QED

Incomplete notes by Ingemar Bengtsson, Fall 2003 (including minor additions by Martin Ericsson). The course is defined by Mandl & Shaw: *Quantum Field Theory* (1984), not by these notes. Other good albeit old fashioned books include Bjorken & Drell: *Relativistic Quantum Mechanics* (1964) and *Relativistic Quantum Fields* (1965) as well as Weinberg: *The Quantum Theory of Fields I* (1995). A more modern book is Peskin and Schroeder: *An Introduction to Quantum Field Theory* (1995).

As long as the Feynman rules are known, and the resulting S-matrix can be shown to be satisfactory, one tends to say, with Alfred E. Neumann: "What, me worry?".

Martinus Veltman

PREREQUISITES

I expect you to know certain things, including elementary facts from analytical mechanics, electrodynamics and quantum mechanics. Not many things—the following list is more or less complete—but if you do not know the things on the list you will probably be in trouble.

First of all you are supposed to recognize an action when you see one, such as

$$S = \int \frac{1}{2}\dot{x}^2 - V(x) \ , \tag{1}$$

and to see immediately what equations of motion you get when varying the action, ignoring time derivatives (why?). Namely in this case

$$\delta S = -\int \delta x(\ddot{x} + \frac{dV}{dx}) \ . \tag{2}$$

You are also supposed to have an idea about classical electrodynamics so that you recognize Maxwell's equations in tensor form,

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \ . \tag{3}$$

You will also know that one can formulate the theory in terms of a vector potential. Indeed this is necessary in order to write down a simple action that leads to Maxwell's equations when its variation is set to zero. The drawback is that you cannot use the theory until you have understood something about gauge invariance, the ambiguity that is always present in the vector potential:

$$\delta A_\alpha = \partial_\alpha \Lambda \ . \tag{4}$$

I expect you to have sufficient experience with relativity so that you know that you have to live with the fact that some people use a spacelike metric $(-+++)$, and some a timelike $(+---)$.

Some background in quantum mechanics is also required. Notably you should know about the harmonic oscillator and its spectrum,

$$H = \frac{1}{2}(p^2 + \omega^2 x^2) = \omega(a^\dagger a + \frac{1}{2}) \ , \tag{5}$$

where

$$x(t) = \frac{1}{\sqrt{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}) \ . \tag{6}$$

This is the general solution both in classical and quantum mechanics (in the Heisenberg picture where the operators are time dependent). Note that we

can look at this as a decomposition of x into positive and negative frequency parts; by convention the positive frequency part is the one where the time dependence is through $e^{-i\omega t}$. It is interesting that this decomposition brings complex numbers in already at the classical stage of the theory.

You should know about the Schrödinger and Heisenberg pictures,

$$\langle \Psi_S(t)|x_S|\Psi_S(t)\rangle = \langle \Psi_S(0)|e^{iHt}x_S e^{-iHt}|\Psi_S(0)\rangle = \langle \Psi_H|x_H(t)|\Psi_H\rangle \ . \tag{7}$$

The time dependence can be shifted around, provided that the time dependence of the matrix elements—that carry all the predictive content of the theory—stays the same. The list of things you have to know is now at an end.

Time dependent perturbation theory may escape your memory, so let me remind you. (In the end this course will become simply an involved exercise in time dependent perturbation theory.) The first step is to split the Hamiltonian in two parts, where the first — H_0 — is something we can handle, say the harmonic oscillator for definiteness. So

$$i\partial_t \Psi_S = H\Psi_S = (H_0 + H_I)\Psi_S \ . \tag{8}$$

Now we go to the interaction or Dirac picture through

$$\Psi_I(t) \equiv e^{iH_0 t}\Psi_S(t) \qquad\qquad x_I(t) \equiv e^{iH_0 t}x_S e^{-iH_0 t} \ . \tag{9}$$

So the operators evolve as "free", that is harmonic oscillator, operators. To compute expectation values we also need to solve for the wave functions; so we look for a formal, perturbative solution of

$$i\partial_t \Psi_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}\Psi_I(t) \equiv H_I(t)\Psi_I(t) \ . \tag{10}$$

Operators such as $H_I(t)$ are now time dependent, although this time dependence is not the "correct" one prescribed by the dynamics of the theory!

Equivalently, if we define the operator

$$\Psi_I(t) = U(t,t_0)\Psi_I(0) \ , \tag{11}$$

we must solve the equation

$$i\partial_t U(t,t_0) = H_I(t)U(t,t_0) \ ; \qquad U(t_0,t_0) = 1 \ . \tag{12}$$

This is not straightforward essentially because $[H_I(t_1), H_I(t_2)] \neq 0$ due to the time dependence. However, one can proceed iteratively:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1)U(t_1,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1) + \tag{13}$$

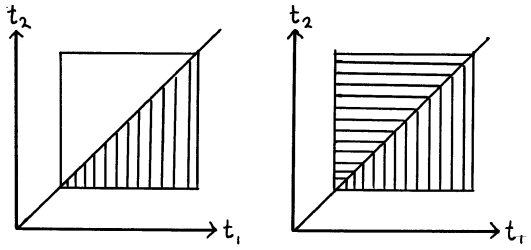

Figure 1: Time ordered perturbation theory: Changing the region of integration.

$$+(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \ ...$$

The first equality is exact and captures the full content of eq. (12), including the initial condition. The second gives a useful perturbative expansion. We now employ a clever trick that will turn out to be absolutely essential if we wish to organize the calculation intelligibly in the relativistic quantum field theory case. It is called time ordering, and uses the observation that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T(H_I(t_1) H_I(t_2)) \ . \tag{14}$$

Instead of integrating over a triangle in the (t_1, t_2) plane we integrate over a square, putting the operators in a particular time order with the "earliest" one occurring first. Actually this is so important that I report the two intermediate steps: First we change the order of integration and then we relabel the integration variables,

$$\int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I(t_1) H_I(t_2)_{|\ t_1 > t_2} = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2) H_I(t_1)_{|\ t_2 > t_1} \ . \tag{15}$$

The trick is easily generalized to higher orders in the operators, so our solution of the Schrödinger equation can be written (formally!) as

$$\begin{aligned} U(t, t_0) = 1 - i \int_{t_0}^{t} H_I(t_1) dt_1 + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T(H_I(t_1) H_I(t_2)) + \ ... \\ \equiv T e^{-i \int_{t_0}^{t} dt H_I(t)} \ . \end{aligned} \tag{16}$$

Each term in the Taylor expansion of the exponential is time ordered. We will deal with this type of expansion in concrete detail later. For the moment, keep

in mind that our interest in time dependent perturbation expansions explains a certain predilection towards time ordered products of operators that will show itself at an early stage.

A *caveat* should also be mentioned. Perturbation theory assumes that we know what Hilbert space we are working in. We know this because the Hilbert space is chosen so that we can represent the algebra of the operators that we have, say x and p. And here we can rely on the Stone-von Neumann theorem, which states that there is (up to unitary equivalence) only one (decent) representation of the Heisenberg algebra

$$[x, p] = i \ . \tag{17}$$

This is true also if we have N independent Heisenberg algebras in the game, say for a point particle in $N = 3$ dimensions. However, it is not true when $N \to \infty$ (basically because representations that one expects to be unitarily equivalent fail to be so), and this is the situation we are in when we deal with quantum field theory. This means that the perturbation expansion that we will embark on comes without any guarantee that it will make any kind of mathematical sense, so that we have no reason to complain when we encounter divergent integrals at an early stage. The attitude that I will take in these notes is the historical one: It all works very well following some trickery when we renormalize the theory. All sorts of more sophisticated responses to this dilemma have been advanced but they are beyond our scope here.

WHAT IS A FIELD THEORY?

A field theory is in a definite sense a dynamical system with an infinite number of degrees of freedom. To see this we begin with a study of N degrees of freedom, say a set of coupled harmonic oscillators given by the quadratic Lagrangian

$$L = \sum_{n=1}^{N} \frac{1}{2}(\dot{\phi}_n^2 - \Omega^2(\phi_{n+1} - \phi_n)^2 - m^2\phi_n^2) \ . \tag{18}$$

The interactions are between nearest neighbours only and the Lagrangian was chosen so that the equations of motion are linear;

$$\ddot{\phi}_n = \Omega^2(\phi_{n+1} - 2\phi_n + \phi_{n-1}) - m^2\phi_n \ . \tag{19}$$

It is convenient to adopt periodic boundary conditions so that the index n can be counted modulo N. The standard philosophy is that the lattice is so large that in the end boundary conditions do not matter much. We can think of the N lattice points as representing positions in space, and we also observe that if $m^2 = 0$ then $\ddot{\phi}_n = 0$ means that the value assumed by ϕ_n equals the average of that assumed by its neighbours. In effect what we have in this case is an approximate description of a sound wave, with ϕ_n standing for the density of the gas at the point labelled n.

Because of linearity the equations are easily solved with a Fourier transformation, that is we define

$$\phi_n = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \Phi_k e^{\frac{i2\pi kn}{N}} \quad \Leftrightarrow \quad \Phi_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \phi_n e^{-\frac{i2\pi kn}{N}} \ . \tag{20}$$

This has the effect of diagonalizing the equations of motion,

$$\ddot{\Phi}_k = -\Omega^2(2\sin\frac{\pi k}{N})^2\Phi_k - m^2\Phi_k \ . \tag{21}$$

Each mode Φ_k can be regarded as a harmonic oscillator. The equations of motion—classical or quantum—are therefore solved by

$$\Phi_k = \frac{1}{\sqrt{2\omega_k}}(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}) \ ; \quad \omega_k^2 \equiv \Omega^2(2\sin\frac{\pi k}{N})^2 + m^2 \ . \tag{22}$$

The momentum canonically conjugate to ϕ_n is $\pi_n = \dot{\phi}_n$. The quantum version of the theory is obtained by postulating the commutation relations

$$[\phi_n, \pi_m] = i\delta_{n,m} \quad \Leftrightarrow \quad [a_k, a_l^\dagger] = \delta_{k,-l} \ . \tag{23}$$

Here $a_{-k} \equiv a_{N-k}$ and so on. The Hamiltonian, when expressed in creation and annihilation operators, is

$$H = \frac{1}{2}\sum_{k=1}^{N} \omega_k (a_k a_{-k}^\dagger + a_k^\dagger a_{-k}) = \sum_{k=1}^{N} \omega_k (a_k^\dagger a_{-k} + \frac{1}{2}) \ . \tag{24}$$

It is a good thing to show this explicitly. The spectrum of this Hamiltonian is easily worked out. The Hilbert space can be built up from a "vacuum" vector $|0\rangle$ and is spanned by the vectors

$$|n_1, n_2, \ \ldots \ , n_N\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \ \ldots \ (a_N^\dagger)^{n_N}}{\sqrt{n_1! n_2! \ \ldots \ n_N!}}|0\rangle \ . \tag{25}$$

To give this Hilbert space a physical interpretation, when we think of this as the quantum version of the theory of sound waves, we say that a state with the "occupation number" $N = \sum_k n_k$ contains N "phonons". These are "particles" in the sense that they can be counted; we can define the number operator

$$N = \sum_{k=1}^{N} a_k^\dagger a_{-k} \tag{26}$$

whose eigenvalues are the occupation numbers of the states,

$$N|n_1, n_2, \ \ldots \ , n_N\rangle = (n_1 + n_2 + \ \ldots \ + n_N)|n_1, n_2, \ \ldots \ , n_N\rangle \ . \tag{27}$$

The phonons also carry energy in discrete amounts. To some extent we can think of them as elementary "sound particles", but we have no particular justification for thinking of phonons as particles localized to a point, or anything like that.

Now what happens if we take the continuum limit, that is to say if we let the lattice spacing a tend to zero, keeping the length of the interval L fixed so that the number of points N goes to infinity? (If we did not believe in atoms we would think that this gives a more exact description of sound waves.) Before taking the limit $a \to 0$ we rescale things a little. In effect we set

$$\sum_n \to \frac{1}{a}\int dx \qquad \phi_n \to \sqrt{a}\phi(x) \qquad \Omega \to \frac{c}{a} \tag{28}$$

and insist that $\phi(x)$ and c shall stay finite when we take the limit. Then we find that

$$L = \frac{1}{2}\sum_n (\dot{\phi}_n^2 - \Omega^2(\phi_{n+1} - \phi_n)^2) \to L = \frac{1}{2}\int dx (\dot{\phi}^2 - c^2\phi'^2) \ , \tag{29}$$

where ϕ' is the spatial derivative of $\phi(x)$, which in itself is a classical field obeying the field equation

$$\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right)\phi(x) \equiv \Box\phi(x) = 0 \ . \tag{30}$$

As our definition we say that a classical field is some kind of function (not necessarily a scalar) of space and time. An important point that this exercise has is that it shows that, in field theory, the role of space is quite different from that of time. The coordinate x plays the same role as that played by the index i in the discrete dynamical system. It is an index, in the sense that it is labelling an infinite set of degrees of freedom. The field equation is *not* analogous to the Schrödinger equation, where x enters in the first place as an operator.

Of course our scalar field can be a function of a three dimensional space just as well as a one dimensional, and is then said to describe $(\infty)^3$ degrees of freedom, one for each spatial point. There will be an action S, a Lagrangian L that you never think of, and a Lagrangian density $\mathcal{L}$ that you think about often since

$$S = \int dt L = \int d^4x \mathcal{L} = -\frac{1}{2}\int d^4x (\partial_\alpha\phi\partial^\alpha\phi + m^2\phi^2) \ . \tag{31}$$

I am using a spacelike metric. There will be a canonical momentum (more precisely, a canonical momentum density)

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{32}$$

and a positive definite Hamiltonian

$$H = \frac{1}{2}\int d^3x (\pi^2 + \partial_i\phi\partial_i\phi + m^2\phi^2) \ . \tag{33}$$

The infinite number of degrees of freedom do not affect the formal structure of the Hamiltionian system very much. There will be canonical Poisson brackets

$$\{\phi(x), \pi(y)\} = \delta^{(3)}(x, y) \tag{34}$$

in terms of which Hamilton's equations of motion take the form

$$\dot{\phi}(x) = \{\phi(x), H\} = \pi \qquad \dot{\pi}(x) = \{\pi(x), H\} = \triangle\phi - m^2\phi \ . \tag{35}$$

(I use $\triangle$ to denote the Laplacian.) There are certain subtle problems that are specific to field theory though, already at the classical stage. In principle we must formulate conditions on the allowed field configurations that ensure that the integral that goes into the definition of H converges, we must check that these conditions hold at all times, and so on. One strategy is to consider the field theory in a finite volume and impose periodic boundary conditions. ("The Universe is a box, anyway.") Another is to work in an infinite volume and insist that the fields fall to zero at infinity, sufficiently fast to ensure that all integrals

exist—with the understanding that this is not a cosmological assumption, it is part of a definition of what we mean by an "isolated system". In particle physics, infinity is where the detectors are.

As long as the action is such that the field equations are linear they can again be solved by means of a Fourier transformation. If we define the theory in a box of volume V we find that the general solution of the Klein-Gordon equation is

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}}(a(k)e^{i(k\cdot x-\omega_k t)} + a^\dagger(k)e^{-i(k\cdot x-\omega_k t)}) \ . \tag{36}$$

Here I am using the conventions of Mandl and Shaw. Because we are in a box $k_i = (k_x, k_y, k_z)$ runs over a discrete set of values only. If we use an infinite volume we get

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2\omega_k}(a(k)e^{i(k\cdot x-\omega_k t)} + a^\dagger(k)e^{-i(k\cdot x-\omega_k t)}) \ . \tag{37}$$

Here I am using my own conventions and k_i takes arbitrary real values. In both cases $\omega_k = \sqrt{k^2+m^2}$, a positive function. The function $a(k)$ can be chosen at will so this is the general solution of the Klein-Gordon equation. It is real provided $a^\dagger(k)$ is the complex conjugate of $a(k)$.

Since the complex function $a(k)$ can be chosen at will we must choose $2 \times (\infty)^3$ numbers to specify the solution, which means that there is one degree of freedom (with "position" and "velocity" to be specified) per spatial point. But note that $a(k)$ is actually a highly non-local function of the field since we must integrate over all space to get it. This observation will become important later on when we use $a(k)$ to extract a particle interpretation for the quantum field. It is also worth noting that eq. (37) gives a (non-local) split of the field into (complex) positive and negative frequency parts,

$$\phi = \phi^{(+)} + \phi^{(-)} \ , \tag{38}$$

where the positive frequency part $\phi^{(+)}$ is the one that is linear in the annihilation operators.

An advantage with the infinite volume formulation is that Lorentz invariance can be made obvious in a simple way. At first sight it seems that the three dimensional integral in the solution is somewhat against the spirit of a four dimensional spacetime, but second thought reveals that

$$\int \frac{d^3k}{2\omega_k} = \int d^4k\delta[(\omega_k + k^0)(\omega_k - k^0)]\theta(k^0) = \int d^4k\delta(k^2+m^2)\theta(k^0) \ . \tag{39}$$

To prove this note that if a function $f(x)$ has simple zeroes at $x = x_i$ then

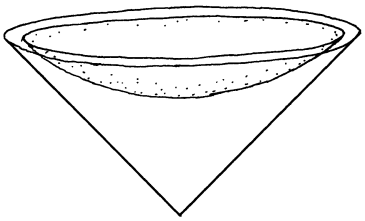

Figure 2: The mass hyperboloid in momentum space.

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{\left|\frac{df}{dx}\right|_{x=x_i}} \ . \tag{40}$$

The measure is now in manifestly Lorentz invariant form. It is seen that the integration is really over the future mass hyperboloid $k^2 + m^2 = 0$ in momentum space, and this hyperboloid is carried onto itself under Lorentz transformations.

QUANTUM THEORY OF THE FREE SCALAR FIELD

We can now pass over to the quantum theory of a free scalar field in a straightforward way. As in the discrete case we can proceed by turning the functions $a(k), a^\dagger(k)$ into hermitian conjugate operators obeying suitable commutation relations. We choose

$$[a(k), a^\dagger(k')] = 2\omega_k \delta^{(3)}(k, k') \ . \tag{41}$$

The somewhat peculiar factor in front of the delta function is allowed because ω_k is positive. My conventions are rigged in such a way that Lorentz invariance is easily discussed. Now the canonical momentum at $t = 0$ is simply the time derivative of ϕ at time zero,

$$\pi(x)_{|t=0} = \frac{i}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2} (a^\dagger(k) e^{-ik\cdot x} - a(k) e^{ik\cdot x}) \ . \tag{42}$$

It is then easy to verify that, at $t = 0$,

$$[\phi(x), \pi(y)] = i\delta^{(3)}(x, y) \ . \tag{43}$$

These are the canonical commutation relations.

Now we need a Hilbert space. First we define the vacuum vector by

$$|0\rangle \ : \qquad a(k)|0\rangle = 0 \ . \tag{44}$$

There is a problem immediately because $a^\dagger(k)$ is not an operator since it takes us out of the Hilbert space we wish to define;

$$||a^\dagger(k)|0\rangle||^2 = \langle 0|a(k)a^\dagger(k)|0\rangle = 2\omega_k \langle 0|0\rangle \delta(k, k) = \infty \ . \tag{45}$$

This is expected. Eigenstates of the momentum do not exist in elementary quantum mechanics either. If we ignore this we can define the "one-particle states" as

$$|k\rangle \equiv a^\dagger(k)|0\rangle \ . \tag{46}$$

If we want to be rigourous we can define wave packets and smeared operators $a^\dagger[\tilde{f}]$ by

$$|\Psi\rangle = \int \frac{d^3k}{2\omega_k} \tilde{f}(k)|k\rangle = a^\dagger[\tilde{f}]|0\rangle \ . \tag{47}$$

We say that $a^\dagger(k)$ is an operator valued distribution. Similarly $\phi(x)$ is an operator valued distribution; $\phi(x)$ is not an operator but $\phi[f] \equiv \int d^4x f(x)\phi(x)$ is an operator for suitable test functions $f(x)$. At the moment this complication

does not matter but it will soon cause problems because it means that objects such as $\phi(x)\phi(x)$—products of fields at the same point—do not exist.

To continue, we can define a Hilbert space called Fock space whose basis vectors are

$$|0\rangle \ , \ |k\rangle \ , \ |k_1, k_2\rangle \ , \ ... \tag{48}$$

that is to say that it is a direct sum of n particle states with all finite n present. Note that the "particles" are bosons. Thus the two-particle states obey

$$|k_1, k_2\rangle = a^\dagger(k_1)a^\dagger(k_2)|0\rangle = a^\dagger(k_2)a^\dagger(k_1)|0\rangle = |k_2, k_1\rangle \ . \tag{49}$$

A general state is a superposition of n-particle states, schematically

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + \ ... \ . \tag{50}$$

We will soon substantiate the interpretation that the "n-particle states" are indeed n-particle states, but before we do so let us say something about the field aspects. First we observe that (schematically)

$$\langle n|\phi(x)|n\rangle \sim \langle n|n-1\rangle + \langle n|n+1\rangle = 0 \ . \tag{51}$$

Actually one can construct so called coherent states for which $\langle\phi(x)\rangle = \phi_{cl}(x)$ is a function behaving as a classical field, but these states are superpositions of n-particle states with all n present. In this sense the field and particle aspects of quantum field theory are complementary. We may also observe that

$$\langle 0|\phi^2(x)|0\rangle - (\langle 0|\phi(x)|0\rangle)^2 = \infty - 0 = \infty \ . \tag{52}$$

This reflects the fact that ϕ^2 is not defined. If we smear the fields with test functions we get a well defined answer, but we find—by choosing test functions that are increasingly peaked—that there are violent "vacuum fluctuations" present.

Let us now look at some important operators in the theory. At first sight the Hamiltonian should be

$$H = \frac{1}{2}\int d^3x \partial_t\phi\partial_t\phi + \partial_i\phi\partial_i\phi + m^2\phi^2 = \frac{1}{2}\int \frac{d^3k}{2\omega_k}\omega_k(a(k)a^\dagger(k) + a^\dagger(k)a(k)) \ . \tag{53}$$

Essentially because $\phi^2(x)$ does not exist this is not an acceptable operator in Fock space. We want all our operators to be normal ordered, that is to say that the annihilation operators should occur to the right of the creation operators. But if we normal order this expression for the Hamiltonian it diverges. This is not too bad though because the divergence is a c-number that does not affect Heisenberg's equations of motion. So we simply omit it and define—note the colons that stand for normal ordering—

$$H = \frac{1}{2}\int d^3x : (\partial_t\phi\partial_t\phi + \partial_i\phi\partial_i\phi + m^2\phi^2) : \equiv \int \frac{d^3k}{2\omega_k}\omega_k a^\dagger(k)a(k) \ . \qquad (54)$$

This is well defined in Fock space.

Another important operator is the Noether charge corresponding to translations in space. Recall that when the action of a classical field theory is invariant under some particular variation of the fields then there necessarily exists a conserved charge in the theory. For our scalar field theory, defined by the action (31),

$$\delta\phi = \partial_i\phi \quad \Rightarrow \quad \delta S = \int d^4x\partial_i\mathcal{L} = 0 \ . \qquad (55)$$

(By assumption the surface term at spatial infinity vanishes.) This is symmetry under spatial translations. On the other hand when the field equations hold, but if we keep careful track of partial integrations with respect to time, we obtain

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \frac{d}{dt}(\dot{\phi}\delta\phi) = \int_{t_1}^{t_2} dt \int d^3x \frac{d}{dt}(\pi\delta\phi) \ , \qquad (56)$$

regardless of the form of the variation $\delta\phi$. But if we insert our particular variation then δS must vanish. The conclusion is that the quantity

$$P_i = -\int d^3x\pi\partial_i\phi \qquad (57)$$

is independent of the time at which it is evaluated, provided that the field equations hold. Hence $\dot{P}_i = 0$ is an automatic consequence of the latter. It is said to be a conserved Noether charge. A nice thing about it is that if we use it instead of H in Hamilton's equations of motion we find that it generates spatial translations, just as the Hamiltonian generarates time translations:

$$\{\phi(x), P_i\} = -\partial_i\phi(x) \ . \qquad (58)$$

The Noether charge always generates the symmetry transformation that is the reason for its existence in the first place—we have simply gone through a particular case of Noether's theorem that holds for any dynamical system defined by an action invariant under some transformation, regardless of whether the number of degrees of freedom is finite or not.

Our Noether charge is conserved also in the quantum theory where

$$P_i = -\int d^3x\pi\partial_i\phi = \int \frac{d^3k}{2\omega_k}k_i a^\dagger(k)a(k) \ . \qquad (59)$$

This time there are no normal ordering difficulties. There is a third operator that is important to us, although it has some funny properties. It is called the number operator and is defined as

$$N = \int \frac{d^3k}{2\omega_k} a^\dagger(k) a(k) \ . \tag{60}$$

It is again a conserved operator (in our free field theory). What is peculiar about is that it can not be written as a local functional of the fields—the creation and annihilation operators being highly non-local functions of the fields. The best we can do is to write it as

$$N = i \int d^3x (\phi^{(-)} \dot{\phi}^{(+)} - \dot{\phi}^{(-)} \phi^{(+)}) \ , \tag{61}$$

using the non-local split of the field into positive and negative frequency parts.

Equipped with these three operators we can finally substantiate the particle interpretation of the theory. First of all

$$H|0\rangle = P_i|0\rangle = N|0\rangle = 0 \ . \tag{62}$$

In this sense the vacuum is empty. Next

$$N|k\rangle = |k\rangle \qquad H|k\rangle = \omega_k|k\rangle \qquad P_i|k\rangle = k_i|k\rangle \ . \tag{63}$$

Hence the one-particle state has the energy and momentum that we expect from a free relativistic particle. The operator N is supposed to count the number of particles, and indeed the eigenvalue of the one-particle state is one. If we need any further verification that one particle is present we could define the angular momentum operator (as the Noether charge corresponding to spatial rotations) and check that the particle behaves as we expect a spin zero particle to behave.

We can now go on in this way. For the two-particle states

$$N|k_1k_2\rangle = 2|k_1k_2\rangle \tag{64}$$

$$H|k_1k_2\rangle = (\omega_{k_1} + \omega_{k_2})|k_1k_2\rangle \qquad P_i|k_1k_2\rangle = (k_1 + k_2)_i|k_1k_2\rangle \ . \tag{65}$$

These are the eigenvalues expected for two non-interacting particles. For the n-particle states we again get what we expect. Hence the n-particle states are indeed states of n particles that can be counted (with the operator N) and whose energies and momenta are additive (as they should be for non-interacting particles). The particle interpretation of quantum field theory is established. On the other hand the "particles" do have some counterintuitive properties, at least if you insist that a particle is something that is localized to a point. We have already seen that the definition of the creation operator is non-local, and so is the definition of the operator N. Indeed at this point we know nothing about the spacetime properties of our particles.

If we consider two real scalar fields collected into one complex scalar ϕ (with complex conjugate $\bar{\phi}$) something new and interesting happens. The action is

$$S = -\int d^4x(\partial_\alpha \bar{\phi}\partial^\alpha \phi + m^2 \bar{\phi}\phi) \tag{66}$$

and the field equation is again just the Klein-Gordon equation, so the two real fields do not interact with each other. But there are twice as many degrees of freedom and the theory admits a new and interesting symmetry transformation, namely

$$\delta\phi = i\epsilon\phi \qquad \Rightarrow \qquad \delta S = 0 \ . \tag{67}$$

(The parameter ϵ is any real number.) The transformation is local, and so is the corresponding conserved Noether charge

$$Q = i\int d^3x(\bar{\phi}\dot{\phi} - \dot{\bar{\phi}}\phi) \ . \tag{68}$$

If we try to couple a scalar field theory to the electromagnetic charge via the vector potential we find that we need a complex valued field to succeed, and then Q will have an interpretation as the electrical charge—an integral of the local charge density.

The general solution of the Klein-Gordon equation now depends on two independent complex numbers $a(k)$ and $b(k)$ and correspondingly there will be two independent pairs of creation and annihilation operators in the quantum theory. Schematically,

$$\phi = \int ae^{-i} + b^\dagger e^i \qquad \bar{\phi} = \int a^\dagger e^i + be^{-i} \ . \tag{69}$$

The number operator (non-local) and the charge operator (local) become, respectively,

$$N = \int \frac{d^3k}{2\omega_k}(a^\dagger(k)a(k) + b^\dagger(k)b(k)) \tag{70}$$

$$Q = \int \frac{d^3k}{2\omega_k}(a^\dagger(k)a(k) - b^\dagger(k)b(k)) \ . \tag{71}$$

All the n-particle states in our Fock space will carry an electrical charge measured by the eigenvalues of Q. Thus

$$Qa^\dagger|0\rangle = a^\dagger|0\rangle \qquad Qb^\dagger|0\rangle = -b^\dagger|0\rangle \ . \tag{72}$$

This is in fact extremely interesting: What we are seeing indicates that it is not possible to write down a quantum field theory that contains negatively charged particles only. We necessarily get anti-particles carrying the opposite charge as well. This indication can in fact be sharpened to a theorem that is a deep consequence of locality and relativistic invariance working together. It also happens to be true (in nature): For each kind of charged particle there is an oppositely charged antiparticle that can exist.

SPACETIME PROPERTIES

We now wish to compute

$$[\phi(x), \phi(y)] = i\Delta(x, y) \tag{73}$$

$$\langle 0|\phi(x)\phi(y)|0\rangle = i\Delta^+(x, y) \tag{74}$$

$$\langle 0|T(\phi(x)\phi(y)|0\rangle = i\Delta_F(x, y) \ . \tag{75}$$

The right hand sides here are c-number distributions known as Green functions, generically denoted by $G(x, y)$—although in a non-linear theory the commutator would be an operator rather than a c-number Green function.

You may wonder why we should compute them? The commutator Green function Δ is clearly of great interest and, if the theory is to make sense, we must show that it vanishes when x and y are spacelike separated so that (roughly speaking) measurements made at spacelike separated points do not interfere. The Wightman function Δ^+ is of interest because it can be shown that it contains all the information in the theory, that is to say that the Hilbert space and the operators can be reconstructed from it (this remains true for interacting theories provided that we include arbitrarily long strings of vacuum expectation values like $\langle 0|\phi(x_1)\phi(x_2) \ ... \ \phi(x_n)|0\rangle$; in the free theory the higher order Wightman functions can be expressed in terms of the second order ones). The Feynman propagator Δ_F is the working horse of time ordered perturbation theory.

Before we compute the Green functions it is helpful to recall that it is allowed to think of the field equation as a matrix equation, where the matrix has continuously many indices. So the "transition" from ordinary matrices to the present case looks like

$$\sum_j C_{ij}V_j \quad \rightarrow \quad \int dx' C(x, x')f(x') \ , \tag{76}$$

and the Klein-Gordon operator can be viewed as a matrix provided that we write

$$C(x, x') = (\Box - m^2)\delta(x, x') \quad \Rightarrow \quad \int dx' C(x, x')\phi(x') = (\Box - m^2)\phi(x) \ . \tag{77}$$

To solve the equation $(\Box - m^2)\phi(x) = j(x)$ it is therefore enough to invert the matrix $C(x, x')$. Its inverse is precisely the inhomogeneous Green function $G(x, x')$;

$$(\Box - m^2)G(x,x') = \delta(x,x') \qquad \Leftrightarrow \qquad \int dx'' C(x,x'')G(x'',x') = \delta(x,x') \ . \quad (78)$$

For the Laplace equation the Green function is unique but for the Klein-Gordon equation we get an entire zoo of Green functions, including homogeneous ones. The various members of the zoo obey either

$$(\Box - m^2)G(x,y) = \begin{array}{lll} \delta^{(4)}(x,y) & \text{inhomogeneous} & \Delta_F \ , \ \Delta_{ret} \\ & & \\ 0 & \text{homogeneous} & \Delta^+ \ , \ \Delta \end{array} \ . \quad (79)$$

The problem is precisely the homogeneous solutions: For the Laplace equation with suitable boundary conditions the only homogeneous solution is zero, while for a wave equation like the Klein-Gordon equation this is not true. In matrix language, the matrix $C(x,x')$ has zero eigenvalues and therefore the inverse is not well defined unless additional conditions are imposed—for instance, the extra condition that the field vanishes to the past of the "disturbance" represented by $j(x)$ leads to the retarded Green function Δ_{ret} familiar from classical electrodynamics. The Feynman propagator is a rather strange animal from a classical point of view and is best characterized by its analyticity properties (as we will see).

We start by deriving some relations between our Green functions. First we define the complex conjugate of the complex Wightman function:

$$\Delta^-(x,y) \equiv (\Delta^+(x,y))^\star = (-i\langle 0|\phi(x)\phi(y)|0\rangle)^\star = -\Delta^+(y,x) \ . \quad (80)$$

Since $[\phi(x),\phi(y)] = \langle 0|[\phi(x),\phi(y)]|0\rangle$ the commutator Green function is

$$\Delta(x,y) = \Delta^+(x,y) - \Delta^+(y,x) = \Delta^+(x,y) + \Delta^-(x,y) = 2\mathrm{Re}[\Delta^+(x,y)] \ . \quad (81)$$

The Feynman propagator is

$$\Delta_F(x,y) = \Theta(t-t')\Delta^+(x,y) - \Theta(-t+t')\Delta^-(x,y) \quad (82)$$

(where $t = x^0$ and $t' = y^0$). It follows that

$$\mathrm{Re}[\Delta_F(x,y)] = \frac{1}{2}\epsilon(t-t')(\Delta^+(x,y) + \Delta^-(x,y)) \quad (83)$$

$$\mathrm{Im}[\Delta_F(x,y)] = \frac{i}{2}(\Delta^-(x,y) - \Delta^+(x,y)) \ . \quad (84)$$

In fact all Green functions can be expressed in terms of the Wightman function and its complex conjugate, or alternatively in terms of the real and imaginary parts of the Feynman propagator.

We would like to calculate the Green functions. Translation invariance actually implies that $G(x,y) = G(x-y)$ for all the Green functions, so it is enough to compute $G(x) \equiv G(x,0)$. In our free field theory (and using the decomposition of the field into negative and positive frequency parts, that is into parts linear in creation and annihilation operators respectively) the Wightman function is

$$\Delta^+(x) = -i\langle 0|\phi(x)\phi(0)|0\rangle = -i\langle 0|\phi^{(+)}(x)\phi^{(-)}(0)|0\rangle = \\ = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{2\omega_k} e^{ik\cdot x} \ . \tag{85}$$

Here the four vector scalar product is $k\cdot x \equiv k_i x_i - k^0 t$ with $k^0 \equiv \omega_k$. But this can be rewritten in an intelligent way. Write

$$-\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2+m^2} = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{(k^0-\omega_k)(k^0+\omega_k)} \ . \tag{86}$$

The integrand is singular at $k^0 = \pm\omega_k$. Choosing a suitable contour in the complex k^0-plane we can perform the integral using the calculus of residues and we obtain

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2+m^2} = -\Delta^+(x) \ , \tag{87}$$

where the choice of contour—in this case, surrounding one of the singularities only—is an essential part of the definition of the integral. It is now evident that

$$(\Box - m^2)\Delta^+(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} = 0 \tag{88}$$

because the integrand is now analytic and the contour can be shrunk to a point. It is also evident that the commutator Green function $\Delta = \Delta^+ + \Delta^-$ is given by the same integral, but this time with a contour that encloses both singularities.

Indeed all Green functions can be written on the form

$$G(x) = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2+m^2} \quad \Rightarrow \quad (\Box - m^2)G(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \ . \tag{89}$$

The integrand of the Green function has poles at $k^0 = \pm\omega_k$ and it is an inhomogeneous or homogeneous Green function depending on whether the contour goes along the real axis or just surrounds one or both of the poles. The integral defining an inhomogeneous Green function can again be evaluated using the calculus of residues because the contour can be closed in the lower half plane when $t > 0$ so that $e^{-ik^0 t} \sim e^{\mathrm{Im}[k^0]t}$ goes to zero when $\mathrm{Im}[k^0]$ is large and negative,

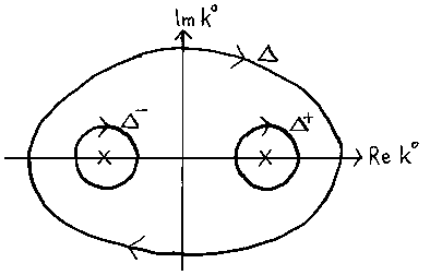

Figure 3: Integration contours for some homogeneous Green functions.

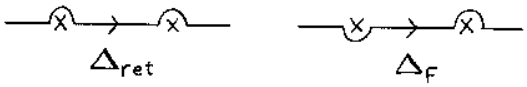

Figure 4: Integration contours for some inhomogeneous Green functions.

and in the upper half plane when $t < 0$. It remains necessary to say whether the contour should pass above, below or through the poles. If it passes above both poles then the integral vanishes for $t < 0$ (since the closed contour surrounds no poles), as appropriate for the retarded Green function Δ_{ret}. A contour that reproduces eq. (82) for the Feynman propagator Δ_F can also be found without much ado.

You may be pleased to know that the Fourier transformation of the Feynman propagator is the object that will mostly concern us later on. But for our present purpose it is interesting to look at the explicit expression in x-space. Let $\sigma \equiv x^2 + y^2 + z^2 - t^2$. After a delicate calculation that I will not repeat one obtains

$$\Delta_F(x) = \frac{im}{8\pi}\frac{1}{\sqrt{\sigma}}H_1^{(1)}(im\sqrt{\sigma}) \ . \tag{90}$$

Here $H_1^{(1)}$ is a Hankel function. It can be expanded for small values of its argument;

$$\begin{aligned}\Delta_F(x) = -\frac{1}{4\pi}\delta(\sigma) - \frac{m^2}{8\pi^2}\Theta(-\sigma)\left(\frac{1}{2} + \frac{m^2}{2^2\cdot 4}\sigma + \ ...\ \right) - \\ -i\left(\frac{1}{4\pi^2}\frac{1}{\sigma} + \frac{m^2}{16\pi^2}\ln|\sigma| + \ ...\ \right) \ .\end{aligned} \tag{91}$$

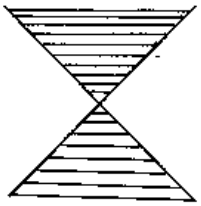

Figure 5: Support of the commutator Green function.

The first observation is that this is singular on the light cone (and the second that the expression becomes very simple when $m = 0$). The real part, and hence the commutator Green function Δ (as well as the retarded Green function Δ_{ret}) has support on and—when the mass is non-zero—inside the light cone. On the other hand the imaginary part has support also outside the light cone. If we expand the Hankel function for large values of the argument we find that when $t = 0$ and $r = \sqrt{x^2 + y^2 + z^2}$ is large then the imaginary part goes to zero like

$$\mathrm{Im}[\Delta_F(x)] \sim \frac{1}{r^{\frac{3}{2}}} e^{-mr} \to 0 \ . \tag{92}$$

This will be true also for the Wightman functions $\Delta^{\pm}$.

What can we make of this? First, for points x and y with spacelike separation we find from eq. (73) that

$$[\phi(x), \phi(y)] = 0 \ . \tag{93}$$

This is relativistic causality. It is consistent with the canonical commutation relations and indeed the latter are easily derived from the commutator Green function. Relativistic causality implies that all local functionals of the field (functions of the field and its derivatives to some finite order) commute at spacelike separation. Thus whenever the points x and y are at spacelike separation the energy density $\mathcal{H}(x)$ obeys

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \quad \text{where} \quad H = \int d^3x \mathcal{H}(x) \ . \tag{94}$$

It is therefore meaningful to talk about the total amount of energy in separate rooms at the same time. The same goes for the momentum and charge densities.

While we are on this subject it may be worth while to remark that although energy is a local concept it does not have all the local properties that we may expect. The vacuum vector is not annihilated by $\mathcal{H}(x)$ since it contains $a^\dagger a^\dagger$ terms. It remains true that the vacuum expectation value of the energy density is zero, but one can find other states in the theory for which $\langle \Psi | \mathcal{H}(x) | \Psi \rangle$ assumes arbitrarily large negative values at a given point. On the other hand negative

energy densities can occur only in ways that are subject to various integral constraints—certainly the integral over all space has to be positive. This issue is still subject to research.

A second point: We have constructed a quantum field theory for scalar particles that are bosons. They are bosons essentially because of the commutation relations assumed for the creation and annihilation operators. Now we could have postulated anti-commutators instead, schematically

$$[a,a]_+ = [a^\dagger, a^\dagger]_+ = 0 \qquad [a,a^\dagger]_+ = 1 \ . \tag{95}$$

The fields would still be

$$\phi = \phi^{(-)} + \phi^{(+)} \sim \int a e^{-it} + a^\dagger e^{+it} \ . \tag{96}$$

They would obey the anti-commutator

$$\begin{aligned} [\phi(x),\phi(y)]_+ = [\phi^{(+)}(x),\phi^{(-)}(y)]_+ + [\phi^{(-)}(x),\phi^{(+)}(y)]_+ = \\ = i\Delta^+(x,y) + i\Delta^+(y,x) = i(\Delta^+(x,y) - \Delta^-(x,y)) \ . \end{aligned} \tag{97}$$

But this is a disaster since it can be expressed by the imaginary part of the Feynman propagator, which does not vanish outside the light cone. So the attempt to build a local relativistic theory of spinless fermions fails. Pursuing this line of thought one can prove a remarkable theorem called the spin-statistics theorem, stating that in a local relativistic theory integer spin particles cannot be fermions and half-integer spin particles cannot be bosons. This is a major triumph for quantum field theory.

A third point: The integrand of the number operator is not a local functional of the fields since it can be expressed only in terms of the positive and negative frequency parts. This means that the "number density" $n(x)$ is non-local, in particular the numbers of particles in two different volumes obey

$$[N_{V_1}, N_{V_2}] \neq 0 \quad \text{where} \quad N_V = \int_V d^3x n(x) \tag{98}$$

even if the two volumes V_1 and V_2 are at spacelike separation. In effect what this means is that a particle cannot be localized to a point. However, since the Wightman function vanishes like e^{-mr} at large spacelike separations r this fact can be ignored at distances much larger than the Compton wavelength h/cm of the particle. For a π-meson—the canonical example of a spinless particle—the Compton wavelength is about 10^{-15} meters.

The concept of "particle" deserves further scrutiny. Clearly the best definition of a particle is "something that causes a click in a detector". A very rough model of a detector takes it to be something that moves along a world line $x^\alpha(\tau)$, not necessarily an inertial one. It has some internal states with different energies E_i. It will interact with the quantum field through a term in the Hamiltonian that we take to be

$$H_I = cm(\tau)\phi(x(\tau)) \ . \tag{99}$$

Here $m(\tau)$ is some operator in the Hilbert space of the detector. Assume that, at the outset, the field is in its vacuum state—already this case has its possibilities as we will see. In first order perturbation theory we will get an amplitude for the transition from a state where the detector is in the state with energy E_i to that with energy E_j which is

$$A(i \to j) = c\langle \Psi E_j | \int_{-\infty}^{\infty} d\tau m(\tau)\phi(x(\tau))|0E_i\rangle \ . \tag{100}$$

The field has gone to a state that we label Ψ—it actually does not concern us much, we are interested in what is happening to the detector. Using

$$m(\tau) = e^{iH_0\tau}m(0)e^{-iH_0\tau} \tag{101}$$

we find that the amplitude factorizes and becomes

$$A(i \to j) = c\langle E_j|m(0)|E_i\rangle \int_{-\infty}^{\infty} d\tau e^{i(E_j - E_i)\tau}\langle \Psi|\phi(x)|0\rangle \ . \tag{102}$$

(The Hilbert space is the product of the Hilbert space of the detector and the Hilbert space of the field.) What we want is the total probability for a transition between two states of the detector so we will perform a sum over all possible states of the field, that is we can use $\sum_\Psi |\Psi\rangle\langle\Psi| = \mathbf{1}$ since we perform a sum over a complete set of states for the field. Thus

$$P(i \to j) = c^2|\langle E_j|m(0)|E_i\rangle|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i(E_j - E_i)(\tau - \tau')}\langle 0|\phi(x)\phi(x')|0\rangle \ . \tag{103}$$

This double integral diverges. We will be looking at situations where the integrand is actually a function of $\Delta\tau = \tau - \tau'$ only. This means that we look at stationary processes, and what we have encountered is a familiar difficulty in such situations. It is easily cured by defining the transition probability per unit time to be

$$\dot{P}(i \to j) = c^2 |\langle E_j | m(0) | E_i \rangle|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i(E_j - E_i)(\Delta\tau)} i\Delta^+(\Delta\tau) \ . \tag{104}$$

Here $\Delta^+(\Delta\tau) = \Delta^+(x(\tau), x(\tau'))$ is the Wightman function restricted so that its arguments lie on the world line of the detector. We need it in x-space rather than in Fourier space. It can be read off from eq. (91). Its analyticity properties, in x-space, will be crucial.

It is interesting to compute the Wightman function from scratch, once more. This will give some further insight into why the wave function admits homogeneous Green functions, and how the analyticity properties arise. We do it for the massless case only. Consider first the function

$$\Psi = -\frac{1}{4\pi^2}\frac{1}{\sigma} \ ; \qquad \sigma = g_{\alpha\beta} x^\alpha x^\beta \ . \tag{105}$$

It solves the wave equation $\Box\phi = 0$ everywhere except where it is ill-defined, namely at $\sigma = 0$. If the metric $g_{\alpha\beta}$ is the Euclidean metric rather than the Minkowski metric it solves the four dimensional Laplace equation everywhere except at $\sigma = 0$ and is in fact the unique Green function of that operator (it is the obvious generalization of $-\frac{1}{4\pi r}$ which is the Green function of the usual Laplace operator). Now we can get a complex solution Ψ_v if we use

$$\sigma_v = g_{\alpha\beta}(x^\alpha - iv^\alpha)(x^\beta - iv^\beta) \ . \tag{106}$$

The solution is regular everywhere except when σ_v vanishes, and this happens only if the real and imaginary parts vanish separately, that is if and only if

$$x^2 - v^2 = x \cdot v = 0 \ . \tag{107}$$

But in the Minkowski case, and if v^α is a timelike vector, this never happens! Let us choose $v^\alpha = (\epsilon, 0, 0, 0)$. This gives an everywhere regular complex solution of the wave equation which is, however, not Lorentz invariant because it depends on a special vector. But we can consider

$$i\Delta^+ = -\lim_{\epsilon\to 0} \Psi_v = \frac{1}{4\pi^2} \lim_{\epsilon\to 0} \frac{1}{\sigma + 2i\epsilon t} = \frac{1}{4\pi^2}\left(\frac{1}{\sigma} - i\pi\epsilon(t)\delta(\sigma)\right) \ . \tag{108}$$

Here we used the mystical formula

$$\frac{1}{z \pm i\epsilon} = \frac{1}{z} \mp i\pi\delta(z) \ . \tag{109}$$

("The merry mathematican won the princess and half the residue." Actually the formula is not mystical; it is a distributional formula to be used inside an integral.) Comparing with eq. (91) it follows that Δ^+ is precisely the Wightman

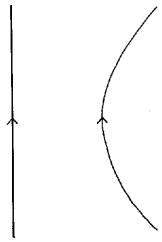

Figure 6: World lines of two detectors, one sitting still and one having constant proper acceleration.

function. This is a distributional solution of the wave equation which is Lorentz invariant because we have taken a limit where the fixed vector v^α disappears. Let us write it out:

$$i\Delta^+ = -\frac{1}{4\pi^2}\frac{1}{(t-i\epsilon)^2 - x^2} \ . \tag{110}$$

When we inspect this expression we see that what remains of our excursion into the complex domain is the fact that the Wightman function is analytic in the lower complex t-plane; indeed it can be defined by that property.

With the Wightman function in hand we can go back to eq. (104) and see what will happen to our detector under various assumptions about the world line that it follows. Let us first suppose that it is in a state of free fall (inertial motion). Without loss of generality we set

$$(t,x,y,z) = (\tau,0,0,0) \quad \Rightarrow \quad i\Delta^+(\Delta\tau) = -\frac{1}{4\pi^2}\frac{1}{(\Delta\tau - i\epsilon)^2} \ . \tag{111}$$

So the integrand is analytic in the lower complex plane. But this means that the detector cannot get excited by the vacuum state; that is if $E_j > E_i$ then the contour of integration can be closed in the lower half plane and since the integrand is analytic within the contour the integral that defines $\dot{P}(i \to j)$ vanishes. This is presumably unsurprising.

It is more interesting to consider the case of a detector whose acceleration a in its own rest frame is constant. It follows the worldline

$$t = \frac{1}{a}\sinh(\tau a) \qquad x = \frac{1}{a}\cosh(\tau a) \qquad \Rightarrow \qquad x^2 - t^2 = \frac{1}{a^2} \ . \tag{112}$$

A little bit of juggling with hyperbolic sines and cosines shows that

$$i\Delta^+(\Delta\tau) = -\frac{a^2}{16\pi^2}\frac{1}{\sinh^2\left(\frac{a\Delta\tau}{2} - i\epsilon\right)} \ . \tag{113}$$

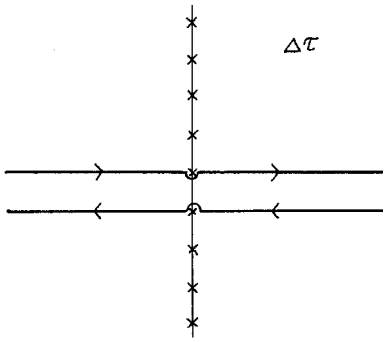

Figure 7: Integration contour for relating transition probabilities.

(If you are curious about this, the identity you need is $\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta = 1 + 2\sinh(\alpha-\beta)/2$.) This time the integrand in eq. (104) has an infinite set of poles on the imaginary axis, namely at

$$\Delta\tau = \frac{2\pi i n}{a} , \tag{114}$$

where n is an arbitary integer. The integral certainly does not vanish; the detector will be excited even though the field is in its vacuum state. To do the integral you need to know that

$$\frac{1}{\sinh^2 x} = \sum_{k=-\infty}^{\infty} \frac{1}{(x + i\pi k)^2} . \tag{115}$$

But we can arrive at the relevant conclusion without doing the integral.

What we need to do is to relate the transition probabilities $\dot{P}(i \to j)$ and $\dot{P}(j \to i)$. We do this by choosing a special contour with no singularities inside so that

$$\int_{-\infty}^{\infty} d\Delta\tau e^{-i(E_j - E_i)\Delta\tau} i\Delta^+(\Delta\tau) + \int_{\infty - i\frac{2\pi}{a}}^{-\infty - i\frac{2\pi}{a}} d\Delta\tau e^{-i(E_j - E_i)\Delta\tau} i\Delta^+(\Delta\tau) = 0 . \tag{116}$$

But the Wightman function is periodic in imaginary time, that is

$$\Delta^+(\Delta\tau) = \Delta^+(\Delta\tau + i\frac{2\pi}{a}) . \tag{117}$$

In the second integral we can therefore shift variables

$$\Delta\tau \to \Delta\tau' = -\Delta\tau - i\frac{2\pi}{a} . \tag{118}$$

This is consistent with the appropriate pole prescription (in the sense that the pole for the second integral lies below the contour in the $\Delta\tau$–plane and above it in the $\Delta\tau'$–plane). Dropping the prime, and introducing d as the factor that characterizes the detector, we can conclude that

$$\dot{P}(i \to j) = d \int_{-\infty}^{\infty} d\Delta\tau e^{-i(E_j - E_i)\Delta\tau} i\Delta^+(\Delta\tau) = \tag{119}$$

$$= d \int_{-\infty}^{\infty} d\Delta\tau e^{-i(E_i - E_j)\Delta\tau} e^{-\frac{2\pi}{a}(E_j - E_i)} i\Delta^+(\Delta\tau) = e^{-\frac{2\pi}{a}(E_j - E_i)} \dot{P}(j \to i) \ .$$

This actually means that the detector behaves as if it were in thermal equilibrium (it expresses the principle of detailed balance). To see this, suppose that the states of the detector are populated according to the Boltzmann distribution

$$N_i = Ne^{-E_i/kT} = Ne^{-2\pi E_i/a} \ . \tag{120}$$

Here k is Boltzmann's constant and we have assigned a temperature $T = \frac{a}{2\pi k}$ to our detector. For the transition rates, that depend on how the levels are populated, we can then deduce that

$$R(i \to j) = N_i \dot{P}(i \to j) = N_j \dot{P}(j \to i) = R(j \to i) \ . \tag{121}$$

So the Boltzmann distribution is the equilibrium distribution for our detector.

What we have seen is a manifestation of the Unruh effect: A detector moving at constant proper acceleration will respond to the vacuum state of a scalar field precisely as if it were in a thermal bath with temperature

$$T = \frac{a}{2\pi k} \approx \frac{a}{10^{19} m/s^2} \ K \ . \tag{122}$$

It would be difficult to observe this effect with a real detector—although perhaps not impossible if we use an elementary particle with some internal states in that role. Conceptually it is nevertheless an important point. The vacuum state is not all that harmless, and the particle concept is observer dependent.

THE DIRAC FIELD

To describe electrons we need a field theory for fermions with mass m and spin 1/2. The particles should also carry an electric charge. Since spin 1/2 particles are two component objects this suggests that we need a field with twice as many components as the complex Klein-Gordon field. This expectation turns out to be basically correct. The construction has to have some clever feature in order to allow fermionic statistics in a way that is consistent with locality.

The solution is the Dirac field equation

$$(i\gamma^{\alpha a}{}_b\partial_\alpha - m\delta^a{}_b)\psi^b = 0 \ . \tag{123}$$

In this equation we have introduced the γ-matrices. Their defining property is that they obey the anti-commutator

$$[\gamma^\alpha, \gamma^\beta]_+ = -2\eta^{\alpha\beta} \ . \tag{124}$$

The sign on the right hand matters. I have chosen it so that my γ-matrices are the "Bjorken and Drell" ones, even though I am using a spacelike metric. The field ψ^a is a spinor, which for our purposes can be defined as a vector in a vector space that carries an irreducible representation of the γ-matrix algebra.

It is reasonable that this could be the solution of our problem, first of all since

$$0 = (i\gamma\cdot\partial + m)(i\gamma\cdot\partial - m)\psi = -(\frac{1}{2}[\gamma^\alpha, \gamma^\beta]_+\partial_\alpha\partial_\beta + m^2)\psi = (\Box - m^2)\psi \ . \tag{125}$$

The mass of the particles should be m. Second, since the equation is first order in derivatives "twice as many components" as the second order Klein-Gordon equation should mean four components, and we will soon see that the γ-matrix algebra requires a four dimensional representation space. Compare eq. (35), where the Klein-Gordon equation is rewritten as a first order equation for two components; in effect

$$\begin{pmatrix} \partial_t & -1 \\ -\triangle + m^2 & \partial_t \end{pmatrix} \begin{pmatrix} \phi \\ \pi \end{pmatrix} = 0 \ . \tag{126}$$

So the number of degrees of freedom is as expected. But we must also check that the equation is Lorentz invariant, that the Hamiltonian is bounded from below, and that the theory has decent locality properties. This will take a little time.

We begin with Lorentz invariance. Recall that a Lorentz transformation is a transformation that takes the point with coordinates x^α to the point with coordinates $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$, where the matrix Λ leaves the Minkowski metric invariant in the sense that

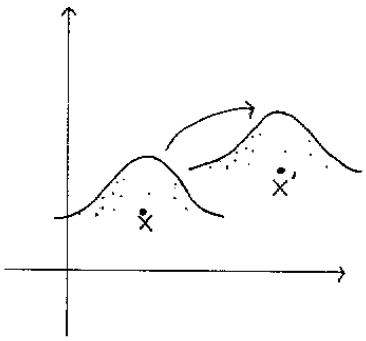

Figure 8: An active transformation and a scalar field.

$$\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta^{\gamma\delta} = \eta^{\alpha\beta} \ . \tag{127}$$

(This includes both rotations and Lorentz boosts.) A scalar field ϕ is by definition a field such that when a Lorentz transformation is performed the function $\phi(x)$ is transformed to a new function $\phi'(x)$ defined by

$$\phi'(x') = \phi(x) \ . \tag{128}$$

If the original function has a peak around the point whose coordinates are x, the new function has a peak around the point whose coordinates are x'. A field equation is Lorentz invariant if and only if the transformed field is a solution to the equation whenever the original field is; a Lorentz transformation takes solutions to solutions. It is easy to see that the Klein-Gordon equation is indeed Lorentz invariant.

A more elaborate example is the vector field A^{α}. By definition it transforms according to

$$A'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta}A^{\beta}(x) \ . \tag{129}$$

It is more convenient—and usually sufficient—to consider "small" Lorentz transformations, that is we set

$$\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} \ ; \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha} \ . \tag{130}$$

Then we ignore all second order and higher terms in ω. The condition on the tensor $\omega_{\alpha\beta}$, with one index lowered using the Minkowski metric, ensures that eq. (127) holds. The transformed vector field is

$$A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x) \approx A^{\mu}(x^{\alpha} - \omega^{\alpha}{}_{\beta}x^{\beta}) + \omega^{\mu}{}_{\nu}A^{\nu}(x) \ . \tag{131}$$

Using a Taylor expansion we then find that

$$\delta A^\mu(x) \equiv A'^\mu(x) - A^\mu(x) = \frac{1}{2}\omega^{\alpha\beta}(J_{\alpha\beta})^\mu{}_\nu A^\nu(x) \ , \tag{132}$$

where $J_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}$ is a tensor and a matrix at the same time;

$$(L_{\alpha\beta})^\mu{}_\nu = (x_\alpha\partial_\beta - x_\beta\partial_\alpha)\delta^\mu{}_\nu \qquad (S_{\alpha\beta})^\mu{}_\nu = \delta^\mu_\alpha\eta_{\beta\nu} - \delta^\mu_\beta\eta_{\alpha\nu} \ . \tag{133}$$

(To prove this, set $\xi^\alpha \equiv \omega^\alpha{}_\beta x^\beta$ so that

$$A^\mu(x-\xi) \approx A^\mu(x) - \xi^\alpha\partial_\alpha A^\mu(x) \quad \Rightarrow \quad \delta A^\mu = \omega^\mu{}_\nu A^\nu - \omega^\alpha{}_\beta x^\beta\partial_\alpha A^\mu \ . \tag{134}$$

Once you have this it is a matter of comparing terms—just insist on having an $\omega^{\alpha\beta}$ in front of everything.) Note that the orbital part $L_{\alpha\beta}$ of the angular momentum tensor commutes with the spin part $S_{\alpha\beta}$. It can be seen that Maxwell's equations are Lorentz invariant in the sense that a solution is transformed to a solution. A key property of the Lorentz transformations is that they form a group, which means that the infinitesimal transformations close in the sense that

$$[\delta_1, \delta_2]A^\mu = \delta_3 A^\mu \ . \tag{135}$$

Explicitly one can see that

$$[\delta_1, \delta_2]A^\mu = \frac{1}{4}\omega_2^{\alpha\beta}\omega_1^{\gamma\delta}[J_{\alpha\beta}, J_{\gamma\delta}]^\mu{}_\nu A^\nu \ . \tag{136}$$

Also that

$$[J_{\alpha\beta}, J_{\gamma\delta}] = \eta_{\beta\gamma}J_{\alpha\delta} - \eta_{\alpha\gamma}J_{\beta\delta} + \eta_{\alpha\delta}J_{\beta\gamma} - \eta_{\beta\delta}J_{\alpha\gamma} \ . \tag{137}$$

From this it follows that the parameter that determines the transformation δ_3 in the equation above is given by an equation of the form $\omega_3 = \omega_3(\omega_1, \omega_2)$. Mathematically this means that the Lorentz group is a Lie group.

We do not need a lot of group theory here, but we can extract some key features that we wish the Dirac theory to have. We need a matrix $S_{\alpha\beta}$ that obeys

$$[S_{\alpha\beta}, S_{\gamma\delta}] = \eta_{\beta\gamma}S_{\alpha\delta} - \eta_{\alpha\gamma}S_{\beta\delta} + \eta_{\alpha\delta}S_{\beta\gamma} - \eta_{\beta\delta}S_{\alpha\gamma} \ . \tag{138}$$

An infinitesimal Lorentz transformation of the spinor field should be given by

$$\delta\psi^a = \frac{1}{2}\omega^{\alpha\beta}(L_{\alpha\beta}\delta^a{}_b + S_{\alpha\beta}{}^a{}_b)\psi^b \ . \tag{139}$$

The Lorentz invariance of the Dirac equation is ensured if solutions transform to solutions, which in infinitesimal terms means that $\delta\psi$ must be such that

$$(i\gamma\cdot\partial - m)\psi = 0 \qquad \Rightarrow \qquad (i\gamma\cdot\partial - m)\delta\psi = 0 \ . \tag{140}$$

And now we have to construct $S_{\alpha\beta}$ out of the building blocks that we have at our disposal, namely the γ-matrices.

The γ-matrix algebra consists, by definition, of all matrices that one can obtain by taking products of γ-matrices, subject to the condition (124), and then all matrices that can be obtained by taking linear combinations of the resulting products. (By definition, an algebra is a vector space equipped with a multiplicative operation.) Hence the γ-matrix algebra consists of linear combinations of the following objects:

$$\mathbf{1} \ , \quad \gamma^\alpha \ , \quad \gamma^{\alpha\beta} \ , \quad \gamma^{\alpha\beta\gamma} \ , \quad \gamma^{\alpha\beta\gamma\delta} \ . \tag{141}$$

Here all the multi-index objects are by definition totally anti-symmetric. The point is that

$$\gamma^\alpha\gamma^\beta = \frac{1}{2}[\gamma^\alpha,\gamma^\beta]_+ + \frac{1}{2}[\gamma^\alpha,\gamma^\beta] = -\eta^{\alpha\beta}\mathbf{1} + \frac{1}{2}[\gamma^\alpha,\gamma^\beta] \equiv -\eta^{\alpha\beta}\mathbf{1} + \gamma^{\alpha\beta} \ . \tag{142}$$

The symmetric pieces of the products can always be removed in this way. Totally anti-symmetric tensors in four dimensions cannot have more than four indices, so our list is complete. Let us now count the number of linearly independent objects in the algebra. It is convenient to begin by defining

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \Rightarrow \qquad (\gamma^5)^2 = 1 \ , \quad [\gamma^\alpha,\gamma^5]_+ = 0 \ . \tag{143}$$

Now we introduce the totally anti-symmetric object $\epsilon_{\alpha\beta\gamma\delta}$ defined by $\epsilon_{0123} = 1$ in all coordinate systems. This is a "pseudo-tensor", that is it behaves like a tensor under rotations and boosts but changes sign under reflections. Now it is easy to see that

$$\gamma^5 = \frac{i}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} \qquad \Leftrightarrow \qquad \gamma^{\alpha\beta\gamma\delta} = i\epsilon^{\alpha\beta\gamma\delta}\gamma^5 \ . \tag{144}$$

If we also trade $\gamma^{\alpha\beta\gamma}$ for $\gamma^5\gamma^\alpha$, the list of objects in the γ-matrix algebra can be rewritten as

$$\mathbf{1} \ , \quad \gamma^\alpha \ , \quad \gamma^{\alpha\beta} \ , \quad \gamma^5\gamma^\alpha \ , \quad \gamma^5 \ . \tag{145}$$

We see that it contains sixteen objects. It is easy to see that—with the obvious exception of $\mathbf{1}$—they must be represented by traceless matrices. For instance

$$\text{Tr}\ \gamma^\alpha = \text{Tr}\ \gamma^\alpha\gamma^5\gamma^5 = -\text{Tr}\ \gamma^5\gamma^\alpha\gamma^5 = -\text{Tr}\ \gamma^\alpha\gamma^5\gamma^5 = -\text{Tr}\ \gamma^\alpha \ . \tag{146}$$

Here we used first eq. (143) and then the cyclic property of the trace. It follows that all the objects in the list must be linearly independent. Suppose that there exist numbers a_1 etc. so that

$$a_1\mathbf{1} + a_{2\alpha}\gamma^\alpha + a_{3\alpha\beta}\gamma^{\alpha\beta} + a_{4\alpha}\gamma^5\gamma^\alpha + a_5\gamma^5 = 0 \ . \tag{147}$$

Taking the trace of this equation implies that $a_1 = 0$. Multiplying the whole expression with γ^0 and again taking the trace implies that $a_{20} = 0$. Going on in this way implies that all the coefficients must vanish. Therefore the γ-matrix algebra contains precisely sixteen linearly independent objects—the algebra is sixteen dimensional. If we have sixteen linearly independent matrices they must be (at least) four by four matrices, so the spinors on which the matrices act must be four component objects. (Clearly this conclusion depends on the dimension D of spacetime. In general the dimension of spinor space is $2^{\frac{D}{2}}$ if D is even, and $2^{\frac{D-1}{2}}$ if D is odd. In fact the Pauli matrices are γ-matrices in three Euclidean dimensions.)

Let us say something about explicit representations—although calculations should be done using nothing but the defining equation (124) as far as possible. The first observation is that since we need sixteen linearly independent matrices altogether the representation must be done with four by four matrices—so the Dirac spinor field is a four component object as advertized. The second observation is that, since

$$(\gamma^0)^2 = 1 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 \ , \tag{148}$$

the matrix γ^0 is hermitian and the matrices γ^i are anti-hermitian (i.e., γ^i has imaginary eigenvalues). Next introduce the two by two Pauli matrices, suitably normalized so that

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k \qquad [\sigma_i, \sigma_j]_+ = \delta_{ij}\mathbf{1} \ . \tag{149}$$

Then an explicit representation (the Dirac representation) is

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ . \tag{150}$$

This representation will prove useful when we discuss the non-relativistic limit of the Dirac equation, essentially because γ^0 is diagonal. Another representation is the Weyl representation given by

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ . \tag{151}$$

Now γ^5 and γ^0 have switched roles and γ^5 is diagonal. This property will turn out to mean that the Weyl representation is useful in situations where the mass

m in the Dirac equation is so small that it can be ignored. Let me just mention a third representation of interest called the Majorana representation, in which all the gamma matrices are purely imaginary—this means that if we had chosen the opposite sign in eq. (124) we could have had a purely real set of gamma matrices, something that is of interest if we want to describe neutral spin one half particles.

Now for some interesting calculations. First

$$[\gamma^{\alpha\beta}, \gamma^{\gamma}] = 2(\eta^{\alpha\gamma}\gamma^{\beta} - \eta^{\beta\gamma}\gamma^{\alpha}) \ . \tag{152}$$

It is crucial to realize that the easy way to prove this is to do it directly from eq. (124). Similarly

$$[\gamma^{\alpha\beta}, \gamma^{\gamma\delta}] = 2(\eta^{\gamma\alpha}\gamma^{\beta\delta} - \eta^{\gamma\beta}\gamma^{\alpha\delta} + \eta^{\delta\beta}\gamma^{\alpha\gamma} - \eta^{\delta\alpha}\gamma^{\beta\gamma}) \ . \tag{153}$$

We have found our spin tensor! We can set

$$S_{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta} \ . \tag{154}$$

We are now in a position to check that the Dirac equation is Lorentz invariant, that is that Lorentz transformations take solutions to solutions.

We are given that ψ^a is a solution of the Dirac equation and we want to show that this implies that $\delta\psi^a$ is too, where $\delta\psi^a$ is given by eqs. (139) and (154). But because ψ^a obeys the Dirac equation we have that

$$(i\gamma\cdot\partial - m)\delta\psi = (i\gamma\cdot\partial - m)\frac{1}{2}\omega^{\alpha\beta}J_{\alpha\beta}\psi = \frac{1}{2}\omega^{\alpha\beta}i[\gamma\cdot\partial, J_{\alpha\beta}]\psi \ . \tag{155}$$

We want to show that this equals zero. The commutator is, using eq. (152),

$$\begin{aligned}[\gamma\cdot\partial, J_{\alpha\beta}] &= \gamma^{\mu}[\partial_{\mu}, L_{\alpha\beta}] + [\gamma^{\mu}, S_{\alpha\beta}]\partial_{\mu} = \\ &= \gamma_{\alpha}\partial_{\beta} - \gamma_{\beta}\partial_{\alpha} - (\delta^{\mu}_{\beta}\gamma_{\alpha} - \delta^{\mu}_{\alpha}\gamma_{\beta})\partial_{\mu} = 0 \ .\end{aligned} \tag{156}$$

This is the proof that the Dirac equation is Lorentz invariant.

We are going to need the action for the Dirac equation as well, and this means that we must learn to construct scalar functions out of spinors. (We are looking for the analogue of raising and lowering vector indices. Recall that under an infinitesimal Lorentz transformation

$$\delta A^{\alpha} = \omega^{\alpha}{}_{\beta}A^{\alpha} \qquad \delta B_{\alpha} = -B_{\beta}\omega^{\beta}{}_{\alpha} \ \Rightarrow \ \delta(A^{\alpha}B_{\alpha}) = 0 \ . \tag{157}$$

The transformation rule for the covariant vector is defined so that the scalar product is a scalar.) Given the spinor ψ^a we must find a "dual" spinor $\bar{\psi}_a$ so

that $\bar{\psi}_a \psi^a$ is a scalar, unaffected by the spin part of the Lorentz transformation. The answer is

$$\bar{\psi}_a = \psi_b^\dagger \gamma^{0b}{}_a \ . \tag{158}$$

The proof uses the hermiticity properties of the γ- matrices that we deduced directly from the algebra. Since γ^0 is hermitian and γ^i anti-hermitian it is true that

$$\gamma_\alpha^\dagger = \gamma^0 \gamma_\alpha \gamma^0 \qquad \Rightarrow \qquad \gamma^0 \gamma_{\alpha\beta}^\dagger \gamma^0 = -\gamma_{\alpha\beta}. \tag{159}$$

Now we proceed with the calculation, remembering that $S_{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}$:

$$\delta\bar{\psi}_a = \frac{1}{2}\omega^{\alpha\beta}\psi^\dagger S_{\alpha\beta}^\dagger \gamma^0 = \frac{1}{2}\omega^{\alpha\beta}\bar{\psi}\gamma^0 S_{\alpha\beta}^\dagger \gamma^0 = -\frac{1}{2}\omega^{\alpha\beta}\bar{\psi}S_{\alpha\beta} \ . \tag{160}$$

It is now immediate that $\delta(\bar{\psi}\psi) = 0$, which is what we wanted to show. Similarly it can be shown that $\bar{\psi}\gamma^\alpha\psi$ transforms as a vector, $\bar{\psi}\gamma_5\gamma^\alpha\psi$ as a pseudo-vector, and so on. A scalar action that leads to the Dirac equation is therefore

$$S = \int \bar{\psi}(i\gamma\cdot\partial - m)\psi \ . \tag{161}$$

The spinor is really a complex field so this action is analogous to the action for the complex scalar field, in particular in the sense that we get the field equations by varying ψ and $\bar{\psi}$ independently.

We have now collected two basically classical arguments that the Dirac field equation has a chance to describe spin 1/2 particles in a relativistic quantum field theory: It is a Lorentz invariant equation with twice as many degrees of freedom as the Klein-Gordon field. It remains to actually construct the (free) quantum field theory and check that the resulting theory is a local theory with a Hamiltonian bounded from below, and also that the spin, statistics and charge are what we want them to be.

To handle the rest of our questions about the Dirac field we proceed to quantize it along the same lines as we did for the free scalar field. That is, we begin by writing down the general solution of the field equation as a (Fourier) sum of plane waves. A single plane wave of (say) positive frequency is of the form

$$\psi^a(x) = u^a(k)e^{ik\cdot x} = u^a(k)e^{i(k_i x^i - \omega_k t)} \ . \tag{162}$$

This is a solution of the Dirac equation provided that

$$(\gamma \cdot k + m)u^a(k) = 0 \ , \tag{163}$$

where it is understood that $k^0 = \omega_k$. The question is how many linearly independent solutions there are? Consider the matrix that multiplies the spinor in eq. (163). One can show that its rank drops from four to two precisely when $k^2 + m^2 = 0$, so the answer is two. The easy way to see this is to study the equation in the limit $k_i = 0 \ \Rightarrow \ \omega_k = m$. Using the Dirac representation of the γ-matrices we find in this limit that

$$\gamma \cdot k = \begin{pmatrix} -m\mathbf{1} & 0 \\ 0 & m\mathbf{1} \end{pmatrix} \ . \tag{164}$$

Therefore there are two linearly independent solutions

$$u_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ . \tag{165}$$

(I write the spinor indices explicitly only on festive occasions. On the other hand I now put a new index r on the spinors that labels these two solutions.) What does this look like at arbitrary k_i? Provided that $k^2 + m^2 = 0$ the following function of k is a solution by construction and in fact gives precisely two linearly independent solutions by a repetition of the preceding argument:

$$u_r(k) = \frac{1}{\sqrt{2m(m+\omega_k)}}(\gamma \cdot k - m)u_r(0) \ , \qquad r = 1, \ 2. \tag{166}$$

When $k_i = 0$ we reproduce the two solutions we have found already. The somewhat strange prefactor is there to ensure Lorentz invariance (which is not trivial because we have brought a constant spinor into the game). In fact, with this factor included,

$$\bar{u}_r(k)u_s(k) = \frac{1}{2m(m+\omega)}\bar{u}_r(0)(\gamma \cdot k - m)(\gamma \cdot k - m)u_s(0) = \delta_{rs} \ . \tag{167}$$

(Use the explicit form of $u(0)$ to show this.) This is a Lorentz invariant because the dependence on ω_k drops out. A drawback of our construction is that it does not have a smooth massless limit but since we are interested in massive electrons this does not matter.

After these preparations we can write down the general solution of the Dirac equation:

$$\psi(x) = \frac{\sqrt{2m}}{(\sqrt{2\pi})^3} \int \frac{d^3k}{2\omega_k} \sum_{r=1}^{2} (u_r^a(k) c_r(k) e^{ik\cdot x} + v_r^a(k) d_r^\dagger(k) e^{-ik\cdot x}) \ , \tag{168}$$

where

$$u_r^a(k) = \frac{\gamma\cdot k - m}{\sqrt{2m(m+\omega_k)}} u_r^a(0) \qquad v_r^a(k) = \frac{\gamma\cdot k + m}{\sqrt{2m(m+\omega_k)}} v_r^a(0) \tag{169}$$

and

$$u_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{170}$$

The solution has twice as many degrees of freedom as a complex Klein-Gordon field, as promised. The extra factor $\sqrt{2m}$ in the solution is there to ensure that the mass dimension of the field differs from that of the scalar field in the appropriate way.

The spinors that enter the general solution obey a number of equations that will turn out to be useful later, so we collect them here:

$$\bar{u}_r u_s = -\bar{v}_r v_s = \delta_{rs} \tag{171}$$

$$\bar{u}_r(k)\gamma^0 u_s(k) = \bar{v}_r(k)\gamma^0 v_s(k) = \frac{\omega}{m}\delta_{rs} \tag{172}$$

$$\sum_{r=1}^{2} u_r \bar{u}_r = \frac{m - \gamma\cdot k}{2m} \qquad -\sum_{r=1}^{2} v_r \bar{v}_r = \frac{\gamma\cdot k + m}{2m} \ . \tag{173}$$

In eqs. (171) we have scalar quantities, the quantities in eqs. (172) transform like the fourth component of a spacetime vector, and eqs. (173) define two projection operators respectively onto the space of positive and negative frequence solutions. To prove eqs. (173) is straightforward γ-matrix algebra, starting from the observation that $\sum u(0)\bar{u}(0) = (1+\gamma^0)/2$.

It is time to quantize. This time we perform the quantization by imposing anti-commutation relations on the operators $(c, c^\dagger)$ and $(d, d^\dagger)$, to see what happens. Thus

$$[c_r(k), c_s^\dagger(k')]_+ = [d_r(k), d_s^\dagger(k')]_+ = 2\omega_k \delta_{rs} \delta^{(3)}(k, k') \ , \tag{174}$$

and

$$[c_r(k), c_s(k')]_+ = [d_r(k), d_s(k')]_+ = 0 \ . \tag{175}$$

Using these operators we can build a Fock space much as we did for the scalar field. As for the complex scalar field there will be two kinds of particles that at this stage we can call "c" and "d" particles. One new thing that happens is that when we build our Fock space using anti-commuting oscillators then the particles will behave like fermions, with anti-symmetric wave functions. For the scalar field this assignment of statistics is inconsistent with locality. Here on the other hand it all works out correctly. In particular the canonical anti-commutation relation for the fields at equal time $t = t' = 0$ is easily computed from eq. (168) and (174 - 175), and found to be

$$[\psi^a(x), \bar{\psi}(y)]_+ = \gamma^0 \delta^{(3)}(x, y) \ , \tag{176}$$

consistent with locality (since it vanishes when $x \neq y$ which are at spacelike separation). In this calculation we make use of eqs. (173).

Actually the various Green functions of the theory can be found without any calculation at all starting with the observation that

$$(i\gamma \cdot \partial - m)(i\gamma \cdot \partial + m) = \Box - m^2 \ . \tag{177}$$

An inhomogeneous Green function of the Dirac equation must obey

$$(i\gamma \cdot \partial - m) G^a{}_b(x, y) = \delta^a{}_b \delta(x, y) \tag{178}$$

so that if $G(x, y)$ is an inhomogeneous Green function for the Klein-Gordon equation then the solution is

$$G^a{}_b(x, y) = (i\gamma \cdot \partial + m) G(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{\gamma \cdot k - m}{k^2 + m^2} e^{ik \cdot x} \ . \tag{179}$$

The various kinds of Green functions now arise from different choices of the pole prescription for the k_0-integral, just as in the scalar case. The canonical (anti)-commutation relation must arise from restricting the commutator Green function to $t = t' = 0$, and it is instructive to compare the spinor and scalar cases, as follows:

$$[\psi^a(x), \bar{\psi}(y)]_+ = i\gamma^0 \partial_t \Delta(x,y)_{|t=t'=0} = -i\gamma^0 \partial_{t'} \Delta(x,y)_{|t=t'=0} = \\ = -i\gamma^0 [\phi(x), \dot{\phi}(y)] = \gamma^0 \delta^{(3)}(x,y) \ . \tag{180}$$

We also need a Hamiltonian operator. Since we know the action, eq. (161), we find only one reasonable candidate. A calculation using eqs. (163) and (172) shows that

$$-\int d^3x \bar{\psi}(i\gamma^i \partial_i - m)\psi = \int \frac{d^3k}{2\omega_k} \omega_k \sum_{r=1}^{2} (c_r^\dagger(k) c_r(k) - d_r(k) d_r^\dagger(k)) \ . \tag{181}$$

We refine this slightly with normal ordering, that is to say we move all creation operators to the left and throw away the divergent c-number that arises in the process. But notice carefully that the sign in front of the resulting $d^\dagger d$ term depends crucially on whether we use commutator or anti-commutators for the oscillators. For the latter choice we get

$$H = -\int d^3x : \bar{\psi}(i\gamma^i \partial_i - m)\psi : = \int \frac{d^3k}{2\omega_k} \omega_k \sum_{r=1}^{2} (c_r^\dagger(k) c_r(k) + d_r^\dagger(k) d_r(k)) \ . \tag{182}$$

This is positive definite, as it should be, and it has the expected eigenvalue ω_k when it acts on the one particle states $c^\dagger(k)|0\rangle$ and $d^\dagger(k)|0\rangle$. Had we used commutators, that is Bose-Einstein statistics, instead we would have obtained a Hamiltonian that was not bounded from below. So Fermi-Dirac statistics is forced on us by consistency with general principles.

There is an electrical charge operator

$$Q = e \int d^3x : \bar{\psi}(x) \gamma^0 \psi(x) : = e \int \frac{d^3k}{2\omega_k} \sum_{r=1}^{2} (c_r^\dagger(k) c_r(k) - d_r^\dagger(k) d_r(k)) \ . \tag{183}$$

We used eqs. (172) once more. It commutes with the Hamiltonian and is therefore conserved under time evolution; it is in fact a Noether charge. It is not positive definite and in fact the "c" and "d" particles have opposite charges. Again, the formalism requires the existence of anti-particles!

The spin of the one particle states is left as an exercise.

VECTOR FIELDS AND GAUGE INVARIANCE

With spins zero and one half behind us we take on spin one. Presumably it should be described by a vector field, so the first guess for the action would be

$$S = -\frac{1}{2}\int d^4x \partial_\alpha A_\beta \partial^\alpha A^\beta = \frac{1}{2}\int d^4x (\dot{A}_i\dot{A}_i - \dot{A}_0\dot{A}_0 - \partial_i A_j \partial_i A_j + \partial_i A_0 \partial_i A_0) \ . \tag{184}$$

But this is wrong, because the Hamiltonian is not bounded from below. If we step back from the problem at hand for a moment we observe that in quantum field theory there is a certain amount of tension between the particle interpretation on the one hand and locality on the other. From the first point of view, if one regards quantum field theory as just a convenient way to treat a quantum theory for an indefinite number of particles, the first problem to solve is "what kind of particles are there". As Wigner showed this can be phrased as a question about the irreducible representations of the Poincaré group. The (interesting) irreducible representations are of two kinds, massive and massless. To get a massive representation, fix a real number $m > 0$ and an irreducible representation of the rotation group (which arises here as the group leaving a timelike momentum vector invariant). Some clever maneouvering now shows that this will give rise to an irreducible representation of the Poincaré group labelled by m and j, where j can take any integer or half integer value. Therefore massive particles of any mass and spin can exist. In particular spin one particles exist (as mathematical objects—they also exist as physical objects, but that is another story). The massless case is less transparent but the interesting cases are again labelled by an integer or half integer number λ, although this number is now a label for a representation of the two dimensional rather than the three dimensional rotation group (and the "internal" Hilbert space of the particle is always two dimensional). The number λ is usually referred to as the helicity of the particle.

Starting from Wigner's description of particles of any allowed spin it is straightforward to build a multi-particle Fock space and also the operators H, P_i, J_i that measure the energy, momentum and angular momentum of the states in Fock space. What is not obvious is how one can construct interactions that are consistent with locality. From the other point of view one starts with local fields whose commutators vanish outside the light cone. Provided that the fields transform like tensors or spinors under the Lorentz group, locality and Lorentz invariance are easily built into the theory (at a formal level, that is) but the particle interpretation of the resulting theory is obscure. Also one must beware of catastrophies like the one that befell our first attempt to build a field theory for spin one particles. The spin one particle should presumably have three times as many degrees of freedom as the scalar field, but there is no spacetime tensor

with three components.

Both points of view have advantages and disadvantages. In the end they do turn out to be consistent with each other. In the case of the Dirac field, the objects that "intertwine" between the particle and field points of view are the spinors $u^a(k)$ and $v^a(k)$ and as we have seen a certain amount of calculation had to be performed in order to see that they had the properties that are required of them. And then spin one half is the easy case ...

To come back to the concrete problem: A field equation appropriate for the description of massive spin one particles is the Proca equation

$$\Box A_\alpha - \partial_\alpha \partial \cdot A - m^2 A_\alpha = 0 \ . \tag{185}$$

The action looks like the action for the electromagnetic field but with a mass term added. How can the four components of the field A_α describe three times as many degrees of freedom as the scalar Klein-Gordon field? To see why, we take the four-divergence of the field equation and observe that

$$0 = \partial^\alpha (\Box A_\alpha - \partial_\alpha \partial \cdot A - m^2 A_\alpha) = -m^2 \partial \cdot A \ . \tag{186}$$

The conclusion—depending crucially on a non-zero mass—is that the four-divergence of the vector field is constrained to be zero. Looking closely at the field equation again we see that its time component is

$$\Box A_0 - \partial_0 \partial \cdot A - m^2 A_0 = \triangle A_0 - \partial_0 \partial_i A_i - m^2 A_0 = 0 \ , \tag{187}$$

where $\triangle$ denotes the Laplace operator. There is no second order time derivative of A_0! To set initial data for the Proca equation we first choose $A_i(x)$ and $\dot{A}_i(x)$ at time $t = 0$ in any way we please. Eqs. (186 - 187) then enable us to compute $A_0(x)$ and $\dot{A}_0(x)$. Finally we solve the equation using these data. The point is that the total amount of freedom that we had in the choice of initial data is precisely three—not four—times the freedom we had for the Klein-Gordon equation.

So the number of degrees of freedom is right. What about the Green functions? They are easily written down starting from the corresponding Green functions for the scalar field. We must find the "inverse of the field equation", that is the inverse of the operator

$$\Box \delta_\alpha^{\ \beta} - \partial_\alpha \partial^\beta - m^2 \delta_\alpha^{\ \beta} \ . \tag{188}$$

The answer is

$$G_\alpha^{\ \beta}(x,y) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \left(\delta_\alpha^{\ \beta} + \frac{k_\alpha k^\beta}{m^2} \right) \ . \tag{189}$$

To get the various Green functions—commutator, Feynman, Wightman, and what not—we just adjust the pole prescription for the k^0-integral suitably. In

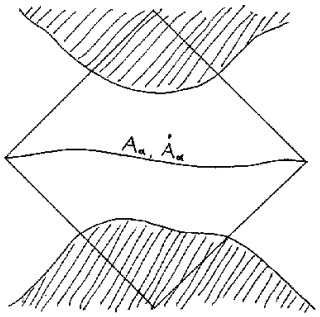

Figure 9: Initial data do not determine the vector potential (Λ is non–zero in the shaded regions).

a way, and in the spirit of the remarks that we made at the beginning of this lecture, this completes the quantization of the massive vector field.

But what happens when $m^2 = 0$? The Green function that we derived diverges in that limit. This is not so surprising in view of the fact that our discussion of the field equation depended in a crucial way on $m^2 \neq 0$; the Lorenz condition $\partial \cdot A = 0$ does not follow from Maxwell's equations without a massterm. Indeed it is well known that there does not exist any choice of initial data that together with Maxwell's equations determines A_α; given a solution $A_\alpha(x)$ the field

$$A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x) \tag{190}$$

is a solution as well, for a quite arbitrary function $\Lambda(x)$. We say that Maxwell's equations are invariant under the gauge transformation (190), and electrodynamics is an example of a gauge theory—indeed the simplest example of a very important concept, so it is worth thinking about.

The key difference between a gauge theory and an ordinary field theory is that the one-to-one correspondence between solutions of the field equations and possible histories of the physical system (that the field theory is assumed to describe) is given up. In a gauge theory we postulate a one-to-one correspondence between histories of the physical system on the one hand, and equivalence classes of solutions of the field equations on the other. Two solutions are said to belong to the same equivalence class if they can be connected with a gauge transformation, such as eq. (190) in the case at hand. The space of solutions will be divided into "gauge orbits", each orbit by definition being such an equivalence class. All solutions belonging to the same orbit represent the same history of the system. The physics of the theory is in its gauge invariant quantities, functions that are unaffected by gauge transformations. A prime example in electrodynamics is the electromagnetic field strength tensor

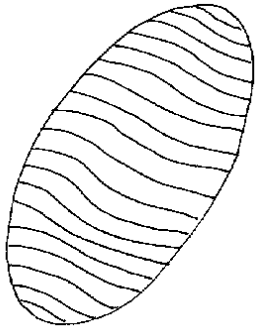

Figure 10: Phase space partitioned into gauge orbits (schematically).

$$F_{\alpha\beta}(x) = \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x) \ . \tag{191}$$

Almost all the physics is in the electromagnetic field, but there are circumstances (such as the Aharonov-Bohm effect) where other, non-local, gauge invariant quantities must be considered.

But what do we actually do? There are two main strategies:

- Choose, by means of some definite prescription, one solution from each orbit, and work with these only. This is called gauge fixing.
- Work directly with the equivalence classes. This will lead us to something called covariant or Gupta-Bleuler quantization.

Before we decide between them, let us gain a little experience with gauge theories. Here is the entire course in one formula:

$$S = \int -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + i\bar{\psi}\gamma \cdot \partial\psi - m\bar{\psi}\psi + eA_\alpha \bar{\psi}\gamma^\alpha \psi \ . \tag{192}$$

This is the action of the Maxwell-Dirac theory and, together with the general principles of quantum mechanics, it contains all of QED. Because of the last (interaction) term its field equations are non-linear and therefore cannot be solved exactly in generic situations. For the electromagnetic field we get

$$\partial_\beta F^{\alpha\beta} + e\bar{\psi}\gamma^\alpha \psi = 0 \ . \tag{193}$$

Now the four-divergence of the first term vanishes identically because of the way that the fields strength tensor was defined, so it must be true that

$$\partial_\alpha (e\bar{\psi}\gamma^\alpha \psi) = 0 \ . \tag{194}$$

Actually this is true, because of the field equation for the spinor field. Was this a fortunate accident? No. It must be so because of gauge invariance. The action is in fact invariant under the transformation

$$\delta\psi = ie\Lambda\psi \qquad \delta\bar{\psi} = -ie\Lambda\bar{\psi} \qquad \delta A_\alpha = \partial_\alpha\Lambda \ , \tag{195}$$

where Λ is some arbitrary function. If you do the calculation you see that this is so only because the interaction term is present, with exactly the right coefficient. So the first observation is that the form of the interaction is dictated by gauge invariance. The second observation is that if we vary the action according to eq. (195), so that $\delta S = 0$, and make the additional assumption that the Dirac field obeys its field equation—so that the terms involving $\delta\psi$ vanish—then we obtain

$$0 = \int \partial_\alpha\Lambda(\partial_\beta F^{\alpha\beta} + e\bar{\psi}\gamma^\alpha\psi) \ . \tag{196}$$

Here Λ is an arbitrary function and the term in brackets is the field equation for the electromagnetic field. Performing a partial integration we conclude that the four-divergence of the field equation must vanish when the matter field obeys its own field equation. So the consistency condition that we noted above is a necessary consequence of gauge invariance.

Before we get carried away with gauge invariance, it is as well to note that the theory remains consistent if we add a mass term for the vector field. It is the massless vector field that demands gauge invariance for consistency. In a different direction, we can generalize the idea of gauge theories in an interesting way. Introduce several Fermi fields, say two for definiteness, and label them with the index a. We can include them all in the action; in the non-interacting case we take the action to be

$$S_0 = \int i\bar{\psi}_a\gamma\cdot\partial\psi^a \ . \tag{197}$$

(A mass term for the fermions is optional.) This is invariant under

$$\delta\psi^a = i\Lambda_i\sigma_i^{a}{}_b\psi^b \qquad \delta\bar{\psi}_a = -i\Lambda_i\bar{\psi}_b\sigma_i^{b}{}_a \ , \tag{198}$$

where σ_i are the Pauli matrices and Λ_i is some spacetime independent vector in some "internal" space. One can show that these transformations close to form a Lie algebra, in fact the $SU(2)$ Lie algebra familiar from quantum mechanics courses. But if we now make the transformations spacetime dependent so that $\Lambda_i = \Lambda_i(x)$ then the action is no longer invariant:

$$\delta(i\bar{\psi}_a\gamma\cdot\partial\psi^a) = -\bar{\psi}_a\sigma_i^{a}{}_b\gamma^\alpha\psi^b\partial_\alpha\Lambda_i \ . \tag{199}$$

We can repair this by adding a term

$$S_I = \int \bar{\psi}_a \sigma_i^a{}_b \gamma^\alpha \psi^b A_{\alpha i} \tag{200}$$

to the action. It describes the interaction of the spinor fields with a triplet of gauge vector fields $A_{\alpha i}$ transforming in a suitable way under gauge transformations. More precisely, we assume that, under gauge transformations,

$$\delta A_{\alpha i} = \partial_\alpha \Lambda_i + \text{non-linear terms} \ . \tag{201}$$

A detailed calculation shows that the non-linear terms must be there; what is new compared to electrodynamics is that

$$\int \left((\delta\bar{\psi}_a)\sigma_i^a{}_b \gamma^\alpha \psi^b + \bar{\psi}_a \sigma_i^a{}_b \gamma^\alpha (\delta\psi^b) \right) A_{\alpha i} \neq 0 \ . \tag{202}$$

Therefore the gauge transformation of the vector field must be a non-linear version of the gauge transformation of the electromagnetic field; the non-linear terms must be chosen so that $\delta S_0 + \delta S_I = 0$. Anyway, once we bring new fields into the action we must provide them with field equations of their own. They must come from a non-linear generalization of the F^2 term in the Maxwell action, because the Maxwell action itself is not invariant under the non–linear version of eq. (201). When the details are fully worked out we have in our hands the simplest example of a Yang-Mills theory—and it is interesting to know that the strong and weak interactions in the Standard Model of particle physics are precisely of this type.

After this excursion into matters beyond QED we return to work, first with the linear Maxwell theory that does not include the Dirac field at all. The first of the two approaches listed above is conceptually simpler even though it leads to messier formulæ, so let us look at this first.[1] One prescription that does what we want is the Coulomb gauge condition $\partial_i A_i = 0$. This is a non-covariant condition, but this is an unavoidable drawback of the first approach. There are two points to check, and we check them under the assumption that space is infinite and everything falls to zero at infinity sufficiently fast, that is to say we assume that the Laplace operator has a unique inverse ($\triangle f = 0 \ \Rightarrow \ f = 0$).

We must check that any solution is gauge equivalent to a solution that obeys the Coulomb condition, and also that two different solutions that obey the Coulomb condition cannot be gauge equivalent. But this is so because if $\partial_i A_i = f$, where f is some function, then the equation

$$0 = \partial_i A_i' = \partial_i (A_i + \partial_i \Lambda) = f + \triangle \Lambda \tag{203}$$

has a solution Λ, which means that there is a gauge equivalent solution A_i' that obeys the Coulomb condition; moreover the solution is unique (by assumption) so that if $f = 0$ then $\Lambda = 0$ and hence $A_i' = A_i$.

[1] It will be a brief look that may be hard to follow if you see this for the first time.

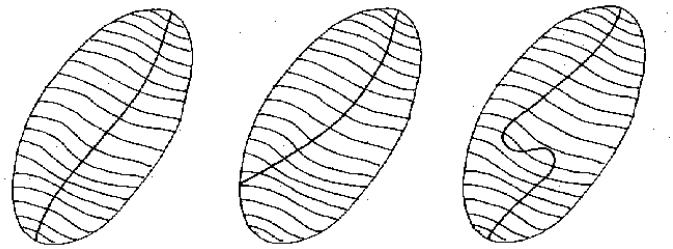

Figure 11: A good gauge choice, and two bad ones.

A systematic way to discuss the Coulomb gauge begins by introducing the projection operators

$$P^T_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\triangle} \qquad P^L_{ij} = \frac{\partial_i \partial_j}{\triangle} \ . \tag{204}$$

Any vector field, and in particular the vector potential, can then be split into a transverse and a longitudinal part,

$$A_i = A^T_i + A^L_i = P^T_{ij} A_j + P^L_{ij} A_j \ ; \qquad \partial_i A^T_i = 0 \qquad A^L_i = \partial_i (\partial_j A_j) \ . \tag{205}$$

While this is well defined, it is also non-local—recall that the inverse $\triangle^{-1}$ is actually a short hand description of the Green function of the Laplace operator:

$$\frac{1}{\triangle} f \equiv \int d^3x' G(x,x') f(x') = -\frac{1}{4\pi} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} f(x') \ . \tag{206}$$

This involves integrating over all of space. In the end this non-locality is harmless, but this is something that has to be shown. Anyway the gauge transformations now take the form

$$\delta A^T_i = 0 \qquad \delta A^L_i = \partial_i \Lambda \ . \tag{207}$$

Imposing the Coulomb gauge means that we set $A^L_i = 0$. Moreover we see from eq. (187) that, in the Coulomb gauge, $A_0 = 0$ (in an interacting theory A_0 will be determined by the electric charge density via the Laplace equation) so that the entire content of the theory is in the transverse vector potential.

Quantization of the theory now proceeds by writing down canonical commutation relations. In the Coulomb gauge the momentum canonically conjugate to A^T_i is the electric field $E_i = \dot{A}^T_i$, which is automatically transverse. But the commutation relations that we write down must be consistent with the fields being transverse. Thus we postulate

$$[A^T_i(x), E_j(y)] = i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\triangle} \right) \delta^{(3)}(x,y) \ . \tag{208}$$

At first, this appears to be in direct contradiction with locality (since the inverse of the Laplacian really involves an integration over all space). At closer scrutiny all is well. The point is that since all the physics is in the gauge invariant quantities the apparent non-locality that we encounter in the Coulomb gauge does not affect the physics. Indeed, if we compute the commutator of the gauge invariant magnetic and electric fields we find, using eq. (208), that

$$[B_i(x), E_j(y)] = \epsilon_{imn}\partial_m[A_n(x), E_j(y)] = i\epsilon_{imj}\partial_m\delta(x,y) \ . \qquad (209)$$

This is local, as it should be.

To summarize this brief look at Coulomb gauge quantization, it made us lose sight of two of our main guiding principles, namely Lorentz invariance and locality. In the end both are still there though. The main virtue of this approach is that it is straightforward to count the number of degrees of freedom. It is two per spatial point, that is twice as many as in the scalar field. Construction of the Fock space is also relatively straightforward. The alternative strategy—"covariant" quantization—turns out to be the other way around. Lorentz invariance and locality is built in from the start, while counting degrees of freedom and making sure that the Fock space is consistent with basic principles is highly non-trivial.

Before we undertake this work, is it necessary? That is, is the photon massless? As we have hinted already QED, when taken in isolation, remains consistent also if the vector field is given a mass. The speed of light would then be less than c, but that is allowed. Experimentally a mass term would show up in various ways, notably in that the familiar Coulomb potential r^{-1} would be modified into a Yukawa potential $r^{-1}e^{-mr}$ since the relevant static Green function would no longer be the Green function of the Laplacian. The best available limit comes from this theoretical fact together with careful studies of the magnetic fields surrounding both the Earth and Jupiter. (The larger scale of Jupiter is compensated by greater accuracy in terrestrial measurements.) The conclusion is that the mass of the photon is $< 2 \cdot 10^{-16}$ eV, which should be compared to the mass of the electron, which is a very light particle weighing in at 511 keV. In view of this it seems very natural to assume that the photon is exactly massless. This conclusion is further strengthened by the way that QED is embedded in the electro-weak $SU(2) \times U(1)$ Yang-Mills theory, where a mass term for the photon can be admitted—if at all—only at the expense of making the theory drastically much more complicated.

COVARIANT QUANTIZATION OF GAUGE THEORIES

Now we turn to covariant quantization of gauge theories. What follows is a brief sketch that aims to make it plausible that the Gupta-Bleuler method works for the free electromagnetic field. You can find the full story elsewhere. You should also be aware that the Gupta-Bleuler method is not the last word on the subject; when quantization of Yang-Mills fields became a concern covariant quantization developed further and acquired catchwords like "BRST symmetry" and "ghost fields". It may be worthwhile to observe that the key differences between Gupta-Bleuler inspired quantization on the one hand, and methods based on gauge fixing (like the Coulomb gauge) on the other, do not have much to do with manifest Lorentz invariance as such—it is just that the former kind of schemes are consistent with manifest covariance, and the latter are not.

The key ingredient in the Gupta-Bleuler method is a gauge choice, but this time the gauge is not "fixed" in quite the same sense as it is when we impose a canonical gauge condition like the Coulomb gauge. Let us look at Maxwell's equations with some arbitrary source term J^α included. For consistency this must be a conserved current—a property that is guaranteed by gauge invariance as we saw above. Thus

$$\partial_\beta F^{\beta\alpha} = \Box A^\alpha - \partial^\alpha \partial \cdot A = J^\alpha \ . \tag{210}$$

If there had been a mass term present we would have had that $\partial \cdot A = 0$. Now we cannot draw that conclusion, but we are free to choose $\partial \cdot A = 0$ as an initial condition. This is called the Lorenz gauge (for the Danish physicist Lorenz). If we insert this condition into Maxwell's equations they become

$$\Box A^\alpha = J^\alpha \qquad \Rightarrow \qquad \Box \partial \cdot A = 0 \tag{211}$$

(since the current is conserved). Hence $\partial \cdot A$ evolves as a free field also in the coupled theory, and if we set it to zero as an initial condition it will remain zero, provided that we agree to perform only those gauge transformations for which $\Box \Lambda = 0$; with this restriction

$$\Box \Lambda = 0 \qquad \Rightarrow \qquad \delta(\partial \cdot A) = 0 \ . \tag{212}$$

There are of course many functions Λ that obey this condition. Hence we can consistently assume that the Lorenz condition is fulfilled at all times, but a restricted kind of gauge transformation is still active. Maxwell's equations now simplify to

$$\Box A^\alpha = J^\alpha \ . \tag{213}$$

This is the approach usually taken in classical radiation problems because it is now trivial to write down Green functions and so on. But we seem to be back

to the theory that we dismissed at the outset of the previous lecture. There is a difference however, in that the Lorenz condition must be built into the theory as a supplementary condition somehow. Quite a bit of work is needed to see precisely how the classical treatment of the Lorenz gauge can be taken over to the quantum theory.

Let us start out by accepting that the field equation for the free electromagnetic field can be written as four copies of the Klein-Gordon equation. Then the general solution of the field equations is

$$A_\alpha(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2\omega_k} (a_\alpha(k) e^{ik\cdot x} + a_\alpha^\dagger e^{-ik\cdot x}) \ . \tag{214}$$

Actually for some purposes it is better to be a little bit more flexible here. We can write the general solution in the form

$$A_\alpha(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2\omega_k} \epsilon_\alpha^r(k) (a_r(k) e^{ik\cdot x} + a_r^\dagger(k) e^{-ik\cdot x}) \ . \tag{215}$$

The index r runs from 0 to 3 and summation over repeated indices is understood. There must be four linearly independent solutions, so $\epsilon_\alpha^r(k)$ denotes a set of four linearly independent spacetime vectors, one for each value of k. They play the same role that the spinors u and v played in the general solution of the Dirac equation and will be chosen to obey

$$\epsilon_\alpha^r \epsilon_\beta^s \eta^{\alpha\beta} = \eta^{rs} \qquad \epsilon_\alpha^r \epsilon_\beta^s \eta_{rs} = \eta_{\alpha\beta}. \tag{216}$$

Here η^{rs} is a matrix having the same form as the Minkowski metric, i.e. with diagonal entries $(1, 1, 1, -1)$. We will use it to raise and lower the Latin indices in the usual way. The solution

$$\epsilon_\alpha^r = \delta_\alpha^r \tag{217}$$

evidently reproduces eq. (214), but the extra flexibility afforded by the k-dependent set of basis vectors turns out to be helpful. Let us observe right away that we cannot set $\epsilon_r \cdot \epsilon_s = \delta_{rs}$ since there cannot exist four linearly independent spacelike vectors in Minkowski space, so eq. (216) is essentially forced upon us from the start.

Using eqs. (215 - 216) we can express the Hamiltonian in terms of the oscillators. In the quantum case we must do some normal ordering to get the following result, namely

$$H = \int d^3x \ \dot{A}_\alpha \dot{A}^\alpha + \partial_i A_\alpha \partial_i A^\alpha = \int \frac{d^3k}{2\omega_k} \omega_k (a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 - a_0^\dagger a_0) \ . \tag{218}$$

This appears to be a disaster since it is not positive definite, as remarked before. Remarkably enough it is not fatal, as we will see when we proceed. Concerning the commutator, we know what we must have, namely

$$[A_\alpha(x), A_\beta(y)] = i\eta_{\alpha\beta}\Delta(x,y) \ , \tag{219}$$

where Δ is the commutator Green function for the Klein-Gordon equation. This is the only reasonable possibility that is both covariant and local. A calculation shows that in order to get this result we must use eq. (216), and moreover on the oscillators we must impose the commutation relations

$$[a_r(k), a_s^\dagger(k')] = 2\omega_k \eta_{rs}\delta^{(3)}(k,k') \ . \tag{220}$$

But this comes at a prize. Explicitly

$$[a_0(k), a_0^\dagger(k')] = -2\omega_k \delta^{(3)}(k,k') \ . \tag{221}$$

The sign is wrong. In the underlying Hilbert space this means that there are one-particle states with negative norm:

$$||\int \frac{d^3k}{2\omega_k} f(k) a_0^\dagger(k)|0\rangle||^2 = -\int \frac{d^3k}{2\omega_k}|f(k)|^2\langle 0|0\rangle < 0 \ . \tag{222}$$

We seem to be going from bad to worse. Certainly the theory as it stands so far—when we think of Maxwell's equations as a collection of four independent Klein-Gordon equations—is no good.

It is tempting to say that because of eq. (221) we should regard a_0 as the creation operator instead of $a_0^\dagger$. This would avoid the negative norms but would bring us in direct conflict with Lorentz covariance—and in the end it would not work. Instead we play the card that we have left, namely the Lorenz condition. This time we can not—as we succesfully did for the massive vector field—impose $\partial \cdot A = 0$ as a condition on the field operators since this would be inconsistent with the commutator (219). What we can do is to impose the Lorenz condition on the states, and say that it defines a "physical subspace" of the Hilbert space. That is to say that any physical system is assumed to be described by a state vector $|\text{phys}\rangle$ that belongs to a subspace of Hilbert space consisting of vectors that obey

$$\langle \text{phys}|\partial \cdot A|\text{phys}\rangle = 0 \tag{223}$$

or equivalently

$$(\partial \cdot A)^{(+)}|\text{phys}\rangle = 0 \ , \tag{224}$$

where $(\partial \cdot A)^{(+)}$ denotes the positive frequency part, that is the part that is linear in annihilation operators. To repeat, this equation is an equation for the state $|\text{phys}\rangle$. The hope is that there are no negative norm states among the solutions. Moreover we hope that the Hamiltonian of the theory is such that if we start the time evolution in a state that belongs to the physical subspace, the system will not evolve out of it. If this works out we can consistently declare

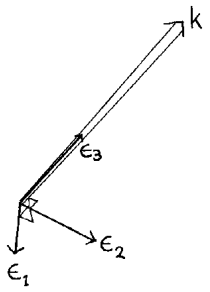

Figure 12: Polarization vectors (in three-space) defined relative to the propagation direction.

that the world always starts in the physical subspace, and consequently that negative norm states are never encountered in reality. This is not so different from the way the Lorenz condition is handled in the classical theory.

The k-dependent set of basis vectors comes in handy when we try to disentangle what the condition means. We observe that a solution of eq. (216) is given by

$$\begin{aligned} \epsilon^0_\alpha &= (-1,0,0,0) \qquad & \epsilon^1_\alpha &= (0,\epsilon^1_i) \\ \epsilon^2_\alpha &= (0,\epsilon^2_i) \qquad & \epsilon^3_\alpha &= (0,\frac{k_i}{\omega_k}) \ , \end{aligned} \tag{225}$$

where

$$k_i \epsilon_i^{1,2} = 0 \tag{226}$$

and the vectors $\epsilon_i^{1,2}$ are assumed to form a basis in the two dimensional space that is orthogonal to the vector k_i. We are going to refer to the one-particle states created by $a_{1,2}$ as transverse photons, those created by a_3 as longitudinal photons, and those created by a_0 as scalar photons. (Namely scalar with respect to rotations. The setup is on the whole a bit less convincing than the corresponding setup for the electron, since the photon does not have a rest frame attached to it—the choice of the time direction is completely arbitrary.) States created with $a_1^\dagger$ are orthogonal to states created with $a_2^\dagger$. They correspond to photons with linear polarization in two perpendicular directions—in general a photon can be in a complex superposition of these two basis states. Note that the language is close to that used in Coulomb gauge quantization, except that in the Coulomb gauge the longitudinal and scalar photons are simply not there.

A little detour: In our special reference frame we can introduce a constant vector n_α and set

$$\epsilon^0_\alpha = n_\alpha \qquad \epsilon^3_\alpha = -\frac{1}{k\cdot n}k_\alpha + n_\alpha \ . \tag{227}$$

Using the completeness relation we now observe that

$$\sum_{r=1}^{2}\epsilon^r_\alpha\epsilon^r_\beta = \eta_{\alpha\beta} - \epsilon^3_\alpha\epsilon^3_\beta + \epsilon^0_\alpha\epsilon^0_\beta = \eta_{\alpha\beta} - \frac{1}{(k\cdot n)^2}k_\alpha k_\beta + \frac{1}{k\cdot n}(k_\alpha n_\beta + n_\alpha k_\beta) \ . \tag{228}$$

This has a consequence that will be useful later on: Suppose that a vector $\mathcal{M}_\alpha$ has the property that $k\cdot\mathcal{M} = 0$. Then

$$\sum_{r=1}^{2}\epsilon^r_\alpha\epsilon^r_\beta\bar{\mathcal{M}}^\alpha\mathcal{M}^\beta = \bar{\mathcal{M}}_\alpha\mathcal{M}^\alpha \ . \tag{229}$$

The bar stands for complex conjugation—later on just such an equation will appear, with $\mathcal{M}_\alpha$ as a gauge invariant scattering amplitude.

But our main interest for the moment is a minor calculation showing that

$$\begin{aligned}(\partial\cdot A)^{(+)} &= \frac{i}{\sqrt{(2\pi)^3}}\int\frac{d^3k}{2\omega_k}k\cdot\epsilon^r(k)a_r(k)e^{ik\cdot x} = \\ &= -\frac{i}{\sqrt{(2\pi)^3}}\int\frac{d^3k}{2\omega_k}\omega_k(a_0(k)-a_3(k))e^{ik\cdot x} \ .\end{aligned} \tag{230}$$

Hence the physical state condition takes the simple form

$$(a_0 - a_3)|\text{phys}\rangle = 0 \ . \tag{231}$$

It is gratifying to observe that at least that particular negative norm state that we encountered in eq. (222) does not obey this condition, so the physical state condition kills at least some negative norm states. We want to show that it kills all of them.

I will not actually do this, but I can make it plausible. The Hilbert space consists of states created from the vacuum by strings of four different kinds of creation operators. There is a basis consisting of product states of the form

$$|\Psi\rangle = |\text{T}\rangle|\text{LS}\rangle \ , \tag{232}$$

where the notation is intended to mean that the first factor is a state containing transverse photons only, the second contains only scalar and longitudinal photons. Now assume that this state belongs to the physical Hilbert space.

This is no restriction on the transverse part, but it is a restriction on the scalar/longitudinal part. Next consider the number operator that counts the number of scalar and longitudinal photons,

$$N_{LS} = \int \frac{d^3k}{2\omega_k}(a_3^\dagger a_3 - a_0^\dagger a_0) \ . \tag{233}$$

(The minus sign must be there!) Suppose that the physical state $|\mathrm{LS}_n\rangle$ contains n scalar and longitudinal photons altogether. Then, using the physical state condition (231) the following calculation can be done:

$$\begin{aligned} n\langle \mathrm{LS}_n|\mathrm{LS}_n\rangle &= \langle \mathrm{LS}_n| \int \frac{d^3k}{\omega_k}(a_3^\dagger a_3 - a_0^\dagger a_0)|\mathrm{LS}_n\rangle = \\ &= \int \frac{d^3k}{\omega_k}\langle \mathrm{LS}_n|a_3^\dagger(a_3 - a_0)|\mathrm{LS}_n\rangle = 0 \ . \end{aligned} \tag{234}$$

Hence

$$\langle \mathrm{LS}_n|\mathrm{LS}_n\rangle = \delta_{n,0} \ . \tag{235}$$

There are no negative norm states (of this type at least) in the physical Hilbert space. That is good. But there are zero norm states, and we have to take an attitude to those.

The solution (I will give a plausibility argument for its viability soon) is similar to the construction of the classical phase space of gauge theories. Consider, in the number basis, a state for longitudinal and scalar photons of the form

$$|\mathrm{LS}\rangle = c_0|\mathrm{LS}_0\rangle + c_1|\mathrm{LS}_1\rangle + c_2|\mathrm{LS}_2\rangle + \ \ldots \ . \tag{236}$$

The idea then is that only the vacuum part really counts, that is to say that the states

$$|\mathrm{T}\rangle|\mathrm{LS}\rangle \quad \text{and} \quad |\mathrm{T}\rangle|\mathrm{LS}_0\rangle \tag{237}$$

describe the same actual state of the physical system, regardless of the values of the coefficients c_1, c_2 and so on. So our final picture of the Hilbert space of QED, in its Gupta-Bleuler version, shows a physical subspace containing no negative norm states and foliated by "gauge orbits" of states that differ only by the amount of zero norm states contained in the superposition.

The plausibility argument in favour of this solution is as follows: Look at an observable, that is some operator whose expectation value we are interested in. The Hamiltonian

$$H = \int \frac{d^3k}{2\omega_k}\omega_k(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 - a_0^\dagger a_0) \tag{238}$$

is the obvious example. Then, following our treatment of the number operator, it is straightforward to check that in a physical state

$$\langle \mathrm{LS}|\langle \mathrm{T}|H|\mathrm{T}\rangle|\mathrm{LS}\rangle = \langle \mathrm{LS}_0|\langle \mathrm{T}|H|\mathrm{T}\rangle|\mathrm{LS}_0\rangle \ , \tag{239}$$

and this confirms that the amount of zero norm states present in the physical state is physically irrelevant. This is actually true for all observables in the theory, not only for the Hamiltonian. In fact it is true by definition since an observable, in a gauge theory, is defined as a Hermitian operator that has this property.

As a final consistency check we must show that if the system starts in a physical state it will not evolve out of the physical subspace. But this is so because

$$[a_0 - a_3, H] = \omega_k (a_0 - a_3) \ . \tag{240}$$

If we apply H to a state that obeys the physical state condition (231) it will still obey the physical state condition;

$$(a_0 - a_3) H|\mathrm{phys}\rangle = \omega_k (a_0 - a_3)|\mathrm{phys}\rangle = 0 \ . \tag{241}$$

A physical state remains physical as time goes by.

From this analysis it may appear that Gupta-Bleuler quantization is just an involved way of doing something that is quite straightforward if we work in the Coulomb gauge—with only transverse photons—to begin with. But please observe that the advantages of Gupta-Bleuler quantization are considerable, notably because the Feynman propagator and the interaction term in the Lagrangian are kept very simple.

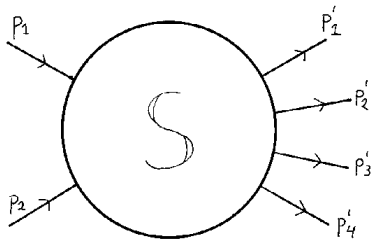

Figure 13: An S-matrix element.

THE S-MATRIX, LSZ, AND THE MAGICAL FORMULA

We have reached a decent understanding of free field theories, classical or quantum. Interacting field theories, with non-linear field equations, is quite another kettle of fish. We no longer have any right to expect that we will ever lay our hands on the general solution of the field equations, and the expectation is that only approximate calculations are feasible. Before turning to approximations it is a good idea to decide what we want to approximate. In particle physics the obvious answer is scattering amplitudes. The physical situation is that the initial state consists of a set of incoming particles with well defined momenta, and the outgoing state consists of another set of incoming particles again with well defined momenta. The theory should describe the relation between these two sets in terms of a probability amplitude. This is given by the S-matrix elements; if we have two incoming particles such matrix elements are $\langle p_1' p_2' \text{out} | p_1 p_2 \text{in} \rangle$, $\langle p_1' p_2' p_3' \text{out} | p_1 p_2 \text{in} \rangle$, $\langle p_1' p_2' p_3' p_4' \text{out} | p_1 p_2 \text{in} \rangle$ and so on—any number of particles can come out so that there is a lot to compute. Our aim is to show that there is a perturbation theory that produces the S-matrix elements as a power series in the coupling constant of the model; the rules for writing down the individual terms in the series are very simple and are called Feynman rules. Unfortunately the actual evaluation of the terms is not easy. To prevent unnecessary complications from obscuring the logic we will derive these rules for the ϕ^4 theory, whose action is

$$S = -\int d^4x \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \ . \tag{242}$$

Such a scalar field theory actually does occupy a corner of the Standard Model—to do with the Higgs field—so it is of direct interest.

In the first step our ambition is to express the S-matrix in terms of vacuum expectation values of the field operators using close to rigourous arguments—or more accurately, using arguments that can be made close to rigourous. In the

second step we set up a perturbation expansion for these vacuum expectation values using arguments that are not rigourous at all. In the first step we will work in the Heisenberg picture. This can be confusing conceptually since the state $|\Psi\rangle$ of the system is time independent. Things can still happen though as we will see. Since the situation in the past is assumed to be that there exists a certain set of particles with well defined momenta we assume that the Hilbert space of the theory has a basis consisting of states

$$|0 \text{ in}\rangle \ , \quad |p \text{ in}\rangle \ , \quad |p_1, p_2 \text{ in}\rangle \ , \quad \ldots \ , \tag{243}$$

and so on. The situation in the past is similar, so we assume that there is another basis of the Hilbert space consisting of the states

$$|0 \text{ out}\rangle \ , \quad |p \text{ out}\rangle \ , \quad |p_1, p_2 \text{ out}\rangle \ , \quad \ldots \ . \tag{244}$$

The in-states and the out-states are distinct, and S-matrix elements can now be interpreted: We describe events that happen not by allowing the states to change in time, but by using different basis states in the past and in the future.

We introduce special field operators called in-fields and out-fields that create such multi-particle states from the vacuum. Thus

$$\phi_{in}(x) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{2\omega_k} (a_{in}(k) e^{ik\cdot x} + a^\dagger_{in}(k) e^{-ik\cdot x}) \tag{245}$$

$$a_{in}(k) = -\frac{i}{\sqrt{(2\pi)^3}} \int d^3x e^{-ik\cdot x} \overset{\leftrightarrow}{\partial_t} \phi_{in}(x) \ . \tag{246}$$

By definition this is a free scalar field that obeys the Klein-Gordon equation (and the double pointed arrow above the derivative means first take the derivative to the right and then subtract the derivative to the left). It describes the kinematics of the theory and does not feel the interaction at all. Classically, the field ϕ that occurs in the Lagrangian obeys the equation

$$(\Box - m^2)\phi(x) = \frac{\lambda}{3!}\phi^3(x) \equiv j(x) \ . \tag{247}$$

We can then use the retarded Green function to construct a classical in-field

$$\phi_{in}(x) = \phi(x) - \int d^4y \Delta_{ret}(x,y) j(y) \ . \tag{248}$$

By construction ϕ_{in} obeys the Klein-Gordon equation. We have chosen the retarded Green function here because if $j(x) \to 0$ as $t \to -\infty$ (say if we assume an "adiabatic cutoff" of the interaction in the distant past) then

$$\lim_{t\to-\infty} j(x) = 0 \qquad \Rightarrow \qquad \lim_{t\to-\infty} (\phi_{in} - \phi) = 0 \ . \tag{249}$$

So the in-field and the true field agree in the far past under these assumptions. Choosing the advanced Green function in eq. (248) produces an "out-field" that agrees with the true field in the far future.

In quantum field theory we assume that it is true that there exists a numerical factor $\sqrt{Z}$ such that we can define

$$\sqrt{Z}\phi_{in}(x) = \phi(x) - \int d^4y \Delta_{ret}(x,y)j(y) \tag{250}$$

and that in some sense (to be specified, if you are careful) it is true that

$$\lim_{t\to-\infty}(\sqrt{Z}\phi_{in} - \phi) = 0 \ . \tag{251}$$

This is called the asymptotic condition. The out-field ϕ_{out} is constructed in a similar way using the advanced Green function.

The numerical factor Z can be explained, but I will just give a hint: The point is that we normalize the fields by insisting that their equal time commutators are just delta functions,

$$[\phi(x),\phi(y)]_{|t=t'} = [\phi_{in}(x),\phi_{in}(y)]_{|t=t'} = i\delta^{(3)}(x,y) \ . \tag{252}$$

But, using translation invariance, we can write

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \sum_n \langle 0|\phi(0)|n\rangle e^{ip_n\cdot(x-y)}\langle n|\phi(0)|0\rangle - (x\leftrightarrow y) \ , \tag{253}$$

where the sum goes over all the n–inparticle states. In the corresponding calculation for the in-field the only matrix element that contributes is $\langle 0|\phi_{in}|1\rangle$. Reasoning along these lines one finds that whenever further matrix elements contribute to eq. (253) it must be true that $0 \le Z < 1$; the result $Z = 0$ would be embarrassing but it is what we will actually compute from perturbation theory later on. I don't give the full argument since this is the only point where I am in the pleasant position of being able to refer you to original work by a Swedish field theorist (Gunnar Källén).

The strategy is now to "remove the ps" from the S-matrix elements in such a way that they become vacuum expectation values of strings of field operators. Thus, if we have n incoming and m outgoing particles

$$\begin{aligned}\langle p_1'p_2' \ldots p_m'|p_1p_2 \ldots p_n\rangle = \langle p_1'p_2' \ldots p_m'|a_{in}^\dagger(p_n)|p_1p_2 \ldots p_{n-1}\rangle = \\ = \langle p_1'p_2' \ldots p_m'|a_{out}^\dagger(p_n)|p_1 \ldots p_{n-1}\rangle + \langle p_1' \ldots p_m'|a_{in}^\dagger(p_n) - a_{out}^\dagger(p_n)|p_1 \ldots p_{n-1}\rangle.\end{aligned} \tag{254}$$

The first term here is non-vanishing only if there is a particle with momentum equal to p_n in the out-state. In effect there is a particle that does not participate

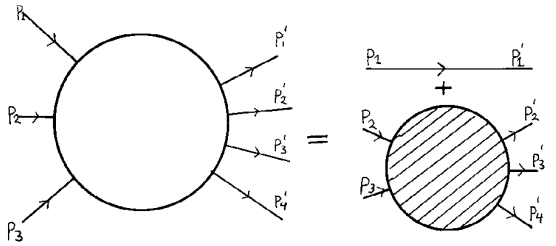

Figure 14: Splitting the S-matrix into connected parts.

in the scattering process. This is not so interesting. Let us dismiss terms of this kind by calling them "forward scattering amplitudes" and then proceed with the evaluation of the "connected" part of the amplitude.

We bring the in- and out-fields into the game. Thus the second term above can be written

$$\frac{i}{\sqrt{(2\pi)^3}} \int d^3x e^{ip_n \cdot x} \overset{\leftrightarrow}{\partial}_t \langle p'_1 p'_2 \ldots p'_m | \phi_{in}(x) - \phi_{out}(x) | p_1 \ldots p_{n-1} \rangle \ . \tag{255}$$

Using the asymptotic condition this is

$$\frac{i}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{Z}} (\lim_{t\to-\infty} - \lim_{t\to\infty}) \int d^3x e^{ip_n \cdot x} \overset{\leftrightarrow}{\partial}_t \langle p'_1 p'_2 \ldots p'_m | \phi(x) | p_1 \ldots p_{n-1} \rangle \ . \tag{256}$$

Next we use an identity that holds for any two functions f and g such that f obeys the Klein-Gordon equation and partial integrations can be done freely:

$$\begin{aligned} &(\lim_{t\to-\infty} - \lim_{t\to\infty}) \int d^3x \ f \overset{\leftrightarrow}{\partial}_t g = \\ &= -\int_{-\infty}^{\infty} dt \int d^3x \ \partial_t (f \overset{\leftrightarrow}{\partial}_t g) = \int d^4x f(\Box - m^2) g \ . \end{aligned} \tag{257}$$

Using this the connected part of the amplitude becomes

$$\begin{aligned} &\langle p'_1 \ldots p'_m | p_1 \ldots p_n \rangle = \\ &= \frac{i}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{Z}} \int d^4x e^{ip_n \cdot x} (\Box - m^2) \langle p'_1 \ldots p'_m | \phi(x) | p_1 \ldots p_{n-1} \rangle \ . \end{aligned} \tag{258}$$

Effectively we have "removed one p" from the amplitude. We go on in this way. An interesting thing happens in the next step:

$$\begin{aligned}\langle p'_1 \ ... \ p'_m|\phi(x)|p_1 \ ... \ p_n\rangle &= \text{forward scattering term} + \\ +\langle p'_1 \ ... \ p'_m|\phi(x)a^\dagger_{in}(p_{n-1}) - a^\dagger_{out}(p_{n-1})\phi(x)|p_1 \ ... \ p_{n-2}\rangle \ .&\end{aligned} \tag{259}$$

Note the ordering here. In both terms that operator which is associated to the later time is placed to the left. Neglecting the forward scattering term and going through the same motions as before we obtain

$$\begin{aligned}&\langle p'_1 \ ... \ p'_m|p_1 \ ... \ p_n\rangle = \\ &= \left(\frac{i}{\sqrt{(2\pi)^3 Z}}\right)^2 \int d^4x e^{ip_n\cdot x}(\Box_x - m^2)\int d^4y e^{ip_{n-1}\cdot y}(\Box_y - m^2)\times \\ &\times\langle p'_1 \ ... \ p'_m|T(\phi(x)\phi(y))|p_1 \ ... \ p_{n-2}\rangle \ .\end{aligned} \tag{260}$$

The two field operators are time ordered. If x_i and x_j are at spacelike separation the time ordering is ambiguous but the operators commute, so the ordering is well defined when it matters.

Repeating the above procedure as many times as necessary we arive at the LSZ formula for the S-matrix:

$$\begin{aligned}&\langle p'_1 \ ... \ p'_m|p_1 \ ... \ p_n\rangle = \text{forward scattering terms} + \\ &+\left(\frac{i}{\sqrt{(2\pi)^3 Z}}\right)^{n+m}\prod_i\int d^4x_i e^{ip_i\cdot x_i}(\Box_i - m^2)\cdot\prod_j\int d^4x_j e^{-ip'_j\cdot x_j}(\Box_j - m^2) \\ &\times\langle 0|T(\phi(x_1)\phi(x_2) \ ... \ \phi(x_{n+m}))|0\rangle \ .\end{aligned} \tag{261}$$

In this way the problem of calculating the S-matrix has been reduced to the problem of calculating the n-point Green functions

$$\tau(x_1, \ ... \ , x_n) = \langle 0|T(\phi(x_1) \ ... \ \phi(x_n))|0\rangle \ . \tag{262}$$

The treatment so far is exact, it is in the Heisenberg picture, and it can be made reasonably rigourous. It is an important point that we have in effect been able to show that the entire content of the theory is contained in the n–point

vacuum correlation functions, in this case vacuum expectation values of strings of time ordered field operators.

Spinor and vector fields can be treated in a quite analogous manner. Of course the expressions will then be decorated with spinors u and v or polarization vectors ϵ_r^α on the "external legs" and, in the case of gauge fields, we will also have to make sure that the in and out states are physical states, so there is a certain amount of additional work to be done.

With the first step of the derivation of the Feynman rules complete we take the second. We will set up a perturbation theory that enables us to compute the Green functions. To do so we switch to the interaction picture, which—for reasons hinted at in the first lecture—means that we take leave of rigour altogether. We will treat the field operators as free fields, obeying linear equations of motion, and the rest of the time dependence will be in the states. We introduce the evolution operator as the time ordered exponential

$$U(t_2,t_1) = Te^{-i\int_{t_1}^{t_2} H_I(t)dt} \; ; \qquad U(t) \equiv U(t,-\infty) \; . \tag{263}$$

We set initial data for the states by means of

$$\lim_{t\to-\infty} |\Psi\rangle_{int} = |\Psi\rangle_H \; , \tag{264}$$

where the subscripts refer to the interaction and Heisenberg pictures, respectively. However, from now on we will drop the subscripts on the interaction picture states. At arbitrary times the interaction picture states and fields are defined by

$$|\Psi(t)\rangle = U(t)|\Psi\rangle_H \quad \text{and} \quad \varphi(x) = U(t)\phi(x)U^{-1}(t) \; . \tag{265}$$

We use φ to denote the field operator in the interaction picture. This field operator evolves like a free field would in the Heisenberg picture, while the state obeys

$$i\partial_t|\Psi(t)\rangle = H_I(t)|\Psi(t)\rangle \; . \tag{266}$$

We will assume (without any kind of justification!) that

$$|0\rangle = |0\text{in}\rangle_H \quad \text{and} \quad U(t)|0\rangle = e^{i\nu}|0\rangle \; , \tag{267}$$

where ν is some physically irrelevant phase. This is the assumption of vacuum stability, that is we assume that the interaction picture vacuum is stable under time evolution.

With these assumptions in hand we can begin to evaluate the Green functions. Suppose that $t_1 > t_2 > \; ... \; > t_n$. Then

$$\tau(x_1,\ \ldots\ ,x_n) = \langle 0|\phi(x_1)\ \ldots\ \phi(x_n)|0\rangle = \tag{268}$$
$$= \langle 0|U^{-1}(t_1)\varphi(x_1)U(t_1,t_2)\varphi(x_2)\ \ldots\ U(t_{n-1},t_n)\varphi(x_n)U(t_n)|0\rangle\ .$$

Now we introduce a time t which is large enough so that

$$t > t_1 \qquad -t < t_n\ . \tag{269}$$

That is, t is later than all the arguments in τ, and $-t$ is earlier than all the arguments in τ. It follows that

$$U(t_n) = U(t_n,-t)U(-t) \qquad U(t_1) = U(t_1,t)U(t)\ . \tag{270}$$

So

$$\tau(x_1,\ \ldots,x_n) = \langle 0|U^{-1}(t)U(t,t_1)\varphi(x_1)U(t_1,t_2)\ \ldots\ \varphi(x_n)U(t_n,-t)U(-t)|0\rangle\ . \tag{271}$$

But this can be written in terms of the time ordered exponential $U(t,-t)$. The result:

$$\tau(x_1,\ \ldots,x_n) = \langle 0|U^{-1}(t)T(\varphi(x_1)\ \ldots\ \varphi(x_n)e^{-i\int_{-t}^{t}H_I(t)dt})U(-t)|0\rangle\ . \tag{272}$$

We are interested in the limit $t\to\infty$. But $U(-\infty)$ is the unit operator while, according to the assumption (267) of vacuum stability,

$$\lim_{t\to\infty}\langle 0|U^{-1}(t) = \lim_{t\to\infty}\langle 0|U^{-1}(t)|0\rangle\langle 0| = \lim_{t\to\infty}\frac{1}{\langle 0|U(t)|0\rangle}\langle 0|\ . \tag{273}$$

We have arrived at the Gell-Mann–Low magical formula

$$\tau(x_1,\ \ldots,x_n) = \frac{\langle 0|T(\varphi(x_1)\ \ldots\ \varphi(x_n)e^{-i\int_{-\infty}^{\infty}H_I(t)dt})|0\rangle}{\langle 0|T(e^{-i\int_{-\infty}^{\infty}H_I(t)dt})|0\rangle}\ . \tag{274}$$

The second step in the derivation of the Feynman rules is complete.

From this platform it will be a straightforward exercise to derive the Feynman rules. It should be noted however that the "derivation" of the Gell-Mann–Low formula is at best heuristic. There is nothing wrong with the logic, only with the assumptions. The logic works perfectly well in non-relativistic quantum mechanics where the representation of the canonical commutation relations is unique. But that is not true here, and the assumption of vacuum stability is false. In fact it is quite respectable to regard the Feynman rules as the definition of the theory (together with the LSZ formula for how to convert the τ-functions into S-matrix elements). Everything else can then be forgotten.

WICK'S THEOREM AND THE FEYNMAN RULES

Thanks to the magical formula, the task of writing down either the τ-functions or the S-matrix elements in an interacting quantum field theory is basically solved. The idea is that you simply look at eq. (274), supplemented by eq. (262) if you are after the S-matrix elements, and immediately write down the terms in a power series expansion (in the coupling constant) that gives the answer. In practice what you do first is to derive something called the Feynman rules of the theory. They give you a strikingly memorable version of this procedure.

Of course the first paragraph here is an overstatement. It is not so easy to actually do this right, especially not if you are working with terms of some high order in the coupling constant. But it is not entirely inaccurate either—what the magical formula shows is that it is enough to be able to deal with strings of time ordered free field operators, and this is doable. The key statement here is known as Wick's theorem. It provides a way to rewrite such strings as sums of normal ordered terms whose vacuum expectation values are trivial to take. In a notation to be explained, the statement is that

$$\begin{aligned} T(\varphi_1 \dots \varphi_n) = {} & :\varphi_1 \dots \varphi_n: + \\ & + :\overline{\varphi_1\varphi_2}\ \varphi_3 \dots \varphi_n: + :\overline{\varphi_1\varphi_2\varphi_3}\ \varphi_4 \dots \varphi_n: + \dots \\ & + :\overline{\varphi_1\varphi_2}\ \overline{\varphi_2\varphi_4}\ \varphi_5 \dots \varphi_n: + \dots \ . \end{aligned} \tag{275}$$

The dots in this equation represent "all possible ways of doing the same thing", and the overlines connecting two field operators means that exactly this pair of operators should be removed from the expression and replaced with $i\Delta_{ij}$, where $\Delta_{ij} = \Delta(x_i, x_j)$ is the Feynman propagator connecting the two points in spacetime that appear as arguments of the field operators.[2] It is easier to explain the meaning of this statement with a few examples rather than trying to make the preceding sentence crystal clear. And anyway the proof of Wick's theorem is by induction on n, so you will need to go through the following motions anyway.

For $n = 2$ it is clear that

$$T(\varphi_1\varphi_2) =: \varphi_1\varphi_2 : + c\ , \tag{276}$$

where c is some classical c-number. We can evaluate it by taking the vacuum expectation value of the expression. Using eq. (75) we see that

[2] The standard notation here is a little different and much better. I cannot do it in Latex.

$$c = \langle 0|T(\varphi_1\varphi_2)|0\rangle = i\Delta_{12} \equiv \overline{\varphi_1\varphi_2} \ . \tag{277}$$

For $n = 3$, assume for the sake of the argument that the point x_3 is earlier than the others. Then

$$\begin{aligned} T(\varphi_1\varphi_2\varphi_3) = T(\varphi_1\varphi_2)\varphi_3 =: \varphi_1\varphi_2 : \varphi_3 + i\Delta_{12}\varphi_3 = \\ =: \varphi_1\varphi_2 : \varphi_3^{(+)} + : \varphi_1\varphi_2 : \varphi_3^{(-)} + i\Delta_{12}\varphi_3 \ , \end{aligned} \tag{278}$$

where we first used the result for $n = 2$ and then decomposed the field φ_3 into positive and negative frequency parts, that is to say into its annihilation and creation parts. If we now decompose also the other field operators in the same way and then move all the positive frequency parts to the right we find, for instance, that

$$\begin{aligned} : \varphi_1\varphi_2 : \varphi_3^{(-)} = : \varphi_1\varphi_2\varphi_3^{(-)} : + \\ + \varphi_1^{(-)}[\varphi_2^{(+)}, \varphi_3^{(-)}] + \varphi_2^{(-)}[\varphi_1^{(+)}, \varphi_3^{(-)}] + \varphi_1^{(+)}[\varphi_2^{(+)}, \varphi_3^{(-)}] + \varphi_2^{(+)}[\varphi_1^{(+)}, \varphi_3^{(-)}] \ . \end{aligned} \tag{279}$$

But the commutators that appear here are c-numbers and can therefore be evaluated by taking vacuum expecation values;

$$[\varphi_1^{(+)}, \varphi_3^{(-)}] = \langle 0|\varphi_1^{(+)}\varphi_3^{(-)}|0\rangle = \langle 0|\varphi_1\varphi_3|0\rangle = \langle 0|T(\varphi_1\varphi_3)|0\rangle = i\Delta_{13} \ , \tag{280}$$

where we used our assumption about the relative time ordering of our points. Collecting things together we arrive in this way to

$$T(\varphi_1\varphi_2\varphi_3) =: \varphi_1\varphi_2\varphi_3 : + i\Delta_{12}\varphi_3 + i\Delta_{23}\varphi_1 + i\Delta_{13}\varphi_2 \ . \tag{281}$$

This is eq. (275) specialized to the case $n = 3$. The meaning of eq. (274) should now have transpired and it should also be clear that a proof by induction is possible for all n.

Actually when we apply Wick's theorem to the magical formula we need only the final (purely c-number) terms, where all the field operators have been "contracted" away—the normal ordered products of field operators do not contribute to the vacuum expectation value.

To see why Wick's theorem helps, let us consider a simple example in a simple theory, such as the ϕ^4 theory of eq. (242). Before doing so, it is perhaps as well to admit that in more complicated theories things are more difficult and sharper tools than Wick's may be needed. Yang-Mills theory is an example

of this. On the other hand for spinorial QED, which is what we are mostly interested in, Wick's theorem works fine. Anyway, the case at hand has

$$H_I = \int d^3x \, \frac{\lambda}{4!} : \varphi^4 : \ . \tag{282}$$

We include the normal ordering to make the resulting formulæ better defined; effectively it means that we will omit all contractions like $\overline{\phi(x)\phi(x)} = i\Delta(x,x)$. This is just as well since this is a divergent expression. Now we focus on the four point function $\tau(x_1, x_2, x_3, x_4)$ for definiteness. When expanded as a power series in the coupling constant the numerator in the magical formula (274) becomes

$$\begin{aligned} &\langle 0|T(\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle - \lambda\frac{i}{4!}\int d^4x \langle 0|T(\varphi_1\varphi_2\varphi_3\varphi_4 : \varphi\varphi\varphi\varphi :)|0\rangle - \\ &-\lambda^2\frac{1}{4!4!}\int d^4x \int d^4x' \langle 0|T(\varphi_1\varphi_2\varphi_3\varphi_4 : \varphi\varphi\varphi\varphi :: \varphi'\varphi'\varphi'\varphi' :)|0\rangle + \dots \ , \end{aligned} \tag{283}$$

where φ is taken at x and φ' at x' (both of which arguments are then integrated over). When we apply Wick's theorem to this expression a collection of products of Feynman propagators between various points will appear. There will also be a contribution from the denominator of the magical formula that actually simplifies matters in the sense that it cancels some of the terms from the numerator. Even without doing the detailed calculation it should be clear that each individual term in the expansion can be described in a graphical manner: If we draw the points, including the points x and x' that we are going to integrate over, and connect pairs of these points with a line if and only if there is a Feynman propagator connecting these two points with each other, then we obtain a pictorial representation of the term in question.

You may feel that my description is getting increasingly schematic. There is a reason for this that I will explain soon.

In the end—when we turn to the LSZ formula for the S-matrix amplitude—we will be more interested in having our formulæ in momentum space. It is therefore convenient to make a Fourier transformation to the momentum space n-point function

$$\tilde{\tau}(p_1, \dots, p_n) = \int d^4x_1 \dots \int d^4x_n e^{\pm ip_1\cdot x_i} \dots e^{\pm ip_n\cdot x_n} \tau(x_1, \dots, x_n) \ , \tag{284}$$

where the signs depend on whether the momenta or out- or ingoing. Then we can state the content of the magical formula as being equivalent to the following Feynman rules: To compute $\tilde{\tau}(p_1, \dots, p_n)$ to any given order in λ use the building blocks in Fig. 15 a, called propagators and vertices respectively. Note

that each propagator has an arrow along which its "momentum" flows, and also that Kirchhoff's rules apply when different momenta meet at a vertex (i.e., the momenta are conserved at the vertices). Using these building blocks, draw all topologically inequivalent diagrams connecting the n external points. If we want to do perturbation theory to order λ^p we include all diagrams containing at most p vertices, and no more. For the 2 and 4 point functions computed to second order this gives the collection of diagrams given in Fig. 15 b. (At higher orders this can get pretty complicated, as shown in Fig. 15 c. Actually, in drawing this set I have started from an interaction Hamiltonian that is not normal ordered. This reflects a more modern attitude to mass renormalization and does not matter at this stage.) All the time we are building a formula corresponding to each diagram, since each propagator in the diagram corresponds to a factor

$$\frac{i}{p^2 + m^2} \tag{285}$$

and each vertex corresponds to a factor $-i\lambda$.

To see if a diagram is topologically inequivalent to another imagine their propagators to be made of rubber (say) and nail down their n free ends on a table. If one configuration can be transformed to another without moving the ends the diagrams are topologically equivalent, otherwise not. Fig. 16 a illustrates this for a theory containing three point vertices only. When a diagram contains closed loops Kirchhoff's rules will leave a momentum running around the loop undetermined. The rule is that the formula corresponding to the diagram gets a prefactor

$$\int \frac{d^4p}{(2\pi)^4} \tag{286}$$

where p is the momentum running around the loop (see Fig. 16 b). Finally the formula should be equipped with an overall momentum conserving delta function

$$(2\pi)^4 \delta^{(4)}(\sum p_{in} - \sum p_{out}) \ . \tag{287}$$

Actually this is not quite the end. Some diagrams will be equipped with an extra combinatorial factor that equals one over the number of ways in which the diagram can be transformed into itself; Fig. 16 c shows what this factor is in some concrete cases. (In practice, the combinatorial factor is the only detail that one is likely to get wrong when using the Feynman rules to low orders. Fortunately QED does not have such factors.)

This is it. In a sense it is all. To get the corresponding rules for how to compute S-matrix elements use the LSZ formula—essentially it just means that we "chop off the external legs" from the τ-functions.

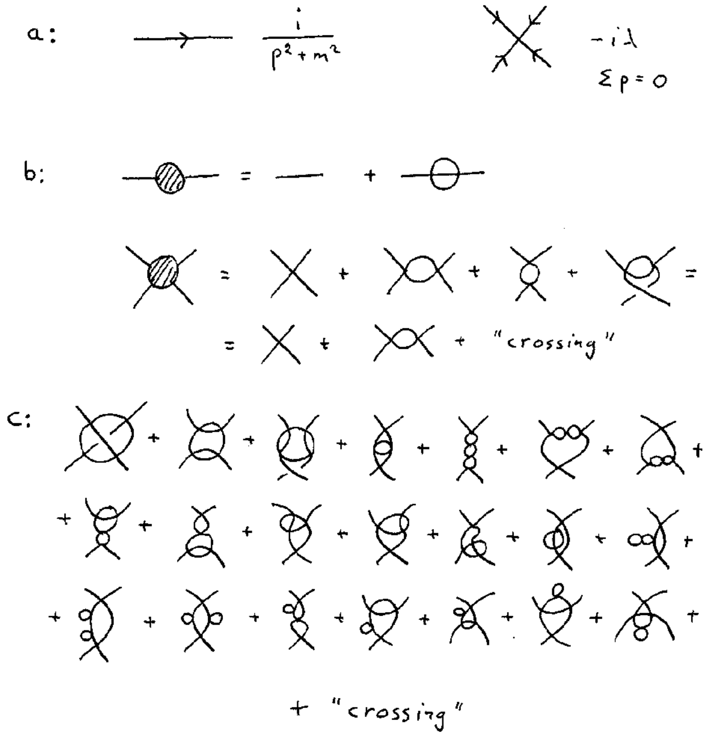

Figure 15: Feynman diagrams.

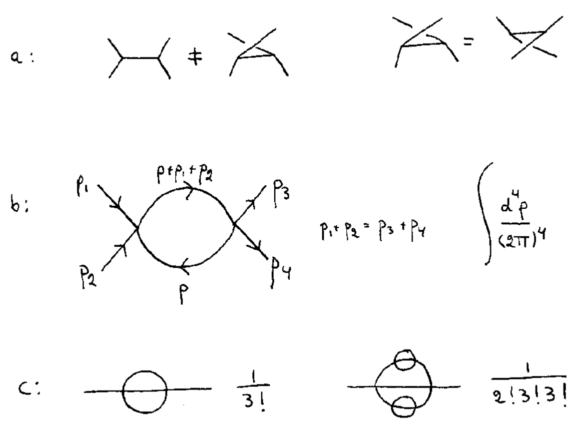

Figure 16: More about Feynman diagrams.

My description has been quite sketchy. The reason is that it is not really very helpful to see the derivation of the Feynman rules described. It is necessary to derive them oneself for a few simple cases at least. Thus we have the following Exercise:

- Derive a formula for the four point $\tilde{\tau}$-function up to second order in λ from scratch, that is, starting from the magical formula. In the process, verify the Feynman rules as described above.

If you do this carefully you will afterwards realize that, except for the combinatorial factor, it is indeed possible to see directly what the answer is. It will also be clear that Feynman rules for any reasonably simple quantum field theory (like QED) can be derived in the same manner.

Of course the evaluation of the τ-function is not complete just because we have derived the Feynman rules. The answer is some function of the external momenta containing lots of terms, as well as integrals over undetermined variables that remain to be computed before we know what the terms really are. So there is work to be done. Some reflection shows that there may well be trouble too. In x-space the τ-functions consist basically of long strings of Feynman propagators, some of them connecting points that we are going to integrate over. But we know that the Feynman propagator is not a function, it is a distribution that has delta function singularities when its two arguments are at lightlike distances from each other. Therefore a typical τ-function will contain products of delta functions. And such things do not really exist. This difficulty is a real one, but it also has a solution called renormalization. We will come to it. Further worries can be raised, like the question whether the power series expansion converges. This we will simply ignore.

CROSS SECTIONS

At this point my notes will become even more sketchy than they used to be, and will in fact rely heavily on Mandl and Shaw. In particular, I will frequently use their notation rather than my own. The main difference is that they normalize the fields using a finite box of side L and volume $V = L^3$. The momentum of the single particle states is then discrete;

$$p_i = \frac{2\pi}{L}(n_1, n_2, n_3) \ . \tag{288}$$

It will become evident very soon why I want to switch to a box.

In principle the calculation of probability amplitudes for arbitrary scattering processes, as power series in the coupling constant, should now be straightforward. In fact it is not—we still have to renormalize the theory. There is another point to worry about: In the end we are not interested in probability amplitudes, we want probabilities. As an intermediate step we compute probability densities by squaring the amplitudes—and immediately there is a catch, because the amplitudes contain unsquarable delta functions. Fortunately this is a trivial difficulty that is easily resolved with a little thought.

Mandl and Shaw give the S-matrix amplitudes in QED on the form

$$\langle p' \mathrm{out} | p \mathrm{in} \rangle = \delta_{fi} + \left[(2\pi)^4 \delta^{(4)}(p_f, p_i) \prod_{fermions} \sqrt{\frac{m}{VE}} \prod_{photons} \sqrt{\frac{1}{2V\omega}} \right] \times \mathcal{M} \ , \tag{289}$$

where the Feynman rules for the amplitude $\mathcal{M}$ are given on their page 129. Now it must be kept in mind that the whole formalism is seriously over–idealized. It describes perfectly sharp momentum states, plane waves in fact. What we really have is a localized beam of incoming particles. Moreover we are unlikely to place any detectors in the beam itself, so that forward scattering is of no interest really. That is to say that we can split $S_{fi} = \delta_{fi} + iT_{fi}$ and concentrate on the transition matrix T, which contains all the off–diagonal elements.

But we must square the delta function. This is where the box (and a finite time interval T) is so helpful: We first approximate the delta functions with the well behaved functions

$$\delta(E, E') \approx \frac{1}{2\pi} \int_{-T/2}^{T/2} dt \ e^{i(E-E')t} \tag{290}$$

$$\delta^{(3)}(p, p') \approx \int_V \frac{d^3x}{(2\pi)^3} e^{i(p_i - p'_i)x_i} = \frac{V}{(2\pi)^3} \delta_{p,p'} \ , \tag{291}$$

where only the discrete set of momenta given in eq. (288) appears in eq. (291). Now we can square the delta function by squaring these well behaved functions. Thus

$$\left(\delta^{(4)}(p_f, p_i)\right)^2 = \frac{VT}{(2\pi)^4}\delta^{(4)}(p_f, p_i) \ . \tag{292}$$

If we take the limits T, $V \to \infty$ this expression diverges, as expected. But we had a similar difficulty when we discussed the Unruh effect, so we know that we are not interested in the probability that something scatters. We are interested in the transition probability per unit time. Let us call it w. We get it by dividing through with the time interval T, so that

$$w = \frac{|T_{fi}|^2}{T} = V(2\pi)^4\delta^{(4)}(p_f, p_i) \prod_{fermions\ in} \frac{m}{VE} \prod_{fermions\ out} \frac{m}{VE_f}|\mathcal{M}|^2 \ . \tag{293}$$

For simplicity we are assuming that there are no external photons. Indeed we will specialize right away to the case of two incoming fermions only. Of course any number of fermions are allowed to come out. (This is an interesting case.)

We can now take the limit $T \to \infty$. The infinite volume limit still looks bad, but then our problem remains over–idealized. Real detectors will have a finite resolution and measure momenta belonging to some interval $[p, p + dp]$. Given eq. (288) the number of states in this interval will be

$$d^3n = \frac{V}{(2\pi)^3}d^3p \ . \tag{294}$$

We are really interested in wd^3n rather than in w as such. Moreover we are not interested in the transition probability per unit time but in the transition probability per unit time and unit incoming flux. The flux of incoming particles is the number of particles coming in per unit of time and area, which is equal to the volume that passes the surface per unit time divided by the total volume V. This works out as

$$\text{flux} = \frac{v_{rel}\Delta t/\Delta t}{V} = \frac{v_{rel}}{V} \ , \tag{295}$$

where v_{rel} is the relative velocity of the colliding particles. With this result we can write down the quantity that we are interested in as

$$d\sigma = \frac{w\prod_f \frac{Vd^3p_f}{(2\pi)^3}}{\text{flux}} = (2\pi)^4\delta^{(4)}(p_f, p_i)\frac{m_1m_2}{E_1E_2v_{rel}}\prod_f \frac{md^3p}{(2\pi)^3E_f}|\mathcal{M}|^2 \ . \tag{296}$$

This interesting quantity is independent of both V and T, so we can now take the limit where these go to infinity.

There is one quibble though, which is that the Lorentz tranformation properties of our expression are a little obscure. However, a calculation in the rest frame of one of the incoming particles shows that

$$(P_1 \cdot P_2)^2 - m_1^2 m_2^2 = m_1^2 m_2^2 \frac{v^2}{1 - v^2} \ . \tag{297}$$

Hence the bothersome factor $E_1 E_2 v_{rel}$ is really a Lorentz scalar;

$$E_1 E_2 v_{rel} = \sqrt{(P_1 \cdot P_2)^2 - m_1^2 m_2^2} \ . \tag{298}$$

So everything is as it should be. Of course our expression is not given as a function of independent variables—thus if we have two particles coming out momentum conservation implies that the only free variables are the angles θ, ϕ describing the scattering. This is why our expression still contains a delta function. We are looking at an integrand, not at a function.

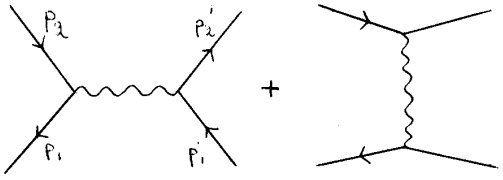

Figure 17: Tree diagrams for lepton-lepton scattering. If the outgoing pair is different from the incoming, the second diagram is absent.

A SIMPLE QED CALCULATION

We would now like to calculate, to lowest non-trivial order, something real. For this reason we turn to QED, take our Feynman rules from Mandl and Shaw, and look at lepton-lepton scattering. The process

$$e^+ + e^- \to e^+ + e^- \tag{299}$$

is known as Bhabha scattering, and to second order in the coupling constant there are two relevant Feynman diagrams. We can simplify matters even further by looking at

$$e^+ + e^- \to \mu^+ + \mu^- \ . \tag{300}$$

Then there is only one relevant diagram, so the task of adding interfering scattering amplitudes trivializes. Using the Feynman rules the task of writing down the amplitude is trivial too. The answer is

$$\mathcal{M}_{r_1 r_2 s_1 s_2} = -ie^2 \bar{u}_{s_2}(p_2')\gamma_\alpha v_{s_1}(p_1')\bar{v}_{r_1}(p_1)\gamma^\alpha u_{r_2}(p_2)\frac{1}{(p_1+p_2)^2} \ . \tag{301}$$

Here we have specified the spin states r_1 and r_2 of the incoming electron-positron pair as well as the spin states s_1 and s_2 of the outgoing muons.

Extracting the cross section requires some work even in this ultra-simple example. The first step must be to square the amplitude. We still have one simplifying (but realistic) assumption up our sleeve. We are going to assume that the incoming beam is unpolarized, and that the spin states of the outgoing particles are not measured. It is perhaps not obvious that this will simplify matters, but it will, as the calculation to follow shows. Summing over outgoing and averaging over incoming spin states gives the amplitude squared as

$$|\mathcal{M}|^2 = \frac{1}{4}\sum_{r_1 r_2 s_1 s_2} |\mathcal{M}_{r_1 r_2 s_1 s_2}|^2 = \frac{e^2}{4}\frac{1}{[(p_1+p_2)^2]^2}A_{\alpha\beta}B^{\alpha\beta} \ , \tag{302}$$

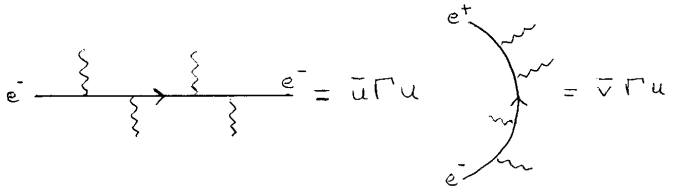

Figure 18: Fermion lines go straight through the diagrams.

where

$$A_{\alpha\beta} = \sum_{s_1 s_2} \bar{u}_{s_2}(p_2')\gamma_\alpha v_{s_1}(p_1')\bar{v}_{s_1}(p_1')\gamma_\beta u_{s_2}(p_2') \tag{303}$$

$$B_{\alpha\beta} = \sum_{r_1 r_2} \bar{v}_{r_1}(p_1)\gamma_\alpha u_{r_2}(p_2)\bar{u}_{r_2}(p_2)\gamma_\beta v_{r_1}(p_1)\ . \tag{304}$$

We can rewrite these expressions as traces over matrices. The good news is that—precisely because we are summing over spin states—the spinors describing the states turn into projection operators that are easy to deal with in such a calculation. In fact we can write

$$\begin{aligned} A_{\alpha\beta} = \mathrm{Tr}\left[\sum_{s_2} u_{s_2}(p_2')\bar{u}_{s_2}(p_2')\right]\gamma_\alpha \left[\sum_{s_1} v_{s_1}(p_1')\bar{v}_{s_1}(p_1')\right]\gamma_\beta = \\ = \mathrm{Tr}\frac{\gamma\cdot p_2' + m_\mu}{2m_\mu}\gamma_\alpha\frac{\gamma\cdot p_1' - m_\mu}{2m_\mu}\gamma_\beta\ . \end{aligned} \tag{305}$$

Here m_μ is the muon mass. Similarly, with m equal to the electron mass,

$$B_{\alpha\beta} = \mathrm{Tr}\frac{\gamma\cdot p_1 - m}{2m}\gamma_\alpha\frac{\gamma\cdot p_2 + m}{2m}\gamma_\beta\ . \tag{306}$$

A little reflection shows that we have come across a general phenomenon. In QED, all fermion lines go "straight through the diagram" and for this reason they will always contribute a trace to the cross section. Of course these traces may be rather long, and if we do not sum over spin states something worse than projectors will occur in them. Still, calculating the trace of a string of gamma-matrices is always doable.

Having done so, we obtain

$$|\mathcal{M}|^2 = \frac{e^4(p_1\cdot p_1'\ p_2\cdot p_2' + p_1\cdot p_2'\ p_2\cdot p_1' + m^2 p_1'\cdot p_2' + m_\mu^2 p_1\cdot p_2 + 2m^2 m_\mu^2)}{2m^2 m_\mu^2 (p_1+p_2)^4}\ . \tag{307}$$

Next we have to think a little about the kinematics of the process. Let us describe it in the center of mass system. The only free parameter is the scattering angle θ, everything else is determined by energy-momentum conservation. Define

$$p_1 = (E, \mathbf{p}) \quad p_2 = (E, -\mathbf{p}) \quad p_1' = (E, \mathbf{p}') \quad p_2' = (E, -\mathbf{p}') \ . \tag{308}$$

We have $(p_1 + p_2)^2 = 4E^2$ and

$$p_1 \cdot p_1' = p_2 \cdot p_2' = E^2 - pp' \cos\theta \tag{309}$$

$$p_1 \cdot p_2' = p_2 \cdot p_1' = E^2 + pp' \cos\theta \tag{310}$$

$$p_1 \cdot p_2 = E^2 + p^2 \qquad p_1' \cdot p_2' = E^2 + p'^2 \ . \tag{311}$$

We must have that $E \geq m_\mu$. When evaluating $|\mathcal{M}|^2$ we will permit ourselves the approximation that the electron mass $m \approx 0$, since $m_\mu = 207 \cdot m$.

We must now stick our expression for the amplitude squared into eq. (296) for the cross section. Because it has not been expressed in terms of independent variables, the latter contains delta functions that we want to get rid off. Fortunately we are looking at a particularly simple case (two particles in, two particles out). The cross section is of the form

$$d\sigma = f(p_1', p_2')\delta(E_1' + E_2' - E_1 - E_2)\delta^{(3)}(p_1' + p_2' - p_1 - p_2)d^3p_1' d^3p_2' \tag{312}$$

with

$$f(p_1', p_2') = (2\pi)^4 \frac{m^2}{\sqrt{(p_1 \cdot p_2)^2 - m^2 m^2}} \frac{m_\mu}{(2\pi)^3 E_1'} \frac{m_\mu}{(2\pi)^3 E_2'} \ . \tag{313}$$

We begin by integrating out p_2'. This gives

$$d\sigma = f(p_1', p_2')\delta(E_1' + E_2' - E_1 - E_2)|p_1'|^2 d|p_1'| d\Omega'_{|\mathbf{p}_2' = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1'} \ . \tag{314}$$

But in our case $E_1' + E_2'$ is some function of $|p_1'|$, and we can make use of the delta function identity $\delta(f(x)) = \left|\frac{df}{dx}\right|^{-1} \delta(x)$ in doing the integration over $|p_1'|$. Thus

$$d\sigma = \frac{f(p_1', p_2')}{\frac{\partial(E_1' + E_2')}{\partial |p_1'|}} d\Omega' \ . \tag{315}$$

In the center of mass system we have $|p_1'| = |p_2'|$ and

$$\frac{\partial(E_1' + E_2')}{\partial|p_1'|} = \frac{|p_1'|}{E_1'} + \frac{|p_2'|}{E_2'} = |p_1'|\frac{E_1' + E_2'}{E_1' E_2'} \ . \tag{316}$$

We are finally getting there. Collecting things together we have

$$d\sigma = \frac{1}{4\pi^2}\frac{m^2 m_\mu^2}{(E_1 + E_2)^2}\frac{|p_1'|}{|p_1|}|\mathcal{M}|^2 d\Omega' \ , \tag{317}$$

which is the integrand of an integral over the only free parameters around, namely the scattering angles.

Fetching our expression for the amplitude squared, ignoring the electron mass as agreed, and doing some algebra, we arrive at

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{c.o.m.}} = \frac{e^4}{4\pi^2}\frac{1}{16}\frac{|p_1'|}{E^5}(E^2 + p'^2\cos^2\theta + m_\mu^2) \ . \tag{318}$$

In the high energy limit where the muon mass can be ignored as well this is

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{c.o.m.}} = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta) \ . \tag{319}$$

There is a good agreement with experiment.

WHAT IS A FEYNMAN DIAGRAM?

The perturbation expansion that we are using is just a form of time ordered perturbation theory. A Feynman diagram is an individual term in such an expansion. But things have been organized in a rather peculiar manner. In a Feynman diagram the lines are assigned "momenta" p^α that are conserved at the vertices. If you think about it, this is a bit strange because "energy" in perturbation theory usually refers to the eigenvalues of the free Hamiltonian H_0, and this energy is not conserved by the interaction. The true—and conserved—energy of the system is an eigenvalue of the total Hamiltonian H_0+H_I. Moreover in perturbation theory we usually keep careful track of the "intermediate states", which are eigenstates of H_0. That is to say, we expect an expansion of the general form

$$S_{fi} = \delta_{fi} + \delta(E_f - E_i)\left(V_{fi} + \sum_\alpha \frac{V_{f\alpha}V_{\alpha i}}{E_i - E_\alpha} + \sum_{\alpha,\beta} \frac{V_{f\beta}V_{\beta\alpha}V_{\alpha i}}{(E_i - E_\beta)(E_i - E_\alpha)} + \dots\right) \tag{320}$$

where we use the matrix elements

$$V_{fi} = \langle f|H_I|i\rangle \tag{321}$$

and so on. The total energy is conserved, $E_i = E_f$, but $E_i \neq E_\alpha$ since the "free" energy is not conserved by the interaction.

In the Feynman diagram expansion things look very different. Choosing a scalar ϕ^3-theory because it is the simplest example, we deal with terms like

$$T_{fi} = \frac{\lambda^2}{(p_1 + p_2)^2 - m^2} + \text{crossing} \ . \tag{322}$$

The "momentum" $p^\alpha = (E, p_i)$ is conserved at the vertices but on the other hand the "energy"

$$E_1 + E_2 = \sqrt{p_1^2 + m^2} + \sqrt{p_2^2 + m^2} \neq \sqrt{(p_1 + p_2)^2 + m^2} \equiv \omega_{1+2} \tag{323}$$

is not the energy of any intermediate particle—if it were the denominator in eq. (322) would be zero. Moreover there seems to be no reference to any global intermediate state of the whole system.

To see how eq. (322) is related to eq. (320) we rewrite the former a little. Observe that

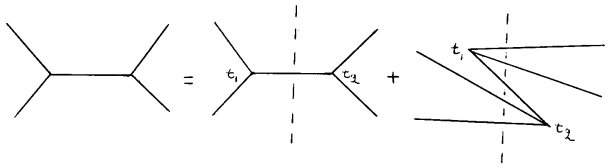

Figure 19: Taking a Feynman diagram apart to keep track of the states.

$$\frac{\lambda^2}{(E_1+E_2)^2-(p_1+p_2)^2-m^2}=\frac{\lambda^2}{(\omega_1+\omega_2)^2-\omega_{1+2}^2}= \\ =\frac{\lambda^2}{2\omega_{1+2}}\left(\frac{1}{\omega_1+\omega_2-\omega_{1+2}}-\frac{1}{\omega_1+\omega_2+\omega_{1+2}}\right)\ . \tag{324}$$

Just a little bit of extra rewriting is needed to see what goes on. We get

$$\frac{\lambda^2}{2\omega_{1+2}}\left(\frac{1}{\omega_1+\omega_2-\omega_{1+2}}+\frac{1}{\omega_1+\omega_2-2\omega_1-2\omega_2-\omega_{1+2}}\right)\ . \tag{325}$$

We see that this really is like eq. (320); the Feynman diagram is in fact a sum of two terms containing rather different intermediate states. In one case there is a single particle with the "free" energy ω_{1+2}, in the other there is a total of five particles.

In this way we can translate the Feynman diagrams into the language familiar from standard quantum mechanics. It only remains to observe that the new language is much better than the old. So we tend to think of the Feynman diagram as describing intermediate "off shell" particles for which $P^2-m^2\neq 0$. Therefore the denominators are not dangerous in general. But there is a case where the intermediate particles come arbitrarily close to being "on shell". This difficulty will occupy us next.

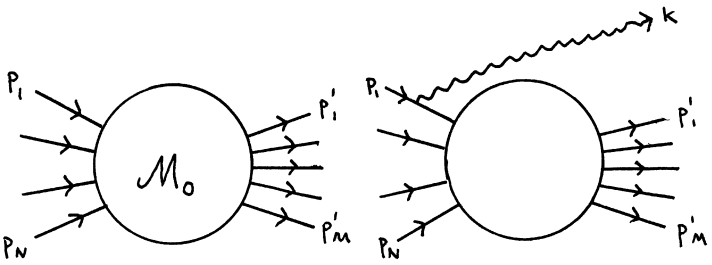

Figure 20: An arbitrary amplitude, to which we add one photon to an external leg.

SOFT PHOTONS

First, consider an arbitrary amplitude at some fixed order in the coupling constant, given by a sum of arbitrarily complicated Feynman diagrams. Let there be N charged particles in and M charged particles out. There could be photons too, but those do not play any role in the argument that follows.

The first observation is that it is peculiar that it is at all possible to have non-trivial amplitudes without any outgoing photons—indeed one expects that charged particles that change their state of motion must radiate. At the end of the argument that is started here, it will be seen that the expectation is correct even though the non-trivial amplitudes do exist to lowest order in the coupling constant.

Now consider an amplitude $\mathcal{M}$ which differs from the given amplitude $\mathcal{M}_0$ only in that there is one extra outgoing photon. We will study this new amplitude in the limit that the extra outgoing photon becomes very soft, that is $k^0 \to 0$ for this particle. Let us first attach the extra photon vertex to an external leg, for definiteness let it be that of an incoming electron. The Feynman rules (from Mandl and Shaw) tell us that

$$\mathcal{M}_0 = (\text{something}) u_r(p) \ . \tag{326}$$

With the extra photon attached we get

$$\begin{aligned} \mathcal{M} &= (\text{something}) \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} ie\gamma^\alpha u_r(p) \epsilon_\alpha(k) = \\ &= e(\text{something}) \frac{\not{p} - \not{k} + m}{2p \cdot k} \not{\epsilon}(k) u_r(p) \ . \end{aligned} \tag{327}$$

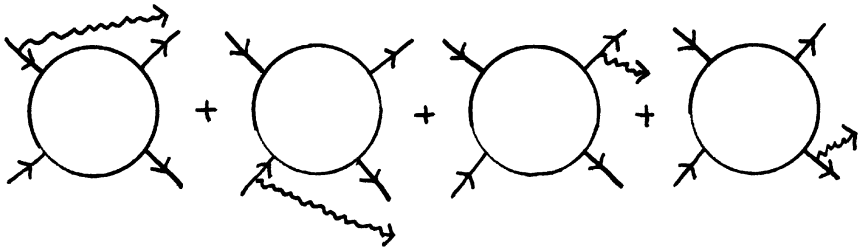

Figure 21: An amplitude that differs from a given one by the presence of one extra soft photon.

In the denominator we used the fact that $p^2 - m^2 = k^2 = 0$. Now in the limit when k^0 becomes very small, that is that the extra photon is very soft, we can make an approximation, use the gamma-matrix algebra, and then the Dirac equation for u:

$$(\not{p} - \not{k} + m)\ \not{\epsilon}(k) u_r(p) \approx (\not{p} + m)\ \not{\epsilon}(k) u_r(p) = 2p \cdot \epsilon u_r(p)\ . \tag{328}$$

We conclude that

$$\mathcal{M} = e(\text{something}) \frac{p \cdot \epsilon}{p \cdot k} u_r(p) = e \frac{p \cdot \epsilon}{p \cdot k} \mathcal{M}_0\ . \tag{329}$$

So this particular contribution to the amplitude with one extra soft photon added is obtained simply by multiplying the original amplitude with a factor. Moreover because of the denominator of that factor it is clear that it blows up in the limit when the extra photon becomes very soft. This blow up happens precisely because the electron line to which the extra photon was attached was an external electron line, obeying $p^2 - m^2 = 0$. Hence diagrams where the extra photon is attached to internal electron lines will not contribute noticeably to the amplitude in the limit when the extra photon becomes soft; in that limit the complete amplitude $\mathcal{M}$ will be given as a sum over all possible ways of attaching the extra photon to the external legs.

In conclusion then, given an amplitude that describes (to some order in e) N particles in and M particles out, the amplitude to one higher order in e describing N particles in and $M + 1$ particles out, with the condition that one of the latter is a soft photon, is

$$\mathcal{M} = \left(\sum_{in} \frac{e_i p_i^\alpha}{p_i \cdot k} - \sum_{out} \frac{e_i' p_i'^\alpha}{p_i' \cdot k} \right) \epsilon_\alpha \mathcal{M}_0\ . \tag{330}$$

One interesting conclusion can be drawn directly. We did not assume anything about the charges here—they were allowed to differ from particle to particle. But we know that the amplitude must be gauge invariant, in the sense that it is unchanged under the transformation $\epsilon_\alpha \to \epsilon_\alpha + k_\alpha$. Applying this transformation to the amplitude $\mathcal{M}$ we find that $\mathcal{M}$ is invariant if and only if

$$0 = \left(\sum_{in} e_i - \sum_{out} e_i' \right) \mathcal{M}_0 \ . \tag{331}$$

The conclusion is that charge is conserved.

It is a straightforward affair to extend this argument to the case of an arbitrary number of extra soft photons going out; again it is only necessary to consider all possible ways of attaching the extra photons to the external legs, and the amplitude will factorize in just the way we have seen already.

There will, however, be a problem with the cross section. Let us consider the case of one outgoing soft photon. Then the cross section will factorize; there will be one factor describing the cross section with no extra photon, one factor coming from the discussion above, one factor because of the extra photon external leg, and one phase space factor for the extra photon. Multiplying these together we get

$$\left(\frac{d\sigma}{d\Omega'} \right) = \left(\frac{d\sigma}{d\Omega'} \right)_o \frac{\alpha}{4\pi^2} \left(\frac{\epsilon \cdot p}{k \cdot p} - \frac{\epsilon \cdot p'}{k \cdot p'} \right)^2 \frac{d^3k}{\omega} \ . \tag{332}$$

(The first factor is the cross section with no extra photon; the notation implies that we are looking at a very simple process but the argument is general.) We sum over polarization states for the extra photon, that is we make use of

$$\mathcal{M}_r = \epsilon_r^\alpha \mathcal{M}_\alpha \qquad \Rightarrow \qquad \sum_{r=1}^{2} |\mathcal{M}_r(k)|^2 = -\mathcal{M}^\alpha \bar{\mathcal{M}}_\alpha \ . \tag{333}$$

Then

$$\left(\frac{d\sigma}{d\Omega'} \right) = \left(\frac{d\sigma}{d\Omega} \right)_o \frac{(-\alpha)}{4\pi^2} \left(\frac{p}{k \cdot p} - \frac{p'}{k \cdot p'} \right)^2 \frac{d^3k}{\omega} \ . \tag{334}$$

The detectors have a finite resolution, so that if we are interested in the cross section for scattering with no extra photon detected we must integrate this expression over the range $0 < k < \Delta E$ for some ΔE characterizing the detector, and assumed to be small enough so that our soft photon approximation applies. The problem is that this integral diverges at the lover end of its range;

$$\int_0^{\Delta E} \left(\frac{d\sigma}{d\Omega'} \right) = +\infty \ . \tag{335}$$

At first sight this looks very bad.

The theory wiggles out of this difficulty in an interesting way. The point turns out to be that there are further contributions to the cross section, coming from amplitudes describing elastic scattering, without soft photons but with radiative corrections. That is to say that their Feynman diagrams have closed loops and the expression for the amplitude involves an integration over the undetermined momenta running around the loops. Examination shows that some of these integrals diverge at the lower, infrared, end of the integration range. (They also diverge at the other end, but that is another story altogether.) After appropriate regularization it will be seen that these divergencies cancel against the divergence that we have already detected in the cross section. Moreover this cancellation occurs—in the cross section—at all orders of the perturbation expansion. And even more, it is possible to sum the relevant terms to all orders of the perturbation expansion and see that the effect is to make the elastic cross section zero. When soft photon emission is included the cross section is not zero. So this solves the riddle that we started out with: Charged particles can change their state of motion, but in doing so they necessarily emit a "cloud" of infinitely many soft photons.

From a fundamental point of view the root of the difficulty is that the assumption that one can use a Fock space to describe both in- and out-states is wrong; if the in-states belong to Fock space then the out-states are necessarily a kind of coherent states that, roughly speaking, contain an infinite set of photons.

When we carry the QED perturbation expansion to higher orders in the coupling constant, we will encounter integrals over undetermined momenta running around the loops. In particular we will come across the electron self energy

$$\Sigma(k) = e^2 \int \frac{d^4p}{(2\pi)^4} \gamma_\mu \frac{i}{\not k + \not p - m} \gamma^\mu \frac{1}{p^2} \ , \tag{336}$$

the vacuum polarization graph

$$\Pi^{\mu\nu}(k) = e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \frac{\gamma^\mu(\not p + \not k + m)\gamma^\nu(\not p + m)}{((p+k)^2 - m^2)(p^2 - m^2)} \ , \tag{337}$$

and the vertex correction

$$\Lambda^\mu(k_1, k_2) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \gamma^\alpha \frac{1}{\not p + \not k_1 - m} \gamma^\mu \frac{1}{\not p - \not k_2 - m} \gamma_\alpha \tag{338}$$

(with external legs amputated). These integrals are in fact divergent. For a rough estimate, use analyticity and a Wick rotation $p_0 \to ip_0$ to convert them into integrals over an Euclidean momentum space. We can do this precisely because of the pole prescitption that went into the definition of the Feynman propagator—the Minkowski space amplitude will be the analytic continuation of an amplitude calculated in Euclidean space! The measure becomes

$$d^4p = r^3 dr d\Omega \ , \tag{339}$$

where $d\Omega$ is the measure on the 3-sphere. Then we can estimate that

$$\Sigma \sim \int^\infty dr \ r^3 \frac{1}{r^3} \ . \tag{340}$$

This appears to be linearly divergent in the ultraviolet, that is to say at the upper end of the integration range. Similarly

$$\Pi^{\mu\nu} \sim \int^\infty dr \ r^3 \frac{1}{r^2} \tag{341}$$

—appears to be quadratically divergent—and

$$\Lambda^\mu \sim \int^\infty dr \ r^3 \frac{1}{r^4} \tag{342}$$

which is logarithmically divergent. Detailed examination will show that Σ and Λ are divergent in the infrared (lower) end as well, but this we will from now on

ignore. It turns out that these are the only divergencies that occur to second order in perturbation theory.

Three divergent Feynman graphs

Now we know that we made unjustified assumptions in the derivation of the Gell-Mann–Low magical formula, on which perturbation theory is based. Hence it is not really surprising that divergencies occur. What is surprising is that the story does not end here. It is possible to save the theory and extract useful information in spite of these divergencies. To get a first inkling of how this can be done, let us look at a toy model of a function that is defined through a divergent integral, and see how the function can be useful in spite of the fact that it is ill defined. Choose

$$f(x) = \int_1^\infty \frac{dy}{x+y} \ . \tag{343}$$

This is logarithmically divergent and hence, on the face of it, it means nothing. However, the definition can be manipulated as follows:

$$f(x) - f(0) = \int_1^\infty \frac{dy}{x+y} - \int_1^\infty \frac{dy}{y} = \int_1^\infty dy \left(\frac{1}{x+y} - \frac{1}{y} \right) \ . \tag{344}$$

We just performed a forbidden manipulation by writing the two integrands under a common integral sign. But if we do it anyway we can then continue as follows:

$$f(x) - f(0) = -x \int_1^\infty \frac{dy}{y(x+y)} = \bar{f}(x) \ . \tag{345}$$

Here $\bar{f}(x)$ is given by a perfectly convergent integral. So we have arrived at the conclusion that

$$f(x) = A + \bar{f}(x) \ . \tag{346}$$

Here A is a divergent constant. We are led to take the point of view that $f(x)$ has an interesting and well defined dependence on x, even though it is in itself not a well behaved function.

We can do an improved version of this argument if we replace the original definition of $f(x)$ with the limit $\Lambda \to \infty$ of the regularized expression

$$f_\Lambda(x) = \int_1^\Lambda \frac{dy}{x+y} = A(\Lambda) + \bar{f}_\Lambda(x) \ . \tag{347}$$

Then the interesting part of $f(x)$ can be defined by the limit

$$\lim_{\Lambda\to\infty} (f_\Lambda(x) - A(\Lambda)) = \bar{f}(x) \ . \tag{348}$$

As long as we keep Λ finite we can isolate the function $\bar{f}_\Lambda(x)$ in a perfectly rigourous manner—and afterwards take the limit to obtain the well defined function $\bar{f}(x)$.

A linearly divergent example is

$$g(x) = \int_1^\infty \frac{ydy}{x+y} \ . \tag{349}$$

Again we try to tame it by means of subtraction. Heuristically

$$g(x) - g(0) = \int_1^\infty ydy \left(\frac{1}{x+y} - \frac{1}{y}\right) = -x\int_1^\infty \frac{ydy}{y(x+y)} = -xf(x) \ . \tag{350}$$

Hence

$$g(x) = B - xA - x\bar{f}(x) \ . \tag{351}$$

Again we see that although the integral that defines the function is divergent, the divergence can be isolated into divergent constants—this time two of them—so that the interesting x–dependence is given by a well defined finite function. The hope then is that the divergent integrals that we encounter in QED are of this type. To do this in a convincing manner requires some careful attention to the regularization procedure and so on.

Above all it requires careful attention to the question of what the answer is supposed to do for us. Let us consider the electron self–energy graph as an example, and imagine that we are calculating the two point function to all orders in perturbation theory. (When the self–energy graph occurs inside some more complicated Feynman graph it is really the complete two point function that should occur there.) We organize the complete expression into one–particle irreducible parts, that is to say into subgraphs that are such that they remain connected if we cut any one of the propagators. Diagrammatically we denote such a part with a blob, and in the equations with the symbol $\Sigma(p)$. If $S_F(p)$ denotes the free electron propagator what we are computing is the complete dressed electron propagator

$$S'_F(p) = S_F(p) + S_F(p)\Sigma(p)S_F(p) + S_F(p)\Sigma(p)S_F(p)\Sigma(p)S_F(p) + \dots \ . \tag{352}$$

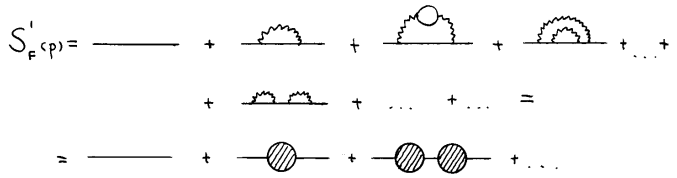

Figure 22: The complete electron propagator.

We can sum this series much as we can sum a geometric series; evidently

$$\begin{aligned} S'_F(p) &= S_F(p) + S_F(p)\Sigma(p)S'_F(p) \\ &\Leftrightarrow \\ (\not{p} - m)S'_F(p) &= 1 + \Sigma(p)S'_F(p) \ . \end{aligned} \tag{353}$$

We have arrived at the expression

$$S'_F(p) = \frac{1}{\not{p} - m - \Sigma(p)} \ . \tag{354}$$

If we had the one-particle irreducible part $\Sigma(p)$ exactly, this would be the exact answer for the propagator in the interacting theory. In practice we will of course have to do with $\Sigma(p)$ computed up to some finite order in perturbation theory.

The interesting thing about eq. (354) is that we see that a divergent constant in $\Sigma(p)$ will not matter very much. The physical information present in the propagator $S'_F(p)$ is the exact location of the pole. (The precise function of p that occurs in the denominator is interesting too, but only when the propagator occurs inside some more complicated amplitude.) For the free propagator the pole occurs at $p^2 = m^2$. Now suppose that $\Sigma(p) = A + \bar{\Sigma}(p)$, where A is constant—that is to say a constant times the unit matrix—and $\bar{\Sigma}(0) = 0$. Then

$$S'_F(p) = \frac{1}{\not{p} - m - A - \bar{\Sigma}(p)} \equiv \frac{1}{\not{p} - m_R - \bar{\Sigma}(p)} \ , \tag{355}$$

where the pole occurs at the renormalized mass $m_R \equiv m + A$. It is the renormalized mass m_R that is actually measured in experiment. The parameter m on the other hand is just a parameter in the theory that is of no significance in itself. And this will be so even in the best of all possible worlds, where $\Sigma(p)$ is a finite function. In a less perfect world where $\Sigma(p)$ is logarithmically divergent the constant A is divergent, but this does not have very dramatic consequences. If things really organize themselves as sketched then this just means that the uninteresting parameter m is divergent too, in precisely the manner needed to produce the finite m_R that is our experimental input.

In 2+1 dimensions things actually do organize themselves as sketched as far as QED is concerned. In 3+1 dimensions additional complications occur. The electron self energy graph then behaves more like the second of the toy integrals that we investigated. One finds

$$\Sigma(p) = B - A\gamma \cdot p + \bar{\Sigma}(p) \ , \tag{356}$$

where both B and A are divergent. (The result is clearly reasonable from the point of view of Lorentz invariance.) This also can be handled; one ends up with a propagator of the form

$$S'_F(p) = Z_1 S^R_F(p) \ , \tag{357}$$

where Z_1 is zero (or one over infinity) and the renormalized propagator $S^R_F(p)$ is well defined after a redefinition of the mass parameter. This is called multiplicative renormalization. The numerical factor Z_1 is not unexpected; indeed we had such a factor already in eq. (250). The embarrassing thing is that its value will be zero. On the other hand this factor will arise in several places in a given amplitude—the aim will be to show that all these factors cancel out. Indeed it is easy to see that if we use $Z = Z_1$ in eq. (262) then this factor will cancel out in the final amplitude for the 2-point function.

Of course, we will have to analyze the integrals (336 - 338) in detail to see how things work, but the discussion so far should have convinced you that there may some leeway for an attempt to wiggle out of the problem with infinite integrals in QED.

REGULARIZATION

We have a delicate argument in front of us. Although the integrals that we have to do are divergent we do not want to actually manipulate any divergent expressions. In an intermediate step we therefore make all integrals finite "by hand". This is the procedure called regularization. Actually there are many such procedures on the market; eventually we must show that the final results are independent of which particular procedure that is chosen. In the toy integral the regularization was performed by simply cutting off the range of the integration, performing the appropriate manipulations, and taking the limit $\Lambda \to \infty$ at the end. The function $\bar{f}(x)$ that we eventually ended up with did not depend on Λ. In quantum field theory an upper limit on the range of integration would violate Lorentz invariance, and this would be a bad thing. Indeed we want our regularization scheme to respect all the symmetries in the problem. There is a reason for this: A detailed examination of our integrals (336 - 337) shows that they are not as divergent as they seem to be. Thus the quadratically and linearly divergent terms in the vacuum polarization graph actually cancel. But this happens because of gauge invariance, and it will happen only if the regularization respects gauge invariance.

It is not altogether straightforward to invent a gauge invariant regularization scheme. The method that I will use is an old one due to Pauli and Villars, and it works well for QED. For Yang-Mills theory it simply fails. Other methods such as dimensional regularization are then used. A final general remark is that in some theories it turns out that there simply does not exist any regularization that preserves all the symmetries of the classical theory; then what happens is that these symmetries are absent from the corresponding quantum theory. There is what is known as an "anomaly". Having said this, let me explain the idea behind Pauli-Villars regularization using a toy integral as an example.

Suppose that we encounter an integral such as

$$I = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} f(p) \ , \tag{358}$$

where $f(p)$ is some function of the momentum such that the integral is logarithmically divergent. The idea is now to change the propagator to

$$\frac{1}{p^2 - m^2} \quad \to \quad \sum_i \frac{c_i}{p^2 - M_i^2} \ , \tag{359}$$

where

$$c_0 = 1 \qquad M_0 = m \tag{360}$$

and where the remaining parameters are at our disposal. We have ensured that we recover the original propagator in the limit that $M_{i>0} \to \infty$. Of course the

idea now is to evaluate the integral with all the regulator masses finite and to take the limit only at the end of the calculation. First we rewrite

$$\sum_i \frac{c_i}{p^2 - M_i^2} = \sum_i \frac{c_i}{p^2} \left(1 + \frac{M_i^2}{p^2} + \left(\frac{M_i^2}{p^2} \right)^2 + \ldots \right) \ . \tag{361}$$

Our integral then becomes

$$I_{reg} = \int \frac{d^4p}{(2\pi)^4} \sum_i \frac{c_i}{p^2} \left(1 + \frac{M_i^2}{p^2} + \left(\frac{M_i^2}{p^2} \right)^2 + \ldots \right) f(p) \ . \tag{362}$$

All our manipulations are above board since they are performed inside the integrand. Next we evaluate the integral and take the regulator masses to infinity at the end. If the original integral is finite we have done nothing, only extra work. But if the original integral is logarithmically divergent it can be saved by choosing

$$\sum_i c_i = 0 \ . \tag{363}$$

We can of course always do this. Inspection of eq. (362) shows that it will also be enough to make I_{reg} finite. The situation will then be analogous to that in eq. (347). The hope is that an interesting renormalized function will emerge after a second step analogous to eq. (348).

If the original integral is quadratically divergent we save the situation by imposing

$$\sum_i c_i = \sum_i c_i M_i^2 = 0 \ . \tag{364}$$

As long as $f(p)$ is a polynomial in the momentum we can always make the integral I_{reg} finite if we go on in this way.

A nice thing about Pauli-Villars regularization is that the peculiar change in the propagator can be understood at the Lagrangian level. If the original action contains a term

$$S = -\frac{1}{2} \int d^4x \ \partial_\alpha \phi \partial^\alpha \phi + m^2 \phi^2 \tag{365}$$

we simply replace it with

$$S_{reg} = -\frac{1}{2} \int d^4x \ \sum_i c_i (\partial_\alpha \phi_i \partial^\alpha \phi_i + M_i^2 \phi_i^2) \ . \tag{366}$$

We are adding new species of particles to the theory, with large (eventually infinite) masses so that they are not excited in experiments conducted at sensible energies. There is a catch though. The regularized theory (with finite regulator

masses) is sick. The reason is that eq. (363) forces at least one of the c_i to be negative. And inspection of the action shows that this implies that the Hamiltonian of the regularized theory cannot be positive definite. This is just a way of saying that we really have to take the limit $M_{i>0} \to \infty$ at the end.

Although Pauli-Villars regularization leads to a sick theory if taken literally, it does respect gauge invariance in QED. Having introduced regulator fields we can couple them to the electromagnetic field through minimal coupling as usual, to ensure that the regulator fields enter the Feynman diagrams they way we want them to. That is to say, there is nothing wrong with using the interaction terms

$$e \int \sum_i \bar{\psi}_i \gamma \cdot A \psi_i \tag{367}$$

in the action. The reason why we cannot go beyond QED is that we will play the same game with the photon propagator. This means that we will perform calculations where massive photons enter the loops. This can be done without too much ado in an abelian gauge theory like QED, because in this case the extra longitudinal degrees of freedom that we introduce in this way will act like free fields—if we keep them away from the external legs they will not enter the loops either. In non-Abelian gauge theories this is no longer true and other measures must be taken.

There is another reason why we want to use massive photon propagotors, namely that some of the loop integrals—such as the electron self energy and the vertex correction graphs—are divergent also in the infrared. This difficulty is separate from the difficulty at hand and we are therefore happy to avoid it through the simple expedient of replacing

$$\frac{1}{p^2} \quad \to \quad \frac{1}{p^2 - \lambda^2} \tag{368}$$

in the photon propagator. We can take the limit $\lambda \to 0$ when we eventually arrive at the cross sections.

The conclusion so far is that we will deal with the regularized integrals

$$-i\Sigma^{(2)}_{reg}(k) = (ie)^2 \int \frac{d^4p}{(2\pi)^4} \sum_i c_i \gamma_\mu \frac{i}{\not k + \not p - m} \gamma^\mu \frac{1}{p^2 - \lambda_i^2} \tag{369}$$

—where only the photon propagator was modified—and

$$\Pi^{\mu\nu}_{reg}(k) = e^2 \int \frac{d^4p}{(2\pi)^4} \sum_i c_i \frac{\mathrm{Tr}\gamma^\mu(\not p + \not k + M_i)\gamma^\nu(\not p + M_i)}{((p+k)^2 - M_i^2)(p^2 - M_i^2)} \tag{370}$$

where the fermion propagator was modified.

Now we look at eq. (369) and realize that we have to actually do it. The first step is to use a trick due to Feynman that cleans the denominator of products of polynomials. The trick is

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \ . \tag{371}$$

Applied to the case at hand, and performing a little algebra, this is

$$\begin{aligned} &\frac{1}{k^2 - \lambda_i^2} \frac{1}{(p+k)^2 - m^2} = \\ &= \int_0^1 \frac{dx}{[(k + (1-x)p)^2 - (1-x)^2 p^2 + (1-x)(p^2 - m^2) - x\lambda_i^2]^2} \ . \end{aligned} \tag{372}$$

If we also perform some gamma matrix algebra in the numerator we then get

$$\Sigma^{(2)}(p) = (ie)^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \sum_i c_i \frac{-2(\not p + \not k) + 4m}{[(k-l)^2 - \mathcal{R}_i^2]^2} \tag{373}$$

where

$$l^\alpha = -(1-x)p^\alpha \tag{374}$$

$$\mathcal{R}_i^2 = (1-x)^2 p^2 - (1-x)(p^2 - m^2) + x\lambda_i^2 \ . \tag{375}$$

Note that we interchanged the order of integration, but that is allowed since all integrals are finite in the regularized theory. We can also change integration variables. In particular, we can change $k^\alpha \to k^\alpha + l^\alpha$. We throw away terms in the integrand that are odd in k^α since they contribute nothing. After a Wick rotation (that contributes an overall factor of i) we can do the integral over the

angular variables in an Euclidean momentum space (leading to an overall factor of $2\pi^2$, which is the area of the 3-sphere). Then

$$-i\Sigma^{(2)}(p) = (ie)^2 \frac{2\pi^2 i}{(2\pi)^4} \int_0^1 \int_0^\infty dk \ \sum_i c_i \frac{k^3}{(k^2+\mathcal{R}_i^2)^2}(-2x \not{p} + 4m) \ . \tag{376}$$

We can do the final momentum integral. So

$$\begin{aligned} -i\Sigma^{(2)}(p) = \\ = (ie)^2 \frac{\pi^2 i}{(2\pi)^4} \int_0^1 dx(-2x \not{p} + 4m) \left[\sum_i c_i \left(\ln (k^2+\mathcal{R}_i^2) + \frac{\mathcal{R}_i^2}{k^2+\mathcal{R}_i^2} \right) \right]_0^\infty \ . \end{aligned} \tag{377}$$

On the face of it this is logarithmically divergent. But now we use our freedom to set $\sum c_i = 0$. Then there will be no contribution from the upper end of the integration range. We make the specific choice

$$c_0 = 1 \quad \lambda_0 = \lambda \qquad c_1 = -1 \quad \Lambda_1 = \Lambda \ , \tag{378}$$

all others zero. In this way we happily arrive at

$$-i\Sigma^{(2)}(p) = (ie)^2 \frac{\pi^2 i}{(2\pi)^4} \int_0^1 dx(-2x \not{p} + 4m) \ln \frac{\mathcal{R}_1^2}{\mathcal{R}_0^2} \ . \tag{379}$$

Explicitly

$$\Sigma^{(2)}(p) = \frac{\alpha}{\pi} \frac{1}{2} \int_0^1 dx \ (2m - x \not{p}) \ln \frac{p^2(1-x)^2 + \Lambda^2 x - (p^2 - m^2)(1-x)}{p^2(1-x)^2 + \lambda^2 x - (p^2 - m^2)(1-x)} \ . \tag{380}$$

This is the regularized result. We should now remove the cutoff by taking the limit $\Lambda \to \infty$, but if we do this the expression diverges logarithmically. Before we proceed it will be necessary to think.

What we are computing is a propagator. The pole of the propagator should occur at the electron mass m. Does it? According to eq. (354) the propagator, to this order in perturbation theory, is

$$S'_F(p) = \frac{1}{\not{p} - m - \Sigma^{(2)}(p)} \ . \tag{381}$$

According to the calculation we just made

$$\Sigma^{(2)}(p)_{|\ p\not=m} = \frac{\alpha}{\pi} \frac{m}{2} \int_0^1 dx(2-x) \ln \frac{(1-x)^2 + \frac{\Lambda^2}{m^2} x}{(1-x)^2 + \frac{\lambda^2}{m^2}} \ . \tag{382}$$

We can set $\lambda = 0$ directly here since this is finite in the infrared. If we also take into account that $\Lambda >> m^2$ we find approximately

$$\frac{\alpha}{\pi}\frac{m}{2}\int_0^1 dx(2-x)\ln\frac{\Lambda^2}{m^2}\frac{x}{(1-x)^2} \approx \frac{\alpha}{\pi}m\int_0^1 dx(2-x)\ln\frac{\Lambda}{m}\ . \tag{383}$$

Hence

$$\Sigma^{(2)}(p)_{|\ \not{p}=m} = \frac{3}{2}\frac{\alpha}{\pi}m\ln\frac{\Lambda}{m}\ . \tag{384}$$

This becomes logarithmically divergent when we remove the cutoff. But it is much more significant that it is non–zero. From eq. (381) we see that the pole of the electron propagator does not sit at $p^2 = m^2$. The parameter m has therefore been robbed of its physical meaning.

We can always write

$$\Sigma(p) = A + (\not{p} - m)B + (\not{p} - m)^2\Sigma_f(p)\ , \tag{385}$$

where

$$A = \frac{3}{2}\frac{\alpha}{\pi}m\ln\frac{\Lambda}{m} \tag{386}$$

$$B = \frac{\partial\Sigma}{\partial\ \not{p}}_{|\ \not{p}=m}\ . \tag{387}$$

The divergent constant A can now be absorbed into the mass m. The pole of the propagator then sits at

$$m_R = m + A = 9.109\cdot 10^{-28}\ \mathrm{g}\ , \tag{388}$$

where the value was taken from experiment, not from theory. The quantity m_R will be held constant when we take the limit $\Lambda \to \infty$. The parameter m will misbehave, but we have no special emotional attachment to m.

We still have to worry about the constant B. First of all we have to explain what the derivative with respect to $\not{p}$ in eq. (387) actually means. It must be true that

$$\frac{\partial}{\partial\ \not{p}}\ \not{p} = 1\ . \tag{389}$$

A little experimentation now shows that a more conventional way of writing the derivative is

$$\frac{\partial}{\partial\ \not{p}} \equiv \frac{1}{4}\gamma^\alpha\frac{\partial}{\partial p^\alpha}\ . \tag{390}$$

So, using $y \equiv 1 - x$, we can spell out that

$$B = \frac{\alpha}{2\pi} \int_0^1 dy(1-y) \ln \frac{y^2 + \frac{\lambda^2}{m^2}(1-y)}{y^2 + \frac{\Lambda^2}{m^2}(1-y)} - \\ -\frac{\alpha}{2\pi} \int_0^1 dy(1+y) \left[\frac{2y^2 - 2y}{y^2 + \frac{\lambda^2}{m^2}(1-y)} - \frac{2y^2 - 2y}{y^2 + \frac{\Lambda^2}{m^2}(1-y)} \right] . \tag{391}$$

In the approximation that Λ is big and λ is small this is

$$B \approx \frac{\alpha}{2\pi} \int_0^1 dy \left((1-y) \ln \frac{y^2}{\frac{\Lambda^2}{m^2}(1-y)} - (1+y) \frac{2y^2 - 2y}{y^2 + \frac{\lambda^2}{m^2}(1-y)} \right) . \tag{392}$$

Here we can do the integration over y and we end up with the result

$$B = \frac{\alpha}{\pi} \left[-\frac{1}{4} \ln \frac{\Lambda^2}{m^2} - \frac{1}{2} \ln \frac{\lambda^2}{m^2} - \frac{9}{8} \right] . \tag{393}$$

This is divergent in the infrared (the limit $\lambda \to 0$) but we do not worry about this. We do worry about the fact that it is divergent in the ultraviolet (the limit $\Lambda \to \infty$), but our prime concern is the mere fact that it is not zero.

Now, to this order in perturbation theory, we can write

$$S'_F(p) = \frac{i}{(1-B)(\not{p} - m_R)(1 - (\not{p} - m_R)\Sigma_f^{(2)}(p))} , \tag{394}$$

where we introduced the renormalized mass m_R. In eq. (385) the bare mass m occurs instead. Actually the difference does not matter here since I work to a fixed order in e only and ruthlessly throw away higher order terms. That is to say, since $m_R = m + o(e^2)$, $B = o(e^2)$ and so on, it will be true that

$$(1-B)(\not{p} - m_R) = p - m - A - B(\not{p} - m) + o(e^4) , \tag{395}$$

and terms of $o(e^4)$ are ignored. What we find is that the residue at the pole is equal to

$$Z_2 \equiv \frac{1}{1-B} \tag{396}$$

and that it happens to be logarithmically divergent. But—and this is really why the numerical factor Z was introduced in the LSZ formalism—the residue at the pole must equal one if the states are correctly normalized. The correct propagator to use then is the renormalized propagator

$$S_F^{ren}(p; m_R) = \frac{1}{Z_2} S'_F(p) \ . \tag{397}$$

At the same time we perform a renormalization of the wave functions. We change from "bare" to "dressed" states according to

$$|bare\rangle = \sqrt{Z_2}|dressed\rangle \ . \tag{398}$$

Later on I will give a sketch that makes it at least vaguely plausible that the structure of the theory is such that the bothersome factor Z_2 now cancels out from all the S-matrix amplitudes. Hence the fact that it happens to vanish can be ignored from a practical point of view. Needless to say that a full proof that this procedure works, and works to any order in perturbation theory, is difficult. But it has been provided.

I should also say that the function $\Sigma_f(p)$ is ultraviolet finite. This function becomes important when the fermion propagator occurs inside some more complicated diagram but to this order in perturbation theory it does not contribute anything to the physics.

VACUUM POLARIZATION

It is more satisfying to discuss the vacuum polarization diagram because here the game is not only to show that the theory is consistent. There is a definite physical effect called the Lamb shift to be harvested as well. We can also see the importance of gauge invariance—and the need for a gauge invariant regularization scheme—in a particularly transparent way.

Let us return to our unregulated expression for the vacuum polarization diagram. As it stands it is an apparently quadratically divergent integral that means nothing. But we can see that if it did mean anything it would have to have certain definite properties. The idea is to attach this graph to an external leg, so that the momentum variable on which it depends is the momentum of a real photon. The amplitude is of the form

$$\epsilon^\alpha \mathcal{M}_\alpha = \epsilon^\alpha \Pi_\alpha^{\ \beta} \mathcal{M}'_\nu \ . \tag{399}$$

Gauge invariance dictates that the amplitude must be unchanged under the substitution $\epsilon \to \epsilon + k$, so that we must have

$$k^2 = 0 \qquad \Rightarrow \qquad k^\alpha \Pi_{\alpha\beta}(k) = 0 \ . \tag{400}$$

The only way to arrange this is to have

$$\Pi_{\alpha\beta}(k) = (k_\alpha k_\beta - g_{\alpha\beta} k^2) A(k) + k^2 B(k) \ . \tag{401}$$

We can now do a little dimensional analysis. Counting mass dimension (that is $[p] = 1$, $[p^n] = n$ and so on) we observe that

$$[\Pi_{\alpha\beta}] = [d^4 p][p^{-2}] = 2 \qquad [A] = [B] = 0 \ . \tag{402}$$

This means that on purely dimensional grounds the functions A and B must be given by integrals that, at the ultraviolet end, must behave like

$$\int d^4 p \ p^{-4} \sim \int^\infty dp \ \frac{1}{p} \ . \tag{403}$$

So we arrive at the conclusion that, because of gauge invariance, the quadratic divergencies must cancel out leaving only logarithmical divergencies in their place.

Explicitly one can convince oneself that

$$\begin{aligned} k^\alpha \Pi_{\alpha\beta} &= \frac{i}{(2\pi)^4} \int d^4 p \mathrm{Tr} \frac{1}{\not{p} - m} \not{k} \frac{1}{\not{p} + \not{k} - m} \gamma_\beta = \\ &= \frac{i}{(2\pi)^4} \int d^4 p \mathrm{Tr} \left(\frac{1}{\not{p} - m} - \frac{1}{\not{p} + \not{k} - m} \right) \gamma_\beta \ . \end{aligned} \tag{404}$$

Now, if this were a well behaved integral, we could perform the shift $p \to p+k$ in the first term and conclude that this expression is zero as it should be. Because it is not we cannot draw this conclusion, but it is clearly advisable to use a regularization scheme that allows this cancellation to happen.

If we use Pauli-Villars regularization and Feynman's trick to simplify the denominator we will get the regularized expression on the form

$$\Pi^{\alpha\beta}_{reg}(k) = i\int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \sum_i c_i \frac{N^{\alpha\beta}}{[p^2 - \mathcal{R}_i^2]^2} \ , \tag{405}$$

where

$$N^{\alpha\beta} = \mathrm{Tr}\gamma^\alpha(\not{p} + \not{k} + \not{l} + M_i)\gamma^\beta(\not{p} + \not{l} + M_i) \tag{406}$$

$$\mathcal{R}_i^2 = M_i^2 - x(1-x)k^2 \tag{407}$$

$$l_\alpha = -(1-x)k_\alpha \ . \tag{408}$$

We can deal with the numerator straight away, using the facts that the trace over an odd number of gamma matrices is zero, that integrals over an odd number of p^α will be zero, and that the angular integration (in Euclidean momentum space) will ensure that

$$p_\alpha p_\beta \to \frac{1}{4} g_{\alpha\beta} p^2 \ . \tag{409}$$

A modest calculation then implies that

$$N^{\alpha\beta} = -(k^\alpha k^\beta - k^2 g^{\alpha\beta}) 8x(1-x) - 2g^{\alpha\beta}(p^2 - 2\mathcal{R}_i^2) \ . \tag{410}$$

In this way we arrive at the preliminary result that

$$\Pi^{\alpha\beta}_{reg}(k) = (k^\alpha k^\beta - k^2 g^{\alpha\beta})\Pi_{reg}(k) + g^{\alpha\beta} D(k) \ , \tag{411}$$

where

$$\Pi_{reg}(k) = -i\int_0^1 dx \ 8x(1-x) \sum_i c_i \int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2 - \mathcal{R}_i^2]^2} \tag{412}$$

$$D(k) = -2i\int_0^1 dx \ \sum_i c_i \int \frac{d^4p}{(2\pi)^4} \frac{p^2 - 2\mathcal{R}_i^2}{[p^2 - \mathcal{R}_i^2]^2} \ . \tag{413}$$

We wish that $D(k) = 0$, otherwise something is wrong with gauge invariance.

Fortunately a detailed evaluation of the integral shows that $D(k)$ is zero. To draw that conclusion the regulator fields must be chosen so that the integral is

finite in the first place. After the Wick rotation the integral can be manipulated to the form

$$D(k) = -e^2 \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dx \left(\sum_i c_i - \sum_i c_i \frac{\mathcal{R}_i^4}{(x + \mathcal{R}_i^2)^2} \right) \ . \tag{414}$$

The requirements for this to be finite, and indeed to vanish, are

$$\sum_i c_i = \sum_i c_i M_i^2 = 0 \ . \tag{415}$$

It is not surprising that we get more conditions now than we got in the calculation of the electron self energy, because this time we had to regularize a quadratically divergent integral. A minimal solution of these equations is

$$c_0 = 1 \qquad c_1 = -1 - \epsilon \qquad c_2 = \epsilon \tag{416}$$

$$M_0^2 = m^2 \qquad M_1^2 = \frac{m^2}{\epsilon} \qquad M_2^2 = \frac{m^2}{\epsilon^2} \ , \tag{417}$$

where we think of

$$\Lambda \equiv M_1 = \frac{m}{\sqrt{\epsilon}} \tag{418}$$

as being larger than any relevant energy scale. Once the regulator fields are chosen in this manner we have a rigourous version of the rough argument that the quadratic divergencies ought to cancel.

The remaining function in our graph is non-zero and contains interesting physics. The result of doing the integral, with the regulator in place already, is

$$\begin{aligned} \Pi_{reg}(k) &= -\frac{e^2}{4\pi^2} \int_0^1 dx \ 2x(1-x) \sum_i c_i \mathcal{R}_i^2 = \\ &= -\frac{\alpha}{\pi} \int_0^1 dx \ 2x(1-x) (\ln (m^2 - x(1-x)k^2) - \\ &\quad - \left(1 + \frac{m^2}{\Lambda^2}\right) \ln (\Lambda^2 - x(1-x)k^2) + \frac{m^2}{\Lambda^2} \ln (\frac{\Lambda^4}{m^2} - x(1-x)k^2)) \approx \\ &\approx -\frac{\alpha}{\pi} \int_0^1 dx 2x(1-x) \ln \frac{m^2 - x(1-x)k^2}{\Lambda^2 - x(1-x)k^2} \ , \end{aligned} \tag{419}$$

where the approximation in the last step took into account that $\Lambda >> m^2$. We see immediately that

$$\Pi_{reg}(0) = \frac{\alpha}{\pi}\frac{2}{3}\ln\frac{\Lambda}{m} \tag{420}$$

is logaritmically divergent as the cutoff goes to infinity, while

$$\lim_{\Lambda\to\infty}\left(\Pi_{reg}(k^2) - \Pi_{reg}(0)\right) = -\frac{\alpha}{\pi}\int_0^1 dx\ 2x(1-x)\ln\frac{m^2 - x(1-x)k^2}{m^2} \tag{421}$$

is finite and independent of Λ. Hence this diagram is finite after a single subtraction. The physics will depend on the perfectly well defined renormalized function

$$\Pi_R(k) = -\frac{\alpha}{\pi}\int_0^1 dx\ 2x(1-x)\ln\frac{m^2 - x(1-x)k^2}{m^2} \tag{422}$$

This time there really are physical consequences already in this order of perturbation theory, as we will now see.

There should be some diagrams here

The physical process that we will look at is scattering of an electron off an external field, to third order in e^2. There are three relevant diagrams, and we will focus on the one that contains the vacuum polarization graph. The external field will be taken to be the static Coulomb field from an atomic nucleus, namely

$$A^{ext}_\alpha(x) = \left(\frac{Ze}{4\pi r}, 0, 0, 0\right)\ , \tag{423}$$

or in momentum space

$$A^{ext}_\alpha(q) = \left(\frac{Ze}{q^2}, 0, 0, 0\right)\ . \tag{424}$$

To first order the amplitude is

$$\mathcal{M}^{(1)} = ie\bar{u}_s(p+q)\ \not{A}^{ext}(q)u_r(p)\ . \tag{425}$$

Taking the vacuum polarization diagram into account this becomes

$$\mathcal{M}^{(3)} = ie\bar{u}_s\gamma^\alpha u_r\left(A^{ext}_\alpha - i\frac{g_{\alpha\beta}}{q^2}\Pi^{\beta\gamma}_{reg}(q)A^{ext}_\gamma(q)\right)\ . \tag{426}$$

Since the external field is in the Lorenz gauge this simplifies to

$$\mathcal{M}^{(3)} = i\bar{u}_s \not{A}^{ext} u_r e \left(1 - \Pi^{(2)}_{reg}(q)\right) \ . \tag{427}$$

There is no propagator in the final expression. Moreover there is an important point, namely that the parameter e occurs only in combination with the factor $(1 - \Pi^2_{reg}(q))$. (As in the discussion of the electron's self–energy, we throw away all terms of higher order in e—hence the fact that e also enters into the function Π_{reg} does not matter.)

The point is that e, like m, does not have any predetermined meaning. Whether the factor that comes together with e is finite or not, we would still set

$$e_R \equiv e(1 - \Pi^2_{reg}(0)) = 1.6 \cdot 10^{-19} \text{ Coulomb} \ , \tag{428}$$

where the numerical value comes from experiment. Having done so we can, to this order in perturbation theory, express the amplitude as

$$\mathcal{M}^{(3)} = i\bar{u}_s \not{A}^{ext}(q) u_r e_R \left(1 - \Pi^{(2)}_R(q^2)\right) \ . \tag{429}$$

To this expression the vacuum polarization contributes a well behaved and finite function. In addition we have performed a multiplicative renormalization of the parameter e,

$$e = \frac{1}{\sqrt{Z_3}} e_R \ . \tag{430}$$

The fact that the renormalization constant is infinite is embarrassing, but not too much embarrassing since it will eventually be argued that it cancels out from all observable quantities.

Physically what eq. (429) shows is that the strength of the interaction between the electron and the electromagnetic field depends on the momentum, or in real space on the distance to the nucleus. To see how, let us assume that

$$q^2 \equiv -\underline{q}^2 << m^2 \ . \tag{431}$$

We can then Taylor expand the logarithm that goes into eq. (422), and approximate

$$\Pi^{(2)}_R(q^2) \approx -\frac{\alpha}{\pi} \int_0^1 dx \ 2x^2(1-x)^2 \frac{\underline{q}^2}{m^2} = -\frac{\alpha}{15\pi} \frac{\underline{q}^2}{m^2} \ . \tag{432}$$

Effectively the potential is changing,

$$\frac{Ze}{\underline{q}^2} \to \frac{Ze}{\underline{q}^2} \left(1 + \frac{\alpha}{15\pi} \frac{\underline{q}^2}{m^2}\right) \ . \tag{433}$$

In real space this means that

$$\frac{Ze}{4\pi r} \to \frac{Ze}{4\pi r} + \frac{Ze\alpha}{15\pi m^2}\delta(x) \ . \tag{434}$$

The delta function piece of a potential will be felt by electrons whose wave functions are non–zero at the origin, that is to say by electrons in s states. Since the "extra" piece of a potential is attractive such electrons will be more tightly bound and their energy levels will be shifted downwards. This is the origin of the Lamb shift that is observed to split the otherwise degenerate levels $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ in the hydrogen atom. The shift is of the order of a 1000 MHz and turns out to be consistent with the above deliberations.

Two final remarks. A general remark is that vacuum polarization will cause the force to grow stronger when the distance shrinks. The wonderful thing is that non–Abelian gauge theories can exhibit the opposite kind of behaviour. A specific remark is that examination of eq. (422) shows that the function $\Pi_R(k^2)$ has a rather interesting structure; notably it will develop an imaginary part when $k^2 \geq 4m^2$. This is due to the fact that electron–positron pairs can be created at this energy.

THE VERTEX GRAPH

There is one more one loop graph that brings divergencies into the theory. It also brings an (ultimately) well defined function of direct relevance for the coupling of an electron to a static magnetic field. This is the vertex graph, given by

$$\Lambda^\mu(p,p') = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2-\lambda^2} \gamma_\alpha \frac{i}{\not p' - \not k - m} \gamma^\mu \frac{i}{\not p - \not k - m} \gamma^\alpha \ . \qquad (435)$$

A new thing happens to us here, which is that the integral is divergent in the infrared end of the integration range. This is why a small photon mass λ has been added as a regulator. In the ultraviolet end the integral is, naively, logarithmically divergent, but the divergence appears in a rather harmless way. To see this, define

$$\Delta \equiv \not P - \not k - m \ , \qquad l \equiv p - P \qquad l' \equiv p' - P \ . \qquad (436)$$

Thus

$$\frac{1}{\not p' - \not k - m} \gamma^\mu \frac{1}{\not p - \not k - m} = \frac{1}{\not l' + \Delta} \gamma^\mu \frac{1}{\not l + \Delta} \ . \qquad (437)$$

Then we use the identity

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \dots \qquad (438)$$

to observe that

$$\Lambda^\mu(p,p') = \qquad (439)$$

$$= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2-\lambda^2} \gamma_\alpha \left(\frac{1}{\Delta} - \frac{1}{\Delta} \not l' \frac{1}{\Delta} + \dots \right) \gamma^\mu \left(\frac{1}{\Delta} - \frac{1}{\Delta} \not l \frac{1}{\Delta} \right) \gamma^\alpha \ .$$

Using this in eq. (435) we find that

$$\Lambda^\mu = \Lambda^\mu(P,P) + \text{finite terms} \ . \qquad (440)$$

The conclusion is that, if we express an amplitude as a function of the momentum transfer $q \equiv p - p'$, then the result is an ill-behaved constant followed by a finite function of q. This is the kind of situation that are by now familiar with, and we should be able to handle it.

Not only that, the divergence turns out to be related to the divergent constant B in the electron propagator. See eq. (385). This happens because of

gauge invariance, and it is true at all orders of (regularized) perturbation theory (provided we use a regularization that respects gauge invariance). Let us define the vertex contribution, including the tree level result, as

$$\Gamma^\mu = \gamma^\mu + \Lambda^\mu \ . \tag{441}$$

Explicitly the relation between Λ^μ and the electron propagator is given by Ward's identity

$$\frac{\partial}{\partial p_\mu}\Sigma(p) = -\Lambda^\mu(p,p) \ . \tag{442}$$

To lowest non-trivial order in perturbation theory it is easy to show this through explicit manipulation of the one loop graphs, using the observation that

$$\frac{\partial}{\partial p_\mu}\frac{1}{\not{p}-m} = -\frac{1}{\not{p}-m}\gamma^\mu\frac{1}{\not{p}-m} \ . \tag{443}$$

Anyway, with this result in hand, we see that if we consider two external electrons ("on the mass shell") and take the limit of zero momentum transfer, we get

$$\bar{u}(p)\Gamma^\mu u(p) = (1-B)\bar{u}\gamma^\mu u = \frac{1}{Z_2}\bar{u}\gamma^\mu u = \bar{u}_{\rm ren}\gamma^\mu u_{\rm ren} \ , \tag{444}$$

where we renormalized the wave functions—as we learned to do when discussing the electron propagator. The final result is independent of the factor Z_2, so the regulator can now be taken away with impunity. The theory produces only finite results.

What are these results in the present case, that is, when we study the scattering of two external electrons interacting with an electromagnetic field? What we are trying to compute, in the Heisenberg picture say, is the expectation value of the current

$$\langle p'|J^\mu(x)|p\rangle = \langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle = ie\bar{u}(p')\Gamma^\mu(p,p')u(p) \ . \tag{445}$$

The current is conserved;

$$\partial_\mu J^\mu = 0 \qquad \Rightarrow \qquad (p'-p)_\mu\langle p'|J^\mu(0)|p\rangle = 0 \ . \tag{446}$$

What can we get? Since u and $\bar{u}$ obey the Dirac equation Lorentz invariance constrains the result to be a sum of three terms only, namely

$$\bar{u}\Gamma^\mu(p,p')u = \bar{u}\left(\gamma^\mu F(q) - \frac{i}{2m}(p+p')^\mu G(q) + \frac{(p-p')^\mu}{2m}H(q)\right)u \ . \tag{447}$$

To see this, write out all possible terms and use the Dirac equation to bring their number down. Current conservation, given that the external electrons are on shell, actually forces

$$H(q) = 0 \ . \tag{448}$$

Hence there are only two terms. It is convenient to rewrite them using Gordon's identity

$$2m\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[(p+p')^\mu + i\gamma^{\mu\nu}(p-p')_\nu\right]u(p) \ . \tag{449}$$

Hence the most general form of the thing we are trying to compute is

$$\Lambda^\mu(p,p') = \gamma^\mu F_{\rm D}(q^2) + \frac{i}{2m}\gamma^{\mu\nu}q_\nu F_{\rm P}(q^2) \ . \tag{450}$$

The two functions here are known as the Dirac and Pauli form factors, respectively. The ultraviolet divergence sits in $F_{\rm D}(0)$. Once this has been isolated and dealt with the renormalized Dirac form factor is finite and obeys

$$F_{\rm D}^{ren}(0) = 0 \ . \tag{451}$$

There is also an infrared divergence in the renormalized Dirac form factor, that has to be dealt with at the level of cross sections. The Pauli form factor will come out finite at both ends, with no special action required.

It only remains to do the integral (435). Since there are three propagators in the denominator we use Feynman's trick in the form

$$\frac{1}{abc} = 2\int_0^1 dx\ x\int_0^1 dy\ \frac{1}{[axy + bx(1-y) + c(1-x)]^3} \ . \tag{452}$$

And so on...

This leads to the anomalous magnetic moment of the electron. But I did not have the time to put it in my notes.

A SKETCH OF RENORMALIZABILITY

What we have seen so far can perhaps be regarded as encouraging signs that maybe the infinities occurring in the Feynman diagrams can be controlled. And moreover that it will never be necessary to actually manipulate infinite integrals. All regularized calculations will be finite, so the real statement will be that in the end all traces of the regularization procedure will disappear from observable quantities. This statement has in fact been proved. A rough sketch follows.

First one shows that it is enough to control the divergencies in the two and three point functions, that is in the fermion propagator S'_F, the photon propagator D'_F, and the vertex contribution $\Gamma^\alpha = \gamma^\alpha + \Lambda^\alpha$. This is not altogether surprising since it is easy to see that graphs do tend to become more convergent as the number of external legs increases—provided that the coupling constant is dimensionless we should get an extra factor of p^2 in the denominator if we add an extra leg. In theories like scalar Φ^6 or Einstein's theory of gravity, where the coupling constant has dimension (length)^2, this is no longer true and indeed such theories are not renormalizable.

There should be lots of pictures in this section

The claim that we are after goes something like this: Once the mass has been appropriately shifted all the divergencies in QED will sit in two scale factors Z_1 and Z_3. There will be well defined renormalized functions given by

$$S'_F(p) = Z_1 S_F^R \qquad D'_F(p) = Z_3 D_F^R \qquad \Gamma^\alpha(p,p') = \frac{1}{Z_1}\Gamma_R^\alpha(p,p') \ . \tag{453}$$

Naively one would have expected three independent scale factors but gauge invariance relates Γ^α and Σ in such a way that only two of them are independent. Now introduce the physical charge e_R through

$$e = \frac{1}{\sqrt{Z_3}} e_R \ , \tag{454}$$

a factor $\sqrt{Z_1}$ for each external fermion, and a factor $\sqrt{Z_3}$ for each external photon. Then the renormalization constants Z_1 and Z_3, whether divergent or no, will cancel out from the S–matrix elements of the theory. The latter will

become well defined functions of the physical charge e_R and the physical mass m_R.

It is instructive to check for one self that the renormalization factors indeed do cancel out from some interesting processes, say Compton scattering and the scattering of light by light. Of course this assumes that the hard part of the work has been done already, namely to show that the two and three point functions are finite. This is actually much more difficult than one might expect, since the electron and photon propagators involve "overlapping divergencies". But it is still true. The main point then is that the divergencies affect only two parameters (e and m) and some irrelevant scale factors.

Stepping back from the problem, it is clear that we got ourselves into trouble by assuming that QED holds at all energy scales. When we sum over intermediate states in the loop diagrams we make use of that assumption, and find divergencies, whereas in Nature the very high energy behaviour is presumably something quite different. But the message is that this does not matter: The interaction of electrons and photons at accessible energies is not affected by the unknown very high energy behaviour, except in so far as it affects the values of e and m, and somehow sets them equal to their observed values. The modern way to phrase what is going on is along such lines, and hence it differs somewhat from the way I have been expressing it so far. The statement is like: The high energy behaviour is experimentally irrelevant and cancels out from observable quantities at low energies—where "low" means those energies where experiments are carried out and "high" presumably means those energies where quantum gravity takes over so that present day theories are not applicable anyway.