions. Then

$$\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4}\frac{u^2}{r^2}\right)dx \tag{5.63}$$
$$= N\omega_N \sum_{k=0}^{\infty}\int_0^1 \left(|u_k'|^2 - \frac{(N-2)^2}{4}\frac{u_k^2}{r^2} + c_k\frac{u_k^2}{r^2}\right)r^{N-1}dr,$$

where $N\omega_N$ is the Hausdorff measure of the $(N-1)$-dimensional unit sphere. By Step 1, the radial term

$$\int_0^1 \left(|u_0'|^2 - \frac{(N-2)^2}{4} \frac{u_0^2}{r^2} \right) r^{N-1} dr \geq C\|u_0\|^2_{W^{1,p}(B)}. \tag{5.64}$$

By Theorem 5.15, it is easy to estimate that

$$\int_0^1 \left(|u_k'|^2 - \frac{(N-2)^2}{4} \frac{u_k^2}{r^2} + c_k \frac{u_k^2}{r^2} \right) r^{N-1} dr \geq \frac{4c_k}{(N-2)^2} \int_0^1 |u_k'|^2 r^{N-1} dr. \tag{5.65}$$

Using the fact that $c_k \geq N-1 > 0$ for $k \geq 1$, the sum in (5.63) over $k = 1, 2, \dots$ is bounded below by $C\|u - u_0\|^2_{H_0^1(B)}$. Joining this and (5.63)–(5.65), the theorem follows in the ball.

Step 3. Assume Ω is a general domain. Assume that $0 \in \Omega$ and $B_a(0) \subset \Omega$. We introduce a smooth cutoff function ϕ such that $0 \leq \phi(x) \leq 1$ with $\phi(x) = 1$ for all $x \in B_{a/2}(0)$ and $\phi(x) = 0$ when $|x| \geq a$. Let $w_1 = u\phi$ and $w_2 = u - w_1$; we have that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4} \frac{u^2}{r^2} \right) dx \\
&\quad = \int_\Omega \left(|\nabla w_1|^2 - \frac{(N-2)^2}{4} \frac{w_1^2}{r^2} \right) dx + \int_\Omega \left(|\nabla w_2|^2 - \frac{(N-2)^2}{4} \frac{w_2^2}{r^2} \right) dx \\
&\quad +2 \int_\Omega \left(\nabla w_1 \cdot \nabla w_2 - \frac{(N-2)^2}{4} \frac{w_1 w_2}{r^2} \right) dx.
\end{aligned} \tag{5.66}$$

Because $(1-2\phi)\nabla\phi = 0$ on $\partial(B_a \backslash B_{a/2})$, we have that

$$\int_\Omega u \nabla u \cdot ((1-2\phi)\nabla\phi) dx = -\frac{1}{2} \int_{B_a \backslash B_{a/2}} u^2 \mathrm{div}((1-2\phi)\nabla\phi) dx.$$

Therefore,

$$\begin{aligned}
&\int_\Omega \nabla w_1 \cdot \nabla w_2 dx \\
&\quad = \int_\Omega \phi(1-\phi)|\nabla u|^2 dx - \int_\Omega |\nabla\phi|^2 u^2 dx + \int_\Omega u\nabla u \cdot ((1-2\phi)\nabla\phi) dx \\
&\quad \geq -C \int_\Omega u^2 dx.
\end{aligned} \tag{5.67}$$

Moreover, note that the support of w_2 is disjoint with the origin; we have that

(5.68)
$$\int_\Omega \frac{w_2^2}{|x|^2}dx + \int_\Omega \frac{w_1 w_2}{|x|^2}dx \le C\int_\Omega u^2 dx.$$

Furthermore, if we apply the result already proved in a ball to $w_1 \in H_0^1(B_a)$, we have that

(5.69)
$$\int_\Omega \left(|\nabla w_1|^2 - \frac{(N-2)^2}{4}\frac{w_1^2}{r^2}\right) dx \ge C\|w_1\|^2_{W^{1,p}(\Omega)}.$$

Combining (5.66)–(5.69), we obtain the conclusion of the theorem. □

The constant $((N-2)^2/4)$ $(N \ge 3)$ is known as the best Hardy constant that is not attained in $H_0^1(\Omega)$.

Notes and Comments. Theorem 5.15 was established by Brézis and Vázquez in [73]. Theorem 5.16 can be found in Vázquez and Zuazua [330]. Related studies were made in Catrina and Wang [83] and Wang and Willem [333] where Caffarelli–Kohn–Nirenberg-type inequalities with remainder terms were established. We also refer the readers to the paper by Adimurthi et al. [3] where the improved Hardy–Sobolev inequality was given in $W_0^{1,p}(\Omega)$.

5.6 Equations with Critical Hardy Constant

We consider the following nonlinear elliptic equation with inverse-square potential and critical Hardy constant.

(5.70)
$$\begin{cases} -\Delta u - \dfrac{(N-2)^2}{4}\dfrac{u}{|x|^2} = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$ $(N \ge 3)$. Assume

($\mathbf{I_1}$) $f : \bar\Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with subcritical growth:
$$|f(x,u)| \le c(1 + |u|^{s-1}) \quad \text{for all } u \in \mathbf{R} \quad \text{and} \quad x \in \bar\Omega,$$
where $s \in (2, 2^*)$; $f(x,u)u \ge 0$ for all $x \in \bar\Omega, u \in \mathbf{R}$ and $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \bar\Omega$.

($\mathbf{I_2}$) There exist $\mu > 2$ and $R > 0$ such that
$$0 < \mu F(x,u) \le f(x,u)u, \qquad x \in \Omega, \qquad |u| \ge R,$$
where $F(x,u) = \int_0^u f(x,v)dv$.

($\mathbf{I_3}$) $f(x,u)$ is odd in u.

Theorem 5.17. *Assume (I_1)–(I_3). Then (5.70) has infinitely many sign-changing solutions.*

To prove the above theorem, we introduce the working space E which is obtained by the completion of $\mathbf{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4} \frac{u^2}{|x|^2} \right) dx \right)^{1/2}$$

associated with the inner product

$$\langle u, v \rangle = \int_\Omega \left(\nabla u \cdot \nabla v - \frac{(N-2)^2}{4} \frac{uv}{|x|^2} \right) dx.$$

Consider the following eigenvalue problem

$$\begin{cases} -\Delta u - \dfrac{(N-2)^2}{4} \dfrac{u}{|x|^2} = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.71}$$

The first eigenvalue of (5.71) is given by

$$\lambda_1 = \inf\{\|u\|^2 : u \in E, \|u\|_2 = 1\}.$$

By Proposition 1.16, $E \hookrightarrow W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ for $p \to 2^-$. The minimizing sequence is compact in $L^2(\Omega)$. By standard argument, we may assume that the first eigenfunction ϕ_1 is positive in Ω. The second eigenvalue is given by

$$\lambda_2 = \inf \left\{ \|u\|^2 : u \in E, \int_\Omega u\phi_1 dx = 0, \|u\|_2 = 1 \right\}$$

which possesses a sign-changing eigenfunction ϕ_2. Similarly, we can characterize the nth eigenvalue λ_n with a sign-changing eigenfunction. By standard elliptic theory, $\lambda_n \to \infty$ as $n \to \infty$.

For s given in (I_1), $s < 2N/(N-2)$, we may choose p such that $s < pN/(N-p), p < 2$. By Proposition 1.16,

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^t(\Omega), \qquad \forall t < Np/(N-p). \tag{5.72}$$

Let

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx, \qquad u \in E.$$

Then G is of $\mathbf{C}^1(E, \mathbf{R})$ and

$$\langle G'(u), v \rangle = \langle u, v \rangle - \int_\Omega f(x,u)v dx, \qquad v \in E,$$

$$G' = \mathbf{id} - K_G.$$

Lemma 5.18. *Assume* (I_1) *and* (I_2) *hold; then* G *satisfies the* (PS) *condition.*

Proof. Assume that $\{u_n\}$ is a (PS) sequence; $\|G'(u_n)\| \to 0$ and $\{G(u_n)\}$ is bounded. A routine argument implies that $\{\|u_n\|\}$ is bounded. By Theorem 5.16 and (5.72), $\{u_n\}$ is compact in $L^s(\Omega)$. By (I_1),

$$\begin{aligned}\|u_n - u_m\|^2 &= \int_\Omega |f(x,u_n) - f(x,u_m)||u_n - u_m|dx + o(1)\\ &\leq c\left(\int_\Omega |u_n - u_m|^s dx\right)^{1/s} + o(1)\\ &\to 0.\end{aligned}$$

This completes the proof of the lemma. □

Let N_k denote the eigenspace of λ_k; then $\dim N_k < \infty$. Let $E_k := N_1 \oplus \cdots \oplus N_k, k \geq 2$.

Lemma 5.19. $G(u) \to -\infty$ *as* $\|u\| \to \infty$*, for all* $u \in E_k$*.*

Proof. Because $\dim E_k < \infty$, then by (I_2),

$$\frac{G(u)}{\|u\|^2} \leq \frac{1}{2} - \int_\Omega \frac{F(x,u)}{\|u\|^2} dx \to -\infty$$

as $\|u\| \to \infty, u \in E_k$. The lemma follows immediately. □

Proof of Theorem 5.17. This is similar to the proof of Theorem 5.7; we leave it to the readers. □

Notes and Comments. If $(N-2)^2/4$ is replaced by a constant μ which is less than the best Hardy constant, Equation (5.70) is called a subcritical potential equation. The existence of solutions for this case was studied in Cao and Peng [78], Chen [102], Egnell [136], and Ferrero and Gazzola [145] (see also Sintzoff and Willem [302] for a more general equation with unbounded coefficients). If $N = 2$, a nonlinear elliptic problem (with singular potentials) was considered in Caldiroli and Musina [77] and Shen et al. [292]. The sign-changing solutions have not been considered there.

5.7 Critical Sobolev–Hardy Exponent Cases

Consider the Dirichlet boundary value problem with critical Sobolev–Hardy exponents and singular terms:

$$(5.73) \qquad -\Delta u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s} u \quad \text{in} \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $2 < r, q \leq 2^*(s)$, Ω is a smooth bounded domain of $\mathbf{R}^N, N > 2$, $0 < s < 2$ and $2^*(s) := (2(N-s))/(N-2)$ is the Sobolev–Hardy exponent.

We have the following main theorems in this section. Theorems 5.20 and 5.21 concern (5.73) with Sobolev–Hardy critical singular terms.

Theorem 5.20. *Assume* $2 < r < 2^*(s), q = 2^*(s)$. *Then there exists a* $\mu_0 > 0$ *such that Equation (5.73) has a sign-changing solution for any* $\lambda > 0$, $\mu \in (0, \mu_0)$.

Theorem 5.21. *Assume* $2 < r < 2^*(s), q = 2^*(s)$. *Then for any* $\lambda > 0$, *Equation (5.73) has an unbounded sequence of sign-changing solutions* (μ_k, u_k) *satisfying*

$$\frac{1}{2}\int_\Omega |\bigtriangledown u_k|^2 dx - \frac{\lambda}{r}\int_\Omega |u_k|^r dx - \frac{\mu_k}{2^*(s)}\int_\Omega \frac{|u_k|^{2^*(s)}}{|x|^s} dx \to \infty, \qquad k \to \infty.$$

Theorems 5.22 and 5.23 concern (5.73) with Sobolev critical nonsingular terms and subcritical singular terms.

Theorem 5.22. *Assume* $2 < q < 2^*(s), r = 2^*(s)$. *Then there exists a* $\lambda_0 > 0$ *such that Equation (5.73) has a sign-changing solution for any* $\mu > 0$, $\lambda \in (0, \lambda_0)$.

Theorem 5.23. *Assume* $2 < q < 2^*(s), r = 2^*(s)$. *Then for any* $\mu > 0$, *Equation (5.73) has an unbounded sequence of sign-changing solutions* (λ_k, u_k) *satisfying*

$$\frac{1}{2}\int_\Omega |\bigtriangledown u_k|^2 dx - \frac{\lambda_k}{2^*(s)}\int_\Omega |u_k|^{2^*(s)} dx - \frac{\mu}{q}\int_\Omega \frac{|u_k|^q}{|x|^s} dx \to \infty, \qquad k \to \infty.$$

Next we provide a result for the existence of infinitely many sign-changing solutions to Equation (5.73) with Sobolev–Hardy subcritical and singular terms.

Theorem 5.24. *Assume* $2 < q, r < 2^*(s)$. *Then for any* $\mu > 0, \lambda > 0$, *Equation (5.73) has an unbounded sequence of sign-changing solutions* (u_k) *satisfying*

$$\frac{1}{2}\int_\Omega |\bigtriangledown u_k|^2 dx - \frac{\lambda}{r}\int_\Omega |u_k|^r dx - \frac{\mu}{q}\int_\Omega \frac{|u_k|^q}{|x|^s} dx \to \infty, \qquad k \to \infty.$$

Let $E := H_0^1(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |\bigtriangledown u|^2 dx\right)^{1/2}$. We define

$$G_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\bigtriangledown u|^2 dx - \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{\mu}{q}\int_\Omega \frac{|u|^q}{|x|^s} dx.$$

Then $G_{\lambda,\mu} \in \mathbf{C}^1(H_0^1(\Omega), \mathbf{R})$. Recall the Sobolev–Hardy inequality, which is essentially due to Caffarelli et al. [76]; there is a constant $C > 0$ such that

$$(5.74)\quad C\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/(2^*(s))} \leq \int_\Omega |\nabla u|^2 dx \quad \text{for all } u \in E := H_0^{1,2}(\Omega).$$

The best Sobolev–Hardy constant (i.e., the largest constant C satisfying the above inequality for all $u \in H_0^{1,2}(\Omega)$) is given by

$$(5.75)\qquad \mu_s := \mu_s(\Omega) := \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\bigtriangledown u|^2 dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}.$$

Lemma 5.25. *Let μ_s be the Sobolev–Hardy constant given by (5.75). Then*

(1) If $2 < r < 2^*(s), q = 2^*(s)$, then for any $\lambda > 0$ and any $\mu > 0$, $G_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all

$$c < \frac{2-s}{2(n-s)}\left(\frac{\mu_s^{n-s}}{\mu^{N-2}}\right)^{1/(2-s)}.$$

(2) If $2 < q < 2^*(s), r = 2^*(s)$, then for any $\lambda > 0$ and any $\mu > 0$, $G_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all

$$c < \frac{1}{N}\left(\frac{\mu_0^N}{\lambda^{N-2}}\right)^{1/2}.$$

(3) If $2 < q, r < 2^*(s)$, then for any $\lambda > 0$ and any $\mu > 0$, $G_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all c.

Proof. (1) Assume that $\{u_n\}$ is a sequence in E satisfying

$$(5.76)\qquad G_{\lambda,\mu}(u_n) \to c < \frac{2-s}{2(N-s)}\left(\frac{\mu_s^{N-s}}{\mu^{N-2}}\right)^{1/(2-s)}, \qquad G'_{\lambda,\mu}(u_n) \to 0.$$

Then

$$(5.77)\quad \langle G'_{\lambda,\mu}(u_n), u_n\rangle = \int_\Omega |\nabla u_n|^2 dx - \mu \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx - \lambda \int_\Omega |u_n|^r dx;$$

hence,

$$\begin{aligned}(5.78)\qquad & 2c + 1 + o(1)\|u_n\| \\ & \geq 2G_{\lambda,\mu}(u_n) - \langle G'_{\lambda,\mu}(u_n), u_n\rangle \\ & \geq \mu\left(1 - \frac{2}{2^*}\right)\int_\Omega \frac{|u_n|^{2^*}}{|x|^s}d + \lambda\left(1 - \frac{2}{r}\right)\int_\Omega |u_n|^r dx.\end{aligned}$$

Combining (5.77) and (5.78), $\{u_n\}$ is bounded in E. By Lemma 5.12 and the Sobolev–Hardy embedding theorem, it is easy to show that

$$\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx - \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx + o(1). \tag{5.79}$$

Furthermore, note that

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx$$

is uniformly bounded in n and that $w/|x|^{s/2^*} \in L^{2^*(s)}(\Omega)$ for any $w \in H_0^{1,2}(\Omega)$. We may show that

$$\int_\Omega \frac{|u_n|^{2^*(s)-2} u_n}{|x|^s} w dx \to \int_\Omega \frac{|u|^{2^*(s)-2} u}{|x|^s} w dx. \tag{5.80}$$

We now assume that $u_n \to u$ weakly in E. For any $v \in E$, by (5.80), we have that $\langle G'_{\lambda,\mu}(u), v\rangle = 0$. Therefore,

$$0 = \langle G'_{\lambda,\mu}(u), u\rangle = \int_\Omega \left(|\nabla u| - \lambda |u|^r - \mu \frac{|u|^{2^*(s)}}{|x|^s} \right) dx.$$

It follows that

$$G_{\lambda,\mu}(u) \geq 0. \tag{5.81}$$

Therefore, by the assumption of item (1),

$$\begin{aligned} G_{0,\mu}(u_n - u) &= G_{\lambda,\mu}(u_n) - G_{\lambda,\mu}(u) + o(1) \\ &\leq G_{\lambda,\mu}(u_n) + o(1) \leq c < \frac{2-s}{2(N-s)} \left(\frac{\mu_s^{N-s}}{\mu^{N-2}} \right)^{1/(2-s)}. \end{aligned} \tag{5.82}$$

Because

$$\begin{aligned} o(1) &= \langle G'_{\lambda,\mu}(u_n), u_n - u\rangle \\ &= \langle G'_{\lambda,\mu}(u_n) - G'_{\lambda,\mu}(u), u_n - u\rangle \\ &= \int_\Omega \left(|\nabla u_n - \nabla u|^2 - \mu \frac{|u_n - u|^{2^*(s)}}{|x|^s} \right) dx + o(1). \end{aligned} \tag{5.83}$$

Combining (5.82) and (5.83),

$$\left(\frac{1}{2} - \frac{1}{2^*(s)} \right) \|\nabla u_n - \nabla u\|_2^2 \leq c < \frac{2-s}{2(N-s)} \left(\frac{\mu_s^{N-s}}{\mu^{N-2}} \right)^{1/(2-s)}. \tag{5.84}$$

By the Sobolev–Hardy inequality and (5.84), we finally have

$$
\begin{aligned}
o(1) &= \int_\Omega \left(|\nabla u_n - \nabla u|^2 - \mu \frac{|u_n - u|^{2^*(s)}}{|x|^s} \right) dx \\
&\geq \int_\Omega |\nabla u_n - \nabla u|^2 dx - \mu(\mu_s)^{(-2^*(s))/2} \left(\int_\Omega |\nabla u_n - \nabla u|^2 dx \right)^{(2^*(s))/2} \\
&\geq c \int_\Omega |\nabla u_n - \nabla u|^2 dx.
\end{aligned}
$$

Thus, $u_n \to u$ in E.

(2) This is similar to Case (1).

(3) It is easier than that of Cases (1) and (2). Because $2 < q, r < 2^*(s)$ are subcritical, the compactness of the Sobolev–Hardy embedding and Sobolev embedding can be applied. $\square$

Next we just give the proof of Theorem 5.24; the others can be done analogously. We leave them to the readers. For simplicity, we write $G_{\lambda,\mu} = G$.

Denote by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$$

the eigenvalues of $-\Delta$ with zero boundary value. Then the principal eigenvalue λ_1 is simple with positive eigenfunction φ_1, and eigenfunction φ_k corresponding to $\lambda_k (k \geq 2)$ is sign-changing. Let N_k denote the eigenspace of λ_k; then $\dim N_k < \infty$. Let $E_k := N_1 \oplus \cdots \oplus N_k,\ k \geq 2$. We use $\mu_{s,q}(\Omega)$ to denote the best Sobolev–Hardy constant:

$$\mu_{s,q} := \inf_{u \in E, u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega \frac{|u|^q}{|x|^s} dx \right)^{2/q}}.$$

Let

$$Q^\star(\rho) = \left\{ u \in E_{k-1}^\perp : \begin{array}{c} \dfrac{\mu}{q} \dfrac{\int_\Omega \frac{|u|^q}{|x|^s} dx}{\|u\|^2} + \mu \dfrac{\|u\| \left(\int_\Omega \frac{|u|^q}{|x|^s} dx \right)^{1/q}}{\|u\| + D_\star \left(\int_\Omega \frac{|u|^q}{|x|^s} dx \right)^{1/q}} \\ + \dfrac{\lambda}{r} \dfrac{\|u\|_r^r}{\|u\|^2} + \lambda \dfrac{\|u\| \|u\|_r}{\|u\| + D_\star \|u\|_r} = \dfrac{1}{4} \end{array} \right\},$$

where $D_\star$ is a fixed constant. Evidently, we have

Lemma 5.26. $\left(\int_\Omega (|u|^q/|x|^s) dx \right)^{1/q} \leq c_1, \|u\|_r \leq c_1,\ \forall u \in Q^\star(\rho)$; *where* $c_1 > 0$ *is a constant.*

Lemma 5.27. *For any $a, b, c > 0$, there is a $d > 0$ such that*

$$G(u) \leq a, \qquad \|u\|_r \leq b \quad \textit{and} \quad \left(\int_\Omega \frac{|u|^q}{|x|^s} dx\right)^{1/q} \leq c \quad \Rightarrow \quad \|u\| \leq d.$$

Lemma 5.28. *For any $a > 0$, we have that*

$$\operatorname{dist}(Q^\star(\rho) \cap G^a, \mathcal{P}) := \delta(a) > 0.$$

Proof. Note that

$$\begin{aligned}
&\int_\Omega \frac{|u|^q}{|x|^s} dx \\
&\leq C_H \|u\|^s \left(\int_\Omega |u|^\sigma dx\right)^{(2-s)/2} \\
&= C_H \|u\|^s \|u\|_\sigma^{q-s}
\end{aligned}$$

and

$$\frac{\int_\Omega \frac{|u|^q}{|x|^s} dx}{\|u\|^2} \leq C'_H \|u\|_\sigma^{q-2},$$

where $\sigma = ((2q - 2s)/(2 - s)) \in (2, 2^*)$, and C_H, C'_H are constants from Hardy and Sobolev inequalities. Then the rest is similar to that of Lemma 5.4. □

Let

$$\Gamma_Y^* = \{h : h \in \mathbf{C}(\Theta_Y, E), \;\; h|_{\partial\Theta_Y} = \mathbf{id}, \;\; h \;\text{ is odd}\},$$

where

$$\Theta_Y := \{u \in Y : \|u\| \leq R_Y\}, \qquad R_Y > 0, Y = E_k.$$

We assume R_Y large enough.

Lemma 5.29. $h(\Theta_Y) \cap Q^\star(\rho) \neq \emptyset, \;\forall\, h \;\in \Gamma_Y^*$.

Proof. Let

$$\begin{aligned}
\beta^*(u) := \frac{\mu}{q} \frac{\int_\Omega \frac{|u|^q}{|x|^s} dx}{\|u\|^2} + \frac{\lambda}{r} \frac{\|u\|_r^r}{\|u\|^2} + \lambda \frac{\|u\| \|u\|_r}{\|u\| + D_\star \|u\|_r} \\
+ \mu \frac{\|u\| \left(\int_\Omega \frac{|u|^q}{|x|^s} dx\right)^{1/q}}{\|u\| + D_\star \left(\int_\Omega \frac{|u|^q}{|x|^s} dx\right)^{1/q}}
\end{aligned}$$

if $u \neq 0$ and $\beta^*(0) = 0$. Then $\beta^* : E \to E$ is continuous. Let

$$U := \{u \in Y : \beta^*(h(u)) < 1/4\} \cap \{u \in Y : \|u\| < R_Y\}.$$

Then U is a neighborhood of zero in Y. Let $P : E \to M_1 := E_{k-1}$ be the projection; then $P \circ h : \partial U \to M_1$ is odd and continuous. By the Borsuk–Ulam theorem, we have that $P \circ h(u) = 0$ for some $u \in \partial U$. Hence, $h(u) \in M = E_{k-1}^{\perp}$. We claim $u \notin \partial\{u \in Y : \|u\| < R_Y\}$. Otherwise, $\|u\| = R_Y$ and then $h(u) = u, P(u) = 0$. It follows that $\beta^*(u) \le \frac{1}{4}$. But this is impossible if we choose R_Y large enough. So, our claim is true. It means

$$u \in \partial \left\{ u \in Y : \beta^*(h(u)) < \frac{1}{4} \right\}, \qquad \|u\| \le R_Y, \qquad u \in Y.$$

Hence, $h(u) \in M, \beta^*(h(u)) = \frac{1}{4}$. Hence, $h(u) \in Q^\star(\rho)$. □

Lemma 5.30. $\lim\limits_{u \in Y, \|u\| \to \infty} G(u) = -\infty,\ \sup\limits_{Y} G := \beta < \infty.$

Consider $G'(u) = G'_{\lambda,\mu}(u) = u - K_G u, u \in E$, where

$$K_G(u) = (-\Delta)^{-1}(\lambda|u|^{r-2}u + \mu|u|^{q-2}u/|x|^s).$$

Lemma 5.31. *Assume* $2 < q, r < 2^*(s)$. *Then there exists a* $\mu_0 \in (0, \delta)$ *such that* $K_G(\pm\mathcal{D}_0(\mu_0)) \subset \pm\mathcal{D}_0(\mu_0)$, *where* $\delta = \delta(\beta)$ *comes from Lemmas 5.28 and 5.30.*

Proof. The proof is similar to that of Lemma 2.29. However, because it involves the Hardy potential, there are still something to be done. First, we have, for any $u \in E_m$, that

$$\begin{aligned} \|u^\pm\|_t &= \min_{w \in (\mp\mathcal{P})} \|u - w\|_t \\ &\le C_t \min_{w \in (\mp\mathcal{P})} \|u - w\| \\ &= C_t \mathrm{dist}(u, \mp\mathcal{P}) \end{aligned}$$

for each $t \in [2, 2^*]$; $C_t > 0$ is a constant depending on t. By the Hardy inequality, we have that

$$\begin{aligned} \left\| \frac{u^\pm}{|x|} \right\|_2 &= \min_{w \in (\mp\mathcal{P})} \left\| \frac{u}{|x|} - w \right\|_2 \\ &\le \min_{w \in (\mp\mathcal{P})} \left\| \frac{u}{|x|} - \frac{w}{|x|} \right\|_2 \\ &\le c\, \mathrm{dist}(u, \mp\mathcal{P}). \end{aligned}$$

Let $v = K_G(u)$. Therefore,

$$
\begin{aligned}
&\operatorname{dist}(v, \mp\mathcal{P})\|v^{\pm}\| \\
&\quad \le \|v^{\pm}\|^2 \\
&\quad = \langle v, v^{\pm}\rangle \\
&\quad = \int_{\Omega} \left(\lambda|u|^{r-2} + \mu\frac{|u|^{q-2}}{|x|^s}\right) uv^{+} dx \\
&\quad \le \int_{\Omega} \left(\lambda|u^{\pm}|^{r-1} + \mu\frac{|u^{\pm}|^{q-1}}{|x|^s}\right) |v^{\pm}| dx \\
&\quad \le c\|u^{+}\|_r^{r-1}\|v^{\pm}\|_r + \mu\left(\int_{\Omega}\frac{|u|^q}{|x|^s}dx\right)^{(q-1)/q}\left(\int_{\Omega}\frac{|v^{\pm}|^q}{|x|^s}dx\right)^{1/q} \\
&\quad \le c\|u^{\pm}\|_r^{r-1}\|v^{\pm}\|_r \\
&\qquad +\mu\left(\left(\int_{\Omega}\frac{|u^{\pm}|^2}{|x|^2}dx\right)^{s/2}\left(\int_{\Omega}|u^{\pm}|^{(2(q-s))/(2-s)}dx\right)^{(2-s)/2}\right)^{(q-1)/q} \\
&\qquad \times\left(\int_{\Omega}\frac{|v^{\pm}|^q}{|x|^s}dx\right)^{1/q} \\
&\quad \le (c\ \operatorname{dist}(u, \mp\mathcal{P})^{r-1} + c\ \operatorname{dist}(u, \mp\mathcal{P})^{(q-1)})\|v^{\pm}\|.
\end{aligned}
$$

Because $r-1>1, q-1>1$, we may choose $\mu_0 < \delta$ small enough so that $\operatorname{dist}(K_G(u), \mp\mathcal{P}) \le \mu_0$ for every $u \in \mp\mathcal{D}_0(\mu_0)$. The conclusion follows. □

Proof of Theorem 5.24. Let

$$\mathcal{D} := -\mathcal{D}_0(\mu_0) \cup \mathcal{D}_0(\mu_0), \qquad \mathcal{S} := E\backslash\mathcal{D}.$$

By Lemma 5.28, we may assume $Q^{\star\star} := Q^{\star}(\rho) \cap G^{\beta} \subset \mathcal{S}$. Here, $Q^{\star\star}$ is a bounded set. Let $\inf_{Q^{\star\star}} G := \gamma$. It is easy to check that $Q^{\star\star} \cap Y \neq \emptyset$. Then $\beta \ge \gamma$. It is easy to see that Theorem 5.6 is also true for the above $Q^{\star\star}$. Similar to the proof of Theorem 5.7, $\inf_{Q^{\star\star}} G := \gamma \to \infty$ as $k \to \infty$ if we choose an appropriate $D_{\star}$. Hence, we may get the conclusion of Theorem 5.24. □

Notes and Comments. Lemma 5.25 was due to Ghoussoub and Yuan [158, Theorem 4.1]. The existence result of infinitely many solutions of Theorem 4.5 was first proved in Ghoussoub and Yuan [158], where the authors claimed that one solution among them is positive and another one is sign-changing; the signs of others had not been decided there. Here we give a confirmation. In [158], the quasilinear type of (5.73) is considered and sign-changing solutions were obtained by Ghoussoub's dual methods. In [158], for semilinear (5.73), they needed either λ large enough or $n > (2r+2)/(r-1)$ corresponding to the case of Theorems 5.20 and 5.21 and that $n > (2(q-s)+2)/(q-1)$ corresponding to the case of Theorems 5.22 and 5.23. Finally, we refer readers to papers by Garcia Azorero and Peral Alonso [155], Ferrero and Gazzola [145], and Ruiz and Willem [265] on the elliptic (parabolic) equations with the Hardy potential, where the existence of positive solutions was studied.

Chapter 6
Parameter Dependence

As we have seen in the previous chapters, the (PS)-type compactness condition (or weak (PS)) plays a crucial role. To verify this condition, one has to prove the boundedness of the (PS) (or weak (PS)) sequence which requires some special assumptions and procedures. These of course are severe restrictions. They strictly control the growth of the nonlinearity. In this chapter, we show the readers how to get a bounded and sign-changing Palais–Smale sequence directly from the linking. The classical Palais–Smale compactness condition and its variants are completely unnecessary.

6.1 Bounded Sign-Changing (PS)-Sequences

Let E be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let A, B be two closed subsets of E. Suppose that $G \in \mathbf{C}^1(E, \mathbf{R})$ is of the form:

$$G(u) := \frac{1}{2}\|u\|^2 - J(u), \qquad u \in E,$$

where $J \in \mathbf{C}^1(E, \mathbf{R})$ maps bounded sets to bounded sets. Define

$$G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - J(u), \qquad \lambda \in \Lambda := \left(\frac{1}{2}, 1\right).$$

Let $\mathcal{K}_\lambda := \{u \in E : G'_\lambda(u) = 0\}$ denote the set of all critical points of G_λ. The gradient $G'_\lambda(u) = \lambda u - J'(u)$, where $J' : E \to E$ is a continuous operator independent of λ. Let $\tilde{E}_\lambda := E \backslash \mathcal{K}_\lambda$. Let $\mathcal{P}$ $(-\mathcal{P})$ denote a closed convex positive (negative) cone of E. Assume

($\mathbf{A_1}$) There exists a $\mu_0 > 0$ such that

$$\operatorname{dist}(J'(u), \pm\mathcal{P}) \leq \frac{1}{5}\operatorname{dist}(u, \pm\mathcal{P})$$

for all $u \in E$ with $\operatorname{dist}(u, \pm\mathcal{P}) < \mu_0$.

W. Zou, *Sign-Changing Critical Point Theory*, doi: 10.1007/978-0-387-76658-4,

For the fixed $\mu_0 > 0$, we define

$$\pm\mathcal{D}_0 := \{u \in E : \mathrm{dist}(u, \pm\mathcal{P}) < \mu_0\}, \tag{6.1}$$

$$\mathcal{D} := \mathcal{D}_0 \cup (-\mathcal{D}_0), \tag{6.2}$$

$$\mathcal{S} = E \backslash \mathcal{D}, \qquad \pm\mathcal{D}_1 := \{u \in E : \mathrm{dist}(u, \pm\mathcal{P}) < \mu_0/2\}. \tag{6.3}$$

Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are open convex, $\mathcal{D}$ is open, $\pm\mathcal{P} \subset \pm\mathcal{D}_1 \subset \pm\mathcal{D}_0$, and $\mathcal{S}$ is closed. Obviously, we have

$$J'(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_1. \tag{6.4}$$

Let Φ be the class of contractions of E defined in (2.1) of Chapter 2. Define

$$\Phi^* := \{\Gamma \in \Phi : \Gamma(t, \mathcal{D}) \subset \mathcal{D} \ \text{ for all } t \in [0,1]\}. \tag{6.5}$$

Then $\Gamma(t,u) = (1-t)u \in \Phi^*$.

($\mathbf{A_2}$) Let A be a bounded subset of E and link a subset B of E; $B \subset \mathcal{S}$ and

$$a_0(\lambda) := \sup_A G_\lambda \leq b_0(\lambda) := \inf_B G_\lambda \qquad \text{for any } \lambda \in \Lambda.$$

Theorem 6.1. *Assume that (A_1) and (A_2) hold. Define*

$$c_0(\lambda) := \inf_{\Gamma \in \Phi^*} \sup_{\Gamma([0,1],A)\cap\mathcal{S}} G_\lambda(u);$$

then

$$c_0(\lambda) \in \Big[b_0(\lambda), \sup_{(t,u)\in[0,1]\times A} G_\lambda((1-t)u) \Big].$$

Moreover, for almost all $\lambda \in \Lambda$,

(1) If $c_0(\lambda) > b_0(\lambda)$, then there is a sequence $\{u_m\}$ depending on λ such that

$$\sup_m \|u_m\| < \infty, \qquad u_m \in \mathcal{S}, \qquad G'_\lambda(u_m) \to 0, \qquad G_\lambda(u_m) \to c_0(\lambda).$$

(2) If $c_0(\lambda) = b_0(\lambda)$, then there is a sequence $\{u_m\}$ depending on λ such that

$$\sup_m \|u_m\| < \infty, \qquad \mathrm{dist}(u_m, \mathcal{S}) \to 0, \qquad G'_\lambda(u_m) \to 0,$$

$$G_\lambda(u_m) \to c_0(\lambda).$$

Proof. Because A links B, we readily have $c_0(\lambda) \geq b_0(\lambda)$. In fact, for any $\Gamma \in \Phi^*$ we first observe that $\Gamma([0,1],A) \cap B \neq \emptyset$; recall that $B \subset \mathcal{S}$. Then we have $\Gamma([0,1],A) \cap \mathcal{S} \neq \emptyset$. Therefore,

$$\begin{aligned}\sup_{\Gamma([0,1],A)\cap\mathcal{S}} G_\lambda &\geq \sup_{\Gamma([0,1],A)\cap B} G_\lambda \\ &\geq \inf_{\Gamma([0,1],A)\cap B} G_\lambda \\ &\geq \inf_B G_\lambda = b_0(\lambda).\end{aligned}$$

Then $c_0(\lambda) \geq b_0(\lambda)$. Evidently,

$$c_0(\lambda) \leq \sup_{(t,u)\in[0,1]\times A} G_\lambda((1-t)u)$$

because $\Gamma(t,u) = (1-t)u \in \Phi^*$. Observe that the map $\lambda \mapsto c_0(\lambda)$ is nondecreasing. Hence, $c_0'(\lambda) := (dc_0(\lambda))/d\lambda$ exists for almost every $\lambda \in \Lambda$.

From now on, we consider those λ where $c_0'(\lambda)$ exists. For a fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda$ be a nonincreasing sequence so that $\lambda_n \to \lambda$ as $n \to \infty$. Then there exists an $\bar{n}(\lambda)$, which depends on λ only, such that

$$c_0'(\lambda) - 1 \leq \frac{c_0(\lambda_n) - c_0(\lambda)}{\lambda_n - \lambda} \leq c_0'(\lambda) + 1 \quad \text{for } n \geq \bar{n}(\lambda). \tag{6.6}$$

We prove the theorem step by step.

Step 1. In this step, we show that there exist $\Gamma_n \in \Phi^*, k_0 := k_0(\lambda) > 0$ such that $\|u\| \leq k_0 := (2c_0'(\lambda) + 6)^{1/2}$ whenever $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$ with

$$G_\lambda(u) \geq c_0(\lambda) - (\lambda_n - \lambda),$$

where $k_0 := (2c_0'(\lambda) + 6)^{1/2}$, is a constant depending on λ and independent of n. In fact, by the definition of $c_0(\lambda)$, there exists a $\Gamma_n \in \Phi^*$ such that

$$\begin{aligned}\sup_{\Gamma_n([0,1],\,A)\cap\mathcal{S}} G_\lambda(u) &\leq \sup_{\Gamma_n([0,1],\,A)\cap\mathcal{S}} G_{\lambda_n}(u) \\ &\leq c_0(\lambda_n) + (\lambda_n - \lambda).\end{aligned} \tag{6.7}$$

If

$$G_\lambda(u) \geq c_0(\lambda) - (\lambda_n - \lambda)$$

for some $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$, then by (6.6) and (6.7), we have that

$$\frac{1}{2}\|u\|^2 = \frac{G_{\lambda_n}(u) - G_\lambda(u)}{\lambda_n - \lambda} \tag{6.8}$$

$$\leq \frac{c_0(\lambda_n) + (\lambda_n - \lambda) - c_0(\lambda) + (\lambda_n - \lambda)}{\lambda_n - \lambda}$$

$$\leq c_0'(\lambda) + 3.$$

It follows that

$$\|u\| \leq (2c_0'(\lambda) + 6)^{1/2} := k_0(\lambda) := k_0; \tag{6.9}$$

here k_0 depends on λ only.

Step 2. In this step, we show that $G_\lambda(u) \leq c_0(\lambda) + (c_0'(\lambda) + 2)(\lambda_n - \lambda)$ for all $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$. By the choice of Γ_n and (6.6) and (6.7), for all $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$, we see that

$$\begin{aligned} &G_\lambda(u) \\ &\leq G_{\lambda_n}(u) \\ &\leq \sup_{\Gamma_n([0,1],\, A) \cap \mathcal{S}} G_{\lambda_n}(u) \\ &\leq c_0(\lambda_n) + (\lambda_n - \lambda) \\ &\leq c_0(\lambda) + (c_0'(\lambda) + 2)(\lambda_n - \lambda). \end{aligned} \tag{6.10}$$

Step 3. In this step, we assume that $c_0(\lambda) > b_0(\lambda)$ and construct the flow. For $\varepsilon > 0$, we define

$$\mathcal{Q}_\varepsilon(n, \lambda) := \left\{ u \in E : \begin{array}{l} \|u\| \leq k_0 + 2, \\ c_0(\lambda) - 2(\lambda_n - \lambda) \leq G_\lambda(u) \leq c_0(\lambda) + 2\varepsilon \end{array} \right\}.$$

Choose $n^*(\lambda) > \bar{n}(\lambda)$ ($\bar{n}(\lambda)$ comes from (6.6)) such that

$$(c_0'(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon, \tag{6.11}$$

$$\lambda_n - \lambda \leq \varepsilon, \tag{6.12}$$

$$\lambda_n - \lambda < c_0(\lambda) - b_0(\lambda) \tag{6.13}$$

for all $n \geq n^*(\lambda)$. We show that

$$\inf_{u \in \mathcal{Q}_\varepsilon(n^*(\lambda),\, \lambda) \cap \mathcal{S}} \|G_\lambda'(u)\| = 0. \tag{6.14}$$

Then the conclusion (1) of the theorem follows from (6.14).

First, $\mathcal{Q}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S} \neq \emptyset$. Indeed, if

$$G_\lambda(u) \leq c_0(\lambda) - (\lambda_n - \lambda)$$

for all $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$, then

$$c_0(\lambda) < c_0(\lambda) - (\lambda_n - \lambda),$$

a contradiction. Therefore, there exists a $u \in \Gamma_n([0,1], A) \cap \mathcal{S}$ such that $G_\lambda(u) \geq c_0(\lambda) - (\lambda_n - \lambda)$; it follows that $\|u\| \leq k_0$. Furthermore, (6.11)–(6.13) imply $G_\lambda(u) \leq c_0(\lambda) + \varepsilon$. Therefore, $u \in \mathcal{Q}_\varepsilon(n, \lambda) \cap \mathcal{S} \neq \emptyset$ for all $n \geq n^*(\lambda)$. Moreover, we observe that

$$\mathcal{Q}_\varepsilon(n, \lambda) \subset \mathcal{Q}_\varepsilon(n^*(\lambda), \lambda) \quad \text{for all } n \geq n^*(\lambda). \tag{6.15}$$

To show (6.14) by negation, we assume that there exists an $\varepsilon^* > 0$ such that

$$\frac{\|G'_\lambda(u)\|^2}{1 + \|G'_\lambda(u)\|} \geq \varepsilon^* \quad \text{for all } u \in \mathcal{Q}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S};$$

here ε^* only depends on $n^*(\lambda), \lambda$, and ε, not on n. Therefore, by (6.15), we still have

$$\frac{\|G'_\lambda(u)\|^2}{1 + \|G'_\lambda(u)\|} \geq \varepsilon^* \quad \text{for all } u \in \mathcal{Q}_\varepsilon(n,\ \lambda) \cap \mathcal{S}, \qquad \forall n \geq n^*(\lambda). \tag{6.16}$$

We seek a contradiction that will confirm the claim of (6.14). Let

$$\Theta_1 := \{u \in E : \|u\| \leq k_0 + 1\}; \tag{6.17}$$

$$\Theta_2 := \{u \in E : \|u\| \geq k_0 + 2\}; \tag{6.18}$$

$$\Theta_3 := \left\{ u \in E : \quad \begin{array}{l} \text{either } G_\lambda(u) \leq c_0(\lambda) - 2(\lambda_n - \lambda) \\ \text{or } \ G_\lambda(u) > c_0(\lambda) + 2\varepsilon \end{array} \right\}; \tag{6.19}$$

$$\Theta_4 := \{u \in E : c_0(\lambda) - (\lambda_n - \lambda) \leq G_\lambda(u) \leq c_0(\lambda) + \varepsilon\}. \tag{6.20}$$

Define

$$\vartheta(u) := \frac{\operatorname{dist}(u, \Theta_2)}{\operatorname{dist}(u, \Theta_1) + \operatorname{dist}(u, \Theta_2)}, \tag{6.21}$$

$$q(u) := \frac{\operatorname{dist}(u, \Theta_3)}{\operatorname{dist}(u, \Theta_3) + \operatorname{dist}(u, \Theta_4)}. \tag{6.22}$$

Recall the definition of $\mathcal{S}$ in (6.3), let

$$\Theta(\alpha) := \{u \in E : \ \operatorname{dist}(u, \mathcal{P}) < \alpha\} \cup \{u \in E : \ \operatorname{dist}(u, -\mathcal{P}) < \alpha\}, \qquad \alpha > 0.$$

Then $\Theta(\alpha)$ is an open neighborhood of the positive and negative cones $-\mathcal{P} \cup \mathcal{P}$. Let $\mathcal{S}^* := E \backslash \overline{\Theta(\mu_0/2)}$, which is an open neighborhood of $\mathcal{S}$, where μ_0 comes from (6.3). Define

$$\pi(u) := \frac{\mathrm{dist}(u, \Theta(\mu_0/4))}{\mathrm{dist}(u, \Theta(\mu_0/4)) + \mathrm{dist}(u, \mathcal{S}^*)}. \tag{6.23}$$

Recall Condition (A_1) and Lemma 2.11; we have a locally Lipschitz continuous map $O_\lambda : \tilde{E}_\lambda \to E$ such that

$$O_\lambda(\pm\mathcal{D}_0 \cap \tilde{E}_\lambda) \subset \pm\mathcal{D}_1$$

and that $V_\lambda(u) := \lambda u - O_\lambda(u)$ is a pseudo-gradient vector field of G_λ.

By (A_1), we observe that $\mathcal{K}_\lambda \subset (-\mathcal{P}) \cup \mathcal{P} \cup \mathcal{S}$ for any $\lambda \in \Lambda$. Hence, $\partial\mathcal{K}_\lambda \subset (-\mathcal{P}) \cup \mathcal{P} \cup \mathcal{S}$ for any $\lambda \in \Lambda$. Therefore, for any $u \in \partial\mathcal{K}_\lambda$, if

$$\|u\| \le k_0 + 2, \qquad c_0(\lambda) - 2(\lambda_n - \lambda) \le G_\lambda(u) \le c_0(\lambda) + 2\varepsilon \quad \text{and} \quad u \notin \pm\mathcal{P}, \tag{6.24}$$

then $u \in \mathcal{Q}_\varepsilon(n,\ \lambda) \cap \mathcal{S}$. By (6.16), there is an open neighborhood U_u of u such that $\|G'_\lambda(w)\|\big|_{U_u} \ge \varepsilon^*/2$, which contradicts the fact that $u \in \partial\mathcal{K}_\lambda$. This means that at least one of the inequalities of (6.24) is not true. It follows that there exists a neighborhood U_u of u such that either $U_u \subset \Theta_2$ or $U_u \subset \Theta_3$ or $U_u \subset \Theta(\mu_0/4)$. Therefore,

$$\vartheta(u)q(u)\pi(u) = 0 \quad \text{ for all } u \in U_u.$$

Consequently, if we define

$$W^*_\lambda(u) := \begin{cases} \dfrac{\vartheta(u)q(u)\pi(u)}{1 + \|V_\lambda(u)\|} V_\lambda(u), & \text{for } u \in \tilde{E}_\lambda, \\ 0, & \text{for } u \in \mathcal{K}_\lambda, \end{cases} \tag{6.25}$$

then W^*_λ is a locally Lipschitz continuous vector field from E to E and $\|W^*_\lambda(u)\| \le 1$ on E. Consider the following Cauchy initial value problem

$$\begin{cases} \dfrac{d\psi(t,u)}{dt} = -W^*_\lambda(\psi(t,u)), \\ \psi(t,u) = u \in E. \end{cases} \tag{6.26}$$

By Theorem 1.41, (6.26) has a unique continuous solution (flow) ψ: $[0,\infty) \times E \to E$.

Step 4. In this step, we also assume that $c_0(\lambda) > b_0(\lambda)$ and show that the flow ψ has some properties. We show that

$$\psi([0,+\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \psi([0,+\infty), \mathcal{D}) \subset \mathcal{D}. \tag{6.27}$$

We first observe that $O_\lambda(\pm\mathcal{D}_0 \cap \tilde{E}_\lambda) \subset (\pm\mathcal{D}_1)$ implies that $O_\lambda(\pm\bar{\mathcal{D}}_0 \cap \tilde{E}_\lambda) \subset (\pm\bar{\mathcal{D}}_1)$. Obviously, $\psi(t,u) = u$ for all $t \ge 0$ and $u \in \bar{\mathcal{D}} \cap \mathcal{K}_\lambda$. Next, we assume that $u \in \bar{\mathcal{D}}_0 \cap \tilde{E}_\lambda$. If there were a $t_0 > 0$ such that $\psi(t_0,u) \notin \bar{\mathcal{D}}_0$, then there would be a number $s_0 \in [0, t_0)$ such that $\psi(s_0, u) \in \partial\bar{\mathcal{D}}_0$ and $\psi(t, u) \notin \bar{\mathcal{D}}_0$ for $t \in (s_0, t_0]$. Consider the following initial value problem

$$\begin{cases} \dfrac{d\psi(t, \psi(s_0, u))}{dt} = -W_\lambda^*(\psi(t, \psi(s_0, u))), \\ \psi(0, \psi(s_0, u)) = \psi(s_0, u) \in E. \end{cases}$$

It has a unique solution $\psi(t, \psi(s_0, u))$. For any $v \in \bar{\mathcal{D}}_0$, if $v \in \mathcal{K}_\lambda$, then $W_\lambda^*(v) = 0$. Hence, $v + \rho(-W_\lambda^*(v)) = v \in \bar{\mathcal{D}}_0$. Assume that $v \in \tilde{E}_\lambda \cap \bar{\mathcal{D}}_0$. Note that $v \in \bar{\mathcal{D}}_0$ implies $\operatorname{dist}(v, \mathcal{P}) \leq \mu_0$. By Lemma 2.11 and a property of the cone $\mathcal{P}$: $x\mathcal{P} + y\mathcal{P} \subset \mathcal{P}$ for all $x, y \geq 0$, we have

$$\begin{aligned}
&\operatorname{dist}(v + \rho(-W_\lambda^*(v)), \mathcal{P}) \\
&\quad = \operatorname{dist}\left(v + \rho\left(-\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) V_\lambda(v), \mathcal{P}\right) \\
&\quad = \operatorname{dist}\left(v + \rho\left(-\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) (\lambda v - O_\lambda(v)), \mathcal{P}\right) \\
&\quad = \operatorname{dist}\left(\left(1 - \rho\lambda\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) v + \rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|} O_\lambda(v), \mathcal{P}\right) \\
&\quad \leq \operatorname{dist}\left(\left(1 - \rho\lambda\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) v + \rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|} O_\lambda(v), \right. \\
&\qquad\quad \left. \rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\mathcal{P} + \left(1 - \lambda\rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right)\mathcal{P}\right) \\
&\quad = \left(1 - \rho\lambda\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) \operatorname{dist}(v, \mathcal{P}) + \rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|} \operatorname{dist}(O_\lambda(v), \mathcal{P}) \\
&\quad \leq \left(1 - \rho\lambda\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|}\right) \mu_0 + \rho\frac{\vartheta(v)q(v)\pi(v)}{1 + \|V_\lambda(v)\|} \frac{\mu_0}{2} \\
&\quad \leq \mu_0 \qquad \left(\text{because } \lambda > \frac{1}{2}\right)
\end{aligned}$$

for ρ small enough. It implies that $v + \rho(-W_\lambda^*(v)) \in \bar{\mathcal{D}}_0$ for $\rho > 0$ small enough. It follows that

$$\lim_{\rho \to 0^+} \frac{\operatorname{dist}(v + \rho(-W_\lambda^*(v)), \bar{\mathcal{D}}_0)}{\rho} = 0, \qquad \forall v \in \bar{\mathcal{D}}_0.$$

By Lemma 1.49, there exists a $\delta > 0$ such that $\psi(t, \psi(s_0, u)) \in \bar{\mathcal{D}}_0$ for all $t \in [0, \delta)$. By the semigroup property, we see that $\psi(t, u) \in \bar{\mathcal{D}}_0$

for all $t \in [s_0, s_0 + \delta)$, which contradicts the definition of s_0. Therefore, $\psi([0,+\infty), \bar{\mathcal{D}}_0) \subset \bar{\mathcal{D}}_0$. Similarly, $\psi([0,+\infty), -\bar{\mathcal{D}}_0) \subset -\bar{\mathcal{D}}_0$. That is, $\psi([0,+\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$. Similar to previous chapters, $\psi([0,+\infty), \mathcal{D}) \subset \mathcal{D}$.

Step 5. In this step, we also assume that $c_0(\lambda) > b_0(\lambda)$. Note that ε^* is independent of n; we may choose $n^{**}(\lambda) > n^*(\lambda)$ such that

$$\lambda_n - \lambda < \frac{\varepsilon^*}{4(c_0'(\lambda)+3)} \qquad \text{for all } n \geq n^{**}(\lambda). \tag{6.28}$$

For each $n > n^{**}(\lambda)$, we define

$$\Gamma_n^*(s,u) := \begin{cases} \psi(2s, u), & 0 \leq s \leq \dfrac{1}{2}, \\ \psi(1, \Gamma_n(2s-1, u)), & \dfrac{1}{2} \leq s \leq 1. \end{cases} \tag{6.29}$$

Then $\Gamma_n^* \in \Phi$. Moreover, by (6.27) of Step 4, $\Gamma_n^* \in \Phi^*$ for all $n \geq n^{**}(\lambda)$.

For each fixed $n > n^{**}(\lambda)$, we consider the following two cases. Both of them lead to a contradiction that confirms (6.14).

If $u \in \Gamma_n^*([0,1/2], A) \cap \mathcal{S}$, then $u = \psi(2s_0, u_0)$ for some $s_0 \in [0, 1/2]$ and $u_0 \in A$. Therefore, by (6.11),

$$\begin{aligned} G_\lambda(u) &= G_\lambda(\psi(2s_0, u_0)) \\ &\leq G_\lambda(u_0) \\ &\leq a_0(\lambda) \\ &\leq b_0(\lambda) \\ &\leq c_0(\lambda) - (\lambda_n - \lambda). \end{aligned} \tag{6.30}$$

If $u \in \Gamma_n^*([\frac{1}{2}, 1], A) \cap \mathcal{S}$, we write $u = \Gamma_n^*(s_1, u_1)$ for some $s_1 \in [\frac{1}{2}, 1]$ and $u_1 \in A$. Then $u = \psi(1, \Gamma_n(2s_1 - 1, u_1)) \in \mathcal{S}$.

If $G_\lambda(\Gamma_n(2s_1 - 1, u_1)) \leq c_0(\lambda) - (\lambda_n - \lambda)$, then

$$\begin{aligned} G_\lambda(u) &= G_\lambda(\psi(1, \Gamma_n(2s_1 - 1, u_1))) \\ &\leq G_\lambda(\psi(0, \Gamma_n(2s_1 - 1, u_1))) \\ &\leq G_\lambda(\Gamma_n(2s_1 - 1, u_1)) \\ &\leq c_0(\lambda) - (\lambda_n - \lambda). \end{aligned} \tag{6.31}$$

If $G_\lambda(\Gamma_n(2s_1 - 1, u_1)) > c_0(\lambda) - (\lambda_n - \lambda)$, we show that (6.31) still holds.

We first observe that $\Gamma_n(2s_1 - 1, u_1) \in \mathcal{S}$. Otherwise, $\Gamma_n(2s_1 - 1, u_1) \in \mathcal{D}$ implies that $u = \psi(1, \Gamma_n(2s_1 - 1, u_1)) \in \mathcal{D}$ by (6.27), which is a contradiction because $u \in \mathcal{S}$. Recall Step 1; we have $\|\Gamma_n(2s_1-1,u_1)\| \le k_0$. Furthermore, by (6.11)–(6.15),

$$\Gamma_n(2s_1-1,u_1) \in \mathcal{Q}_\varepsilon(n,\lambda)\cap\mathcal{S} \subset \mathcal{Q}_\varepsilon(n^*(\lambda),\ \lambda)\cap\mathcal{S}.$$

Therefore, by (6.16),

$$\frac{\|G'_\lambda(\Gamma_n(2s_1-1,u_1))\|^2}{1+\|G'_\lambda(\Gamma_n(2s_1-1,u_1))\|} \ge \varepsilon^*. \tag{6.32}$$

On the other hand, because $\|W^*_\lambda(u)\| \le 1$ for all $u \in E$, we have that

$$\|\psi(t,\Gamma_n(2s_1-1,u_1)) - \psi(0,\Gamma_n(2s_1-1,u_1))\| \le t$$

and that

$$\begin{aligned}\|\psi(t,\Gamma_n(2s_1-1,u_1))\| & \\ &\le t + \|\Gamma_n(2s_1-1,u_1)\| \\ &\le k_0+1.\end{aligned} \tag{6.33}$$

for all $t \in [0,1]$. There are two subcases again.

If $G_\lambda(\psi(t,\Gamma_n(2s_1-1,u_1))) \le c_0(\lambda) - (\lambda_n - \lambda)$ for some $t \in [0,1]$, then

$$\begin{aligned}G_\lambda(u) &= G_\lambda(\psi(1,\Gamma_n(2s_1-1,u_1))) \\ &\le G_\lambda(\psi(t,\Gamma_n(2s_1-1,u_1))) \\ &\le c_0(\lambda) - (\lambda_n-\lambda).\end{aligned} \tag{6.34}$$

Hence, we have an inequality as (6.31).

If $G_\lambda(\psi(t,\Gamma_n(2s_1-1,u_1))) > c_0(\lambda) - (\lambda_n - \lambda)$ for all $t \in [0,1]$. By (6.11)–(6.13),

$$\begin{aligned}G_\lambda(\psi(t,\Gamma_n(2s_1-1,u_1))) & \\ &\le G_\lambda(\Gamma_n(2s_1-1,u_1)) \\ &\le c_0(\lambda)+\varepsilon.\end{aligned} \tag{6.35}$$

On the other hand, insert (6.27) again; $\psi(1,\Gamma_n(2s_1-1,u_1)) \in \mathcal{S}$ implies that $\psi(t,\Gamma_n(2s_1-1,u_1)) \in \mathcal{S}$ for all $t \in [0,1]$. Combining (6.15), (6.33), and (6.35), we have that

$$\psi(t,\Gamma_n(2s_1-1,u_1)) \in \mathcal{Q}_\varepsilon(n,\lambda)\cap\mathcal{S} \subset \mathcal{Q}_\varepsilon(n^*(\lambda),\ \lambda)\cap\mathcal{S} \tag{6.36}$$

for all $t \in [0, 1]$. By (6.16) and (6.36),

$$\frac{\|G'_\lambda(\psi(t, \Gamma_n(2s_1 - 1, u_1)))\|^2}{1 + \|G'_\lambda(\psi(t, \Gamma_n(2s_1 - 1, u_1)))\|} \geq \varepsilon^* \quad \text{for all } t \in [0, 1]. \tag{6.37}$$

Moreover, by (6.21)–(6.23), (6.33)–(6.35),

$$\begin{aligned} \vartheta(\psi(t, \Gamma_n(2s_1 - 1, u_1))) \\ &= q(\psi(t, \Gamma_n(2s_1 - 1, u_1))) \\ &= \pi(\psi(t, \Gamma_n(2s_1 - 1, u_1))) \\ &= 1 \end{aligned} \tag{6.38}$$

for all $t \in [0, 1]$. Combining the definition of the pseudo-gradient vector field and (6.37) and (6.38), it follows that

$$\begin{aligned} &G_\lambda(\psi(t, \Gamma_n(2s_1 - 1, u_1))) - G_\lambda(\Gamma_n(2s_1 - 1, u_1)) \\ &\quad \leq \int_0^t \frac{dG_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))}{ds} ds \\ &\quad \leq \int_0^t -\left\langle G'_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1))), \frac{V_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))}{1 + \|V_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))\|} \right\rangle ds \\ &\quad \leq -\frac{1}{2} \int_0^t \frac{\|G'_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))\|^2}{1 + \|V_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))\|} ds \\ &\quad \leq -\frac{1}{4} \int_0^t \frac{\|G'_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))\|^2}{1 + \|G'_\lambda(\psi(s, \Gamma_n(2s_1 - 1, u_1)))\|} ds \\ &\quad \leq -\frac{1}{4} \varepsilon^* t. \end{aligned}$$

It follows that

$$\begin{aligned} G(u) \\ &= G_\lambda(\psi(1, \Gamma_n(2s_1 - 1, u_1))) \\ &\leq G_\lambda(\Gamma_n(2s_1 - 1, u_1)) - \frac{1}{4}\varepsilon^* \\ &\leq c_0(\lambda) + (c'_0(\lambda) + 2)(\lambda_n - \lambda) - \frac{1}{4}\varepsilon^* \qquad \text{(by (6.11))} \\ &\leq c_0(\lambda) - (\lambda_n - \lambda). \qquad \text{(by (6.29))} \end{aligned} \tag{6.39}$$

Summing up (6.30), (6.31), (6.34), and (6.39), we have

$$G(u) \le c_0(\lambda) - (\lambda_n - \lambda)$$

for all $u \in \Gamma_n^*([0,1], A) \cap \mathcal{S}$ and all $n > n^{**}(\lambda)$. This contradicts the definition of $c_0(\lambda)$ because $\Gamma_n^* \in \Phi^*$. The contradiction guarantees the truth of (6.14) which deduces Conclusion (1) of the theorem.

♣ In the next steps 6–11, we consider the case of $c_0(\lambda) = b_0(\lambda)$. We prove that the Conclusion (2) of Theorem 6.1 is true.

Step 6. A is bounded, therefore $d_A := \max\{\|u\| : u \in A\} < \infty$. For $\varepsilon > 0$, $T > 0$, we define

$$\Omega(\varepsilon, T, \lambda) := \left\{ u \in E : \begin{array}{l} \|u\| \le k_0(\lambda) + 4 + d_A, \\ |G_\lambda(u) - c_0(\lambda)| \le 3\varepsilon, \;\; d(u, \mathcal{S}) \le 4T \end{array} \right\}. \tag{6.40}$$

We claim that $\Omega(\varepsilon, T, \lambda) \ne \emptyset$ for any $\varepsilon > 0, T > 0$. Indeed, by (6.11), we choose n large enough such that

$$\begin{aligned} \sup_{u \in \Gamma_n([0,1],\, A) \cap \mathcal{S}} G_\lambda(u) &\le \sup_{u \in \Gamma_n([0,1],\, A) \cap \mathcal{S}} G_{\lambda_n}(u) \\ &\le c_0(\lambda) + 3\varepsilon. \end{aligned} \tag{6.41}$$

Because A links B, there exists a pair of numbers $(s_0, u_0) \in [0,1] \times A$ such that $\Gamma_n(s_0, u_0) \in B \subset \mathcal{S}$. Hence, $\operatorname{dist}(\Gamma_n(s_0, u_0), \mathcal{S}) = 0$ and

$$\begin{aligned} G_\lambda(\Gamma_n(s_0, u_0)) &\ge b_0(\lambda) = \inf_B G_\lambda \\ &= c_0(\lambda) > c_0(\lambda) - (\lambda_n - \lambda) \ge c_0(\lambda) - 3\varepsilon. \end{aligned} \tag{6.42}$$

By Step 1, $\|\Gamma_n(s_0, u_0)\| \le k_0$. Hence, $\Gamma_n(s_0, u_0) \in \Omega(\varepsilon, T, \lambda) \ne \emptyset$.

Step 7. We prove that

$$\inf\{\|G'_\lambda(u)\| : u \in \Omega(\varepsilon, T, \lambda)\} = 0 \quad \text{for all } \varepsilon, T \in (0,1), \tag{6.43}$$

which implies Conclusion (2) of the theorem.

By a contradiction, we assume that there exist $\delta > 0, \varepsilon_1 > 0, T_1 \in (0,1)$ such that

$$\|G'_\lambda(u)\| \ge 3\delta \quad \text{for all } u \in \Omega(\varepsilon_1, T_1, \lambda). \tag{6.44}$$

Choose $n^*(\lambda)$ so large again that

$$(\lambda_n - \lambda) \le \varepsilon_1, \qquad (c_0'(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon_1, \qquad (\lambda_n - \lambda) < \delta T_1 \tag{6.45}$$

and

$$(\lambda_n - \lambda) \le \frac{\delta^2}{(c_0'(\lambda)+2)(1+3\delta)} T_1, \qquad \text{for all } n \ge n^*(\lambda). \tag{6.46}$$

Define

$$\Omega^*(n, \varepsilon_1, T_1, \lambda) := \left\{ u \in E : \begin{array}{c} \|u\| \le k_0 + 4 + d_A, \\ c_0(\lambda) - (\lambda_n - \lambda) \le G_\lambda(u) \le c_0(\lambda) + 3\varepsilon_1, \\ \mathrm{dist}(u, \mathcal{S}) \le 4T_1. \end{array} \right\}. \tag{6.47}$$

By (6.45) and (6.46) and the same arguments as those in (6.41) and (6.42),

$$\Omega^*(n, \varepsilon_1, T_1, \lambda) \ne \emptyset; \qquad \Omega^*(n, \varepsilon_1, T_1, \lambda) \subset \Omega(\varepsilon_1, T_1, \lambda). \tag{6.48}$$

Define

$$\Theta_5 := \Theta_5(\varepsilon_1, T_1, \lambda) := \left\{ u \in E : \begin{array}{c} \|u\| \le k_0 + 3 + d_A, \\ |G_\lambda(u) - c_0(\lambda)| \le 2\varepsilon_1, \\ \mathrm{dist}(u, \mathcal{S}) \le 3T_1 \end{array} \right\}. \tag{6.49}$$

Then, by the same arguments as in Step 6, $\Theta_5 \ne \emptyset$. Let

$$\xi(u) := \frac{\mathrm{dist}(u, E \backslash \Omega(\varepsilon_1, T_1, \lambda))}{\mathrm{dist}(u, \Theta_5) + \mathrm{dist}(u, E \backslash \Omega(\varepsilon_1, T_1, \lambda))} \tag{6.50}$$

and define

$$\Theta_6 := \Theta_6(\varepsilon_1, T_1, \lambda) := \left\{ u \in E : \begin{array}{c} \|u\| \le k_0 + 2 + d_A, \\ |G_\lambda(u) - c_0(\lambda)| \le \varepsilon_1, \\ \mathrm{dist}(u, \mathcal{S}) \le 2T_1 \end{array} \right\}. \tag{6.51}$$

Then, by a similar argument as that in Step 6, $\Theta_6 \ne \emptyset$ and

$$\Theta_6 \subset \Theta_5 \subset \Omega(\varepsilon_1, T_1, \lambda). \tag{6.52}$$

Define

$$\zeta(u) := \frac{\operatorname{dist}(u, E\backslash\Theta_5)}{\operatorname{dist}(u,\Theta_6) + \operatorname{dist}(u, E\backslash\Theta_5)}. \tag{6.53}$$

For any $u \in \partial\mathcal{K}_\lambda$, if $u \in \Omega(\varepsilon_1, T_1, \lambda)$ (a closed subset), then by (6.44), there exists an open neighborhood U_u of u such that $\|G'_\lambda(w)\| \geq 2\delta$ for all $w \in U_u$. This is impossible because $u \in \partial\mathcal{K}_\lambda$. So, $u \notin \Omega(\varepsilon_1, T_1, \lambda)$. Hence, we may find a neighborhood U_u of u such that $U_u \subset E\backslash\Omega(\varepsilon_1, T_1, \lambda)$. By (6.50),

$$\xi(w) = 0 \quad \text{for all } w \in U_u.$$

Therefore,

$$W^{**}_\lambda(u) := \begin{cases} \dfrac{\xi(u)\zeta(u)}{1+\|V_\lambda(u)\|}V_\lambda(u), & \text{for } u \in \tilde{E}_\lambda, \\ 0, & \text{for } u \in \mathcal{K}_\lambda, \end{cases} \tag{6.54}$$

is a locally Lipschitz continuous vector field from E to E. Moreover,

(1) $\|W^{**}_\lambda(u)\| \leq 1$ for all $u \in E$,
(2) $\langle G'_\lambda(u), W^{**}_\lambda(u)\rangle \geq 0$ for all $u \in E$,
(3) For any $u \in \Theta_5$, then $u \in \Theta_5 \subset \Omega(\varepsilon_1, T_1, \lambda) \subset \tilde{E}_\lambda$ and $\xi(u) = 1$, $\|G'(u)\| \geq 3\delta$ (by (6.44)).

Hence,

$$\begin{aligned} &\left\langle G'_\lambda(u), \frac{\xi(u)}{1+\|V_\lambda(u)\|}V_\lambda(u)\right\rangle \\ &= \left\langle G'_\lambda(u), \frac{V_\lambda(u)}{1+\|V_\lambda(u)\|}\right\rangle \\ &\geq \frac{\|G'_\lambda(u)\|^2}{4(1+\|G'_\lambda(u)\|)} \\ &\geq \frac{9\delta^2}{4(1+3\delta)}. \end{aligned} \tag{6.55}$$

Consider the following Cauchy initial value problem

$$\begin{cases} \dfrac{d\psi_1(t,u)}{dt} = -W^{**}_\lambda(\psi_1(t,u)), \\ \psi_1(0,u) = u \in E. \end{cases}$$

It has a unique continuous solution ψ_1: $[0,\infty) \times E \to E$. Note that if $\zeta(\psi_1(t,u)) \neq 0$, then by (6.53), $\psi_1(t,u) \in \Theta_5$. Therefore, by (6.55), we have that

$$\frac{dG_\lambda(\psi_1(t,u))}{dt} \le -\frac{9\delta^2}{4(1+3\delta)}\zeta(\psi_1(t,u)) \le 0 \tag{6.56}$$

for all $u \in E$ and $t \ge 0$.

Step 8. We show that

$$\psi_1([0,+\infty),\bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \psi_1([0,+\infty),\mathcal{D}) \subset \mathcal{D}. \tag{6.57}$$

The idea is similar to that in Step 4. We give a brief proof for completeness.

We first observe that $O_\lambda(\pm\mathcal{D}_0 \cap \tilde{E}_\lambda) \subset (\pm\mathcal{D}_1)$ implies that $O_\lambda(\pm\bar{\mathcal{D}}_0 \cap \tilde{E}_\lambda) \subset (\pm\bar{\mathcal{D}}_1)$. Obviously, by (2.47), $\psi_1(t,u) = u$ for all $t \ge 0$ and $u \in \bar{\mathcal{D}} \cap \mathcal{K}_\lambda$. Next, we assume that $u \in \bar{\mathcal{D}}_0 \cap \tilde{E}_\lambda$. If there were a $t_0 > 0$ such that $\psi_1(t_0,u) \notin \bar{\mathcal{D}}_0$, then there would be a number $s_0 \in [0,t_0)$ such that $\psi_1(s_0,u) \in \partial\bar{\mathcal{D}}_0$ and $\psi_1(t,u) \notin \bar{\mathcal{D}}_0$ for $t \in (s_0,t_0]$. Consider the following initial value problem

$$\begin{cases} \dfrac{d\psi_1(t,\psi_1(s_0,u))}{dt} = -W_\lambda^{**}(\psi_1(t,\psi_1(s_0,u))), \\ \psi_1(0,\psi_1(s_0,u)) = \psi_1(s_0,u) \in E. \end{cases}$$

It has a unique solution $\psi_1(t,\psi_1(s_0,u))$. For any $v \in \bar{\mathcal{D}}_0$, if $v \in \mathcal{K}_\lambda$, then $W_\lambda^{**}(v) = 0$. Hence, $v + \beta(-W_\lambda^{**}(v)) = v \in \bar{\mathcal{D}}_0$. Assume that $v \in \tilde{E}_\lambda \cap \bar{\mathcal{D}}_0$. Note that $v \in \bar{\mathcal{D}}_0$ implies that $\mathrm{dist}(v,\mathcal{P}) \le \mu_0$. By Lemma 2.11, we have

$$\begin{aligned}
&\mathrm{dist}(v + \beta(-W_\lambda^{**}(v)),\mathcal{P}) \\
&= \mathrm{dist}\left(v + \beta\left(-\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)V_\lambda(v),\mathcal{P}\right) \\
&= \mathrm{dist}\left(v + \beta\left(-\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)(\lambda v - O_\lambda(v)),\mathcal{P}\right) \\
&= \mathrm{dist}\left(\left(1 - \beta\lambda\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)v + \beta\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}O_\lambda(v),\mathcal{P}\right) \\
&\le \mathrm{dist}\left(\left(1 - \beta\lambda\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)v + \beta\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}O_\lambda(v),\right. \\
&\qquad \left.\beta\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\mathcal{P} + \left(1 - \lambda\beta\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)\mathcal{P}\right) \\
&= \left(1 - \beta\lambda\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)\mathrm{dist}(v,\mathcal{P}) + \beta\frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\mathrm{dist}(O_\lambda(v),\mathcal{P})
\end{aligned}$$

$$\leq \left(1 - \beta\lambda \frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\right)\mu_0 + \beta \frac{\xi(v)\zeta(v)}{1+\|V_\lambda(v)\|}\frac{\mu_0}{2}$$

$$\leq \mu_0 \qquad \left(\text{because } \lambda > \frac{1}{2}\right).$$

Hence, $v + \beta(-W_\lambda^{**}(v)) \in \bar{\mathcal{D}}_0$ for $\beta > 0$ small enough. It follows that

$$\lim_{\beta\to 0^+} \frac{\operatorname{dist}(v + \beta(-W_\lambda^{**}(v)), \bar{\mathcal{D}}_0)}{\beta} = 0, \qquad \forall v \in \bar{\mathcal{D}}_0.$$

By Lemma 1.49, $\psi([0,+\infty), \bar{\mathcal{D}}_0) \subset \bar{\mathcal{D}}_0$. Similarly, $\psi([0,+\infty), -\bar{\mathcal{D}}_0) \subset -\bar{\mathcal{D}}_0$. That is, $\psi([0,+\infty), \bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$. The same as the proof of Step 4, we have $\psi([0,+\infty), \mathcal{D}) \subset \mathcal{D}$.

Step 9. We claim: $\psi_1(t,u) \notin B$ for all $t \in [0, T_1]$ and $u \in A$.

For $u \in A$, by (6.56), we have that

$$G_\lambda(\psi_1(t,u)) \leq G_\lambda(u) \leq a_0(\lambda) \leq b_0(\lambda) = c_0(\lambda), \qquad \forall\, t \in [0, T_1] \tag{6.58}$$

and

$$\begin{aligned} G_\lambda(\psi_1(t,u)) &\\ &= G_\lambda(u) + \int_0^t \frac{dG_\lambda(\psi_1(\sigma,u))}{d\sigma} d\sigma \\ &\leq G_\lambda(u) - \int_0^t \frac{9\delta^2}{4(1+3\delta)} \zeta(\psi_1(\sigma,u)) d\sigma \end{aligned} \tag{6.59}$$

for all $t \in [0, T_1]$. If the claim of this step is not true, then there are $t_0 \in [0, T_1]$ and $u \in A$ such that $\psi_1(t_0, u) \in B$. Then

$$G_\lambda(\psi_1(t_0,u)) \geq c_0(\lambda) = b_0(\lambda) = \inf_B G_\lambda.$$

By (6.58) and (6.59), we see that

$$\int_0^{t_0} \frac{9\delta^2}{4(1+3\delta)} \zeta(\psi_1(\sigma,u)) d\sigma = 0.$$

Hence, $\zeta(\psi_1(\sigma,u)) = 0$ for $\sigma \in [0, t_0]$; that is $\psi_1(\sigma, u) \notin \Theta_6$, $\forall \sigma \in [0, t_0]$. In particular, $\psi_1(t_0, u) \notin \Theta_6$. Therefore, one of the following three cases occurs.

$$\|\psi_1(t_0,u)\| > k_0 + 2 + d_A, \tag{6.60}$$

$$|G_\lambda(\psi_1(t_0,u)) - c_0(\lambda)| > \varepsilon_1, \tag{6.61}$$

$$\operatorname{dist}(\psi_1(t_0, u), \mathcal{S}) > 2T_1. \tag{6.62}$$

Because

$$\|\psi_1(t_0, u) - \psi_1(\sigma', u)\| \le |t_0 - \sigma'|,$$

then

$$\|\psi_1(t_0, u)\| \le \|\psi_1(0, u)\| + T_1 \le d_A + 1;$$

it implies that (6.60) can never be true. If (6.61) holds, then $G_\lambda(\psi_1(t_0, u)) < c_0(\lambda) - \varepsilon_1$ (by (6.58)). Hence, $\psi_1(t_0, u) \notin B$ because $\inf_B G_\lambda = b_0(\lambda) = c_0(\lambda)$. Evidently, (6.62) implies that $\psi_1(t_0, u) \notin B$. Therefore, the claim of Step 9 is true.

Step 10. We claim: $\psi_1(T_1, \Gamma_n(2s-1, u)) \notin B$ for all $u \in A$ and all $s \in \left[\frac{1}{2}, 1\right]$.

For any fixed $u \in A$ and $s \in \left[\frac{1}{2}, 1\right]$, if $\Gamma_n(2s-1, u) \in \mathcal{D}$, then by (6.57),

$$\psi_1(T_1, \Gamma_n(2s-1, u)) \in \mathcal{D} \subset E \backslash \mathcal{S} \subset E \backslash B.$$

Next, we assume

$$\Gamma_n(2s-1, u) \in \mathcal{S} \tag{6.63}$$

and divide the proof of the claim into two cases.

Case 1. If $\psi_1(\sigma, \Gamma_n(2s-1, u)) \in \Theta_6$ for all $\sigma \in [0, T_1]$, we have that

$$\begin{aligned}
&G_\lambda(\psi_1(T_1, \Gamma_n(2s-1, u))) \\
&\quad = G_\lambda(\Gamma_n(2s-1, u)) + \int_0^{T_1} \frac{dG_\lambda(\psi_1(\sigma, \Gamma_n(2s-1, u)))}{d\sigma} d\sigma \\
&\quad \le G_\lambda(\Gamma_n(2s-1, u)) \\
&\qquad - \int_0^{T_1} \frac{9\delta^2}{4(1+3\delta)} \zeta(\psi_1(\sigma, \Gamma_n(2s-1, u))) d\sigma \qquad \text{(by (6.56))} \\
&\quad = G_\lambda(\Gamma_n(2s-1, u)) - \frac{9\delta^2}{4(1+3\delta)} T_1 \qquad \text{(by (6.53))} \\
&\quad \le c_0(\lambda) + (c_0'(\lambda) + 2)(\lambda_n - \lambda) - \frac{9\delta^2}{4(1+3\delta)} T_1 \qquad \text{(by (6.11) and (6.63))} \\
&\quad \le c_0(\lambda) - \frac{5\delta^2}{4(1+3\delta)} T_1, \qquad \text{(by (6.46))}
\end{aligned}$$

which implies that $\psi_1(T_1, \Gamma_n(2s-1, u)) \notin B$ because $c_0(\lambda) = b_0(\lambda)$.

Case 2. If there exists a $t_0 \in [0, T_1]$ such that

$$\psi_1(t_0, \varGamma_n(2s-1,u)) \notin \varTheta_6,$$

then one of the following alternatives holds.

$$\|\psi_1(t_0, \varGamma_n(2s-1,u))\| > k_0 + 2 + d_A. \tag{6.64}$$

$$|G_\lambda(\psi_1(t_0, \varGamma_n(2s-1,u))) - c_0(\lambda)| > \varepsilon_1. \tag{6.65}$$

$$\operatorname{dist}(\psi_1(t_0, \varGamma_n(2s-1,u)), \mathcal{S}) > 2T_1. \tag{6.66}$$

Assume that (6.64) holds. We show that $\psi_1(T_1, \varGamma_n(2s-1,u)) \notin B$. Otherwise, if $\psi_1(T_1, \varGamma_n(2s-1,u)) \in B$, then

$$\begin{aligned} b_0(\lambda) &= c_0(\lambda) \\ &\le G_\lambda(\psi_1(T_1, \varGamma_n(2s-1,u))) \\ &\le G_\lambda(\varGamma_n(2s-1,u)). \end{aligned} \tag{6.67}$$

By (6.63), (6.67), and Step 1, $\|\varGamma_n(2s-1,u)\| \le k_0$. Furthermore, because

$$\|\psi_1(t_0, \varGamma_n(2s-1,u)) - \psi_1(0, \varGamma_n(2s-1,u))\| \le t_0,$$

it follows that

$$\|\psi_1(t_0, \varGamma_n(2s-1,u))\| \le k_0 + t_0 \le k_0 + 1,$$

which contradicts (6.64). Hence, $\psi_1(T_1, \varGamma_n(2s-1,u)) \notin B$.

Assume that (6.65) holds, we show that $\psi_1(T_1, \varGamma_n(2s-1,u)) \notin B$. Note, by (6.11), (6.63), and (6.47), that

$$\begin{aligned} &G_\lambda(\psi_1(t_0, \varGamma_n(2s-1,u))) \\ &\quad \le G_\lambda(\psi_1(0, \varGamma_n(2s-1,u))) \\ &\quad = G_\lambda(\varGamma_n(2s-1,u)) \\ &\quad \le c_0(\lambda) + \varepsilon_1. \end{aligned}$$

Therefore, (6.56) and (6.65) imply that

$$\begin{aligned} &G_\lambda(\psi_1(T_1, \varGamma_n(2s-1,u))) \\ &\quad \le G_\lambda(\psi_1(t_0, \varGamma_n(2s-1,u))) \\ &\quad \le c_0(\lambda) - \varepsilon_1. \end{aligned}$$

It follows that $\psi_1(T_1, \varGamma_n(2s-1,u)) \notin B$ because $c_0(\lambda) = b_0(\lambda)$.

Assume that (6.66) holds. Note that

$$\|\psi_1(t,u) - \psi_1(t',u)\| \le |t - t'|.$$

It therefore follows that

$$\begin{aligned} &\|\psi_1(t, \Gamma_n(2s-1,u)) - w\| \\ &\quad \ge \|\psi_1(t_0, \Gamma_n(2s-1,u)) - w\| - |t - t_0|, \end{aligned}$$

for all $w \in B, t \in [0, T_1]$. Hence,

$$\operatorname{dist}(\psi_1(t, \Gamma_n(2s-1,u)), B) \ge T_1$$

for all $t \in [0, T_1]$. In particular,

$$\psi_1(T_1, \Gamma_n(2s-1,u)) \notin B. \tag{6.68}$$

That is, each case of (6.64)–(6.66) implies (6.68).

Cases 1 and 2 complete the proof of the claim in Step 10.

Step 11. To get the final contradiction, we have to introduce a new mapping:

$$\Gamma_1^*(s,u) := \begin{cases} \psi_1(2sT_1, u), & 0 \le s \le 1/2, \\ \psi_1(T_1, \Gamma_n(2s-1,u)), & 1/2 \le s \le 1. \end{cases}$$

Then $\Gamma_1^* \in \Phi^*$ (in view of Step 8). However, by Steps 9 and 10, $\Gamma_1^*(s,A) \cap B = \emptyset$ for all $s \in [0,1]$, which contradicts the fact that A links B. We get the final contradiction. This justifies (6.43) in Step 7 which implies Conclusion (2) of the theorem. □

The first case of Theorem 6.1 implies that $\{u_m\}$ is a sign-changing bounded (PS) sequence. For the second case, note that we may choose an open neighborhood $\mathcal{S}_0$ of $\mathcal{S}$ such that $B \subset \mathcal{S} \subset \mathcal{S}_0 \subset E\backslash(-\mathcal{P} \cup \mathcal{P})$ (see the definition of $\mathcal{S}$ in (6.3)); hence, $\{u_m\}$ is still a sign-changing bounded (PS) sequence. For both cases, if $\{u_m\}$ has a convergent subsequence, then its limit must belong to $\mathcal{S}$ because $\mathcal{S}$ is closed. That is, G_λ has a sign-changing critical point in $\mathcal{S}$ for almost all $\lambda \in \Lambda$. Note that the classical Palais–Smale compactness condition is not needed.

Theorem 6.2. *Under the assumptions of Theorem 6.1, if J' is compact, then for almost all $\lambda \in \Lambda$, G_λ has a sign-changing critical point in $\mathcal{S}$.*

For a special choice of linking sets A and B, we obtain the following weaker version.

Theorem 6.3. *Assume that (A_1) and (A_2) hold. Suppose that $E = Y \oplus M$, $1 < \dim Y < \infty$ and that*

(1) $G_\lambda(v) \le \delta_\lambda$ *for all* $v \in Y$ *and* $\lambda \in \Lambda$, *where* $\delta_\lambda \ge 0$ *is a constant,*
(2) $G_\lambda(w) \ge \delta_\lambda$ *for all* $w \in \{w : w \in M, \|w\| = \rho_\lambda\} \subset \mathcal{S}$ *and* $\lambda \in \Lambda$; *where* ρ_λ *is a positive constant,*
(3) $G_\lambda(sw_0 + v) \le C_0$ *for all* $s \ge 0, v \in Y$, *and* $\lambda \in \Lambda$, $w_0 \in M$ *with* $\|w_0\| = 1$ *is a fixed element, and* C_0 *is a constant.*

If J' *is compact, then for almost all* $\lambda \in \Lambda$, G_λ *has a sign-changing critical point in* $\mathcal{S}$.

Proof. Define $\chi \in \mathbf{C}^\infty(\mathbf{R})$ such that $\chi = 0$ in $\left(-\infty, \frac{1}{2}\right)$ and $\chi = 1$ in $(1, \infty), 0 \le \chi \le 1$. Write $u \in E$ as $u = v + w, v \in Y, w \in M$. Let

$$G_{\lambda,n}(u) = G_\lambda(u) - \left(C_0 + \frac{1}{n}\right)\chi\left(\frac{\|u\|^2}{n}\right), \qquad n = 1, 2, \ldots.$$

Then

$$\begin{aligned} G'_{\lambda,n} &= G'_\lambda(u) - 2\left(C_0 + \frac{1}{n}\right)\chi'\left(\frac{\|u\|^2}{n}\right)\frac{u}{n} \\ &= \lambda u - J'(u) - 2\left(C_0 + \frac{1}{n}\right)\chi'\left(\frac{\|u\|^2}{n}\right)\frac{u}{n} \\ &:= \lambda u - \tilde{J}'(u) \end{aligned} \tag{6.69}$$

and

$$\|G'_\lambda(u) - G'_{\lambda,n}(u)\| \le C_1 n^{-1/2}. \tag{6.70}$$

Moreover, $G_{\lambda,n}(v) \le \delta_\lambda$ for all $v \in Y$. For any $w \in M$, if $\|w\| = \rho_\lambda$, then $\chi(\|w\|^2/n) = 0$ for $n > 2\rho_\lambda^2$ and consequently

$$G_{\lambda,n}(w) = G_\lambda(w) \ge \delta_\lambda.$$

Choose $\|sw_0 + v\| := n^{1/2} := R_n$. Then $R_n > \rho_\lambda$ if n is large enough, and

$$\begin{aligned} &G_{\lambda,n}(sw_0 + v) \\ &\quad = G_\lambda(sw_0 + v) - (C_0 + 1/n)\chi\left(\frac{\|sw_0 + v\|^2}{n}\right) \\ &\quad \le -\frac{1}{n}. \end{aligned}$$

Let

$$B := \{w \in M : \|w\| = \rho_\lambda\},$$

and

$$A_n := \{v \in Y : \|v\| \le R_n\} \cup \{sw_0 + v : s \ge 0, v \in Y, \|sw_0 + v\| = R_n\}.$$

Then A_n links B (cf. Chapter 2), $B \subset \mathcal{S}$, and $\sup_{A_n} G_\lambda \le \inf_B G_\lambda, \forall \lambda \in \Lambda$. Furthermore, for any $u \in E$ with $\mathrm{dist}(u, \mathcal{P}) < \mu_0$, by (A_1) and (6.69), we have

$$\begin{aligned}
&\mathrm{dist}(\tilde{J}'(u), \mathcal{P}) \\
&\quad \le \mathrm{dist}(J'(u), \mathcal{P}) + \mathrm{dist}\left(2\left(C_0 + \frac{1}{n}\right)\chi'\left(\frac{\|u\|^2}{n}\right)\frac{u}{n}, \mathcal{P}\right) \\
&\quad \le \mathrm{dist}(J'(u), \mathcal{P}) + 2\left(C_0 + \frac{1}{n}\right)\chi'\left(\frac{\|u\|^2}{n}\right)\frac{1}{n}\mathrm{dist}(u, \mathcal{P}) \\
&\quad \le \frac{1}{4}\mathrm{dist}(u, \mathcal{P})
\end{aligned}$$

for n large enough. Then $G_{\lambda,n}$ satisfies all the conditions of Theorem 6.1. Because J' is compact, by standard argument, the bounded (PS) sequence has a convergent subsequence. Therefore, for almost all $\lambda \in \Lambda$, there exists a $u_n \in \mathcal{S}$ such that

$$G'_{\lambda,n}(u_n) = 0,$$

$$G_{\lambda,n}(u_n) \in [\,\delta_\lambda, \sup_{(t,u)\in[0,1]\times A_n} G_{\lambda,n}((1-t)u)].$$

Evidently,

$$\|G'_\lambda(u_n) - G'_{\lambda,n}(u_n)\| = \|G'_\lambda(u_n)\| \le C_1 n^{-1/2} \to 0, \quad (\text{by } (6.70))$$

$$\delta_\lambda \le G_{\lambda,n}(u_n) \le G_\lambda(u_n) \le G_{\lambda,n}(u_n) + C_0 + 1/n,$$

$$\sup_{(t,u)\in[0,1]\times A_n} G_{\lambda,n}((1-t)u) \le C_0.$$

Therefore, $G_\lambda(u_n) \to c \in [\delta_\lambda, 2C_0]$ as $n \to \infty$. Finally,

$$\begin{aligned}
G'_\lambda(u_n) &= G'_\lambda(u_n) - G'_{\lambda,n}(u_n) \\
&= 2\left(C_0 + \frac{1}{n}\right)\chi'\left(\frac{\|u_n\|^2}{n}\right)\frac{u_n}{n} \\
&= \frac{C_n}{n}u_n \to 0,
\end{aligned}$$

where $\{C_n\}$ is a bounded sequence. That is, for almost all $\lambda \in \left(\frac{1}{2}, 1\right)$, we find $u_n \in \mathcal{S}$ such that

$$G'_{\lambda-(C_n/n)}(u_n) = 0,$$

which implies the conclusion of the theorem. $\square$

We now assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \leq C_\star\|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. In the sequel, all properties are with respect to the norm $\|\cdot\|$ if without specific indication. Write $E = M \oplus Y$, where $Y, M := Y^\perp$ are closed subspaces with $\dim Y < \infty$ and $(M\backslash\{0\}) \cap (-\mathcal{P} \cup \mathcal{P}) = \emptyset$; that is, the nontrivial elements of M are sign-changing. Let $y_0 \in M\backslash\{0\}$ with $\|y_0\| = 1$ and $0 < \rho < R$ with

$$R^{p-2}\|y_0\|_\star^p + \frac{R\|y_0\|_\star}{1 + D_\star\|y_0\|_\star} > \rho, \qquad D_\star > 0, p > 2 \text{ are constants.}$$

Let

$$A := \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$

$$B := \left\{u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star\|u\|_\star} = \rho\right\}.$$

Then by Proposition 2.10, A links B in the sense of Definition 2.1. Choose

$$a_\star > \sup_{(t,u)\in[0,1]\times A,\ \lambda\in[1/2,1]} G_\lambda((1-t)u) + 2. \tag{6.71}$$

Define

$$B^\star := B \cap G_{1/2}^{a_\star}. \tag{6.72}$$

Choose $\Gamma(t,u) = (1-t)u \in \Phi^*$ (cf. (2.22)); then $\Gamma(t,a) \in B$ for some $(t,a) \in [0,1] \times A$. Moreover, $\Gamma(t,a) \in G_{1/2}^{a_\star}$, hence, $B^\star := B \cap G_{1/2}^{a_\star} \neq \emptyset$. Set

$$\Phi_\lambda^{**} := \{\Gamma \in \Phi^* : \Gamma([0,1], A) \subset G_\lambda^{a_\star}\}. \tag{6.73}$$

Then $\Gamma(t,u) = (1-t)u \in \Phi^* \cap \Phi_\lambda^{**}$. Note that $G_\lambda^{a_\star} \subset G_{1/2}^{a_\star}$.

Lemma 6.4. $\|u\|_\star \leq c_1,\ \forall u \in B$; *here* c_1 *is a constant.*

Lemma 6.5. *Assume that for any* $a, b > 0$, *there is a* $c = c(a,b) > 0$ *such that*

$$G_{1/2}(u) \leq a \quad \textit{and} \quad \|u\|_\star \leq b \qquad \Rightarrow \|u\| \leq c.$$

Then we have that

$$\operatorname{dist}(B^\star := B \cap G_{1/2}^{a_\star}, \mathcal{P}) := \delta_1 > 0.$$

Proof. This is the same as that of Lemma 2.16. □

Therefore, we may assume that $B^\star \subset \mathcal{S}$ as long as the μ_0 of Condition (A_1) is small enough; this is indeed true in our applications.

($\mathbf{A_2^*}$) Assume

$$a_0(\lambda) := \sup_A G_\lambda \le b_0(\lambda) := \inf_{B^\star} G_\lambda \quad \text{for any } \lambda \in \Lambda.$$

($\mathbf{A_3^*}$) Assume that for any $a, b > 0$, there is a $c = c(a, b) > 0$ such that

$$G_{1/2}(u) \le a \quad \text{and} \quad \|u\|_\star \le b \Rightarrow \quad \|u\| \le c.$$

Theorem 6.6. *Assume that* (A_1), $(A_2^\star)$, and $(A_3^\star)$ *hold. Define*

$$c_0(\lambda) := \inf_{\Gamma \in \Phi^{**}_\lambda} \sup_{\Gamma([0,1],A)\cap\mathcal{S}} G_\lambda(u);$$

then

$$c_0(\lambda) \in \Big[b_0(\lambda), \sup_{(t,u)\in[0,1]\times A} G_\lambda((1-t)u) \Big].$$

Moreover, for almost all $\lambda \in \Lambda$,

(1) If $c_0(\lambda) > b_0(\lambda)$, *then there is a sequence* $\{u_m\}$ *depending on* λ *such that*

$$\sup_m \|u_m\| < \infty, \qquad u_m \in \mathcal{S}, \qquad G'_\lambda(u_m) \to 0, \qquad G_\lambda(u_m) \to c_0(\lambda),$$

(2) If $c_0(\lambda) = b_0(\lambda)$, *then there is a sequence* $\{u_m\}$ *depending on* λ *such that*

$$\sup_m \|u_m\| < \infty, \qquad \operatorname{dist}(u_m, \mathcal{S}) \to 0, \qquad G'_\lambda(u_m) \to 0,$$

$$G_\lambda(u_m) \to c_0(\lambda).$$

Proof. Keep in mind that the flow is descending. If $\lambda_1 \le \lambda_2$, then we have

$$\Phi^{**}_{\lambda_2} \subset \Phi^{**}_{\lambda_1}.$$

Therefore, $c_0(\lambda)$ is nondecreasing. Replace B by $B^\star$ and Φ^* by Φ^{**}_λ; then the proof is the same as that of Theorem 6.1. □

Notes and Comments. The novelty of Theorem 6.3 is the sign-changing property of the critical point via a weaker linking geometry. It should be noted that in the original form of the saddle point theorem (cf. Rabinowitz [255]), it is required that

$$G(sw_0 + v) \le 0 \quad \text{for all } s \ge 0, \qquad v \in Y, \qquad \|sw_0 + v\| = R$$

holds for some $R \ge \rho$. To get this, one has to show that

$$\limsup_{R\to\infty} \{G(sw_0 + v) : s \ge 0, v \in Y, \|sw_0 + v\| = R\} < 0.$$

This is much more demanding than that of Theorem 6.3. This was also observed in Schechter [268, 275] where the Palais–Smale condition was required and no nodal structure of the critical point was obtained. In the proof of Theorem 6.1, we have used the idea of the so-called "monotonicity method", because G_λ is monotonically depending on λ. This trick was introduced by Struwe in [312, 313] for minimization problems and essentially developed by Jeanjean [178, 180–182] for one positive solution of the mountain pass type. Some ideas adopted here and in the sequel come from Jeanjean in [178]. Further developments were made by Schechter, Szulkin, Willem, and Zou in [282, 286, 287, 320, 336, 344, 346] for (only the existence of solutions of) elliptic systems, homoclinic orbits of Hamiltonian systems, Schrödinger equations, and so on.

6.2 Bounded (PS) Sequences via Symmetry

In this section, assume that $E = \overline{\oplus_{j\in \mathbf{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of all positive integers. Set $E_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$, and

$$B_k = \{u \in E_k : \|u\| \le \rho_k\}.$$

We write $Y = E_k, M = Z_k$. Then $\dim Y < \infty$; $\dim Y - \mathrm{codim} M \ge 1$. Assume that $(M\backslash\{0\}) \cap (-\mathcal{P} \cup \mathcal{P}) = \emptyset$; that is, the nontrivial elements of M are sign-changing. We assume that $\mathcal{P}$ is weakly closed; that is, if $\mathcal{P} \ni u_k \rightharpoonup u$ weakly in $(E, \|\cdot\|)$, then $u \in \mathcal{P}$. In all applications in this book, this is satisfied automatically. As before, we assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \le C_\star \|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = M_1 \oplus M$. Let

$$Q^\star(\rho) := \left\{ u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star \|u\|_\star} = \rho \right\},$$

where $\rho > 0, D_\star > 0, p > 2$ are fixed constants. Evidently, we have

Lemma 6.7. $\|u\|_\star \le c_1, \forall u \in Q^\star(\rho)$, *where* $c_1 > 0$ *is a constant.*

Lemma 6.8. *Assume* $(A_3^\star)$. *For any* $a > 0$, *we have that*

$$\mathrm{dist}(Q^\star(\rho) \cap G_{1/2}^a, \mathcal{P}) := \delta(a) > 0.$$

Proof. Similar to Lemma 5.4. ∎

Let

$$\Gamma_k(\lambda) = \left\{ \gamma : \begin{array}{l} \gamma \in \mathbf{C}([0,1] \times B_k, E) \\ \gamma(t,\cdot)|_{\partial B_k} = \mathbf{id} \ \text{ for each } t \in [0,1] \\ \gamma(t,u) \ \text{ is odd in } u; \gamma(t,\mathcal{D}) \subset \mathcal{D} \ \text{ for all } t \in [0,1] \\ \sup G_\lambda(\gamma([0,1], B_k)) \leq \max_{B_k} G_1(u) := a_0 \end{array} \right\};$$

then $\gamma = \mathbf{id} \in \Gamma_k(\lambda)$. Note that both $\|\cdot\|_\star$ and $\|\cdot\|$ are equivalent in Y; we have a constant ϱ_Y such that

$$\|u\| \leq \varrho_Y \|u\|_\star, \quad \text{for all } u \in Y.$$

We assume that $\rho_k \geq \varrho_Y + 2$ and

$$\frac{\left(\frac{\rho_k}{\varrho_Y}\right)^p}{\rho_k^2} + \frac{\rho_k\left(\frac{\rho_k}{\varrho_Y}\right)}{\rho_k + D_\star C_\star \rho_k} > \rho. \tag{6.74}$$

Lemma 6.9. $h(t, B_k) \cap Q^\star(\rho) \neq \emptyset, \quad \forall\, h \in \Gamma_k(\lambda), \quad \forall\, t \in [0,1].$

Proof. Let

$$\beta^*(u) := \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star \|u\|_\star}$$

if $u \neq 0$ and $\beta^*(0) = 0$. Then $\beta^* : E \to E$ is continuous. Set $h(t,\cdot) = h(\cdot)$. Let

$$U := \{u \in Y : \beta^*(h(u)) < \rho\} \cap \{u \in Y : \|u\| < \rho_k\};$$

then U is a neighborhood of zero in Y. Let $P : E \to M_1$ be the projection; then $P \circ h : \partial U \to M_1$ is odd and continuous. By the Borsuk–Ulam theorem, we have that $P \circ h(u) = 0$ for some $u \in \partial U$. Hence, $h(u) \in M$. We claim $u \notin \partial\{u \in Y : \|u\| < \rho_k\}$. Otherwise, $\|u\| = \rho_k$ and then $h(u) = u, P(u) = 0$. It follows that

$$\frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star \|u\|_\star} \leq \rho.$$

Note that $\|u\|_\star \leq C_\star \|u\| \leq C_\star \varrho_Y \|u\|_\star$ in Y. Therefore,

$$\begin{aligned} &\frac{\left(\frac{\rho_k}{\varrho_Y}\right)^p}{\rho_k^2} + \frac{\rho_k\left(\frac{\rho_k}{\varrho_Y}\right)}{\rho_k + D_\star C_\star \rho_k} \\ &= \frac{\left(\frac{\|u\|}{\varrho_Y}\right)^p}{\|u\|^2} + \frac{\|u\|\left(\frac{\|u\|}{\varrho_Y}\right)}{\|u\| + D_\star C_\star \|u\|} \\ &\leq \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\|\|u\|_\star}{\|u\| + D_\star \|u\|_\star} \\ &\leq \rho. \end{aligned}$$

This is impossible in view of (6.74). So, our claim is true. It means

$$u \in \partial\{u \in Y : \beta^*(h(u)) < \rho\}, \qquad \|u\| \le \rho_k, \qquad u \in Y.$$

Hence,

$$h(u) \in M,$$

$$\frac{\|h(u)\|_\star^p}{\|h(u)\|^2} + \frac{\|h(u)\|\|h(u)\|_\star}{\|h(u)\| + D_\star\|h(u)\|_\star} = \rho \Rightarrow h(u) \in Q^\star(\rho).$$

□

Let

$$N_k := Q^\star(\rho) \cap G_{1/2}^{a_0}, \qquad a_0 := \max_{B_k} G_1.$$

By Lemma 6.8, we have that

$$\mathrm{dist}(N_k, \mathcal{P}) := \delta(a_0) > 0.$$

Therefore, we may assume that $N_k \subset \mathcal{S}$. Define

$$a_k(\lambda) := \max_{\partial B_k} G_\lambda, \qquad b_k(\lambda) := \inf_{N_k} G_\lambda,$$

$$c_k(\lambda) := \inf_{\gamma\in\Gamma_k(\lambda)} \max_{\gamma([0,1],B_k)\cap\mathcal{S}} G_\lambda.$$

($\mathbf{A_3}$) Assume $a_k(\lambda) < b_k(\lambda)$ for any $\lambda \in \Lambda$.

Theorem 6.10. *Assume that (A_1), $(A_3^\star)$, and (A_3) hold and that G_λ is even for all $\lambda \in \Lambda$. Then, for almost all $\lambda \in \Lambda$, there is a sequence $\{u_m\}$ depending on λ such that*

$$\sup_m \|u_m\| < \infty, \qquad u_m \in \mathcal{S}, \qquad G'_\lambda(u_m) \to 0,$$

$$G_\lambda(u_m) \to c_k(\lambda) \in \Big[b_k(\lambda),\ \max_{u\in B_k} G_1(u)\Big].$$

In particular, if J' is compact, then for almost all $\lambda \in [1/2, 1]$, G_λ has a sign-changing critical point in $\mathcal{S}$ with critical value in $[b_k(\lambda),\ \max_{u\in B_k} G_1(u)]$.

Proof. By the intersection lemma 6.9, for any $\gamma \in \Gamma_k(\lambda)$, we have that

$$\gamma([0,1], B_k) \cap N_k \ne \emptyset;$$

then $\gamma([0,1], B_k) \cap \mathcal{S} \ne \emptyset$. Therefore,

$$\sup_{\gamma([0,1],B_k)\cap\mathcal{S}} G_\lambda \ge \sup_{\gamma([0,1],B_k)\cap N_k} G_\lambda$$

$$\geq \inf_{\gamma([0,1],B_k)\cap N_k} G_\lambda$$

$$\geq \inf_{N_k} G_\lambda$$

$$= b_k(\lambda).$$

Then $c_k(\lambda) \geq b_k(\lambda)$. Evidently,

$$c_k(\lambda) \leq \sup_{u\in B_k} G_\lambda(u) \leq \max_{u\in B_k} G_1(u),$$

the right-hand side is a constant independent of λ.

Note that $\Gamma_k(\lambda_2) \subset \Gamma_k(\lambda_1)$ if $\lambda_1 \leq \lambda_2$. Then $c_k(\lambda)$ is still nondecreasing. Similar to the proof of Theorem 6.1, $c_k'(\lambda) := (dc_k(\lambda))/d\lambda$ exists for almost every $\lambda \in (\frac{1}{2}, 1)$. From now on, we consider those λ where $c_k'(\lambda)$ exists. For a fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda$ be a nonincreasing sequence so that $\lambda_n \to \lambda$ as $n \to \infty$. Then there exists an $\bar{n}(\lambda)$, which depends on λ only, such that

$$c_k'(\lambda) - 1 \leq \frac{c_k(\lambda_n) - c_k(\lambda)}{\lambda_n - \lambda} \leq c_k'(\lambda) + 1 \quad \text{for } n \geq \bar{n}(\lambda). \tag{6.75}$$

We prove the theorem step by step.

Step 1. We show that there exists a $\gamma_n \in \Gamma_k(\lambda)$ and $d_0 := d_0(\lambda) > 0$ such that

$$\|u\| \leq d_0 \quad \text{whenever } u \in \gamma_n([0,1], B_k) \cap \mathcal{S} \quad \text{with } G_\lambda(u) \geq c_k(\lambda) - (\lambda_n - \lambda),$$

where d_0 is dependent on λ and independent of n.

In fact, by the definition of $c_k(\lambda)$, there exists a $\gamma_n \in \Gamma_k(\lambda)$ such that

$$\begin{aligned} \sup_{u\in\gamma_n([0,1],\,B_k)\cap\mathcal{S}} G_\lambda(u) &\leq \sup_{u\in\gamma_n([0,1],\,B_k)\cap\mathcal{S}} G_{\lambda_n}(u) \\ &\leq c_k(\lambda_n) + (\lambda_n - \lambda). \end{aligned} \tag{6.76}$$

If $G_\lambda(u) \geq c_k(\lambda) - (\lambda_n - \lambda)$ for some $u \in \gamma_n([0,1], B_k) \cap \mathcal{S}$, then by (6.75) and (6.76), we have that

$$\frac{1}{2}\|u\|^2 = \frac{G_{\lambda_n}(u) - G_\lambda(u)}{\lambda_n - \lambda} \leq c_k'(\lambda) + 3. \tag{6.77}$$

It follows that

$$\|u\| \leq (2c_k'(\lambda) + 6)^{1/2} := d_0(\lambda) := d_0; \tag{6.78}$$

here d_0 is dependent on λ only.

Step 2. By the choice of γ_n and (6.75) and (6.76), we see that

$$(6.79)\qquad \begin{aligned} G_\lambda(u) &\le G_{\lambda_n}(u) \\ &\le c_k(\lambda) + (c_k'(\lambda) + 2)(\lambda_n - \lambda) \end{aligned}$$

for all $u \in \gamma_n([0,1], B_k) \cap \mathcal{S}$.

Step 3. For $\varepsilon > 0$, we define

$$\mathcal{H}_\varepsilon(n,\lambda) := \left\{ u \in E : \begin{array}{l} \|u\| \le d_0 + 2, \\ c_k(\lambda) - 2(\lambda_n - \lambda) \le G_\lambda(u) \le c_k(\lambda) + 2\varepsilon \end{array} \right\}.$$

Choose $n^*(\lambda) > \bar{n}(\lambda)$ ($\bar{n}(\lambda)$ comes from (6.75)) such that

$$(6.80)\quad (c_k'(\lambda)+2)(\lambda_n-\lambda) < \varepsilon, \qquad \lambda_n-\lambda \le \varepsilon, \qquad 2(\lambda_n-\lambda) < c_k(\lambda)-a_k(\lambda)$$

for all $n \ge n^*(\lambda)$. Then,

$$\mathcal{H}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S} \ne \emptyset$$

for all $n \ge n^*(\lambda)$. Indeed, by the definition of $c_k(\lambda)$, there exists at least one $u \in \gamma_n([0,1], B_k) \cap \mathcal{S}$ such that $G_\lambda(u) > c_k(\lambda) - (\lambda_n - \lambda)$; it follows that $\|u\| \le d_0$. Furthermore, (6.79) and (6.80) imply $G_\lambda(u) \le c_k(\lambda) + \varepsilon$. Therefore, $u \in \mathcal{H}_\varepsilon(n,\lambda) \cap \mathcal{S} \ne \emptyset$ for all $n \ge n^*(\lambda)$. Evidently,

$$(6.81)\qquad \mathcal{H}_\varepsilon(n,\lambda) \subset \mathcal{H}_\varepsilon(n^*(\lambda),\lambda) \quad \text{for all } n \ge n^*(\lambda).$$

We show that

$$(6.82)\qquad \inf_{u \in \mathcal{H}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S}} \|G_\lambda'(u)\| = 0.$$

Then the conclusion of the theorem follows from (6.82).

To prove (6.82) by negation, we assume that there exists an $\varepsilon^* > 0$ such that

$$(6.83)\qquad \|G_\lambda'(u)\| \ge \varepsilon^* \quad \text{for all } u \in \mathcal{H}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S};$$

here ε^* only depends on $n^*(\lambda), \lambda$, and ε, not on n. Therefore, by (6.81),

$$(6.84)\qquad \|G_\lambda'(u)\| \ge \varepsilon^* \quad \text{for all } u \in \mathcal{H}_\varepsilon(n,\ \lambda) \cap \mathcal{S},\ \ \forall n \ge n^*(\lambda).$$

We now proceed to seek a final contradiction. Let

$$\Omega_1 := \{u \in E : \|u\| \le d_0 + 1\};$$

$$\Omega_2 := \{u \in E : \|u\| \ge d_0 + 2\};$$

$$\Omega_3 := \left\{ u \in E : \begin{array}{l} \text{either } G_\lambda(u) \le c_k(\lambda) - 2(\lambda_n - \lambda) \quad \text{or} \\ G_\lambda(u) > c_k(\lambda) + 2\varepsilon \end{array} \right\};$$

$$\Omega_4 := \{u \in E : c_k(\lambda) - (\lambda_n - \lambda) \le G_\lambda(u) \le c_k(\lambda) + \varepsilon\}.$$

Define

$$\beta(u) := \frac{\mathrm{dist}(u, \Omega_2)}{\mathrm{dist}(u, \Omega_1) + \mathrm{dist}(u, \Omega_2)}, \tag{6.85}$$

$$\xi(u) := \frac{\mathrm{dist}(u, \Omega_3)}{\mathrm{dist}(u, \Omega_3) + \mathrm{dist}(u, \Omega_4)}. \tag{6.86}$$

Recall the definition of $\mathcal{S}$ in (6.3); let

$$\Theta(\alpha) := \{u \in E : \ \mathrm{dist}(u, \mathcal{P}) < \alpha\} \cup \{u \in E : \ \mathrm{dist}(u, -\mathcal{P}) < \alpha\}, \qquad \alpha > 0.$$

Then $\Theta(\alpha)$ is an open neighborhood of the positive and negative cones $-\mathcal{P} \cup \mathcal{P}$. Let $\mathcal{S}^* := E \backslash \overline{\Theta(\mu_0/2)}$, which is an open neighborhood of $\mathcal{S}$, where μ_0 comes from (6.3). Define

$$\pi(u) := \frac{\mathrm{dist}(u, \Theta(\mu_0/4))}{\mathrm{dist}(u, \Theta(\mu_0/4)) + \mathrm{dist}(u, \mathcal{S}^*)}. \tag{6.87}$$

Recall condition (A_3) and Lemmas 2.11 and 2.12; we have a locally Lipschitz continuous map $O_\lambda : \tilde{E}_\lambda \to E_\lambda$ such that $O_\lambda(\pm\mathcal{D}_0 \cap \tilde{E}_\lambda) \subset \pm\mathcal{D}_0$ and that $V_\lambda(u) := \lambda u - O_\lambda(u)$ is a pseudo-gradient vector field of G_λ. In particular, we may choose O_λ, hence V_λ, to be odd because G_λ is even for all λ. By (A_1), $\partial\mathcal{K}_\lambda \subset (-\mathcal{P}) \cup \mathcal{P} \cup \mathcal{S}$ for all $\lambda \in \Lambda$. Then for any $u \in \partial\mathcal{K}_\lambda$, if

(6.88)
$$\|u\| \le d_0 + 2, \qquad c_k(\lambda) - 2(\lambda_n - \lambda) \le G_\lambda(u) \le c_k(\lambda) + 2\varepsilon \quad \text{and} \quad u \notin \pm\mathcal{P},$$

then, $u \in \mathcal{H}_\varepsilon(n, \ \lambda) \cap \mathcal{S}$. By (6.84), there exists a neighborhood U_u of u such that $\|G'_\lambda(w)\| |_{U_u} \ge \varepsilon^*/2$. This is impossible because $u \in \partial\mathcal{K}_\lambda$. This means that at least one of the inequalities of (6.88) is not true. It follows that there exists a neighborhood U_u of u such that either $U_u \subset \Omega_2$ or $U_u \subset \Omega_3$ or $U_u \subset \Theta(\mu_0/4)$. Therefore, $\beta(u)\xi(u)\pi(u) = 0$ for all $u \in U_u$. Consequently,

$$W^*_\lambda(u) := \begin{cases} \dfrac{\beta(u)\xi(u)\pi(u)}{1 + \|V_\lambda(u)\|} V_\lambda(u), & \text{for } u \in \tilde{E}_\lambda, \\ 0, & \text{for } u \in \mathcal{K}_\lambda, \end{cases}$$

is a locally Lipschitz continuous vector field from E to E and $\|W^*_\lambda(u)\| \le 1$ on E. Moreover, note that $\Omega_i (i = 1, 2, 3, 4)$ and $\Theta(\alpha)$ are symmetric sets; we see that $\beta(u), \xi(u), \pi(u)$ are even. Therefore, $W^*_\lambda(u)$ is odd in u. Let ψ:

$[0,\infty)\times E \to E$ be the unique continuous solution of the following Cauchy initial value problem

$$\frac{d\psi(t,u)}{dt} = -W_\lambda^*(\psi(t,u)), \qquad \psi(0,u) = u \in E.$$

By the same arguments as in Step (4) of the proof of Theorem 6.1,

$$\psi([0,+\infty),\bar{\mathcal{D}}) \subset \bar{\mathcal{D}}, \qquad \psi([0,+\infty),\mathcal{D}) \subset \mathcal{D}. \tag{6.89}$$

Step 5. Choose $n^{**}(\lambda) > n^*(\lambda)$ such that

$$\lambda_n - \lambda < \frac{(\varepsilon^*)^2}{4(c_k'(\lambda)+3)(1+\varepsilon^*)} \tag{6.90}$$

for all $n \geq n^{**}(\lambda)$. For each $n > n^{**}(\lambda)$, we define

$$\gamma_n^*(s,u) := \psi(1,\gamma_n(s,u)), \qquad s \in [0,1]. \tag{6.91}$$

Then by (6.80), (6.86), and (6.89), $\gamma_n^* \in \mathit{\Gamma}_k(\lambda)$. Take any $u \in \gamma_n^*([0,1],B_k) \cap \mathcal{S}$; we write $u = \gamma_n^*(s_1,u_1) \in \mathcal{S}$ for some $s_1 \in [0,1]$ and $u_1 \in B_k$. Then $u = \psi(1,\gamma_n(s_1,u_1)) \in \mathcal{S}$.

If $G_\lambda(\gamma_n(s_1,u_1)) \leq c_k(\lambda) - (\lambda_n - \lambda)$, then

$$\begin{aligned} G_\lambda(u) &= G_\lambda(\psi(1,\gamma_n(s_1,u_1))) \\ &\leq G_\lambda((0,\gamma_n(s_1,u_1))) \\ &= G_\lambda(\gamma_n(s_1,u_1)) \\ &\leq c_k(\lambda) - (\lambda_n - \lambda). \end{aligned} \tag{6.92}$$

If $G_\lambda(\gamma_n(s_1,u_1)) > c_k(\lambda) - (\lambda_n - \lambda)$, we show that (6.92) still holds. We first observe that $\gamma_n(s_1,u_1) \in \mathcal{S}$. Otherwise, $\gamma_n(s_1,u_1) \in \mathcal{D}$ implies that $u = \psi_1(1,\gamma_n(s_1,u_1)) \in \mathcal{D}$ by (6.89), which is a contradiction because $u \in \mathcal{S}$. Recall Step 1; we have $\|\gamma_n(s_1,u_1)\| \leq d_0$. Furthermore, by (6.79) and (6.80),

$$\gamma_n(s_1,u_1) \in \mathcal{H}_\varepsilon(n,\lambda) \cap \mathcal{S} \subset \mathcal{H}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S}.$$

Therefore, by (6.84),

$$\|G_\lambda'(\gamma_n(s_1,u_1))\| \geq \varepsilon^*. \tag{6.93}$$

On the other hand, because $\|W_\lambda^*(u)\| \leq 1$ for all $u \in E$, we have that

$$\|\psi(t,\gamma_n(s_1,u_1)) - \psi(0,\gamma_n(s_1,u_1))\| \leq t$$

and that

$$\|\psi(t,\gamma_n(s_1,u_1))\| \le t + \|\gamma_n(s_1,u_1)\| \le d_0 + 1 \quad \text{for all } t \in [0,1]. \tag{6.94}$$

If $G_\lambda(\psi(t,\gamma_n(s_1,u_1))) \le c_k(\lambda) - (\lambda_n - \lambda)$ for some $t \in [0,1]$, then

$$G_\lambda(u) = G_\lambda(\psi(1,\gamma_n(s_1,u_1))) \le c_k(\lambda) - (\lambda_n - \lambda); \tag{6.95}$$

then we have an inequality as (6.92).

Now we assume $G_\lambda(\psi(t,\gamma_n(s_1,u_1))) > c_k(\lambda) - (\lambda_n - \lambda)$ for all $t \in [0,1]$. By (6.79)–(6.81),

$$G_\lambda(\psi(t,\gamma_n(s_1,u_1))) \le G_\lambda(\gamma_n(s_1,u_1)) \le c_k(\lambda) + \varepsilon. \tag{6.96}$$

By (6.89), $\psi(1,\gamma_n(s_1,u_1)) \in \mathcal{S}$ implies that $\psi(t,\gamma_n(s_1,u_1)) \in \mathcal{S}$ for all $t \in [0,1]$. Combining (6.81), (6.94), and (6.96), we have that

$$\psi(t,\gamma_n(1,u_1)) \in \mathcal{H}_\varepsilon(n,\lambda) \cap \mathcal{S} \subset \mathcal{H}_\varepsilon(n^*(\lambda),\ \lambda) \cap \mathcal{S} \tag{6.97}$$

for all $t \in [0,1]$. By (6.84),

$$\|G'_\lambda(\psi(t,\gamma_n(s_1,u_1)))\| \ge \varepsilon^* \quad \text{for all } t \in [0,1]. \tag{6.98}$$

Moreover, by (6.85), (6.86), (6.94), and (6.96),

$$\begin{aligned} \beta(\psi(t,\gamma_n(1,u_1))) &= \xi(\psi(t,\gamma_n(1,u_1))) \\ &= \pi(\psi(t,\gamma_n(1,u_1))) \\ &= 1 \end{aligned} \tag{6.99}$$

for all $t \in [0,1]$. Combining the definition of the pseudo-gradient vector field and (6.98) and (6.99), it follows that

$$\begin{aligned} &G_\lambda(\psi(t,\gamma_n(s_1,u_1))) - G_\lambda(\gamma_n(s_1,u_1)) \\ &\quad \le \int_0^t \frac{dG_\lambda(\psi(s,\gamma_n(s_1,u_1)))}{ds} ds \\ &\quad \le \int_0^t -\left\langle G'_\lambda(\psi(s,\gamma_n(s_1,u_1))), \frac{V_\lambda(\psi(s,\gamma_n(s_1,u_1)))}{1+\|V_\lambda(\psi(s,\gamma_n(s_1,u_1)))\|} \right\rangle ds \\ &\quad \le -\frac{1}{2}\int_0^t \frac{\|G'_\lambda(\psi(s,\gamma_n(s_1,u_1)))\|^2}{1+\|V_\lambda(\psi(s,\gamma_n(s_1,u_1)))\|} ds \\ &\quad \le -\frac{1}{4}\int_0^t \frac{\|G'_\lambda(\psi(s,\gamma_n(s_1,u_1)))\|^2}{1+\|G'_\lambda(\psi(s,\gamma_n(s_1,u_1)))\|} ds \\ &\quad \le -\frac{1}{4}\frac{(\varepsilon^*)^2}{1+\varepsilon^*} t. \end{aligned}$$

It follows that

$$
\begin{aligned}
(6.100)\qquad G(u) &= G_\lambda(\psi(1,\gamma_n(s_1,u_1))) \\
&\le G_\lambda(\gamma_n(s_1,u_1)) - \frac{1}{4}\frac{(\varepsilon^*)^2}{1+\varepsilon^*} \\
&\le c_k(\lambda) + (c_k'(\lambda)+2)(\lambda_n-\lambda) - \frac{1}{4}\frac{(\varepsilon^*)^2}{1+\varepsilon^*} \qquad \text{(by (6.79))} \\
&\le c_k(\lambda) - (\lambda_n - \lambda). \qquad \text{(by (6.90))}
\end{aligned}
$$

Summing up (6.92), (6.95), and (6.100), we have

$$G(u) \le c_k(\lambda) - (\lambda_n - \lambda)$$

for all $u \in \gamma_n^*([0,1], B_k) \cap \mathcal{S}$ and all $n > n^{**}(\lambda)$. This contradicts the definition of $c_k(\lambda)$ because $\gamma_n^* \in \Gamma_k(\lambda)$ (cf. (6.91)). The contradiction guarantees the truth of (6.82) from which the conclusion of Theorem 6.10 follows. □

Notes and Comments. A classical theorem for the existence of critical points of even functionals is the so-called symmetric mountain pass theorem due to Ambrosetti and Rabinowitz [15] (see also Rabinowitz [251, 253, 254, 257]) by which we may get infinitely many critical points without nodal structure. A related result to Theorem 6.10 is known as the fountain theorem because the critical points spout out like a fountain. The earlier form of the fountain theorem and its dual were established by Bartsch in [28] and by Bartsch and Willem in [50] (see also Willem [335]), respectively. Other applications can be found in Bartsch and Willem [48, 49] and Bartsch and de Figueiredo [35]), a variant version of which is given in Zou [342]. In all these papers, the sign of the critical point cannot be decided.

6.3 Positive and Negative Solutions

In this section, we establish a parameter depending on the mountain pass theorem inside the cones, by which we may get positive and negative solutions directly. Assume $e_0^\pm \in \pm\mathcal{P}$. Let

$$\Psi^\pm := \{\phi \in \mathbf{C}([0,1], \pm\mathcal{D}_0) : \phi(0) = 0, \phi(1) = e_0^\pm\}.$$

We introduce the following assumption.

(A$_4$) $\beta_\pm(\lambda) := \inf_{\phi\in\Psi^\pm} \sup_{\phi([0,1])} G_\lambda > \max\{G_\lambda(0), G_\lambda(e_0^\pm)\} := \rho_\pm \quad$ for all $\lambda \in \Lambda$.

Theorem 6.11. *Assume that (A_1) and (A_4) hold. Then for almost all $\lambda \in \Lambda$, there are two sequences $\{u_m(\pm)\} \subset \pm\mathcal{D}_0$ depending on λ such that*

$$\sup_m \|u_m(\pm)\| < \infty, \qquad G'_\lambda(u_m(\pm)) \to 0,$$

$$G_\lambda(u_m(\pm)) \to \beta_\pm(\lambda) \quad \textit{as } m \to \infty,$$

where

$$\beta_\pm(\lambda) \in [\rho_\pm, \Pi_\pm(\lambda)], \qquad \Pi_\pm(\lambda) := \max_{t\in[0,1]} G_\lambda(te_0^\pm).$$

Furthermore, if both $\{u_m(\pm)\}$ *have convergent subsequences (for instance,* J' *is compact), then for almost all* $\lambda \in \left(\frac{1}{2}, 1\right)$, G_λ *have two critical points* $u^\pm \in \pm P$.

Proof. We only consider the case of "+". As before, $\beta'_+(\lambda)$ exists for almost all $\lambda \in \Lambda$. For this kind of λ, choose $\lambda_n \in (\lambda, 2\lambda) \cap \left(\frac{1}{2}, 1\right)$ such that $\lambda_n \to \lambda$ and

$$\beta'_+(\lambda) - 1 \le \frac{\beta_+(\lambda_n) - \beta_+(\lambda)}{\lambda_n - \lambda} \le \beta'_+(\lambda) + 1, \quad \text{as } n \text{ large enough.} \tag{6.101}$$

By the definition of $\beta_+(\lambda)$, there exists a $\phi_n \in \Psi^+$ such that

$$\sup_{\phi_n([0,1])} G_\lambda \le \sup_{\phi_n([0,1])} G_{\lambda_n} \le \beta_+(\lambda_n) + (\lambda_n - \lambda). \tag{6.102}$$

Hence,

$$\begin{aligned} G_\lambda(u) &\le G_{\lambda_n}(u) \\ &\le \sup_{\phi_n([0,1])} G_{\lambda_n} \\ &\le \beta_+(\lambda) + (c'_+(\lambda) + 2)(\lambda_n - \lambda) \end{aligned} \tag{6.103}$$

for all $u \in \phi_n([0,1])$. By (6.102), it is easy to check that

$$\|u\| \le k_0 := (2\beta'_+(\lambda) + 6)^{1/2} \tag{6.104}$$

if $G_\lambda(u) \ge \beta_+(\lambda) - (\lambda_n - \lambda)$ and $u \in \phi_n([0,1])$. Define

$$\mathcal{A}(\varepsilon, \lambda) := \{u \in \mathcal{D}_0 : \|u\| \le k_0 + 3, |G_\lambda(u) - \beta_+(\lambda)| \le \varepsilon\}.$$

We first observe that $\mathcal{A}(\varepsilon, \lambda) \ne \emptyset$. To see this, choose n large enough such that

$$(c'_+(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon.$$

By (6.103), $G_\lambda(u) \le \beta_+(\lambda) + \varepsilon$ for all $u \in \phi_n([0,1])$. Evidently, by the definition of $\beta_+(\lambda)$, we cannot have $G_\lambda(u) < \beta_+(\lambda) - (\lambda_n - \lambda)$ for all $u \in \phi_n([0,1])$. That is, there exists at least one $u \in \phi_n([0,1])$ such that

$$G_\lambda(u) \ge \beta_+(\lambda) - (\lambda_n - \lambda) \ge \beta_+(\lambda) - \varepsilon; \tag{6.105}$$

hence, $\|u\| \le k_0$, $u \in \mathcal{A}(\varepsilon, \lambda) \ne \emptyset$. We just need to prove that

$$\inf\{\|G'_\lambda(u)\| : u \in \mathcal{A}(\varepsilon, \lambda)\} = 0. \tag{6.106}$$

By way of negation, assume that there exists an $\varepsilon_0 > 0$ such that $\|G'_\lambda(u)\| \ge \varepsilon_0$ for $u \in \mathcal{A}(\varepsilon_0, \lambda)$. Without loss of generality, we may assume $\varepsilon_0 < (\beta_+(\lambda) - \rho_+)/3$. Choose n large enough such that

$$(\lambda_n - \lambda) \le \varepsilon_0/5, \qquad (c'_+(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon_0/5,$$

$$\lambda_n - \lambda < 2(\beta_+(\lambda) - \rho_+), \qquad \lambda_n - \lambda < \frac{\varepsilon_0^2}{4(1+\varepsilon_0)(\beta'_+(\lambda)+3)}.$$

Then by (6.103),

$$\begin{aligned} G_\lambda(u) &\le \beta_+(\lambda) + (c'_+(\lambda) + 2)(\lambda_n - \lambda) \\ &< \beta_+(\lambda) + \varepsilon_0/5 \end{aligned} \tag{6.107}$$

for all $u \in \phi_n([0,1])$. Define

$$\mathcal{A}^*(\varepsilon_0, \lambda) := \left\{ u \in \mathcal{D}_0 : \begin{array}{c} \|u\| \le k_0 + 3, \\ \beta_+(\lambda) - (\lambda_n - \lambda) \le G_\lambda(u), \\ G_\lambda(u) \le \beta_+(\lambda) + \varepsilon_0 \end{array} \right\}. \tag{6.108}$$

Then, by a similar argument, $\mathcal{A}^*(\varepsilon_0, \lambda) \ne \emptyset$ and $\mathcal{A}^*(\varepsilon_0, \lambda) \subset \mathcal{A}(\varepsilon_0, \lambda)$. Define

$$M_1 := \left\{ u \in \mathcal{D}_0 : \begin{array}{l} \|u\| \le k_0 + 2, \\ \beta_+(\lambda) - \frac{\lambda_n - \lambda}{2} \le G_\lambda(u) \le \beta_+(\lambda) + \frac{\varepsilon_0}{2} \end{array} \right\},$$

$$M_2 := \left\{ u \in \mathcal{D}_0 : \begin{array}{l} \|u\| \le k_0 + 1, \\ \beta_+(\lambda) - \frac{\lambda_n - \lambda}{4} \le G_\lambda(u) \le \beta_+(\lambda) + \frac{\varepsilon_0}{4} \end{array} \right\}.$$

Then $M_2 \subset M_1 \subset \mathcal{A}^*(\varepsilon_0, \lambda)$. Let

$$\kappa(u) := \frac{\operatorname{dist}(u, E \backslash M_1)}{\operatorname{dist}(u, M_2) + \operatorname{dist}(u, E \backslash M_1)},$$

$$V^*_\lambda(u) = \begin{cases} \kappa(u) \dfrac{V_\lambda(u)}{1 + \|V_\lambda(u)\|}, & u \in \tilde{E}_\lambda, \\ 0, & u \in \mathcal{K}_\lambda, \end{cases}$$

where V_λ comes from Lemma 2.12. Then $V_\lambda^*(u)$ is a locally Lipschitz continuous vector field on E. Consider the following initial value problem,

$$\begin{cases} \dfrac{d\vartheta(t,u)}{dt} = -V_\lambda^*(\vartheta(t,u)), \\ \vartheta(0,u) = u \in E. \end{cases}$$

Its unique solution $\vartheta(t,u)(t \geq 0)$ satisfies

$$\|\vartheta(t,u) - u\| \leq t; \tag{6.109}$$

$$\frac{dG_\lambda(\vartheta(t,u))}{dt} \leq -\frac{1}{4}\kappa(\vartheta(t,u))\frac{\|G'_\lambda(\vartheta(t,u))\|^2}{1+\|G'_\lambda(\vartheta(t,u))\|} \leq 0. \tag{6.110}$$

Therefore, by (6.107) and (6.110),
(6.111)

$$G_\lambda(\vartheta(t,u)) \leq G_\lambda(u) \leq \beta_+(\lambda) + \varepsilon_0/5, \qquad \forall\, u \in \phi_n([0,1]), \qquad \forall t \geq 0.$$

If $u \in \phi_n([0,1])$ such that $G_\lambda(u) > \beta_+(\lambda) - (\lambda_n - \lambda)/4$, then by (6.104), $\|u\| \leq k_0$; hence, $u \in M_2$. Therefore, by (6.110) and (6.111), we must have

$$G_\lambda(\vartheta(t,u)) \leq G_\lambda(u) \leq \beta_+(\lambda) - (\lambda_n - \lambda)/4, \tag{6.112}$$

for all $u \in \phi_n([0,1])$ and $u \notin M_2, \forall t \geq 0$.

If $u \in \phi_n([0,1]) \cap M_2$, we show that (6.112) is still true for $t = 1$.

Suppose that t_1 is the largest number (may equal ∞) such that $\vartheta(t,u) \in M_2$ for $0 \leq t \leq t_1$. If $t_1 < 1$, because $\vartheta(t_1+s,u) \notin M_2$ for $s > 0$ small enough and in view of (6.111), we have either

$$\|\vartheta(t_1+s,u)\| > k_0 + 1 \tag{6.113}$$

or

$$G_\lambda(\vartheta(t_1+s,u)) < \beta_+(\lambda) - (\lambda_n - \lambda)/4. \tag{6.114}$$

We claim that the second conclusion of (6.113) is true. Otherwise,

$$\begin{aligned} G_\lambda(u) &\geq G_\lambda(\vartheta(t_1+s,u)) \\ &\geq \beta_+(\lambda) - (\lambda_n - \lambda)/4 \\ &\geq \beta_+(\lambda) - (\lambda_n - \lambda), \end{aligned}$$

which implies by (6.102) that $\|u\| \leq k_0$. By (6.109),

$$\|\vartheta(t_1+s,u)\| \leq \|u\| + t_1 + s \leq k_0 + 1;$$

that is, the first alternative of (6.113) fails too. Now, use (6.110) again; we must have

$$G_\lambda(\vartheta(1,u)) \le G_\lambda(\vartheta(t_1+s,u)) \le \beta_+(\lambda) - (\lambda_n - \lambda)/4. \tag{6.115}$$

If $t_1 \ge 1$, then $\vartheta(t,u) \in M_2$ for $0 \le t \le 1$. If

$$G_\lambda(\vartheta(1,u)) > \beta_+(\lambda) - (\lambda_n - \lambda)/4, \tag{6.116}$$

then by (6.110), $G_\lambda(u) \ge \beta_+(\lambda) - (\lambda_n - \lambda)/4$, it implies that $\|u\| \le k_0$ and then by (6.109), $\|\vartheta(t,u)\| \le k_0 + 1$ for all $t \in [0,1]$. Combining (6.110), (6.111), and (6.116), we observe that

$$\vartheta(t,u) \in M_2 \subset \mathcal{A}(\varepsilon_0, \lambda)$$

for all $t \in [0,1]$. Hence, $\kappa(\vartheta(t,u)) = 1$ for all $t \in [0,1]$. It follows that

$$\begin{aligned} &G_\lambda(\vartheta(1,u)) - G_\lambda(u) \\ &\quad = \int_0^1 \frac{dG_\lambda(\vartheta(s,u))}{ds} ds \\ &\quad \le -\frac{1}{4}\int_0^1 \kappa(\vartheta(s,u)) \frac{\|G'_\lambda(\vartheta(s,u))\|^2}{1 + \|G'_\lambda(\vartheta(s,u))\|} ds \\ &\quad \le -\frac{\varepsilon_0^2}{4(1+\varepsilon_0)}. \end{aligned}$$

Furthermore, by (6.107),

$$\begin{aligned} &G_\lambda(\vartheta(1,u)) \\ &\quad \le -\frac{\varepsilon_0^2}{4(1+\varepsilon_0)} + \beta_+(\lambda) + (\lambda_n - \lambda)(\beta'_+(\lambda) + 2) \\ &\quad \le \beta_+(\lambda) - \frac{(\lambda_n - \lambda)}{4}, \end{aligned} \tag{6.117}$$

by the way we have chosen n. Anyway, (6.116) cannot be true. Recall (6.112), (6.115), and (6.117); we have that

$$G_\lambda(\vartheta(1,u)) \le \beta_+(\lambda) - (\lambda_n - \lambda)/4, \qquad \forall u \in \phi_n([0,1]). \tag{6.118}$$

Similar to the proof of Step 4 of Theorem 6.1, we have

$$\vartheta([0,+\infty), \mathcal{D}_0) \subset \mathcal{D}_0. \tag{6.119}$$

If we define

$$\phi^*(t) := \vartheta(1, \phi_n(t)),$$

then because $0, e_0^+ \notin M_1$, we have that

$$\phi^*(0) = \vartheta(1, \phi_n(0)) = \vartheta(1, 0) = 0,$$

$$\phi^*(e_0^+) = \vartheta(1, \phi_n(e_0^+)) = \vartheta(1, e_0^+) = e_0^+,$$

Combining this and (6.119), we see that $\phi^* \in \Psi^+$. But (6.118) contradicts the definition of $\beta_+(\lambda)$. Then we get the first part of Theorem 6.11. As for the second part, we just note that (A_1) implies that $\mathcal{K}_\lambda \cap \mathcal{D}_0 \subset P$. $\square$

Notes and Comments. The classical mountain pass theorem can be found in Ambrosetti and Rabinowitz [15] (see also Rabinowitz [251, 253, 254]). A version of the parameter-depending mountain pass theorem on whole space was first established in Jeanjean [178] which also produces a bounded (PS) sequence. But the sign of the limits was not decided there. In general, to get positive or negative solutions to elliptic equations, more techniques and theory (e.g., the trick of truncating function and the maximum principle) are needed. Here, we may get both positive and negative solutions directly. A version of the mountain pass theorem in an open subset was established in Sun [316] (see also Sun and Hu [317]) where the (PS) condition was necessary. Readers are referred to Chang [97, 96, 98], Li and Wang [198], Schechter [267, 270, 275], Schechter and Zou [289], and Tintarev [327] for other variants of the mountain pass theorem.

6.4 Subcritical Schrödinger Equation

Consider the existence of sign-changing solutions to the Schrödinger equation:

$$-\Delta u + V(x)u = f(x, u), \qquad x \in \mathbf{R}^N, \tag{6.120}$$

where $2^* \big(= 2N/(N-2)$ if $N \geq 3$ or $2^* = \infty$ if $N = 1, 2\big)$ is the critical Sobolev exponent; $f(x,t) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with a subcritical growth. Throughout this chapter, we always assume the following geometric condition and growth hypotheses.

$(\mathbf{S}_1)$ Let $V(x) \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N), V_0 := \operatorname{essinf}_{\mathbf{R}^N} V(x) > 0$. For any $M', r > 0$,

$$\operatorname{meas}(\{x \in B_r(y) : V(x) \leq M'\}) \to 0 \quad \text{as } |y| \to \infty, \tag{6.121}$$

where $B_r(y)$ denotes the ball centered at y with radius r.

$(\mathbf{S}_2)$ $f : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with a subcritical growth:

$$|f(x,u)| \leq c(1 + |u|^{s-1}) \quad \text{for all } u \in \mathbf{R} \text{ and } x \in \mathbf{R}^N,$$

where $s \in (2, 2^*)$; $f(x,u)u \geq 0$ for all (x,u) and $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \mathbf{R}^N$.

Let E be the Hilbert space

$$E := \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} V(x)u^2 dx < \infty \right\}$$

endowed with the inner product

$$\langle u, v \rangle := \int_{\mathbf{R}^N} (\nabla u \nabla v + V(x)uv) dx$$

for $u, v \in E$ and norm $\|u\| := \langle u, u \rangle^{1/2}$. The role of (S_1) ensures the compactness of certain embeddings of the working spaces. The limit in (6.121) can be replaced by one of the following stronger conditions.

- $(\bar{\mathbf{S}}_1)$ $\mathrm{meas}(\{x \in \mathbf{R}^N : V(x) \leq M'\}) < \infty$ for any $M' > 0$ (see Bartsch and Wang [39]),
- $(\tilde{\mathbf{S}}_1)$ $V(x) \to \infty$ as $|x| \to \infty$. (see Rabinowitz [260] and Omana and Willem [238]).

A classical result due to Molchanov [228] (see also Kondrat'ev and Shubin [184, Corollary 6.2] and Bartsch et al. [38]) says that $E \hookrightarrow\hookrightarrow L^2(\mathbf{R}^N)$ and hence, by the Gagliardo–Nirenberg inequality (cf. Lemma 1.18), $E \hookrightarrow\hookrightarrow L^p(\mathbf{R}^N)$ for $p \in [2, 2^*)$. By standard elliptic theory (see, e.g., Evans [141] and Reed and Simon [262]) we have the following lemma.

Lemma 6.12. *The hypothesis (S_1) implies that the eigenvalue problem*

$$-\Delta u + V(x)u = \lambda u, \qquad x \in \mathbf{R}^N$$

possesses a sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$$

with finite multiplicity for each λ_k. The principal eigenvalue λ_1 is simple with a positive eigenfunction φ_1, and the eigenfunctions φ_k corresponding to λ_k $(k \geq 2)$ are sign-changing.

Let X_k denote the eigenspace of λ_k; then $\dim X_k < \infty$. Let $E_k := X_1 \oplus \cdots \oplus X_k$ and

$$\mathcal{P} := \{u \in E : u(x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^N\}.$$

Then $\mathcal{P}$ $(-\mathcal{P})$ is the positive (negative) cone of E. Let $\mathcal{S}$ be defined as (6.3).

Consider

$$G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x, u) dx, \qquad \lambda \in \left(\frac{1}{2}, 1\right) := \Lambda. \tag{6.122}$$

Then

$$G_\lambda \in \mathbf{C}^1(E, \mathbf{R}), \qquad G'_\lambda = \lambda \mathbf{id} - J',$$

where $J' := (-\Delta + V)^{-1} f$. We seek the critical points of $G_1 := G$. Consider Equation (6.120) with superlinear and subcritical growth. We are interested in the existence of infinitely many sign-changing solutions. Assume

($\mathbf{P_1}$) $\liminf\limits_{|t|\to\infty} \dfrac{f(x,t)}{t} = \infty$ uniformly for $x \in \mathbf{R}^N$.

($\mathbf{P_2}$) $f(x,t)$ is odd in t.

($\mathbf{P_3}$) $\dfrac{f(x,t)}{t}$ is nondecreasing in $t > 0$.

Theorem 6.13. *Assume (S_1) and (S_2) and (P_1)–(P_3). Then Equation (6.120) has one positive solution, one negative solution, and infinitely many sign-changing solutions.*

Condition (P_3) can be replaced by

($\mathbf{P_4}$) $H(x,t) := f(x,t)t - 2F(x,t) \geq 0$ for all $x \in \mathbf{R}^N$; $H(x,t)$ is convex in $t > 0$.

Theorem 6.14. *Assume (S_1), (S_2), (P_1), (P_2), and (P_4).*
Then Equation (6.120) has one positive solution, one negative solution, and infinitely many sign-changing solutions.

We prove Theorems 6.13 and 6.14 by applying Theorem 6.10. Given $k \geq 2$, set

$$E_k = \oplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

Then $E = E_{k-1} \oplus Z_k$. Let

$$B_k := \{u \in E_k : \|u\| \leq \rho_k\},$$

where $\rho_k > 0$ is a constant to be determined.

By (S_2), there exist $C_F > 0, s \in (2, 2^*)$ such that

$$|F(x,u)| \leq \frac{\lambda_1}{8} u^2 + C_F |u|^s, \quad \text{for } x \in \mathbf{R}^N, \qquad u \in \mathbf{R}. \tag{6.123}$$

Recall the Gagliardo–Nirenberg inequality:

$$\|u\|_s \leq c_s \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha} \quad \text{for all } u \in H^1(\mathbf{R}^N), \tag{6.124}$$

where $\alpha = N\left(\frac{1}{2} - 1/s\right) \in (0,1), c_s > 1$ is a constant depending on s, N.

Lemma 6.15. *Under the assumptions of Theorems 6.13 and 6.14, there exists a constant $\rho_k > 0$ independent of λ such that*

$$\max_{\partial B_k} G_\lambda \leq a_k \leq 0 \tag{6.125}$$

for all $\lambda \in \left(\frac{1}{2}, 1\right)$. Here a_k is independent of λ.

Proof. Because $\dim E_k < \infty$, then by (P_1),

$$\frac{G_\lambda(u)}{\|u\|^2} \le \frac{1}{2} - \int_{\mathbf{R}^N} \frac{F(x,u)}{\|u\|^2} dx \to -\infty$$

as $\|u\| \to \infty, u \in E_k$ uniformly for $\lambda \in \left(\frac{1}{2}, 1\right)$. Then there exists a $\rho_k > 0$ such that $\max_{\partial B_k} G_\lambda \le a_k \le 0$, where $\rho_k > 0, a_k$ are independent of λ. □

For $s \in (2, 2^*)$ given in (6.123), let

$$Q^\star(\rho) := \left\{ u \in E_{k-1}^\perp : \frac{\|u\|_s^s}{\|u\|^2} + \frac{\|u\|\|u\|_s}{\|u\| + \lambda_k^{\beta'}\|u\|_s} = \rho \right\},$$

where $\beta' = (1-\alpha)(s-2), \rho := 1/16C_F > 0$ are fixed constants, $\alpha = N\left(\frac{1}{2} - 1/s\right)$ comes from (6.124). On the other hand, we have a constant $\Lambda_s > 1$ such that

$$\|u\|_s \le \Lambda_s \|u\|, \qquad u \in E. \tag{6.126}$$

For $u \in Q^\star(\rho)$, by (6.126) we have

$$\begin{aligned}
\rho &= \frac{\|u\|_s^s}{\|u\|^2} + \frac{\|u\|\|u\|_s}{\|u\| + \lambda_k^{\beta'}\|u\|_s} \\
&\le \frac{\|u\|\|u\|_s}{2(\|u\|\lambda_k^{\beta'}\|u\|_s)^{1/2}} + \frac{\|u\|_s^2}{\|u\|^2}\|u\|_s^{s-2} \\
&\le \frac{(\|u\|\|u\|_s)^{1/2}}{2(\lambda_k^{\beta'})^{1/2}} + \Lambda_s^2 \|u\|_s^{s-2} \\
&\le \frac{(\Lambda_s)^{1/2}\|u\|}{2(\lambda_k^{\beta'})^{1/2}} + \Lambda_s^2 \|u\|_s^{s-2}.
\end{aligned}$$

By the Gagliardo–Nirenberg inequality in (6.124),

$$\|u\|_s^{s-2} \le c_s^{s-2} \|u\|^{\alpha(s-2)} \|u\|_2^{(1-\alpha)(s-2)}. \tag{6.127}$$

But $u \in E_{k-1}^\perp$; we see that $\lambda_k \|u\|_2^2 \le \|u\|^2$. Hence, by (6.127),

$$\|u\|_s^{s-2} \le c_s^{s-2} \|u\|^{s-2} \lambda_k^{-(((1-\alpha)(s-2))/2)}. \tag{6.128}$$

Therefore,

$$\rho \le \frac{(\Lambda_s)^{1/2}\|u\|}{2(\lambda_k^{\beta'})^{1/2}} + (\Lambda_s)^2 c_s^{s-2}\|u\|^{s-2}\lambda_k^{-(((1-\alpha)(s-2))/2)}$$
$$\le \left(\frac{1}{(\lambda_k^{\beta'})^{1/2}} + \frac{1}{\lambda_k^{((1-\alpha)(s-2))/2}}\right)(2\Lambda_s^2 c_s^{s-2})\max\{\|u\|, \|u\|^{s-2}\}.$$

Then we have that

$$\frac{\lambda_k^{((1-\alpha)(s-2))/2}}{(4\Lambda_s^2 c_s^{s-2})}\rho \le \max\{\|u\|, \|u\|^{s-2}\}. \tag{6.129}$$

Hence, we have

Lemma 6.16. *For all $u \in Q^\star(\rho)$,*

$$\|u\| \ge \Lambda_s^* \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\}\min\{\rho, \rho^{1/(s-2)}\},$$

where Λ_s^* is a constant independent of k.

Lemma 6.17. $\dfrac{\|u\|_s^s}{\|u\|^2} \le \rho, \quad \forall u \in Q^\star(\rho).$

Lemma 6.18. $\|u\|_s \le c_1, \quad \forall u \in Q^\star(\rho).$

Proof. Actually,

$$\|u\|_s \le \rho(1 + \lambda_k^{\beta'}\Lambda_s), \qquad \rho = \frac{1}{16C_F}.$$

□

Consider the functional

$$G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x,u)dx, \qquad \lambda \in \left(\frac{1}{2}, 1\right) := \Lambda.$$

Then by (6.123),

$$G_\lambda(u) \ge \frac{1}{4}\|u\|^2 - \frac{\lambda_1}{8}\|u\|_2^2 - C_F\|u\|_s^s$$
$$\ge \frac{1}{8}\|u\|^2 - C_F\|u\|_s^s$$
$$= \|u\|^2\left(\frac{1}{8} - C_F\frac{\|u\|_s^s}{\|u\|^2}\right).$$

Combine Lemma 6.16 and Lemma 6.17; we have the following.

Lemma 6.19. *For any $u \in Q^\star(\rho)$, we have that*

$$G_\lambda(u) \geq \delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\},$$

where $\delta_s > 0$ is a constant independent of k and λ.

By Lemma 6.9 and Lemma 6.15, we may choose ρ_k large such that Lemma 6.9 holds. Now let

$$N_k := Q^\star(\rho) \cap G_{1/2}^{a_0}, \qquad a_0 := \max_{B_k} G_1.$$

By Lemma 6.8, we have that

$$\operatorname{dist}(N_k, \mathcal{P}) := \delta(a_0) > 0. \tag{6.130}$$

For $\mu_0 > 0$, we define

$$\mathcal{D}_0(\mu_0) := \{u \in E : \operatorname{dist}(u, \mathcal{P}) < \mu_0\},$$

$$\mathcal{D}_1(\mu_0) := \{u \in E : \operatorname{dist}(u, \mathcal{P}) < \mu_0/2\}.$$

$$\mathcal{D} := -\mathcal{D}_0(\mu_0) \cup \mathcal{D}_0(\mu_0), \qquad \mathcal{S} := E\backslash\mathcal{D}. \tag{6.131}$$

Lemma 6.20. *Consider G_λ with*

$$G'_\lambda(u) = \lambda u - J'(u). \tag{6.132}$$

There exists a $\mu_0 < 1/2$ (may be chosen small enough) such that

$$\operatorname{dist}(J'(u), \pm\mathcal{P}) \leq \frac{1}{5}\operatorname{dist}(u, \pm\mathcal{P})$$

for all $u \in E$ with $\operatorname{dist}(u, \pm\mathcal{P}) < \mu_0$. That is, (A_1) is satisfied for G_λ.

Proof. This is the same as the proof of Lemmas 2.29. □

By Lemma 6.20 and (6.130), we may assume that

$$N_k \subset \mathcal{S}.$$

By Lemma 6.19, we have the following.

Lemma 6.21. *Under the assumptions of Theorems 6.13 and 6.14, there exist constants $r_k > 0$ independent of λ such that*

$$b_k \leq \inf_{N_k} G_\lambda \tag{6.133}$$

for all $\lambda \in (\frac{1}{2}, 1)$. Here b_k are independent of λ. Moreover, $b_k \to \infty$ as $k \to \infty$.

Note Lemma 6.20 and $N_k \subset \mathcal{S}$; we may use Theorem 6.10 to G_λ on E for almost all λ. Then G_λ has a sign-changing critical point $u(\lambda)$ in $\mathcal{S}$ with critical value in $[b_k, \max_{B_k} G_1(u)]$, an interval independent of λ.

Lemma 6.22. *Let $\lambda_m \in \left(\frac{1}{2}, 1\right)$ and $\lambda_m \to 1$. Set $u_m = u(\lambda_m)$. Under the assumptions of Theorems 6.13 and 6.14, $\{u_m\}$ is bounded.*

Proof. Assume that $\|u_m\| \to \infty$ as $m \to \infty$. We consider $w_m := u_m/\|u_m\|$. Then, up to a subsequence, we get that

$$\begin{aligned} &w_m \rightharpoonup w \quad \text{in } E, \\ &w_m \to w \quad \text{in } L^t(\mathbf{R}^N) \text{ for } 2 \le t < 2^*, \\ &w_m(x) \to w(x) \text{ a.e. } x \in \mathbf{R}^N. \end{aligned}$$

Case 1. $w \neq 0$ in E. Because $G'_\lambda(u_m) = 0$, we have that

$$\int_{\mathbf{R}^N} \frac{f(x, u_m)u_m}{\|u_m\|^2} dx \le c.$$

On the other hand, by Fatou's lemma and Conditions (P_1) and (S_2),

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{f(x, u_m)u_m}{\|u_m\|^2} dx \\ &\quad = \int_{\{w(x)\neq 0\}} |w_m(x)|^2 \frac{f(x, u_m)u_m}{|u_m|^2} dx \to \infty, \end{aligned}$$

a contradiction.

Case 2. $w = 0$ in E. Define

$$G'_\lambda(t_m u_m) := \max_{t\in[0,1]} G_\lambda(t u_m).$$

For any $c > 0$ and $\bar{w}_n := (4c)^{1/2} w_n$, we have, for n large enough, that

$$G_\lambda(t_m u_m) \ge G_\lambda(\bar{w}_m) = c - \int_{\mathbf{R}^N} F(x, \bar{w}_m) dx \ge c/2,$$

which implies that $\lim_{m\to\infty} G_\lambda(t_m u_m) = \infty$. Evidently, $t_m \in (0, 1)$; hence, $\langle G'_\lambda(t_m u_m), t_m u_m\rangle = 0$. It follows that

$$\int_{\mathbf{R}^N} \left(\frac{1}{2} f(x, t_m u_m) t_m u_m - F(x, t_m u_m)\right) dx \to \infty.$$

If Condition (P_3) holds, $h(t) = \frac{1}{2} t^2 f(x, s)s - F(x, ts)$ is increasing in $t \in [0, 1]$; hence $\frac{1}{2} f(x, s)s - F(x, s)$ is increasing in $s > 0$. Combining the oddness of f, we have that

$$\begin{aligned}(6.134) \qquad & \int_{\mathbf{R}^N} \left(\frac{1}{2} f(x,u_m)u_m - F(x,u_m)\right) dx \\ & \geq \int_{\mathbf{R}^N} \left(\frac{1}{2} f(x,t_m u_m)t_m u_m - F(x,t_m u_m)\right) dx \\ & \to \infty,\end{aligned}$$

If Condition (P_4) holds, then (6.134) is still true. Therefore, we get a contradiction inasmuch as

$$\begin{aligned}& \int_{\mathbf{R}^N} \left(\frac{1}{2} f(x,u_m)u_m - F(x,u_m)\right) dx \\ & = G_\lambda(u_m) \in [b_k, \max_{B_k} G].\end{aligned}$$

Once we have proved the boundedness of $\{u_m\}$, it is easy to get a sign-changing critical point u of G_1 with critical value in $[b_k, \max_{B_k} G_1(u)]$ (independent of λ). Because $b_k \to \infty$ as $k \to \infty$, we get infinitely many sign-changing critical points of G_1.

Finally, we apply Theorem 6.11 to prove the existence of positive and negative solutions. Evidently, $G_\lambda(u) \to -\infty$ for $u \in E_k$ and $\|u\| \to \infty$ uniformly for $\lambda \in \left(\frac{1}{2}, 1\right)$. Because $E_1 \subset \mathcal{P}$, we may choose $e_0^+ \in \mathcal{P}$ such that $G_\lambda(e_0^+) \leq G_\lambda(0) = 0$. Obviously, by (S_2), all the conditions of Theorem 6.11 are satisfied and therefore, G_λ has a critical point $v_\lambda \in \mathcal{P}$:

$$G'_\lambda(v_\lambda) = 0, \qquad G_\lambda(v_\lambda) = \beta_+(\lambda) \geq c^* > 0;$$

here we may choose $c^* > 0$ independent of λ (by (S_2)). Similarly, we may show that $\{v_\lambda\}_{\lambda \in \Lambda}$ is bounded and has a convergent subsequence whose limits $v^* \in \mathcal{P}$ satisfy $G'(v^*) = 0, G(v^*) \geq c^* > 0$. Analogously, we may get a negative critical point of G. $\square$

Notes and Comments. In recent years many existence results have been obtained for (6.120) under various conditions on $V(x)$ and $f(x,t)$. In Rabinowitz [260] the author had obtained one positive and one negative solution to (6.120) under the assumption that $V(x) \to \infty$ as $|x| \to \infty$. A generalization of the main result in [260] can be found in Bartsch et al. [38]. In Bartsch and Wang [42], the existence and multiplicity results were obtained under an assumption stronger than (S_1). One sign-changing solution had been obtained in Bartsch and Wang [43, 44] for Dirichlet problems and Schrödinger equations. In a recent paper Bartsch et al. [37] studied (6.120) with superlinear $f(x,u)$. In the case where $f(x,u)$ is odd in u, infinitely many sign-changing solutions were obtained in [37] by using genus and by imposing a global (PS) compactness condition. An estimate of the number of nodal domains was given there. All the papers mentioned above deal with superlinear cases. In Furtado et al. Silva [152, 153], and Perera and Schechter [247], double resonance was considered but no information concerning the sign-changing solutions was obtained. Under the Ambrosetti–Rabinowitz super-quadratic (ARS, for short), Wang [331] obtained the existence result of three solutions (one is positive, another one is negative) on a superlinear Dirichlet elliptic equation and later in Bartsch and Wang [40], the authors proved for semilinear Dirichlet problems that the third solution is sign-changing. This result was generalized to nonlinear Schrödinger equations in Bartsch and Wang [41] where the (ARS) condition plays an important role. Recall the papers of Coti Zelati and Rabinowitz [121, 122], where $V(x)$ and $f(x,t)$ were periodic for each x variable and infinitely many sign-changing solutions were obtained by a totally different theory. Hence, the (ARS) condition was demanded there. See Bartsch and Weth [46, 47] and Wang [332] for related papers on sign-changing solutions, and see also Lupo and Micheletti [219] and Perera [241, 242] for existence results on multiple solutions. Conditions (P_1) and (P_3) above are essentially different from the (ARS) condition. Some weakened (ARS) conditions can be found in Jeanjean [178], Liu and Wang [213], Schechter and Zou [285], Willem and Zou [336], and Zhou [341].

6.5 Critical Cases

Consider the Schrödinger equation with critical Sobolev exponent growth:

$$-\Delta u + V(x)u = \beta|u|^{2^*-2}u + f(x,u), \qquad x \in \mathbf{R}^N. \tag{6.135}$$

It is well known that the equation

$$-\Delta u + u = |u|^{2^*-2}u, \qquad x \in \mathbf{R}^N,$$

has no positive solution (cf. Benci and Cerami [54]). This is a consequence of the Pohozaev-type identity (cf. Berestycki and Lions [59]). The equation

$$-\Delta u + V(x)u = |u|^{2^*-2}u, \qquad x \in \mathbf{R}^N, \tag{6.136}$$

has a positive solution provided $V(x) \geq 0$ and its $L^{N/2}$-norm is small (see Benci and Cerami [54] and Ben-Naoum et al. [56]). If $V(x) \equiv 0$ on $\mathbf{R}^N$, this equation has the positive solution

$$\frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}.$$

In particular, all positive solutions of (6.136) can be obtained from this solution by dilations and translations. If $V(x) \to \infty$ as $|x| \to \infty$ and $f(x,u) \equiv 0$, Chabrowski and Yang [89] obtained one nontrivial solution to (6.135), where β can be a function.

We mention a Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = \mu u + |u|^{2^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.137}$$

where Ω is a bounded smooth domain of $\mathbf{R}^N$. The existence of solutions of (6.137) had been studied extensively after the celebrated paper of Brézis and Nirenberg [71].

$(\mathbf{P_5})$ $f(x,t)t - 2F(x,t) \geq 0$ for all $x,\ t$.

Theorem 6.23. *Assume* (S_1), (S_2), (P_1), (P_2), *and* (P_5). *Then there exists a* $\beta_0 > 0$ *such that, for any* $\beta \in (0, \beta_0)$, *Equation (6.135) has one sign-changing solution, one positive solution, and one negative solution.*

The following theorem concerns the sign-changing solution u of (6.135) with respect to parameter $\beta > 0$. We say that $\{(\beta_k,\ u_k)\}$ are sign-changing and unbounded if $\{u_k\}$ are sign-changing and unbounded.

Theorem 6.24. *Assume* (S_1), (S_2), (P_1), (P_2), *and* (P_5).
Then Equation (6.135) has a sequence of positive solutions (β_k, v_k), *a sequence of negative solutions* (β_k, w_k), *and an unbounded sequence of sign-changing solutions* $\{(\beta_k,\ u_k)\}$ *satisfying*

$$\frac{1}{2}\int_{\mathbf{R}^N} (|\bigtriangledown u_k|^2 + V(x)|u_k|^2)dx - \frac{\beta_k}{2^*}\int_{\mathbf{R}^N} |u_k|^{2^*} dx - \int_{\mathbf{R}^N} F(x,u_k)dx \to \infty$$

as $k \to \infty$.

We use the same notations E_k, Z_k, N_k, B_k, as in the preceding section. Let

$$S := \inf_{u\neq 0, u\in H^1(\mathbf{R}^N)} \frac{\|\bigtriangledown u\|^2}{\|u\|_{2^*}^2}$$

be the best Sobolev constant and define

$$S_E := \inf_{u\neq 0, u\in E} \frac{\|u\|^2}{\|u\|_{2^*}^2}. \tag{6.138}$$

Then $S_E \geq S$. Define

$$G_{\lambda,\beta}(u) = \frac{\lambda}{2}\|u\|^2 - \frac{\beta}{2^*}\int_{\mathbf{R}^N} |u|^{2^*} dx - \int_{\mathbf{R}^N} F(x,u)dx \tag{6.139}$$

for all $\lambda \in (\frac{1}{2},\ 1),\ \beta > 0$. Then

$$G_{\lambda,\beta} \in \mathbf{C}^1(E, \mathbf{R}), \qquad G'_{\lambda,\beta} = \lambda\, \mathbf{id} - J_\beta,$$

where $J_\beta(u) := (-\Delta + V)^{-1}[\beta|u|^{2^*-2}u + f(x,u)]$.

Lemma 6.25. *There is a constant ρ_k depending on k, independent of λ, β, such that*

$$\rho_k > 0, \qquad G_\lambda(u) \le 0,$$

for all $u \in E_k$ with $\|u\| = \rho_k$ uniformly for $\lambda \in (\frac{1}{2}, 1)$ and $\beta > 0$.

Proof. Because $\dim E_k < \infty$, then by (P_1),

$$\frac{G_{\lambda,\beta}(u)}{\|u\|^2} \le \frac{1}{2} - \int_{\mathbf{R}^N} \frac{F(x,u)}{\|u\|^2} dx \to -\infty$$

as $\|u\| \to \infty, u \in E_k$ uniformly for $\lambda \in (\frac{1}{2}, 1)$ and $\beta > 0$. The lemma follows immediately. □

Let $B_k, Q^\star(\rho)$ be as defined in the previous section. Let

$$N_k = Q^\star(\rho) \cap G^{a_0}_{\frac{1}{2},0}, \qquad a_0 := \max_{B_k} G_{1,0}.$$

Lemma 6.26. *There exists a constant $\tilde{\beta}_0 > 0$ such that*

$$G_{\lambda,\beta}(u) \ge \frac{1}{2}\delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\} := b_k$$

for all $u \in N_k$ and all $\lambda \in (\frac{1}{2}, 1), \beta \in (0, \tilde{\beta}_0)$. Where $\delta_s > 0, \alpha \in (0,1)$ are independent of λ, β, k; $\tilde{\beta}_0$ depends on k, s, α. In particular, $b_k \to \infty$ as $k \to \infty$.

Proof. By Lemma 6.19, for any $u \in Q^\star(\rho)$, we have that

$$G_{\lambda,0}(u) \ge \delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\},$$

where $\delta_s > 0$ is a constant independent of λ, k, β. By Lemma 6.18,

$$\|u\|_s \le \rho(1 + \lambda_k^{\beta'} \Lambda_s), \qquad \rho = \frac{1}{16 C_F}, \qquad u \in Q^\star(\rho).$$

Therefore, on $N_k = Q^\star(\rho) \cap G^{a_0}_{1/2,0}$ with $a_0 := \max_{B_k} G_{1,0}$, we have that

$$\|u\| \le \Big(8 C_F (16 C_F)^{-s} (1 + \lambda_k^{\beta'} \Lambda_s)^s + 8 \max_{B_k} G_{1,0}\Big)^{1/2} := \Xi. \tag{6.140}$$

On N_k, we have the following estimates:

$$\begin{aligned} G_{\lambda,\beta}(u) &= G_{\lambda,0}(u) - \frac{\beta}{2^*}\int_{\mathbf{R}^n} |u|^{2^*} dx \\ &\ge \delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\} - \frac{\beta}{2^*} S_E^{-2^*/2} \|u\|^{2^*} \end{aligned}$$

$$\geq \delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\} - \frac{\beta}{2^*} S_E^{-2^*/2} \Xi^{2^*}$$

$$\geq \frac{1}{2}\delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\}$$

for $\beta \leq (2^*/2) S_E^{2^*/2} \Xi^{-2^*} \delta_s \min\{\lambda_k^{((1-\alpha)(s-2))/2}, \lambda_k^{(1-\alpha)/2}\} := \tilde{\beta}_0$. □

Note that

$$G'_{\lambda,\beta}(u) = \lambda u - J'_\beta(u) = \lambda u - J'_\beta u, \qquad u \in E.$$

Lemma 6.20 is still valid for J'_β.

By Theorem 6.10, for almost all $\lambda \in \left(\frac{1}{2}, 1\right)$, and any $\beta \in (0, \tilde{\beta}_0)$, $G_{\lambda,\beta}$ has a sign-changing critical point $u(\lambda, \beta)$ such that

$$G'_{\lambda,\beta}(u(\lambda,\beta)) = 0, \qquad G_{\lambda,\beta}(u(\lambda,\beta)) \in \left[b_k, \max_{B_k} G_{1,0}\right]; \tag{6.141}$$

here b_k (defined in Lemma 6.26) and $\max_{B_k} G_{1,0}$ are two constants depending on k (independent of λ, β).

Lemma 6.27. *Assume* (P_1) *and* (P_5). *Let* $\lambda_m \to 1$ *as* $m \to 1$ *and denote* $u_m := u(\lambda_m, \beta)$. *If*

$$\beta \in \left(0, \frac{S_E^{2^*/2}}{\left(N \max_{B_k} G_{1,0}\right)^{(2^*-2)/2}}\right),$$

then $\{u_m\}$ *has a convergent subsequence.*

Proof. Because

$$G'_{\lambda_m,\beta}(u_m) = 0, \tag{6.142}$$

$$G_{\lambda_m,\beta}(u_m) \in \left[b_k, \max_{B_k} G_{1,0}\right]; \tag{6.143}$$

here $[b_k, \max_{B_k} G_{1,0}]$ is a finite interval depending on k only. We first prove that $\{u_m\}_m^\infty$ is bounded. Assume $\{u_m\}_m^\infty$ is unbounded for a contradiction. We observe that

$$\int_{\mathbf{R}^N} \frac{\frac{2\beta}{2^*}|u_m|^{2^*}}{\|u_m\|^2} dx \tag{6.144}$$

$$\leq \int_{\mathbf{R}^N} \frac{\frac{2\beta}{2^*}|u_m|^{2^*} + 2F(x,u_m)}{\|u_m\|^2} dx$$

$$\to 1 \quad \text{as } m \to \infty. \tag{6.145}$$

Let $w_m = u_m/\|u_m\|$; then $w_m \to w^*$ weakly in E, strongly in $L^2(\mathbf{R}^N)$, and a.e. x in $\mathbf{R}^N$. Denote $\Omega_1 = \{x \in \mathbf{R}^N : w^*(x) \neq 0\}$. Then $(|u_m|^{2^*}/u_m^2)w_m^2 \to \infty$ for $x \in \Omega_1$. If Ω_1 has a positive measure, then

$$\begin{aligned}\int_{\mathbf{R}^N} \frac{|u_m|^{2^*}}{\|u_m\|^2} dx \\ &= \int_{\mathbf{R}^N} \frac{|u_m|^{2^*}}{u_m^2} w_m^2 dx \\ &\geq \int_{\Omega_1} \frac{|u_m|^{2^*}}{u_m^2} w_m^2 dx \\ &\to \infty;\end{aligned}$$

this contradicts (6.144). Thus, the measure of Ω_1 must be zero; that is, $w^* \equiv 0$ a.e. $x \in \mathbf{R}^N$. On the other hand, choose $\omega \in (2, 2^*)$; then

$$\begin{aligned}\int_{\mathbf{R}^N} \left(\frac{(\omega/2^* - 1)\beta|u|^{2^*} + \omega F(x, u_m) - u_m f(x, u_m)}{u_m^2} \right) w_m^2 dx \\ \to \lambda_m \left(\frac{\omega}{2} - 1 \right).\end{aligned} \tag{6.146}$$

However,

$$\begin{aligned}\limsup_{m\to\infty} \frac{(\omega/2^* - 1)\beta|u_m|^{2^*} + \omega F(x, u_m) - u_m f(x, u_m)}{u_m^2} w_m^2 \\ \leq \limsup_{m\to\infty} \frac{c(1 + |u_m|^2)}{u_m^2} w_m^2 \\ = 0.\end{aligned} \tag{6.147}$$

We observe that (6.146) and (6.147) imply $\omega - 2 \leq 0$; it is a contradiction. Therefore, $\{u_m\}$ is bounded and we may assume that $u_m \to u(1, \beta) := u$ weakly in E, strongly in $L^2(\mathbf{R}^N)$, and a.e. x in $\mathbf{R}^N$. Using Brezis–Lieb's Lemma 5.12, we get

$$\int_{\mathbf{R}^N} (|u_m|^{2^*} - |u|^{2^*} - |u - u_m|^{2^*}) dx \to 0.$$

Furthermore, by Lemma 5.14,

$$\int_{\mathbf{R}^N} (F(x, u_m) - F(x, u) - F(x, u_m - u)) dx \to 0.$$

Therefore, we have that

$$G_{\lambda_m,\beta}(u_m) - G_{\lambda_m,\beta}(u) - G_{\lambda_m,\beta}(u_m - u) = o(1).$$

Furthermore,

$$\begin{aligned}
&\lambda_m\|u\|^2 + \lambda_m\|u_m - u\|^2 - \int_{\mathbf{R}^N}(\beta|u|^{2^*} + \beta|u - u_m|^{2^*} + f(x,u_m)u_m)dx \\
&\quad = \langle u_m, u_m\rangle - \int_{\mathbf{R}^N}(\beta|u_m|^{2^*} + f(x,u_m)u_m)dx + o(1) \\
&\quad = \langle G'_{\lambda_m,\beta}(u_m), u_m\rangle + o(1) \\
&\quad = o(1).
\end{aligned}$$

Noting that $G'_{\lambda_m,\beta}(u) = 0$, we have

$$\begin{aligned}
(6.148)\quad &\lambda_m\|u_m - u\|^2 \\
&\quad = \beta\int_{\mathbf{R}^N}|u - u_m|^{2^*}dx + \int_{\mathbf{R}^N}(f(x,u_m)u_m - f(x,u)u)dx + o(1).
\end{aligned}$$

Next, we have to estimate $\int_{\mathbf{R}^N}(f(x,u_m)u_m - f(x,u)u)dx$. We first estimate

$$\int_{|x|\le R}(f(x,u_m)u_m - f(x,u)u)dx$$

for any $R > 0$. Note $\{\|u_m\|\}$ is bounded; we have

$$\begin{aligned}
&\text{meas}\{x \in \mathbf{R}^N : |x| \le R, |f(x,u_m)u_m| \ge k\} \\
&\quad \le k^{-1}\int_{\{x\in\mathbf{R}^N:|x|\le R,|f(x,u_m)u_m|\ge R\}}|f(x,u_m)u_m|dx \\
&\quad \le k^{-1}c(\text{meas}B_R(0)) \\
&\quad \to 0
\end{aligned}$$

as $k \to \infty$. That is, $\{f(x,u_m)u_m\}$ is uniformly integrable. Hence,

$$(6.149)\qquad \int_{|x|\le R}(f(x,u_m)u_m - f(x,u)u)dx \to 0$$

as $m \to \infty$ for any $R > 0$. On the other hand, by the Hölder inequality,

$$(6.150)\quad \int_{|x|>R}(f(x,u_m)u_m - f(x,u)u)dx$$

$$\leq \int_{|x|>R} |f(x,u_m)u_m - f(x,u_m)u + f(x,u_m)u - f(x,u)u| dx$$

$$\leq c\|u_m - u\|_2 + c\int_{|x|>R} (|u|^{s-1}|u_m - u|)dx + c\left(\int_{|x|>R} |u|^2 dx\right)^{1/2}$$

$$+ c\int_{|x|>R} |u_m|^{s-1}|u|dx + \int_{|x|>R} |u|^2 dx + \int_{|x|>R} |u|^s dx.$$

Because

$$\int_{|x|>R} (|u|^{s-1}|u_m - u|)dx \leq \int_{|x|>R} |u|^s dx + \|u_m\| \left(\int_{|x|>R} |u|^{2^*} dx\right)^{(s-1)/2^*}, \tag{6.151}$$

and

$$\int_{|x|>R} |u_m|^{s-1}|u|dx \leq c\|u_m\|^{s-1} \left(\int_{|x|>R} |u|^{2^*/(2^*-s+1)} dx\right)^{(2^*-s+1)/2^*}, \tag{6.152}$$

then (6.149)–(6.152) imply that

$$\int_{|x|>R} (f(x,u_m)u_m - f(x,u)u)dx \to 0 \quad \text{as } R \to \infty. \tag{6.153}$$

By combining (6.149) and (6.153) ,

$$\int_{\mathbf{R}^N} (f(x,u_m)u_m - f(x,u)u)dx \to 0 \quad \text{as } m \to \infty.$$

By (6.148), we have

$$\lambda_m\|u_m - u\|^2 = \beta\int_{\mathbf{R}^N} |u_m - u|^{2^*} dx + o(1) \leq \beta S_E^{-2^*/2}\|u_m - u\|^{2^*} + o(1). \tag{6.154}$$

If, up to a subsequence, $\lim_{m\to\infty} \|u_m - u\|^2 = t > 0$, then (6.154) implies

$$t \geq \beta^{-2/(2^*-2)} S_E^{2^*/(2^*-2)}.$$

Note $G'_{\lambda_m,\beta}(u) = 0$ and (P_5); we know that $G_{\lambda_m,\beta}(u) \geq 0$. It follows from (6.154) that

$$\begin{aligned}
&\max_{B_k} G_{1,0} + o(1) \\
&\quad \geq G_{\lambda_m,\beta}(u_m) \\
&\quad = G_{\lambda_m,\beta}(u) + G_{\lambda_m,\beta}(u_m - u) + o(1) \\
&\quad \geq G_{\lambda_m,\beta}(u_m - u) + o(1) \\
&\quad = \frac{\lambda_m}{2}\|u_m - u\|^2 - \frac{\beta}{2^*}\int_{\mathbf{R}^N} |u_m - u|^{2^*} dx - \int_{\mathbf{R}^N} F(x, u_m - u) dx + o(1) \\
&\quad = \frac{\lambda_m}{N}\|u_m - u\|^2 + o(1).
\end{aligned}$$

Therefore,

$$\max_{B_k} G_{1,0} \geq \frac{1}{N}\beta^{-(2/(2^*-2))} S_E^{2^*/(2^*-2)};$$

that is,

$$\beta \geq \frac{S_E^{2^*/2}}{(N \max_{B_k} G_{1,\,0})^{(2^*-2)/2}}.$$

This is a contradiction. This means that $\lim_{m\to\infty} \|u_m - u\| = 0$. $\square$

Proofs of Theorems 6.23 and 6.24. By Lemmas 6.26 and 6.27, we now assume that

$$0 < \beta < \min\left\{ \frac{S_E^{2^*/2}}{(N \max_{B_k} G_{1,0})^{(2^*-2)/2}},\ \ \tilde{\beta}_0 \right\} := \beta_0.$$

Then, for any $\beta \in (0, \beta_0)$, there exists a sign-changing critical point $u(1,\beta)$ such that (see (6.141))

$$G'_{1,\beta}(u(1,\beta)) = 0, \qquad G_{1,\beta}(u(1,\beta)) \in \left[b_k, \max_{B_k} G_{1,0}\right]. \tag{6.155}$$

That is, $u(1,\beta)$ is a sign-changing solution of (6.135). The proofs of the existence of positive and negative solutions are trivial. This is a case of critical exponents, therefore we have to adopt the methods of Lemma 6.27 to prove that the positive and negative (PS)-sequences have convergent subsequences. Condition (S_2) may guarantee the nontriviality of the limit. We omit the details.

Theorem 6.24 is a straightforward consequence of Theorem 6.23 and Lemma 6.26. □

Notes and Comments. Readers may consult the results on sign-changing solutions of Dirichlet boundary value problems obtained in Cerami et al. [85], where the dimension $N \geq 6$. The ideas of Chabrowski [86], Ghoussoub [156], Hirano et al. [170], and Tarantello [324] (see also Ekeland and Ghoussoub [139]) are also worthy reading for finding sign-changing solutions of elliptic equations with a critical exponent. We also refer readers to the papers by Brézis and Nirenberg [71], Chabrowski [88], Chabrowski and Szulkin [87], Chabrowski and Yang [89, 90], de Figueiredo et al. [148, 149], Li [205], Silva and Xavier [301], and Schechter and Zou [284] for (semi- and quasilinear) elliptic problems involving critical Sobolev exponents, where the existence of (positive) solutions was studied.

As we have seen in this book, we have mainly applied the ideas of finite-dimensional linking to the sign-changing solutions. There are some infinite-dimensional linking theorems which were established for the existence of critical points of the strongly indefinite functionals; see Bartsch and Clapp [32], Benci [52], Benci and Rabinowitz [55], Buffoni et al. [74], Hofer [174], Hulshof and van der Vorst [177], Kryszewski and Szulkin [185], Schechter [274], Schechter and Zou [281, 286], and Szulkin and Zou [320]. We also refer the readers to Chang et al. [100], Fei [143], Long [217], Szulkin and Zou [319], and Zou [345] for strongly indefinite functionals by Morse theory.

Finally, before closing the main matter of the book, we would like to introduce some other papers on sign-changing solutions to concrete elliptic equations. The papers of Chen et al. [101], Ding et al. [135], Neuberger [236], and Neuberger and Swift [237], are mainly based on numerical methods that are totally different from the theory of the current book. By use of topological methods (critical groups) the Morse indices of sign-changing solutions for nonlinear elliptic problems can be determined in the paper by Bartsch et al. [31]. Some earlier results on this aspect can be observed in Bahri and Lions [25], Lazer and Solimini [193], Perera and Schechter [243], and Solimini [307] where the Morse indices of critical points of minimax type were estimated.

Chapter 7
On a Bartsch–Chang–Wang–Weth Theory

In this chapter, we are interested in the further properties of the sign-changing critical points, namely the Morse index and the number of nodal domains. The results dealt with in this chapter are mainly borrowed from Bartsch, Chang, and Wang [31] and Bartsch and Weth [45]. Uniformly, we call them a Bartsch–Chang–Wang–Weth theory. We are only concerned with some fundamental ideas and applications. Actually, these topics deserve to be treated in a specific book. Readers are referred to the papers of Bartsch, among others.

7.1 Some Basic Results on Morse Theory

Let E be an infinite-dimensional Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. Let $G \in \mathbf{C}^1(E, \mathbf{R})$.

Definition 7.1. Let u_0 be an isolated critical point of G with $G(u_0) = c$; then the kth critical group of G at u_0 is defined by

$$C_k(G, u_0) := H_k(G^c, G^c \backslash \{u_0\}), \qquad k \in \mathbf{Z}.$$

Here H_k denotes the kth singular homology group with coefficient in a field $\mathcal{F}$, $\mathbf{Z} = \{0, 1, 2, \ldots\}$.

If U is a neighborhood of the critical point u_0 such that u_0 is the unique critical point of G in U, then by excision we have

$$C_k(G, u_0) \cong H_k(G^c \cap U, G^c \cap U \backslash \{u_0\}).$$

If $G \in \mathbf{C}^2(E, \mathbf{R})$, we use $G''(u)$ to denote the unique bounded self-adjoint linear operator $T : E \to E$ such that

$$\langle G''(u)v, w \rangle = \langle Tw, v \rangle, \qquad \forall\, u, v, w \in E.$$

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Recall that a linear operator $T : E \to E$ is Fredholm if T has a finite-dimension kernel (denoted by $ker(T)$) and the range of T (denoted by $range(T)$) is closed and has finite codimension. When T is a self-adjoint Fredholm operator, then because the range of T is the orthogonal complement of the kernel of T, we have $E = ker(T) \oplus range(T)$. Assume that $G''(u_0)$ is Fredholm. Because $G''(u_0)$ is self-adjoint, E can be decomposed as $E = ker(G''(u_0)) \oplus range(G''(u_0))$. The restriction of $(G''(u_0))$ to its range is invertible, thus by the basic spectral theory there exist closed subspaces E_+ and E_- of the range of $G''(u_0)$ and a constant $b > 0$ such that E_+ and E_- are orthogonal and

$$\langle G''(u_0)u, u\rangle \geq b\|u\|^2, \qquad \forall u \in E_+;$$

$$\langle G''(u_0)u, u\rangle \leq -b\|u\|^2, \qquad \forall u \in E_-.$$

Definition 7.2. We call $\dim E_-$ the Morse index of the critical point u_0 and $\dim E_- + \dim kerG''(u_0)$ the generalized Morse index of u_0. If $G''(u_0)$ is invertible, that is, $kerG''(u_0) = \{0\}$, then u_0 is called a nondegenerate critical point.

Set $V := kerG''(u_0)$ and denote by $\nu := \nu(G, u_0) := \dim V$ the nullity of G at u_0. By the generalized Morse lemma, we have a diffeomorphism $h : U_0 \to U$ from a neighborhood U_0 of 0 in E to a neighborhood U of u_0 in E and a $\mathbf{C}^1$-function $\Psi_0 : U_0 \cap V \to \mathbf{R}$ such that

$$G(h(v + w)) = G(u_0) + \frac{1}{2}\langle G''(u_0)w, w\rangle + \Psi_0(v)$$

for $v \in V, w \in V^\perp$ with $v + w \in U_0$. The next proposition is known as the shifting lemma of Gromoll–Meyer (see Chang [94]). Its proof can be found in Chang [94] and Mawhin and Willem [225].

Proposition 7.3. *Assume that $G \in \mathbf{C}^2(E, \mathbf{R})$. Let u_0 be an isolated critical point of G with Morse index $\mu := \mu(G, u_0)$ and nullity $\nu := \nu(G, u_0)$. Assume that $G''(u_0)$ is a Fredholm operator with index 0. Then*

$$C_k(G, u_0) \cong C_{k-\mu}(\Psi_0, 0), \qquad k \in \mathbf{Z},$$

where Ψ_0 is the function from the generalized Morse lemma. In particular, $C_k(G, u_0) = 0$ for all $k \notin [\mu, \mu + \nu]$. Moreover, if $\nu = 0$, then $C_k(G, u_0) \cong \mathcal{F}$ for $k = \mu$, and $C_k(G, u_0) \cong 0$ otherwise.

The following proposition is well known.

Proposition 7.4. *Assume that $G \in \mathbf{C}^2(E, \mathbf{R})$ satisfies the (PS) condition. Let u_0 be an isolated critical point of G. The following are equivalent.*

(1) u_0 *is a local minimum.*

(2) $C_k(G, u_0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$

(3) $C_0(G, u_0) \neq 0$.

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The proof for the implication of (3) $\Rightarrow$ (1) can be found in Chang [94].

Definition 7.5. A critical point u_0 of G is said to be of the mountain pass type if for any sufficiently small and open neighborhood U of u_0 the set $G^c \cap U \backslash \{u_0\}$ is not path connected, where $c = G(u_0)$.

Theorem 7.6. *Assume that* $G \in \mathbf{C}^2(E, \mathbf{R})$ *satisfies the (PS) condition. Let* u_0 *be an isolated critical point of the mountain pass type and* $G(u_0) = c$*;* $G''(u_0)$ *is a Fredholm operator with index 0 and* $\dim ker G''(u_0) = 1$ *if* $0 \in \sigma(G''(u_0))$*, the spectrum of* $G''(u_0)$*. Then the Morse index* μ *of* u_0 *is less than or equal to* 1.

Proof. Let $V := ker G''(u_0)$ and $W := V^{\perp} = W^+ \oplus W^-$, where $W^{\pm}$ is the generalized eigenspace of $G''(u_0)$ corresponding to $\sigma(G''(u_0)) \cap \mathbf{R}^{\pm}$, respectively. Then the Morse index $\mu = \dim W^-$ and the nullity $\nu = \dim V$. By the generalized Morse lemma, there is a diffeomorphism $h : U_0 \to U$ and a $\mathbf{C}^1$-function $\Psi_0 : U_0 \cap V \to \mathbf{R}$, where U_0 is a neighborhood of 0 and U is a neighborhood of u_0, such that

$$G(h(v + w)) = G(u_0) + \frac{1}{2}\langle G''(u_0)w, w\rangle + \Psi_0(v)$$

for all $v + w \in U_0, v \in V, w \in W$. Thus,

$$\begin{aligned} &G^c \cap U \\ &\quad = \{h(v + w) : \langle G''(u_0)w, w\rangle + 2\Psi_0(v) \leq 0\} \\ &\quad \cong \{(v + w) \in U_0 : \langle G''(u_0)w, w\rangle + 2\Psi_0(v) \leq 0\}. \end{aligned}$$

Denote $B_\varepsilon V := \{u \in V : \|u\| \leq \varepsilon\}$. Choose $\varepsilon > 0, \delta > 0$ small enough such that $B_\varepsilon V \times B_\delta W \subset U_0$ and that

$$(7.1) \qquad 2 \sup_{B_\varepsilon V} |\Psi_0(v)| < |\langle G''(u_0)w, w\rangle|, \qquad \forall w \in W^- \qquad \text{with } \|w\| = \delta.$$

For $0 \neq v + w \in B_\varepsilon V \times B_\delta W \subset U_0$ satisfying $G(h(v + w)) \leq c$, it is connected to $v + w^- \in B_\varepsilon V \times B_\delta W^-$ by the path

$$(7.2) \qquad \gamma(t) := v + w^- + (1 - t)w^+, \qquad t \in [0, 1].$$

Note that $G(h(\gamma(t))) \leq c$ and $\gamma(t) \in U_0 \backslash \{0\}$ for all $t \in [0, 1]$.

If $\dim W^- > 1$, we show that the set

$$T := G^c \cap h(B_\varepsilon V \times B_\delta W)\backslash\{u_0\} \tag{7.3}$$

is path connected; this contradicts the fact that u_0 is of the mountain pass type. To this end, choose

$$w_1^- := \begin{cases} \delta \dfrac{w^-}{\|w^-\|}, & \text{if } w^- \neq 0, \\ \text{any element } w_1^- \text{ of } W^- \text{ with } \|w_1^-\| = \delta, & \text{if } w^- = 0. \end{cases}$$

Now $\gamma(1)$ defined in (7.2) is connected to $v + w_1^-$ along the path given by

$$\gamma(t) := v + (2-t)w^- + (t-1)w_1^-$$

for all $t \in [1,2]$. Note that $G(h(\gamma(t))) \leq c$ and $\gamma(t) \in U_0\backslash\{0\}$ for all $t \in [1,2]$. Define

$$\gamma(t) := (3-t)v + w_1^-, \qquad t \in [2,3];$$

it connects $\gamma(2)$ to $\gamma(3) = w_1^-$. Moreover, $\gamma(t) \in U_0\backslash\{0\}$ and $G(h(\gamma(t))) \leq c$ for $t \in [2,3]$ (see (7.1)). Let

$$G^c \cap h(B_\varepsilon V \times B_\delta W)\backslash\{u_0\}. \tag{7.4}$$

The above arguments show that every $u \in T$ can be connected by a path inside the set T to a point in $h(\{0\} \times \partial B_\delta W^-) \subset T$. Hence, T is path connected. This contradicts the fact that u_0 is of the mountain pass type. Therefore, $\mu \leq 1$. □

Theorem 7.7. *Under the assumptions of Theorem 7.6, the following statements are equivalent.*

(1) u_0 *is of the mountain pass type.*

(2) $C_k(G, u_0) \cong \begin{cases} \mathcal{F}, & \textit{if } k = 1, \\ 0, & \textit{otherwise.} \end{cases}$

(3) $C_1(G, u_0) \neq 0$.

Proof. (1) ⇒ (2). If $\mu = 0$, by hypothesis, we see that $\nu = 1$; hence $V \cong \mathbf{R}$. Because u_0 is isolated, Ψ_0 does not change sign on $(-\varepsilon, 0)$ nor on $(0, \varepsilon)$. Because T (see 7.3) is not path connected, we observe that $\Psi_0(v) < 0$ for $0 < |v| < \varepsilon$. It follows that

$$C_k(G, u_0) \cong C_k(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu = 1$, we have that

$$C_k(G, u_0) \cong C_{k-1}(\Psi_0, 0).$$

If $C_0(\Psi_0, 0) \neq 0$, then by Proposition 7.4, we have that

$$C_k(G, u_0) \cong C_{k-1}(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $C_0(\Psi_0, 0) = 0$, similar to the above arguments, we see that we may find $\varepsilon > 0, \delta > 0$ such that $B_\varepsilon V \times B_\delta W \subset U_0$ and that every point $u \in T$ can be connected within T to a point $h(w_1^-)$ with $w_1^- \in W^-, \|w_1^-\| = \delta$. Note that $C_0(\Psi_0, 0) = 0$ implies that there is a path $\gamma : [0, 1] \to \Psi_0^0 \cap B_\varepsilon V$ with $\gamma(0) = 0, \gamma(1) \neq 0$. Thus the path $h(\gamma(t) + w_1^-)$ deforms $h(w_1^-)$ to $h(\gamma(1) + w_1^-)$ within T. Note that $\Psi_0(\gamma(1)) \leq 0$ and $h(\gamma(1) + w_1^-)$ can be connected to $h(\gamma(1))$ within T by the path $h\big(\gamma(1) + (1 - t)w_1^-\big)$; we see that T is also path connected. This is a contradiction.

Now we show that (2) $\Rightarrow$ (1).

If $\mu = 0$, then $\nu = 1$. By using the generalized Morse lemma we have that

$$C_1(\Psi_0, 0) = C_1(G, u_0) \neq 0,$$

which implies that $\Psi_0(v) < 0$ for $0 < |v| \leq \varepsilon$ and $\varepsilon > 0$ small enough. It follows that

$$\begin{aligned} & h(B_\varepsilon V \times \{0\}) \backslash \{u_0\} \\ & \quad \subset T \\ & \quad \subset h((B_\varepsilon V \backslash \{0\}) \times B_\delta W), \end{aligned}$$

where T is defined in (7.4). Hence, if $U \subset h(B_\varepsilon V \times B_\delta W)$ is a neighborhood of u_0 then $G^c \cap U \backslash \{u_0\}$ cannot be connected.

If $\mu = 1$, then $C_0(\Psi_0, 0) \cong C_1(G, u_0) \neq 0$, it follows that 0 is a strict local minimum of Ψ_0. Then the generalized Morse lemma implies that

$$\begin{aligned} & h(\{0\} \times B_\delta W^-) \backslash \{u_0\} \\ & \quad \subset T \\ & \quad \subset G^c \cap h(B_\varepsilon V \times (B_\delta W \backslash B_\delta W^+)); \end{aligned}$$

by this we get the same conclusion. □

Obviously, we have (2) $\Rightarrow$ (3). Now we show that (3) $\Rightarrow$ (2). If u_0 is nondegenerate, we are done. Otherwise, by the shifting lemma,

$$C_k(G, u_0) \cong C_{k-\mu}(\Psi_0, 0), \qquad k \in \mathbf{Z}.$$

If $\mu = 1$, we see that $C_0(\Psi_0, 0) \neq 0$, which means that 0 is an isolated local minimum of Ψ_0; then

$$C_k(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$C_k(G, u_0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu = 0$, then

$$C_k(G, u_0) \cong C_k(\Psi_0, 0), \qquad k \in \mathbf{Z}.$$

The assumptions of the theorem imply that Ψ_0 is defined on a 1-manifold. Then $C_1(\Psi_0, 0) \neq 0$ implies that 0 is a local maximum of Ψ_0. Thus,

$$C_k(G, u_0) \cong C_k(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Notes and Comments. Readers may consult the books [94] by Chang and [225] by Mawhin and Willem for more results on the Morse theory. Proposition 7.4 is due to Chang [94]. Theorems 7.6 and 7.7 are due to Bartsch et al. [31]. They can also be found in Bartsch [30]. In particular, the proof for the case "(3) $\Rightarrow$ (2)" of Theorem 7.7 is borrowed from Chang [94]. Definition 7.5 is due to Hofer [175]. A slightly modified version of the definition of a mountain pass point can be found in Pucci and Serrin [248, 249]. In Bartsch [30] (see also Bartsch and Wang [40]), a critical point theory on partial-order Hilbert spaces was established; we give more notes or comments in the following sections.

7.2 Critical Groups of Sign-Changing Critical Points

Assume $\mathcal{P} \subset E$ is a closed convex cone of E. It induces a partial order of E defined by:

$$u \geq v \Leftrightarrow u - v \in \mathcal{P}; \qquad u > v \Leftrightarrow u \geq v \quad \text{and} \quad u \neq v.$$

A mapping $h : E \to E$ is called order preserving if $u \geq v$ implies that $h(u) \geq h(v)$ for all $u, v \in E$.

As we show in our applications, $\mathcal{P}$ may have an empty interior in the topology of E. Now we assume that there is a Banach space E_0 which is densely embedded into E such that

$$\mathcal{P}_0 := E_0 \cap \mathcal{P}$$

has a nonempty interior in E_0, denoted by $\overset{\circ}{\mathcal{P}}_0$. The elements of $\overset{\circ}{\mathcal{P}}_0$ are called positive. We write $u \gg v$ if $u - v \in \overset{\circ}{\mathcal{P}}_0$. A mapping $h : E \to E$ is called strongly order preserving if $h(u) \gg h(v)$ as long as $u, v \in E_0$ with $u > v$. We assume that there is an element $e_0 \in \overset{\circ}{\mathcal{P}}_0$ such that $\langle u, e_0 \rangle > 0$ for all $u \in \mathcal{P}_0 \setminus \{0\}$. Let $G : E \to \mathbf{R}$ satisfy the following hypotheses.

($\mathbf{A_1}$) $G \in \mathbf{C}^2(E, \mathbf{R})$, $G(0) = 0, G'(0) = 0$; $\mathcal{K} := \{u \in E : G'(u) = 0\} \subset E_0$. The (PS) condition holds for G.

($\mathbf{A_2}$) The gradient of G is of the form $G'(u) = u - \Theta_G(u)$, where $\Theta_G : E \to E$ is a compact operator satisfying $\Theta_G(E_0) \subset E_0$. The restriction $\Theta := \Theta_G|_{E_0} : E_0 \to E_0$ is of $\mathbf{C}^1$ and strongly order preserving.

($\mathbf{A_3}$) For any $u_0 \in \mathcal{K}$, we assume that any eigenvector of the Fréchet derivative $\Theta_G'(u_0) \in \mathcal{L}(E)$ (the set of all bounded linear operators) lies in E_0. The largest eigenvalue of $\Theta_G'(u_0)$ is simple and its eigenspace is spanned by a positive eigenvector.

($\mathbf{A_4}$) One of the following holds.

(1) G is bounded below.

(2) For any $u \in E, u \neq 0$, we have $G(tu) \to -\infty$ as $t \to \infty$. There is a $c < 0$ such that $\langle G'(u), u \rangle < 0$ whenever $G(u) \leq c$.

(3) There is a compact self-adjoint linear operator $\Pi_G \in \mathcal{L}(E)$ such that $G'(u) = u - \Pi_G u + o(\|u\|)$ as $\|u\| \to \infty$. All eigenvectors of Π_G lie in E_0, the largest eigenvalue is simple, and its eigenspace is spanned by a positive eigenvector $e_\infty \in \overset{\circ}{\mathcal{P}}_0$ such that $\langle u, e_\infty \rangle > 0$ for every $u \in \mathcal{P}_0 \setminus \{0\}$. Furthermore, $\Pi := \Pi_G|_{E_0} \in \mathcal{L}(E_0)$.

Definition 7.8. Let μ_0, ν_0 denote the Morse index and the nullity of 0, respectively. Under the conditions (A_1)–(A_4), we define by μ_∞, ν_∞ the Morse index and nullity of infinity as the following.

$$\mu_\infty = \nu_\infty = 0, \qquad \text{if } (A_4)\text{-}(1)\text{holds;}$$

$$\mu_\infty = \infty, \nu_\infty = 0, \qquad \text{if}(A_4)\text{-}(2) \text{ holds;}$$

$$\begin{aligned} \mu_\infty = {} & \text{the number of negative eigenvalues of} \\ & \mathbf{id} - \Pi_G \text{counted with multiplicities and} \end{aligned}$$

$$\nu_\infty = \dim\ker(\mathbf{id} - \Pi_G)\text{if}(A_4)\text{-}(3)\text{holds.}$$

Theorem 7.9. *If $\mu_0 \geq 2$ and $\mu_\infty + \nu_\infty \leq 1$, then G has a sign-changing critical point u_1. If all sign-changing critical points with negative critical values are isolated, then there exists a sign-changing critical point u_1 which is of mountain pass type and*

$$G(u_1) < 0, \qquad C_k(G, u_1) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider the negative gradient flow $\phi(t,\cdot)$ of G on E defined by

$$\frac{d}{dt}\phi(t,u) = -G'(\phi(t,u)), \qquad \phi(0,u) = u \in E. \tag{7.5}$$

By (A_2) we have that $\phi(t,u) \in E_0$ for $u \in E_0$ and $\phi(t,u)$ induces a continuous local flow on E_0. Obviously, $G\big(\phi(t,u)\big)$ is strictly decreasing in t if u is not a critical point of G. Moreover, for $v \in \mathcal{P}_0\backslash\{0\}$, by (A_2) we have that

$$v - G'(v) = \Theta_G(v) \gg \Theta_G(0) = 0.$$

This implies that the vector field $-G'$ points at v inside the cone $\overset{\circ}{\mathcal{P}}_0$: $v - G'(v) \in \overset{\circ}{\mathcal{P}}_0$. Setting

$$D = \mathcal{P}_0 \cup (-\mathcal{P}_0), \qquad D^\star = D\backslash\{0\}.$$

Then

$$\phi(t,v) \in \overset{\circ}{D}, \qquad \forall v \in D^\star, \qquad v \neq 0, \qquad t > 0.$$

For any $r \in \mathbf{R}$, let

$$l_r : H_1(E_0, D^\star) \to H_1(E_0, G^r \cup D^\star) \tag{7.6}$$

be the homeomorphism induced by the inclusion $(E_0, D^\star) \rightsquigarrow (E_0, G^r \cup D^\star)$ on the first homology level. Because E_0 is contractible and $D^\star$ is homotopy equivalent to a two-point space, we have that

$$H_1(E_0, D^\star) \cong \mathcal{F}. \tag{7.7}$$

Set

$$\Lambda := \{r \in \mathbf{R} : l_r \neq 0\}.$$

Claim 1. $\Lambda \neq \emptyset$. Because $\mu_\infty + \nu_\infty \leq 1$, Condition (A_4)-(1) or (A_4)-(3) applies. In Case (1), we set $e_\infty := e_0 \in \overset{\circ}{\mathcal{P}}_0$. In Case (3) we choose $e_\infty \in \overset{\circ}{\mathcal{P}}_0$ as the positive eigenvector of $\mathbf{id} - \Pi$ with $\|e_\infty\| = 1$ belonging to the largest eigenvalue of Π. Note that G is bounded below on $E_1 := E_0 \cap (e_\infty)^\perp$. For $r < \inf_{E_1} G$, we have the inclusion:

$$(E_0, D^\star) \overset{j}{\rightsquigarrow} (E_0, G^r \cup D^\star) \overset{i}{\rightsquigarrow} (E_0, E_0\backslash E_1).$$

Because $j \circ i$ is a homotopy equivalence we see that $r \in \Lambda$.

Claim 2. $c := \sup \Lambda < 0$. Because $\mu_0 \geq 2$, let $\lambda_1 < \lambda_2 < 0$ be the two smallest eigenvalues of $G''(0)$ and e_1, e_2 be the normalized eigenvectors corresponding to λ_1 and λ_2, respectively. By (A_3), we know that $e_1 \in \overset{\circ}{D}$. Let S_ε be the sphere of radius ε in $\mathrm{span}\{e_1, e_2\}$. Then $\max_{S_\varepsilon} G < 0$ for $\varepsilon > 0$ small enough.

We show that $r \notin \Lambda$ for all $r \geq \max_{S_\varepsilon} G$ and all $\varepsilon > 0$ small enough. If $r \geq \max_{S_\varepsilon} G$, we have that

$$(E_0, S_\varepsilon \cup D^\star) \subset (E_0, G^r \cup D^\star).$$

Then we just have to show that

$$H_1(E_0, S_\varepsilon \cup D^\star) \cong 0.$$

But this is an immediate consequence of the long exact sequence of the pair $(E_0, S_\varepsilon \cup D^\star)$ and the fact that $H_1(E_0) = 0$ and that $H_0(S_\varepsilon \cup D^\star) \to H_0(E_0)$ is an isomorphism because $D^\star \cup S_\varepsilon$ is path connected.

By a standard deformation argument, it is easy to see that c is a critical value (cf. Chang [94]). Suppose that G has only finitely many sign-changing critical points $u_1, \ldots, u_m$ at the level c. We choose $\delta > 0$ and neighborhoods $U_i \subset E_0 \backslash D$ of u_i with the following properties.

(a) $U_i \cap U_j = \emptyset$ for $i \neq j$.
(b) u_i is the only critical point of G in U_i.
(c) $G^{c-\delta} \cup U_i$ is positive invariant under $\phi(t, \cdot)$.
(d) There is a $T \geq 0$ such that $\phi(T, G^{c+\delta}) \subset G^{c-\delta} \cup U_1 \cup \cdots \cup U_m$.

Because $c + \delta \notin \Lambda$ and $c - \delta \in \Lambda$, the long exact sequence of the triple

$$(E_0, G^{c+\delta} \cup D^\star, G^{c-\delta} \cup D^\star)$$

yields $H_1(G^{c+\delta} \cup D^\star, G^{c-\delta} \cup D^\star) \neq 0 :$

$$\begin{tikzcd} & H_1(E_0, D^\star) \arrow[d, "\neq 0"] \arrow[dr, "0"] & \\ H_1(G^{c+\delta} \cup D^\star, G^{c-\delta} \cup D^\star) \arrow[r] & H_1(E_0, G^{c-\delta} \cup D^\star) \arrow[r] & H_1(E_0, G^{c+\delta} \cup D^\star). \end{tikzcd}$$

Let $U := U_1 \cup U_2 \cup \cdots \cup U_m$. By (c) and (d), $G^{c-\delta} \cup U \cup D^\star$ is a strong deformation retract of $G^{c+\delta} \cup D^\star$. Hence,

$$H_1(G^{c-\delta} \cup U \cup D^\star, G^{c-\delta} \cup D^\star) \cong H_1(G^{c+\delta} \cup D^\star, G^{c-\delta} \cup D^\star).$$

By the excision property, we get that

$$\begin{aligned} &H_1(U, U \cap G^{c-\delta}) \\ &\quad \cong H_1(G^{c-\delta} \cup U, G^{c-\delta}) \\ &\quad \cong H_1(G^{c-\delta} \cup U \cup D^\star, G^{c-\delta} \cup D^\star). \end{aligned}$$

Now properties (a) and (b) imply that

$$H_1(U, U \cap G^{c-\delta}) \cong \bigoplus_{i=1}^m H_1(U_i, U_i \cap G^{c-\delta}) \cong \bigoplus_{i=1}^m C_1(G, u_i).$$

Therefore, there exists an $i \in \{1, 2, \dots, m\}$ such that $C_1(G, u_i) \neq 0$. By Theorem 7.6, u_i is of the mountain pass type and $C_k(G, u_i) \cong \delta_{k1}\mathcal{F}$. $\square$

Theorem 7.10. *If $\mu_\infty \geq 2$ and $\mu_0 + \nu_0 \leq 1$, then G has a sign-changing critical point u_1 with $G(u_1) > 0$. If all sign-changing solutions are isolated, then there is a sign-changing solution u_1 with $G(u_1) > 0$, Morse index $\mu \in \{1, 2\}$, and*

$$C_0(G, u_1) = C_1(G, u_1) = 0, \qquad C_2(G, u_1) \neq 0.$$

In particular, u_1 is neither a local minimum nor of mountain pass type. If $\mu = 2$ or the nullity $\nu \leq 1$, then

$$C_k(G, u_1) \cong \begin{cases} \mathcal{F}, & \text{if } k = 2, \\ 0, & \text{otherwise} \end{cases} \qquad k \in \mathbf{Z}.$$

Thus, u_1 looks homologically like a nondegenerate critical point with Morse index 2.

Proof. Because $\mu_\infty \geq 2$, either (A_4)-(2) or (A_4)-(3) applies. In both cases, there are $v_\infty \in \overset{\circ}{\mathcal{P}}_0$ and $w_\infty \perp v_\infty$ with $\|v_\infty\| = \|w_\infty\| = 1$ and $G(u) < 0$ for any $u \in \operatorname{span}\{v_\infty, u_\infty\}$ with $\|u\| \geq R$ for $R > 0$ large enough. Let

$$Q := \{sv_\infty + tw_\infty : |s| \leq R, t \in [0, R]\}$$

and

$$\partial Q := \{sv_\infty + tw_\infty \in Q : |s| = R \ \text{ or } \ t \in \{0, R\}\}.$$

Then $\partial Q \subset G^0 \cup D$. Let $\beta := \max_Q G$ so that

$$(Q, \partial Q) \rightsquigarrow (G^\beta \cup D, G^0 \cup D).$$

Let $\eta(\beta) \in H_2(G^\beta \cup D, G^0 \cup D)$ be the image of $1 \in \mathcal{F} \cong H_2(Q, \partial Q)$ under the homeomorphism

$$\mathcal{F} \cong H_2(Q, \partial Q) \to H_2(G^\beta \cup D, G^0 \cup D)$$

induced by the inclusion. For any $r \leq \beta$, let

$$l_r : H_2(G^r \cup D, G^0 \cup D) \to H_2(G^\beta \cup D, G^0 \cup D)$$

be induced by the inclusion. Define

$$\Lambda := \{r \leq \beta : \eta(\beta) \in \operatorname{image}(l_r)\}, \qquad c := \inf \Lambda.$$

Claim. $\eta(\beta) \neq 0$. To show this, we let $e_1 \in \overset{\circ}{\mathcal{P}}_0$ be the first eigenvector of $G''(0)$ and set

$$E_1 := \operatorname{span}\{e_1\}, E_2 := E_1^\perp \cap E_0.$$

Because $\mu_0 + \nu_0 \leq 1$ we obtain $\inf_{S_\varepsilon E_2} G \geq \alpha > 0$ for some $\varepsilon > 0$ small enough. This implies that

$$(Q, \partial Q) \subset (G^\beta \cup D, G^0 \cup D) \subset (E_0, E_0 \backslash S_\varepsilon E_2).$$

The inclusion induces the homeomorphism

$$H_2(Q, \partial Q) \to H_2(E_0, E_0 \backslash S_\varepsilon E_2).$$

We just have to show that the homeomorphism is not 0. Choose $w_0 \in E_2$ with $\|w_0\| = 1$ and let

$$Q_0 := \{sv_\infty + tw_0 : |s| \leq R, t \in [0, R]\}.$$

Rotating w_∞ into w_0, we may deform $(Q, \partial Q)$ into $(Q_0, \partial Q_0)$ within $(E_0, E_0 \backslash S_\varepsilon E_2)$. Therefore, it suffices to show that

$$H_2(Q_0, \partial Q_0) \to H_2(E_0, E_0 \backslash S_\varepsilon E_2)$$

is not 0. Let

$$Q_1 := \{se_1 + tw_0 : |s| \leq R, t \in [0, R]\}.$$

The rotation from v_∞ to e_1 inside $\overset{\circ}{\mathcal{P}}_0$ deforms $(Q_0, \partial Q_0)$ into $(Q_1, \partial Q_1)$ within $(E_0, E_0 \backslash S_\varepsilon E_2)$. Thus, we just have to show that

$$H_2(Q_1, \partial Q_1) \to H_2(E_0, E_0 \backslash S_\varepsilon E_2)$$

is not 0. To prove this, we define

$$Q_2 := \{se_1 + tw_0 \in Q_1 : |s| \leq R, t = 0, R\} \subset \partial Q_1$$

and

$$Q_3 := \{se_1 + tw_0 \in Q_1 : |s| = R, t \in [0, R]\} \subset \partial Q_1.$$

Then $\partial Q_1 = Q_2 \cup Q_3$ and the inclusion $(Q_2, Q_2 \cap Q_3) \rightsquigarrow (\partial Q_1, Q_3)$ is an excision. Similarly, we define

$$W_1 := E_1 \times (E_2 \backslash S_\varepsilon E_2), \qquad W_2 := (E_1 \backslash \{0\}) \times E_2.$$

Then $E_0 \backslash S_\varepsilon E_2 = W_1 \cup W_2$. Moreover, the inclusion $(W_1, W_1 \cap W_2) \rightsquigarrow (E_0 \backslash S_\varepsilon E_2, W_2)$ is an excision. Consider the commutative diagram

$$\begin{array}{ccccc}
H_2(Q_1, Q_3) \to & H_2(Q_1, \partial Q_1) & \xrightarrow{\partial} & H_1(\partial Q_1, Q_3) \\
& \downarrow i_1 & & \downarrow i_2 \\
& H_2(E_0, E_0 \backslash S_\varepsilon E_2) & \to & H_1(E_0 \backslash S_\varepsilon E_2, W_2).
\end{array}$$

The top row is exact as part of the long exact sequence of the triple $(Q_1, \partial Q_1, Q_3)$. Note that

$$H_2(Q_1, Q_3) \cong H_1(Q_3) = 0$$

and that i_2 is an injective; we see that $i_1 \neq 0$. By using the excision isomophisms i_2 is an injective if and only if

$$i_3 : H_1(Q_2, Q_3 \cap Q_2) \to H_1(W_1, W_1 \cap W_2)$$

is injective. This is an immediate consequence because the inclusions $Q_2 \rightsquigarrow W_1$ and $Q_2 \cap Q_3 \rightsquigarrow W_1 \cap W_2$ are homotopy equivalences. Hence, they induce isomorphisms on the homotopy level. Thus, i_3 is an isomorphism. The claim is true.

By the above claim, we know that $0 \notin \Lambda$ because $l_0 = 0$ (see (7.6)). Note that $\beta \in \Lambda$; hence $c \in [0, \beta]$. By the assumption, $\mu_0 + \nu_0 \leq 1$, the sign-changing solutions cannot accumulate at zero. Assume that the sign-changing solutions are isolated; there are only finitely many sign-changing solutions with values in $[0, \beta]$. It implies that $G^0 \cup D$ is a strong deformation retract of $G^r \cup D$ for $r > 0$ small enough. Therefore, $c > 0$. Let $u_1, \ldots, u_m$ be all sign-changing critical points at the level $c \in (0, \beta]$. We choose $\delta > 0$ and neighborhoods U_i of $u_i, i = 1, \ldots, m$, as in the proof of Theorem 7.9. Consider the commutative diagram

$$\begin{array}{ccc}
 & H_2(G^{c-\delta} \cup D, G^0 \cup D) & \\
 & \downarrow j \qquad\qquad\qquad \searrow (j_{c-\delta}) & \\
H_2(G^{c+\delta} \cup D, G^0 \cup D) & \overset{j_{c+\delta}}{\to} & H_2(G^{\beta} \cup D, G^0 \cup D) \\
\downarrow & & \\
H_2(G^{c+\delta} \cup D, G^{c-\delta} \cup D). & &
\end{array}$$

Because $c + \delta \in \Lambda$, there is an $\eta(c+\delta) \in H_2(G^{c+\delta} \cup D, G^0 \cup D)$ with $j_{c+\delta}(\eta(c+\delta)) = \eta(\beta)$. Now, $\eta(c+\delta)$ cannot lie in the image of j because $c - \delta \notin \Lambda$. That is, $\eta(\beta) \notin \text{image}(j_{c-\delta})$. Therefore, the exactness of the left column yields

$$H_2(G^{c+\delta} \cup D, G^{c-\delta} \cup D) \neq 0.$$

Similar to the proof of Theorem 7.9, we have that

$$C_2(G, u_i) \neq 0, \quad \text{for some } i \in \{1, \ldots, m\}.$$

This is the required sign-changing solution which is neither a local minimum nor of mountain pass type . If $\mu = 2$, by the shifting lemma, we see that

$$C_k(G, u_i) \cong C_{k-2}(\Psi_0, 0) \quad \text{for all } k \in \mathbf{Z}.$$

Thus, $C_0(\Psi_0, 0) \neq 0$. By Proposition 7.4, 0 is a local minimum of Ψ_0 and moreover,

$$C_k(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we obtain

$$C_k(G, u_i) \cong \begin{cases} \mathcal{F}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Next we assume that $\mu \leq 1$ and $\nu = \nu(G, u_i) \leq 1$; hence $\mu = 1, \nu = 1$ because $C_2(G, u_i) \neq 0$. Then we have by the shifting lemma that

$$C_k(G, u_i) \cong C_{k-1}(\Psi_0, 0).$$

It follows that $C_1(\Psi_0, 0) \neq 0$; hence Ψ_0 is a local maximum of Ψ_0. Therefore,

$$C_k(\Psi_0, 0) \cong \begin{cases} \mathcal{F}, & \text{if } k = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad C_k(G, u_i) \cong \begin{cases} \mathcal{F}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

□

Theorem 7.11. *Assume that $\nu_0 = \nu_\infty = 0, \mu_\infty \geq 1$ and $\mu_0 \neq \mu_\infty$. Suppose that all sign-changing critical points are isolated.*

(1) If $\mu_0 \geq 1$, then G has a sign-changing critical point u_1 such that either $G(u_1) > 0$, $C_{\mu_0+1}(G, u_1) \neq 0$, or $G(u_1) < 0$, $C_{\mu_0-1}(G, u_1) \neq 0$.
(2) If (A_4)-(3) holds with $\mu_\infty \geq 2$, then G has a sign-changing critical point u_1 satisfying $C_{\mu_\infty}(G, u_1) \neq 0$.

Proof. *Case 1.* By Theorems 7.9 and 7.10, we just have to consider the cases $\mu_0 \geq 2$ and $\mu_\infty \geq 2$. Using excision and the fact that 0 is a nondegenerate critical point, we obtain for each $\varepsilon > 0$ small enough that

$$H_{\mu_0}(G^\varepsilon \cup D^\star, G^{-\varepsilon} \cup D^\star) \cong H_{\mu_0}(G^\varepsilon, G^{-\varepsilon}) \neq 0. \tag{7.8}$$

It is easy to see (cf. Bartsch and Wang [43] and Chang [95]) that for each $\alpha > 0$ large enough,

$$H_k(E_0, G^{-\alpha} \cup D^\star) \cong H_k(E_0, G^{-\alpha}) \cong \delta_{k\mu_\infty}\mathcal{F}, \qquad k \in \mathbf{Z}. \tag{7.9}$$

Note that $\mu_\infty \geq 1$ and $\nu_\infty = 0$. We fix such a small $\varepsilon > 0$ and such a large $\alpha > 0$. Consider the following diagram

(7.10)

$$\begin{array}{c} H_{\mu_0+1}(E_0, G^\varepsilon \cup D^\star) \xrightarrow{\partial} H_{\mu_0}(G^\varepsilon \cup D^\star, G^{-\alpha} \cup D^\star) \to H_{\mu_0}(E_0, G^{-\alpha} \cup D^\star) \\ \downarrow j \\ H_{\mu_0}(G^\varepsilon \cup D^\star, G^{-\varepsilon} \cup D^\star) \end{array}$$

$$\downarrow$$

$$H_{\mu_0-1}(G^{-\varepsilon} \cup D^\star, G^{-\alpha} \cup D^\star).$$

Note that the vertical maps are part of the long exact sequence of the triple

$$(G^\varepsilon \cup D^\star, G^{-\varepsilon} \cup D^\star, G^{-\alpha} \cup D^\star).$$

Combining (7.8) we see that either

(i) $H_{\mu_0}(G^\varepsilon \cup D^\star, G^{-\alpha} \cup D^\star) \neq 0$ or
(ii) $H_{\mu_0-1}(G^{-\varepsilon} \cup D^\star, G^{-\alpha} \cup D^\star) \neq 0$.

For Case (i), by (7.9) we have that $H_{\mu_0}(E_0, G^{-\alpha} \cup D^\star) = 0$. By (7.10), we see that it is a part of the exact sequence of the triple $(E_0, G^\varepsilon \cup D^\star, G^{-\alpha} \cup D^\star)$; we deduce that

$$H_{\mu_0+1}(E_0, G^\varepsilon \cup D^\star) \neq 0.$$

Choose a nontrivial element $\vartheta \in H_{\mu_0+1}(E_0, G^\varepsilon \cup D^\star)$. For $r \geq \varepsilon$, let

$$l_r : H_{\mu_0+1}(E_0, G^\varepsilon \cup D^\star) \to H_{\mu_0+1}(E_0, G^r \cup D^\star)$$

be the homeomorphism induced by the corresponding inclusion. Consider the set $\Lambda := \{r \geq \varepsilon : l_r(\vartheta) \neq 0\}$ and define $c := \sup \Lambda$. Then, $c < \infty$. That is, if σ is a singular chain representing ϑ then the carrier $|\sigma| \subset E_0$ of σ is compact and contained in G^r for $r \geq \max_{|\sigma|} G$. Hence, $l_r(\vartheta) = 0$ for $r \geq \max_{|\sigma|} G$ and $c < \max_{|\sigma|} G < \infty$. Thus $c \in [\varepsilon, \infty)$. We proceed as before to obtain a sign-changing critical point u_1 of G such that $G(u_1) = c$ and $C_{\mu_0+1}(G, u_1) \neq 0$.

On the other hand, if $H_{\mu_0-1}(G^{-\varepsilon} \cup D^\star, G^{-\alpha} \cup D^\star) \neq 0$, we consider

$$\Lambda := \{r \in [-\alpha, -\varepsilon] : H_{\mu_0-1}(G^r \cup D^\star, G^{-\alpha} \cup D^\star) = 0\}$$

and $c := \sup \Lambda \in [-\alpha, -\varepsilon]$. It is easy to see that G has a sign-changing critical point u_1 on the level c with $C_{\mu_0-1}(G, u_1) \neq 0$.

Case 2. Consider $\alpha > 0$ as in (7.9) and let

$$\Lambda := \{r \geq -\alpha : H_{\mu_\infty}(G^r \cup D^\star, G^{-\alpha} \cup D^\star) \neq 0\}.$$

We first show that $\Lambda \neq \emptyset$. To show this, we choose a nontrivial element $\vartheta \in H_{\mu_\infty}(E_0, G^{-\alpha} \cup D^\star)$; this is possible because of (7.9). If σ is a singular chain representing ϑ and if $r \geq \max_{|\sigma|} G$, then ϑ comes from

$$H_{\mu_\infty}(G^r \cup D^\star, G^{-\alpha} \cup D^\star).$$

This implies that $[\max_{|\sigma|} G, \infty) \subset \Lambda$. Thus $c := \inf \Lambda \in [-\alpha, \infty)$ is finite. As before, G has a critical point u_1 with $C_{\mu_\infty}(G, \mu_1) \neq 0$ and $u_1 \in E_0 \backslash D^\star$. Because $C_{\mu_\infty}(G, 0) = 0$, we see that u_1 is a nontrivial sign-changing critical point. $\square$

Notes and Comments. The results of this section are due to Bartsch et al. [31]. Conditions (A_1)–(A_4) and Definition 7.8 were introduced in Bartsch and Wang [40]. The hypothesis that all sign-changing critical points must be isolated can be weakened. It suffices to assume that all sign-changing critical points with values in a finite closed interval are isolated. By imposing some stronger assumptions, it is possible to find more sign-changing critical points (cf. Bartsch et al. [31]). Theorem 7.11 can be extended to the degenerate case (i.e., ν_0 and ν_∞ may be nontrivial). In this case, the following Landesmann–Lazer condition should be assumed (cf. [31]).

$$G(u) \to \pm\infty, \quad \text{for } u \in ker(\mathbf{id} - \varPi_G) \text{ as } \|u\| \to \infty. \tag{7.11}$$

Readers may find related results and methods in Bartsch and Li [36] and Chang [94] for degenerate situations without the sign-changing conclusions.

In Bartsch and Wang [40], it was shown by using the maximum principle and the critical group that certain solutions change sign. The philosophy is that if the behavior of the energy functional near zero and near infinity implies a nontrivial critical group of a critical point, then this point can be neither positive nor negative. Similar to Theorem 7.9, Theorem 2.1 of Bartsch and Wang [40] got a sign-changing critical point u_1 such that any positive critical point u_2 must have $u_2 \gg u_1$; any negative critical point u_3 must have $u_3 \ll u_1$. Moreover, if the functional is bounded below, then it indeed has strongly maximal negative and strongly minimal positive critical points. Theorem 2.2 of Bartsch and Wang [40] obtained a sign-changing critical point u_1 such that any critical point $u_2 < u_1$ implies that $u_2 \ll 0$ and $u_2 > u_1$ implies that $u_2 \gg 0$. Also, in Bartsch [30], the extremality properties of the positive, negative, and sign-changing solutions were obtained where the Borel cohomology was involved. Finally, we would like to mention the paper [99] due to Chang and Jiang where sign-changing solutions were found for Dirichlet problems with indefinite nonlinearities by the Morse theory (an earlier result was obtained in Li and Wang [198] by a mountain pass theorem in order intervals).

7.3 Sign-Changing Solutions of Mountain Pass Type

Let $\varOmega \subset \mathbf{R}^N$ be a Lipschitz bounded domain. Consider the following Dirichlet problem

$$\begin{cases} -\varDelta u = f(u), & \text{in } \varOmega, \\ u = 0, & \text{on } \partial\varOmega. \end{cases} \tag{7.12}$$

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the eigenvalues of the $-\varDelta$ with zero boundary value condition. Assume

(B₁) $f \in \mathbf{C}^1(\mathbf{R}), f(0) = 0.$

(B₂) $\limsup_{|t|\to\infty} \frac{f(t)}{t} < \lambda_1.$

(B₃) $f'(t) \to \beta \in \mathbf{R}$ as $|t| \to \infty$. If $\beta = \lambda_k$, we assume that

$$f(t) - \beta t \quad \text{is bounded in } t \in \mathbf{R} \text{ and either}$$

$$\int_\Omega \left(F(u) - \frac{\beta}{2} u^2 \right) dx \to \infty,$$

$$\text{for } u \in V := \{u \in \mathbf{C}_0^\infty(\Omega) : -\Delta u = \beta u\} \quad \text{with} \quad \|u\| \to \infty \qquad \text{or}$$

$$\int_\Omega \left(F(u) - \frac{\beta}{2} u^2 \right) dx \to -\infty, \qquad \text{for } u \in V \quad \text{with } \|u\| \to \infty.$$

Let $E := H_0^1(\Omega)$ be the usual Sobolev space with the norm and inner product:

$$\|u\| = \left(\int_\Omega |\nabla u| dx \right)^{1/2}, \qquad \langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v dx.$$

Theorem 7.12. *Suppose that either* (B_1), (B_2), *or* (B_1) *and* (B_3) *with* $\beta < \lambda_2$ *hold. Moreover,* $f'(0) \geq \lambda_2$. *Then (7.12) has a sign-changing solution. If all sign-changing solutions with negative energy are isolated, then (7.12) has a solution that changes sign, has Morse index at most* 1, *and is of the mountain pass type.*

Proof. Consider the functional

$$G(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u) dx, \qquad u \in E = H_0^1(\Omega).$$

Under the conditions of Theorem 7.12, G may not satisfy the smoothness condition in (A_1) and the order preserving condition in (A_2). But in the case of (B_2), by a standard argument due to Hofer [173], there exists a $\mathbf{C}^1$-modification $\tilde{f}$ of f satisfying

(a) $\tilde{f}(0) = 0.$

(b) $\limsup_{|t|\to\infty} \frac{\tilde{f}(t)}{t} < \lambda_1.$

(c) $|\tilde{f}'(t)| < a$ for all $t \in \mathbf{R}$ and some $a > 0$.

(d) $\tilde{f}'(0) > \lambda_2.$

Moreover, solutions of (7.12) are precisely the solutions of the modified Dirichlet problem

$$\begin{cases} -\Delta u = \tilde{f}(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.13}$$

Therefore, we may always assume that $|f'(t)| < a$ for all $t \in \mathbf{R}$. On E, we use the following equivalent norm $\|u\|_E = \big(\int_\Omega (|\nabla u| + au^2)dx\big)^{1/2}$. Let

$$\mathcal{P} := \{u \in E : u(x) \geq 0 \ \text{ a.e. in } \ \Omega\}$$

be the positive cone of E. Choose the Banach space $E_0 := C_0^1(\Omega)$ with the usual norm. Then E_0 is dense in E and $\mathcal{P}_0 := E_0 \cap \mathcal{P}$ has nonempty interior $\overset{\circ}{\mathcal{P}_0}$. Let e_0 be the unique normalized positive eigenfunction of $-\Delta$ on Ω with zero boundary value condition, then

$$e_0 \in \overset{\circ}{\mathcal{P}_0} \quad \text{and} \quad \langle u, e_0 \rangle > 0 \quad \text{for all } u \in \mathcal{P}_0 \backslash \{0\}.$$

Rewrite

$$G(u) = \frac{1}{2}\|u\|_E^2 - \int_\Omega H(u)dx, \quad \text{for } u \in E,$$

where $h(t) = f(t) + at, H(t) = \int_0^t h(s)ds$. Then it is easy to see that $G \in \mathbf{C}^2(E, \mathbf{R})$. It is trivial to check that the (PS) condition is satisfied by G in the case of (B_2). If (B_3) holds with $\beta \neq \lambda_k$, the (PS) condition also can be proved easily. But if $\beta = \lambda_k$, we can also prove the (PS) condition because the Landesman–Lazer condition (see (7.11)) is satisfied. It follows from the standard regularity theory that all critical points of G lie in E_0. Thus (A_1) of the previous section is satisfied. Because $h'(t) > 0$, we observe that (A_2) and (A_3) hold. Furthermore, if (B_2) holds then (A_4)-(1) is satisfied. If (B_3) holds and $\beta \in (\lambda_1, \lambda_2)$, then (A_4)-(3) is satisfied. Note that the Morse index μ_0 of G at 0 is ≥ 2 because $f'(0) > \lambda_2$. These arguments imply that $\mu_\infty = \nu_\infty = 0$ if (B_2) holds and $(\mu_\infty, \nu_\infty) = (0, 0)$, or $(0, 1)$ or $(1, 0)$ if (B_3) holds. That is, $\mu_\infty + \nu_\infty \leq 1$. The conclusion of the theorem follows from Theorem 7.9. □

Definition 7.13. Let u be defined on a domain Ω (may or may not be bounded); the components of $\Omega \backslash u^{-1}(0)$ are called the nodal domains (or sets) of u.

Next we consider the following superlinear cases. Assume

(B₄) There exists a p with $2 < p < (2N)/(N-2)$ and there is a $c > 0$ such that

$$|f(t)| \leq c(1 + |t|^{p-1}), \qquad \forall \ t \in \mathbf{R}.$$

(B₅) There are $R > 0, \gamma > 2$ such that

$$0 < \gamma F(t) \leq tf(t), \qquad \forall \ |t| \geq R.$$

(B₆) $f'(t) > f(t)/t, \quad \forall \ t \neq 0.$

Theorem 7.14. *Assume that $f'(0) < \lambda_2$ and all the sign-changing solutions are isolated. Suppose that either (B_1) and (B_3) with $\beta > \lambda_2$ hold or (B_1), (B_4), and (B_5) hold. Then (7.12) has a sign-changing solution u_1 which is neither a local minimum nor of mountain pass type.*

If in addition (B_6) holds, then u_1 has precisely two nodal sets, its Morse index is 2, and

$$C_k(G, u_1) = \begin{cases} \mathcal{F}, & \textit{if } k = 2, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For the asymptotically linear case, $f'(t)$ is bounded. The proof is simpler. We just prove the superlinear case by using Theorem 7.10. We have to show that G has a sign-changing critical point u satisfying

$$C_0(G, u) = C_1(G, 0) = 0, \qquad C_2(G, u) \neq 0.$$

Furthermore, we want to show, if in addition (B_6) holds, that

$$C_k(G, u) \cong \begin{cases} \mathcal{F}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

First, by (B_5) there exists a $c > 0$ such that

$$F(u) \geq C|u|^\gamma, \qquad \forall |u| \geq R. \tag{7.14}$$

By (7.14) and (B_4), it is easy to check that there exist $A > 0$, $R_n \to> r_n \to \infty$ such that

$$0 \leq \liminf_{n\to\infty} \frac{f'(R_n)}{|R_n|^{p-2}} \leq \limsup_{n\to\infty} \frac{f'(R_n)}{|R_n|^{p-2}} \leq A, \tag{7.15}$$

$$0 \leq \liminf_{n\to\infty} \frac{f'(-r_n)}{|r_n|^{p-2}} \leq \limsup_{n\to\infty} \frac{f'(-r_n)}{|r_n|^{p-2}} \leq A. \tag{7.16}$$

Define

$$f_n(t) = \begin{cases} f(t), & -r_n \leq t \leq R_n, \\ f(R_n) + f'(R_n)(t - R_n) + |t - R_n|^{p-2}(t - R_n), & t \geq R_n, \\ f(-r_n) + f'(-r_n)(t + r_n) + |t + r_n|^{p-2}(t + r_n), & t \leq -r_n, \end{cases} \tag{7.17}$$

Observe that $p \geq \gamma$ by (7.14). Because of (7.15) and (7.16) we have that

$$|f_n(t)| \leq a_0(1 + |t|^{p-1}) \tag{7.18}$$

and

$$\gamma F_n(t) \leq f_n(t)t, \qquad \forall |t| \geq R \tag{7.19}$$

uniformly for all n. There also exists $a_n > 0$ such that

$$|f_n'(t)| \leq a_n(1 + |t|^{p-2}), \qquad \forall t \in \mathbf{R} \tag{7.20}$$

and

$$f_n'(t) \geq -a_n, \qquad \forall t \in \mathbf{R}. \tag{7.21}$$

Next, for each n we use an equivalent norm on E given by

$$\|u\|_{E_n}^2 = \int_\Omega (|\nabla u|^2 + a_n u^2)dx.$$

We write E_n for the space E with this norm. Define

$$g_n(t) = f_n(t) + a_n t, G_n(t) = \int_0^t g_n(s)ds$$

and

$$G_n(u) = \frac{1}{2}\|u\|_{E_n}^2 - \int_\Omega G_n(u)dx. \tag{7.22}$$

Then $G_n \in \mathbf{C}^2(E_n, \mathbf{R})$ by (7.20). It is easy to check by using (7.18) and (7.19) that (A_1), (A_2), and (A_3) are satisfied. With (7.19), we also get that for each $u \in E\backslash\{0\}$ there exists $M(u) > 0$ such that $G_n(tu) < 0$ for all $t > M(u)$ and there exists $a < 0$ such that $\langle G_n'(u), u\rangle < 0$ if $G_n(u) \leq a$. Therefore, we may apply Theorem 7.10 to G_n. Moreover, by the proof of Theorem 7.10, we see that the critical values c_n are bounded above by $\beta_n = \max_{B_n} G_n$, where

$$B_n := \{sv_\infty + tw_\infty : |s| \leq T_n, t \in [0, T_n]\}$$

for some $T_n > 0$ such that

$$G_n(u) < 0, \quad \text{for all } u \in \operatorname{span}\{v_\infty, w_\infty\} \text{with } \|u\|_{E_n} \geq T_n,$$

where v_∞, w_∞ are normalized eigenfunctions corresponding to the first and second eigenvalues. From the above arguments, we see that there exists a $T_0 > 0$ independent of n such that

$$G_n(u) < 0 \quad \text{for all } u \in \operatorname{span}\{v_\infty, w_\infty\} \text{with } \|u\|_{E_n} \geq T_0.$$

Hence, we may choose a fixed B_0 such that

$$c_n \leq \beta_0 = \max_{B_0} G_n.$$

Thus c_n is uniformly bounded. Next we observe that

$$
\begin{aligned}
c_n &= G_n(u_n) - \frac{1}{\gamma}\langle G_n'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|_{E_n}^2 - \int_\Omega \left(F_n(u_n) - \frac{1}{\gamma} f_n(u_n)u_n\right) dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|^2 - c,
\end{aligned}
$$

where $c > 0$ is independent of n because of (7.19). Therefore, u_n is bounded in E. Then by (7.18), Equation (7.12), and standard elliptic estimates we get

$$\|u_n\|_{C_0^1(\Omega)} \leq c$$

independent of n. Thus for n large, u_n is a sign-changing solution of the original problem (7.12). We fix n large enough and write $u_n = u$. We denote the restriction of G and G_n to $\mathbf{C}_0^1(\bar{\Omega})$ by $\tilde{G}, \tilde{G}_n$, respectively. By a theorem due to Chang [94], we have for $k = 0, 1$,

$$0 = C_k(G_n, u) = C_k(\tilde{G}_n, u) = C_k(\tilde{G}, u) = C_k(G, u)$$

and

$$0 \neq C_2(G_n, u) = C_2(G, u).$$

If in addition (B_6) holds, we also get

$$C_k(G, u) = C_k(G_n, u) = \begin{cases} \mathcal{F}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Note that a sign-changing solution has Morse index $\mu \geq 2$. Combining this with the fact $\mu \leq 2$ we see that $\mu = 2$. □

The following theorem is a straightforward consequence of Theorem 7.11.

Theorem 7.15. *Suppose that all the sign-changing solutions are isolated. Assume that (B_1) holds and that*

$$\lambda_k < f'(0) < \lambda_{k+1}, \quad \textit{for some } k \geq 0;$$

here $\lambda_0 = -\infty$. Moreover, suppose that (B_3) or (B_4) and (B_5) hold. In the case of (B_3) we assume that there are $\lambda_m \neq \lambda_k, m \geq 1$ with $\lambda_m < \beta < \lambda_{m+1}$.

(1) If $k \geq 1$ then there exists a sign-changing solution u_1 such that either $G(u_1) > 0, C_{k+1}(G, u_1) \neq 0$ or $G(u_1) < 0, C_{k-1}(G, u_1) \neq 0$.
(2) If (B_3) holds with $\beta \in (\lambda_m, \lambda_{m+1}), k \neq m \geq 2$, then (7.12) has a sign-changing solution u_1 with $C_m(G, u_1) \neq 0$.

Notes and Comments. The Morse index of a mountain pass point has been found independently in Ambrosetti [11] for the nondegenerate case and in Hofer [176] for the general case of possibly degenerate critical points.

The results of this section are due to Bartsch et al. [31]. The nondegeneracy assumptions in Theorem 7.15 can be dropped by using the ideas of Bartsch and Li [36] and Chang [94]. The information on the Morse indices of sign-changing solutions can be used to obtain new multiplicity results. The readers can combine Theorems 7.12–7.15 (even degree theory) to get the existence results of two sign-changing solutions (see, e.g., Corollary 2.4 of Bartsch et al. [31]). In [31], the results were extended to elliptic systems. The trick on the modification $\tilde{f}$ of f used in the proof of Theorem 7.12 was due to Hofer [173]. Similar ideas for constructing the function f_n in (7.17) was applied in Castro and Cossio [79]. We refer readers to Bartsch and Wang [40], where some order relations among positive (negative) solutions, local minimizers, and sign-changing solutions were given. In particular, it was first shown that the third solution obtained in Wang [331] was indeed sign-changing as expected for a long time. Finally, we mention that in Li and Wang [198] and Dancer and Yan [131], sign-changing solutions of the mountain pass type were obtained.

Before closing this section, we refer readers to the following papers on the estimates of Morse indices: Fang and Ghoussoub [142] (Morse type information on Palais–Smale sequences), Hofer [175], Lazer and Solimini [193] (non-symmetric functionals), Ramos and Sanchez [261], and Tanaka [323] (even functionals).

7.4 Nodal Domains

Consider the following equation.

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7.23}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is a smooth and bounded domain. In this section we are concerned with the number of nodal domains of the weak solutions to (7.23). Assume

(C_1) $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and $f(x,0) = 0$ a. e. $x \in \Omega$.

(C_2) There exist a $p \in (2, (2N/(N-2))]$ if $N \geq 3$ (otherwise, $p > 2$ if $N = 2$) and $c > 0$ such that $|f(x,t)| \leq c(|t| + |t|^{p-1})$ for all $t \in \mathbf{R}$ and a.e. $x \in \Omega$.

(C_3) The function $t \to (f(x,t))/|t|$ is nondecreasing on $\mathbf{R}\backslash\{0\}$ for a.e. $x \in \Omega$.

The energy functional of (7.23) is

$$G(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x,u)dx, \qquad u \in E := H_0^1(\Omega).$$

Then $G \in \mathbf{C}^1(E, \mathbf{R})$.

Lemma 7.16. *Assume* (C_1)*–*(C_3)*. If* $\langle G'(u), u\rangle = 0$ *for some* $u \in E$ *with* $u \neq 0$*, then*

$$0 \le G(u) = \sup_{t\ge 0} G(tu). \tag{7.24}$$

In particular, if the function in (C_3) *is strictly increasing, then* $G(u) > 0$*.*

Proof. Let $\phi(t) := G(tu), t \ge 0$. Then

$$\phi'(t) = \langle G'(tu), u\rangle = t\int_{\Omega}\left(|\nabla u|^2 - \frac{f(x,tu)}{tu}u^2\right)dx, \qquad t > 0.$$

Hence (C_3) implies that $\phi'(t)/t$ is nonincreasing on $(0,\infty)$. Furthermore, the set

$$S := \{t > 0 : \phi'(t) = 0\}$$

is a subinterval of $(0,\infty)$ and $1 \in S$. Let $b < \infty$ be the right endpoint of S. Then ϕ is strictly decreasing on (b,∞). Note that

$$0 \le \max_{t\in[0,b]} \phi(t) \le \max_{t\in S}\phi(t) = \phi(1).$$

We get (7.24). If the function in (C_3) is strictly increasing, then $\phi'(t)/t$ is strictly decreasing on $(0,\infty)$; hence $S = \{1\}$ and $\phi'(t) > 0$ for all $t \in (0,1)$. We then have that $G(u) = \phi(1) > \phi(0) = 0$. □

Theorem 7.17. *Assume* (C_1)*–*(C_3)*. If* f *is odd in* u*, then every weak solution of* (7.23) *with* $0 < G(u) \le \beta_n$ *has at most* n *nodal domains, where*

$$\beta_n := \inf_{X\subseteq E, \dim V \ge n} \sup_X G, \qquad n > 0.$$

Proof. By negation; assume that u has more than n nodal domains. Let $\Omega_1, \ldots, \Omega_n$ be a choice of such domains; we define functions w_i $(i = 1, 2, \ldots, n)$ as the following.

$$w_i(x) := \begin{cases} u(x), & \text{if } x \in \Omega_i, \\ 0, & \text{if } x \in \Omega\backslash\Omega_i. \end{cases} \tag{7.25}$$

Then $w_i \in E$ (see Müller-Pfeiffer [230]). For $v := u - \sum_{i=1}^n w_i$, we have

$$0 < G(u) = G(v) + \sum_{i=1}^n G(w_i).$$

Up to an appropriate choice of $\Omega_i, i = 1, \ldots, n$, we may assume that $G(v) > 0$. Note that

$$\langle G'(w_i), w_i \rangle = \langle G'(u), w_i \rangle = 0;$$

we have by Lemma 7.16 that $G(w_i) = \sup_{t \in \mathbf{R}} G(tw_i)$ because G is even. Let

$$X = \operatorname{span}\{w_1, \ldots, w_n\}.$$

We obtain

$$\beta_n \leq \sup_X G = \sum_{i=1}^n G(w_i) = G(u) - G(v) < G(u), \tag{7.26}$$

which contradicts the assumption. □

Consider the nodal Nehari set

$$M := \{u \in E : u_+ \neq 0, u_- \neq 0, \langle G'(u), u_+ \rangle = \langle G'(u), u_- \rangle = 0\}; \tag{7.27}$$

here

$$u_\pm = \pm \max\{\pm u, 0\}. \tag{7.28}$$

Let

$$\beta := \inf_M G. \tag{7.29}$$

Theorem 7.18. *If* (C_1)–(C_3) *hold and if the function in* (C_3) *is strictly increasing, then every weak solution* $u \in M$ *of* (7.23) *with* $G(u) = \beta$ *has precisely two nodal domains.*

Proof. By negation; we assume that u has at least three nodal domains. We choose nodal domains Ω_1, Ω_2 such that $w_1 \geq 0, w_2 \leq 0$ for the associated functional $w_1, w_2 \in H_0^1(\Omega)$ defined as in (7.25). Clearly, $w_1 + w_2 \in M$. The function $v := u - w_1 - w_2$ satisfies $\langle G'(v), v \rangle = 0$. This implies that $G(v) > 0$ by Lemma 7.16. Then we have that $\beta \leq G(w_1 + w_2) < G(u)$ (see (7.26)), which contradicts the assumption. □

Notes and Comments. The results of this section are due to Bartsch and Weth [45]. A classical result of Courant and Hilbert [124] states that the number $\#(e)$ of nodal domains of a Dirichlet eigenfunction e of the Laplacian in Ω is bounded above by $\mu(e) + 1$, where $\mu(e)$ is the Morse index of e. By Benci and Fortunato [53], we also have the inequality $\#(u) \leq \mu(u)$ for any solution of (7.12) provided $f(0) = 0$ and $f'(t) > f(t)/t$ for all $t \neq 0$. Similar estimates as Theorems 7.17 and 7.18 can also be found in Bartsch et al. [37], where the results were established for Schrödinger equations with potentials yielding compactness.

7.5 Sign-Changing Solutions with Least Energy

Consider again the semilinear Dirichlet problem (7.23). We now study the least energy solution related to the nodal Nehari sets defined in (7.27). We need the following conditions throughout this section.

($\mathbf{D_1}$) $f \in \mathbf{C}^1(\Omega \times \mathbf{R}, \mathbf{R}), f(x,0) = 0$ for all $x \in \Omega$.
($\mathbf{D_2}$) There exist $p \in (2, (2N/(N-2))$ if $N \geq 3$ (if $N \geq 2$, we let $p > 2$) and $c > 0$ such that $|f'(x,t)| \leq c(1 + |t|^{p-2})$ for all $t \in \mathbf{R}$ and all $x \in \Omega$, where $f'(x,t) = \partial f / \partial t$.
($\mathbf{D_3}$) $f'(x,t) > f(x,t)/t > 0$ for all $x \in \Omega, t \neq 0$.
($\mathbf{D_4}$) There exist $R > 0$ and $\gamma > 2$ such that

$$0 < \gamma F(x,t) \leq t f(x,t), \quad \text{for all } x \in \Omega, |t| \geq R.$$

Consider the Hilbert space $H := E \cap H^2(\Omega)$ endowed with the scalar product from $H^2(\Omega)$ and the induced norm $\|\cdot\|_H$, where $E := H_0^1(\Omega)$. Define the functionals $\Pi_\pm : E \to \mathbf{R}$ given by

$$\Pi_\pm(u) := \int_\Omega |\nabla u_\pm|^2 dx = \int_\Omega \nabla u \cdot \nabla u_\pm dx;$$

here $u_\pm$ is defined in (7.28).

Lemma 7.19. *$\Pi_\pm$ is differentiable at $u \in H$ with derivative $\Pi'_\pm(u) \in E'$ given by*

$$\langle \Pi'_\pm(u), v \rangle = \int_{\pm u > 0} ((-\Delta u)v + \nabla u \nabla v) dx. \tag{7.30}$$

Moreover, $\Pi_\pm|_H \in \mathbf{C}^1(H)$.

Proof. Let $u, v \in H$ and $t \neq 0$; we define

$$K_1(t) := \{x \in \Omega : u(x) + tv(x) \geq 0, u(x) > 0\},$$

$$K_2(t) := \{x \in \Omega : u(x) + tv(x) \geq 0, u(x) < 0\},$$

$$K_3(t) := \{x \in \Omega : u(x) + tv(x) < 0, u(x) > 0\},$$

$$K_4 := \{x \in \Omega : u(x) = 0\}.$$

Let $\pi_i(x)$ and $\pi(x)$ be the characteristic functions associated with $K_i(t)$ and K_4, respectively; $i = 1, 2, 3$. Because $\nabla u = 0$ and $-\Delta u = 0$ a.e. on K_4 (cf. Gilbarg and Trudinger [160]), we have

$$\frac{1}{t}(\Pi_+(u + tv) - \Pi_+(u))$$

$$
\begin{aligned}
&= \frac{1}{t} \int_{\Omega} (\nabla(u+tv)\nabla(u+tv)^{+}, \nabla u \nabla u^{+}) dx \\
&= \frac{1}{t} \int_{\Omega} ((-\Delta u)(u+tv)^{+} + (\Delta u)u^{+}) dx + \int_{\Omega} \nabla v \nabla (u+tv)^{+} dx \\
&= \frac{1}{t} \int_{\Omega} \pi_1((-\Delta u)(u+tv) + (\Delta u)u) dx + \int_{\Omega} \pi_1 \nabla v \nabla (u+tv) dx \\
&\quad + \frac{1}{t} \int_{\Omega} \pi_2((-\Delta u)(u+tv)) dx + \int_{\Omega} \pi_2 \nabla v \nabla (u+tv) dx \\
&\quad + \frac{1}{t} \int_{\Omega} \pi_3 (\Delta u) u dx + t \int_{\Omega} \pi_4 \nabla v \nabla v^{+} dx \\
&= \int_{\Omega} \pi_1 (-\Delta u) v dx + \int_{\Omega} \pi_1 \nabla u \nabla v + o(1)
\end{aligned}
$$

as $t \to 0$. The last equality is a consequence of the fact that $\pi_1, \pi_3 \to 0$ pointwise a.e. on Ω for $t \to 0$. By the Lebesgue theorem,

$$\int_{\Omega} \pi_2((-\Delta u)v + \nabla v \nabla u) dx \to 0.$$

Using the definition of π_2, π_3, we have that

$$\left| \int_{\Omega} \frac{(\pi_3 - \pi_2)(\Delta u) u}{t} dx \right| \leq \int_{\Omega} (\pi_2 + \pi_3) |\Delta u| |v| dx \to 0$$

for $t \to 0$. Thus

$$
\begin{aligned}
\langle \Pi_{+}'(u), v \rangle &= \lim_{t \to 0} \left(\int_{\Omega} \pi_1 (-\Delta u) v dx + \int_{\Omega} \pi_1 \nabla u \nabla v dx \right) \\
&= \int_{u>0} ((-\Delta u) v + \nabla u \nabla v) dx,
\end{aligned}
$$

as claimed. The proof for Π_{-}' is similar.

For the second part of the lemma, we take a sequence $u_n \to u$in H; then

$$
\begin{aligned}
&|\langle \Pi_{+}'(u_n) - \Pi_{+}'(u), v \rangle| \\
&\quad = \left| \int_{u_n > 0} ((-\Delta u_n) v + \nabla u_n \nabla v) dx - \int_{u>0} ((-\Delta u) v + \nabla u \nabla v) dx \right| \\
&\quad \leq 2 \|u_n - u\|_H \|v\|_H \\
&\qquad + \left| \int_{u \leq 0 < u_n} ((-\Delta u) v + \nabla u \nabla v) dx + \int_{u_n \leq 0 < u} ((-\Delta u) v + \nabla u \nabla v) dx \right|
\end{aligned}
$$

$$\leq 2\|u_n - u\|_H \|v\|_H$$
$$+ \left(\int_{u \leq 0 < u_n} (|\Delta u|^2 + |\nabla u|^2) dx + \int_{u_n \leq 0 < u} (|\Delta u|^2 + |\nabla u|^2) \right)^{1/2} \|v\|_H$$
$$= o(1)\|v\|_H.$$

Note that $\nabla u = 0$ and $\Delta u = 0$ a.e. on the zero set of u. Thus, $(\Pi_+|_H)'$ is continuous and the proof is complete for Π_+. The proof for Π'_- is similar. □

Define $F_\pm : E \to \mathbf{R}$ by

$$F_\pm(u) = \int_\Omega f(x, u) u_\pm dx, \qquad u \in E.$$

Then we have the following.

Lemma 7.20. $F_\pm \in \mathbf{C}^1(E, \mathbf{R})$ *with the derivative given by*

$$\langle F'_\pm(u), v \rangle = \int_\Omega f'(x, u_\pm) u^\pm v dx + \int_\Omega f(x, u_\pm) v dx.$$

Proof. It is quite similar to that of Lemma 7.19. □

Lemma 7.21. *The set $M \cap H$ is a $\mathbf{C}^1$-manifold of codimension two in H, where M is defined in (7.27).*

Proof. Define $\rho_\pm : E \to \mathbf{R}$ by

$$\rho_\pm(u) = \langle G'(u), u_\pm \rangle.$$

Then

$$M \cap H = \{u \in H : u_\pm \neq 0, \rho_\pm(u) = 0\}.$$

By Lemmas 7.19 and 7.20, $\rho_\pm$ is differentiable in $u \in H$ with

$$\langle \rho'_\pm(u), u_\pm \rangle = \int_\Omega (|\nabla u_\pm|^2 - f'(x, u)(u_\pm)^2) dx, \qquad \langle \rho'_\pm(u), u_\mp \rangle = 0.$$

Moreover, $\rho_\pm|_H \in \mathbf{C}^1(H, \mathbf{R})$. Using (D_3) implies that $\langle \rho'_\pm(u), u_\pm \rangle < 0$ for all $u \in M \cap H$. Approximating $u_\pm$ by functions in H, we conclude that $\big(\rho'_+(u), \rho'_-(u)\big) \in \mathcal{L}(H, \mathbf{R}^2)$ is onto for each $u \in M \cap H$. This completes the proof of the theorem. □

Theorem 7.22. *Let $u \in M$ be a critical point of G with $G(u) = \inf_M G$. Then the Morse index of u is precisely 2.*

Proof. By (D_3),

$$\langle G''(u)u_\pm, u_\pm\rangle = \int_\Omega (|\nabla u_\pm|^2 - f(x,u)(u_\pm)^2)dx < 0.$$

Hence the Morse index of u is at least 2. On the other hand, by the elliptic regularity theory, $u \in H$. Denote by $T_0 \subset H$ the tangent space of the manifold $M \cap H$ at u. We claim $\langle G''(u)v, v\rangle \geq 0$ for all $v \in T_0$. Actually, by Lemma 7.21, for each $v \in T_0$, we may find a $\mathbf{C}^1$-curve $\gamma : [-1,1] \to M \cap H$ such that $\gamma(0) = u$ and $\dot\gamma(0) = v$. Because $\langle G'(u), v\rangle = 0$, we observe that $G \circ \gamma : [-1,1] \to \mathbf{R}$ is twice differentiable at 0 and

$$\frac{\partial^2}{\partial t^2}(G\circ\gamma)\Big|_{t=0} = \langle G''(u)v, v\rangle.$$

Note that $G(u) = \min\limits_{v \in M\cap H} G(v)$, we infer that $(\partial^2/\partial t^2)(G\circ\gamma)\big|_{t=0} \geq 0$. Hence, the claim above is true. Because $T_0 \subset H$ has codimension two and H is dense in E, we conclude that the Morse index of u is at most 2. □

Obviously, $G' = \mathbf{id} - \Theta_G$, where $\Theta_G : E \to E$ is compact and strongly order preserving. Due to (D_3), we may use the usual scalar product

$$\langle u, v\rangle_E = \int_\Omega (\nabla u \nabla v + uv)dx, \qquad u, v \in E = H_0^1(\Omega).$$

We denote the corresponding norm by $\|\cdot\|_E$. By integrating $-\nabla G$ we obtain a flow $\eta(t,u) : \mathcal{O} \to E$, where $\mathcal{O} \subset \mathbf{R} \times E$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}\eta(t,u) = -\nabla G(\eta(t,u)), \\ \eta(0,u) \quad\;\; = u, \quad (t,u) \in \mathcal{O}. \end{cases} \tag{7.31}$$

We introduce the following notations.

(7.32) $\Upsilon^- := \{u \in E_0 : u \text{ is a sign-changing subsolution of } (7.23)\}$.

(7.33) $\Upsilon^+ := \{u \in E_0 : u \text{ is a sign-changing supersolution of } (7.23)\}$.

(7.34) $\Upsilon^\star := \{(0,0)\} \cup \{(u,v) \in \Upsilon^- \times \Upsilon^+ : u < v\} \subset E_0 \times E_0$.

(7.35) $\Upsilon^* := \cup_{(u,v)\in\Upsilon^\star}((u + \mathcal{P}_0) \cup (v - \mathcal{P}_0)) \subset E_0$.

It is easy to see by the strong order preservingness of Θ_G that if $(u,v) \in \Upsilon^\star$, then $(u + \mathcal{P}_0) \cup (v - \mathcal{P}_0)$ is positive invariant under $\eta(t,\cdot)$ for all $t \geq 0$. Combining this and the standard deformation as used in previous chapters, we may easily prove the following lemma. Readers may also consult the paper of Bartsch [30] for details.

Lemma 7.23. *Suppose that $\mathcal{K}_c \subset \Upsilon^*$ for some $c > 0$. Then there is an $\varepsilon > 0$ and a homotopy $\eta : (G_{E_0}^{c+\varepsilon} \cup \Upsilon^*) \times [0,1] \to G_{E_0}^{c+\varepsilon} \cup \Upsilon^*)$ such that*

(a) $\eta(t, G_{E_0}^d \cup \Upsilon^*) \subset G_{E_0}^d \cup \Upsilon^*$ for all $d \leq c+\varepsilon, t \in [0,1]$.
(b) $\eta(1, G_{E_0}^{c+\varepsilon} \cup \Upsilon^*) \subset G_{E_0}^{c-\varepsilon} \cup \Upsilon^*$.

Let

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_k^* < \cdots \tag{7.36}$$

denote the Dirichlet eigenvalues of the operator $-\Delta - f'(x,0)$ on Ω.

Theorem 7.24. *Assume that (D_1)–(D_4) hold and $\lambda_2^* > 0$. Then (7.23) has a sign-changing solution u^* with the following properties.*

(1) $G(u^*) = \inf_M G$ (M is defined in (7.27)); u^* has precisely two nodal domains.
(2) u^* has Morse index 2.
(3) If $u < u^*$ is a subsolution of (7.23), then $u \leq 0$.
(4) If $u > u^*$ is a supersolution of (7.23), then $u \geq 0$.

Proof. By (D_4), it is routine to show that

$$\lim_{t\to\infty} G(tu) = -\infty \tag{7.37}$$

for every $u \in E\backslash\{0\}$. Note that any critical point $u \notin \Upsilon^*$ of G is a minimal element of Υ^- and a maximal element of Υ^+; that is, it has the properties (3) and (4) of Theorem 7.24.

Let $e_1 \in \mathcal{P}_0$ be the normalized first Dirichlet eigenfunction of $-\Delta u - f'(x,0)$ on Ω. Because $\lambda_2 > 0$, there exists $r > 0$ and a $\mathbf{C}^1$-map $\kappa^* : \{e_1\}^\perp \cap B_r(0) \to \{e_1\}$ such that for every $u = w + \kappa^*(w) \in Graph(\kappa^*)$ we have that $\eta(t,u) \to 0$ as $t \to \infty$. In fact, if $\lambda_1 > 0$ we may take $\kappa^* \equiv 0$, if $\lambda_1 \leq 0$, $Graph(\kappa^*)$ is the E-local stable manifold of 0. Set

$$S^* := \{u = w + \kappa^*(w) : w \in \{e_1\}^\perp, \|w\|_E = r\} \subset E.$$

Note that

$$\alpha := \inf_{S^*} G > 0. \tag{7.38}$$

Furthermore, by Lemma 4.5 of Bartsch [30] we have that

$$S^* \cap \Upsilon^* = \emptyset. \tag{7.39}$$

Put $\gamma := \alpha/2$ and consider the inclusion

$$j_c : \quad (G_{E_0}^c \cup \Upsilon^*, G_{E_0}^\gamma \cup \Upsilon^*) \rightsquigarrow (E, E\backslash S^*), \qquad \forall c \geq \gamma,$$

which is well defined by (7.38). It induces a homeomorphism

$$j_c^* : H^2(E, E\backslash S^*) \to H^2(G_{E_0}^c \cup \Upsilon^*, G_{E_0}^\gamma \cup \Upsilon^*).$$

Here and in the following $H^*(C, D)$ denotes the Alexander–Spanier cohomology of the pair $D \subset C$ with the integer coefficients. We claim

$$H^2(E, E\backslash S^*) \cong \mathbf{Z}. \tag{7.40}$$

For this, we set

$$E_1 := \mathbf{R}e_1, \qquad S_r E_1^\perp := \{u \in E_1^\perp : \|u\|_E = r\}.$$

Note that the pair $(E, E\backslash S^*)$ is a homeomorphism to the pair $(E, E\backslash S_r E_1^\perp)$. Hence

$$H^2(E, E\backslash S^*) \cong H^2(E, E\backslash S_r E_1^\perp).$$

Now the pair $(E, E\backslash S_r E_1^\perp)$ is the same as the product pair

$$(E_1, E_1\backslash\{0\}) \times (E_1^\perp, E_1^\perp\backslash S_r E_1^\perp).$$

The Künneth theorem shows that

$$H^2(E, E\backslash S_r E_1^\perp) \cong H^1(E_1^\perp, E_1^\perp\backslash S_r E_1^\perp) \cong \tilde{H}^0(E_1^\perp\backslash S_r E_1^\perp) \cong \mathbf{Z},$$

which proves (7.40); the claim is true. Now we define

$$\bar{c} := \inf\{c \geq \gamma : j_c^* \text{ is injective}\}.$$

Then $\bar{c} \geq \alpha$, because $G_{E_0}^c \cup \Upsilon^* \subset E\backslash S^*$; hence $j_c^* = 0$ for $c < \alpha$. Next we show that

$$\bar{c} \leq \beta \tag{7.41}$$

with β given by (7.29). For this, let $\varepsilon > 0$ and choose $u \in M$ such that $G(u) < \beta + \varepsilon/2$. By Lemma 7.16 we have that

$$G(\lambda u_+ + \mu u_-) \leq G(u), \qquad \forall \lambda, \mu \geq 0.$$

By (7.37), there exists $R > 0$ such that

$$G(\lambda u_+ + \mu u_-) \leq 0, \quad \text{for all } \max\{\lambda, \mu\} \geq R.$$

Approximating $u^+, -u^-$ with suitable functions $v_1, v_2 \in \mathcal{P}_0$, we have that

$$G(\lambda v_1 - \mu v_2) \leq \beta + \varepsilon, \quad \text{for all } 0 \leq \lambda, \mu \leq R,$$

and

$$G(\lambda v_1 - \mu v_2) \leq \gamma, \quad \text{for all } \max\{\lambda, \mu\} \geq R.$$

Define

$$C := \{\lambda v_1 - \mu v_2 : 0 \le \lambda, \mu \le R\},$$

then $C \subset \mathrm{span}\{v_1, v_2\} \subset E_0$ and

$$\partial C := \{\lambda v_1 - \mu v_2 \in C : \min\{\lambda, \mu\} = 0, \text{ or } \max\{\lambda, \mu\} = R\}.$$

We have the following inclusions

$$(C, \partial C) \overset{i}{\rightsquigarrow} (G_{E_0}^{\beta+\varepsilon} \cup \Upsilon^*, G_{E_0}^{\gamma} \cup \Upsilon^*) \overset{j_{\beta+\varepsilon}}{\rightsquigarrow} (E, E\backslash S^*).$$

We claim that the induced map

$$i^* \circ j^*_{\beta+\varepsilon} : H^2(E, E\backslash S^*) \to H^2(C, \partial C) \tag{7.42}$$

is an isomorphism. Using the notation $E_1 = \mathbf{R}e_1$ from above it is easy to construct a homeomorphism

$$h : (E, E\backslash S^*) \to (E, E\backslash S_r E_1^{\perp})$$

such that $h \circ j_{\beta+\varepsilon} \circ i$ is homotopic to the inclusion

$$i_0 : (C, \partial C) \rightsquigarrow (E, E\backslash S_r E_1^{\perp}).$$

Let $e_2 \in E_1^{\perp}$ with $\|e_2\|_E = 1$ and define

$$C_1 := \{\lambda e_1 + \mu e_2 : |\lambda| \le R, \mu \in [0, R]\};$$

then $C_1 = B_R E_1 \times [0, R]e_2$ and

$$\partial C_1 = \{\lambda e_1 + \mu e_2 : |\lambda| = R \quad \text{or} \quad \mu \in \{0, R\}\}.$$

Note that $(C, \partial C)$ can be deformed to $(C_1, \partial C_1)$ within $(E, E\backslash S_r E_1^{\perp})$. This implies that $i^* \circ j^*_{\beta+\varepsilon}$ is an isomorphism if and only if the inclusion

$$i_1 : (C_1, \partial C_1) \rightsquigarrow (E, E\backslash S_r E_1^{\perp})$$

induces an isomorphism. Now

$$(C_1, \partial C_1) \cong (B_R E_1, S_R E_1) \times ([0, R]e_2, \{0, Re_2\})$$

and

$$(E, E\backslash S_r E_1^{\perp}) \cong (E_1, E_1\backslash\{0\}) \times (E_1^{\perp}, E_1^{\perp}\backslash S_r E_1^{\perp}).$$

Because the inclusions

$$(B_R E_1, S_R E_1) \rightsquigarrow (E_1, E_1\backslash\{0\})$$

and

$$([0, R]e_2, \{0, Re_2\}) \rightsquigarrow (E_1^{\perp}, E_1^{\perp}\backslash S_r E_1^{\perp})$$

induce isomorphisms on cohomology levels, the claim of (7.42) follows by the naturality of the Künneth maps.

Now because $i^* \circ j^*_{\beta+\varepsilon}$ is an isomorphism, $j^*_{\beta+\varepsilon}$ is an injective and thus $\bar{c} \leq \beta + \varepsilon$. Hence the claim of (7.41) is true. Now we prove that

$$\mathcal{K}_{\bar{c}} \not\subset \Upsilon^*. \tag{7.43}$$

By negation, if $\mathcal{K}_{\bar{c}} \subset \Upsilon^*$, then by Lemma 7.23, there is an $\varepsilon > 0$ and a homotopy

$$h : (G_{E_0}^{\bar{c}+\varepsilon} \cup \Upsilon^*) \times [0,1] \to (G_{E_0}^{\bar{c}+\varepsilon} \cup \Upsilon^*)$$

such that $h_1^* \circ j^*_{\bar{c}-\varepsilon} = j^*_{\bar{c}+\varepsilon}$, where

$$h_1^* : \quad H^2(G_{E_0}^{\bar{c}-\varepsilon} \cup \Upsilon^*, G_{E_0}^{\gamma} \cup \Upsilon^*) \to H^2(G_{E_0}^{\bar{c}+\varepsilon} \cup \Upsilon^*, G_{E_0}^{\gamma} \cup \Upsilon^*)$$

is induced by

$$h_1 : (G_{E_0}^{\bar{c}+\varepsilon} \cup \Upsilon^*, G_{E_0}^{\gamma} \cup \Upsilon^*) \to (G_{E_0}^{\bar{c}-\varepsilon} \cup \Upsilon^*, G_{E_0}^{\gamma} \cup \Upsilon^*).$$

Hence, because $j^*_{\bar{c}+\varepsilon}$ is injective, $j^*_{\bar{c}-\varepsilon}$ has to be injective as well. This, however, contradicts the definition of $\bar{c}$ and thus (7.43) is proved.

Take $u^* \in \mathcal{K}_{\bar{c}}, u^* \notin \Upsilon^*$. Then u^* is a sign-changing solution of (7.23) having the properties (3) and (4) of Theorem 7.24. In particular, $u^* \in M$ and therefore, $\bar{c} = G(u^*) \geq \beta$. In fact, the equality holds by (7.41) and hence the remaining properties (1) and (2) are established by Theorems 7.17 and 7.22. □

Theorem 7.25. *Assume that (D_1)–(D_4) hold and that f is odd in u. Then there exists a sequence of distinct solutions $\pm u_k, k \geq \min\{l, \lambda_l > 0\}$ of (7.23) with the following properties.*

(1) $\|u_k\| \to \infty$ as $k \to \infty$.
(2) u_k is sign-changing for $k \geq 2$.
(3) u_k has at most k nodal domains.
(4) If $u < u_k$ is a subsolution of (7.23), then $u \leq 0$.
(5) If $u > u_k$ is a supersolution of (7.23), then $u \geq 0$.

Proof. We set

$$m_0 := \min\{l : \lambda_l > 0\} - 1.$$

If $m_0 > 0$, then we put $W := V^\perp$, where V is the generalized Dirichlet eigenspace of $-\Delta - f'(x,0)$ associated with the eigenvalues $\lambda_1, \ldots, \lambda_{m_0}$. If $m_0 = 0$ we set $W := \{e_1\}^\perp$, where e_1 is the eigenvector of λ_1. Then $d_0 :=$ codim $W = \max\{1, m_0\}$. By the stable mainifold theorem there exists a Lipschitz continuous map $g^* : B_r W = W \cap B_r(0) \to W^\perp$ for $r > 0$ small enough such that

$$S^* := \{u = g^*(w) + w : w \in W, \|w\|_E = r\} \tag{7.44}$$

is contained in the local stable manifold of 0. Then

$$S^* \cap \Upsilon^* = \emptyset. \tag{7.45}$$

Recall that G is even; then $S^* = -S^*$ and then g^* is odd. Let h^* denote the Borel cohomology for the group $\mathbf{Z}/2$ with coefficient ring $h^*(pt) \cong \mathcal{F}_2[\omega]$. If $B \subset \Upsilon^*$ are $\mathbf{Z}/2$-spaces, $A' \subset \Upsilon^*, B' \subset B$ are invariant subspaces and $\xi \in h^*(\Upsilon^*, B)$; then we write $\xi|_{(A',B')}$ for the image of ξ under the homeomorphism $h^*(\Upsilon^*, B) \to h^*(A', B')$ induced from the inclusion. Letting

$$\alpha := \frac{1}{2} \inf_{S^*} G > 0$$

and using (7.45), we have an inclusion

$$j_c : (G^c_{E_0} \cup \Upsilon^*, G^c_{E_0} \cup \Upsilon^*) \rightsquigarrow (E_0, E_0 \backslash S^*) \overset{\simeq}{\rightsquigarrow} (E, E \backslash S^*) \tag{7.46}$$

for $c \geq \alpha$. According to Lemma 6.1 of Bartsch [30], there exists an element $\psi \in h^{d_0+1}(E, E\backslash S^*)$ with the following property.

If $R > 0$ such that $S^* \subset \ \mathrm{int}_E B_R(0)$ and if $Y \subset E$ is a finite-dimensional subspace with $d = \dim Y > \mathrm{codim}\ W = d_0$, then

$$0 \neq \omega^{d-d_0-1} \cdot \psi\big|_{(B_RY, \{0\} \cup S_RY)} \in h^d(B_RY, \{0\} \cup S_RY). \tag{7.47}$$

Using this cohomology class we may consider the values

$$c_k := \inf\{c \geq \alpha : j_c^*(\omega^{k-d_0-1} \cdot \psi) \neq 0 \in h^k(G^c_{E_0} \cup \Upsilon^*, G^c_{E_0} \cup \Upsilon^*)\} \tag{7.48}$$

for all $k \geq d_0 + 1$. In Bartsch [30], it is shown that there is a sequence of critical points $(u_k)_{k \geq d_0+1}$ of G satisfying Properties (1)–(4) of Theorem 7.25 and such that $G(u_k) = c_k$ for all $k \geq d_0 + 1$. In view of Theorem 7.17, it suffices to show that

$$c_k \leq \beta_k, \qquad \forall k \geq d_0 + 1. \tag{7.49}$$

We fix k. By (7.37), for any given k-dimensional subspace $Y \subset E_0$, we may find a positive number $R > 0$ such that $G(u) \leq 0, \forall u \in Y, \|u\| \geq R$. Hence for $\beta_0 := \max_Y G$, we obtain that

$$(B_RY, \{0\} \cup S_RY) \subset (G^\beta_{E_0} \cup \Upsilon^*, G^\alpha_{E_0} \cup \Upsilon^*) \subset (E_0, E_0 \backslash S^*).$$

Combining (7.47) and (7.48), it follows that $c_k \leq \beta$. We finish the proof of the theorem by recalling that $E_0 \subset E$ is dense. □

Notes and Comments. The results of this section are due to Bartsch and Weth [45]. Theorem 7.24 improves a result of Bartsch [30] (see also Bartsch et al. [31]) as well as Castro et al. [80]. In [80], no extremality properties such as (3) and (4) of Theorem 7.24 are studied. In Bartsch [30], properties (3) and (4) of Theorem 7.24 are obtained, but (1) and (2) could only be proved by assuming that all sign-changing solutions are isolated which can almost not be checked. In particular, in the proof of Theorem 7.24, one need not calculate the Morse index. Theorem 7.25 is an extension of Theorems 1.1 and 7.3 of Bartsch [30], where sign-changing solutions are supposed to be isolated. Finally, as observed in Bartsch [30], the ideas of [30] and of this chapter can be used to prove the existence of connecting orbits between the sign-changing stationary solutions for the parabolic equation.

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