Ground States of the ϕ^4 Double-Well QFT

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Abstract

At second order in perturbation theory, we find the ground states of the ϕ^4 double-well quantum field theory in 1+1 dimensions. The operators which create these ground states from the free vacuum are constructed explicitly at this order, as is the operator which interpolates between the ground states. As a warmup we perform the analogous calculation in quantum mechanics, where the true ground state is unique but in perturbation theory there are also two ground states.

1 Introduction

One of the greatest triumphs in nonperturbative quantum field theory was the calculation of the leading order correction to the ϕ^4 kink mass in Ref. [1]. This theory was regularized by placing it in a periodic box and fixing the number of modes considered. The calculation was so successfully that it was performed again by a number of groups, many of whom found different results. In Ref. [2] it was explained that the reason for this discrepancy is that the leading correction depends on the regularization scheme, with a fixed mode number and sharp energy cutoff leading to distinct results. Since then even more schemes have been proposed, such as demanding that the correction be independent of the topological sector when the mass vanishes [3]. While various consistency checks have been performed, it remains unclear which scheme, if any, is correct or whether perhaps all are correct but with different definitions of the bare theory or renormalization conditions.

Almost immediately it had been noted [4] that the regularization and renormalization of the ϕ^4 theory were not necessary at all, as scalar theories in 2d with canonical kinetic terms are manifestly finite when normal-ordered, because the only divergences arise from

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loops involving a single vertex. A manifestly finite calculation was therefore possible. Such a calculation has recently appeared [5]. From the point of view of the original theory, one might argue that normal ordering is just another scheme. We will take a different point of view, that the normal-ordered theory is a finite quantum field theory with a kink solution in which quantum corrections can be calculated free from any ambiguities. Therefore the normal-ordered theory will be the starting point of our investigation.

Now that all ambiguities have been removed, one can for the first time meaningfully press deeper into the quantum regime. It is now possible to unambiguously calculate the subleading corrections to the kink mass and indeed to the kink state itself. However, to do this, one must first calculate the ϕ^4 vacua and their energies to the same order in perturbation theory. This is the goal of the present paper.

This is a straightforward exercise in perturbation theory. But it has so far escaped attention for several reasons. First of all, the perturbation theory does not converge [6]. Therefore to describe such models rigorously, one convolves the Hamiltonian with a spatial cutoff [7]. However were one to include such a cutoff in the double-well theory, it would imply that beyond the cutoff there is a unique minimum of the potential. One might still introduce two states, one with an expectation value of the scalar field near each minimum within the cutoff. However, there would be finite action instantons which mix these states and the true ground state would be a unique, symmetric combination[1]. Therefore there is no stable kink in this theory.

We will ignore these problems, because our goal is eventually to understand the supersymmetric theory where the divergent vacuum energy at the heart of these problems does not appear. Therefore we will proceed with a perturbative approach with no spatial cutoff.

We begin in Sec. 2 with a description of the corresponding problem in quantum mechanics. Here, just like in the position cutoff quantum field theory, there is a unique ground state [8]. However, at any order in perturbation theory there are in fact two distinct ground states, one localized in each well. We will calculate these ground states and the operators which create them from the harmonic oscillator ground state. Then in Sec. 3 we perform the full computation in quantum field theory, calculating the corrections to the states and energy at second order. We see that the solution is a straightforward generalization of that in quantum mechanics. In Sec. 4 we construct the operator which interpolates between the two ground states. This operator is, in a sense, the $x_0 \to \infty$ limit of the operator that creates a kink centered at x_0.

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[1]Of course the usual logic is to begin by assuming the existence of a unique vacuum.

2 The Double Well In Quantum Mechanics

The double-well model in quantum mechanics has been solved many times, exactly and approximately. In this section we will review the construction of the ground state(s) in second order perturbation theory, as a warm-up for the very similar calculation in quantum field theory in Sec. 3. There are several reasons not to solve this model in perturbation theory. One problem is that the perturbation theory does not converge [9]. Of course this is a standard problem in quantum theories, where one is used to having only asymptotic expansions. However in the present case it means that one can expect significant errors in the wave function whenever one is not close to the spatial minimum of the potential about which one defines the expansion.

A more serious problem is that while the Schrodinger equation with no extra degrees of freedom always has a unique ground state solution [8] this is not necessarily true in perturbation theory. In the present case, in perturbation theory there are in fact two ground states, one localized at each minimum of the well. In the full, nonperturbative theory these two states are mixed by instantons with only the symmetric combination being the ground state. However no instantons appear in perturbation theory. This problem is actually an advantage for us, as it mimics the case in quantum field theory, where any such instantons would have infinite action and so there really are two distinct vacua. Recall that in 1+1 dimensions global symmetries, such as our reflection symmetry, can be spontaneously broken if they are discrete.

We consider a particle in a double-well potential. The corresponding Hamiltonian is

$$H = \frac{p^2}{2m} + k \left(x^2 - v^2 \right)^2,\tag{2.1}$$

where m is the particle's mass, k and v are positive constants and we have set $\hbar = 1$. If the well-separation $2v$ is sufficient, then in perturbation theory there will be two ground states, centered approximately on $x = v$ and $x = -v$, both with the same energy E. We will denote these two states by $|+v\rangle$ and $|-v\rangle$

$$H |\pm v\rangle = E |\pm v\rangle .\tag{2.2}$$

Our task is to find E and also the wave functions of the two ground states.

To find the state $|+v\rangle$, we can rewrite the Hamiltonian as an anharmonic oscillator centered on $x = v$

$$H = \frac{p^2}{2m} + k \left((x - v)^4 + 4v(x - v)^3 + 4v^2(x - v)^2 \right).\tag{2.3}$$

This can be simplified using the spatial displacement operator

$$\mathcal{D}_\alpha = \exp(-i\alpha p) \tag{2.4}$$

to define a new Hamiltonian H' by

$$H' = \mathcal{D}_{-v} H \mathcal{D}_{-v}^{-1} = \frac{p^2}{2m} + 4kv^2 \left(x^2 + \frac{1}{v} x^3 + \frac{1}{4v^2} x^4 \right), \tag{2.5}$$

which again has two ground states. We will be interested in that localized at $x = 0$

$$|g_1\rangle = \mathcal{D}_{-v} |+v\rangle, \qquad H' |g_1\rangle = \mathcal{D}_{-v} H \mathcal{D}_{-v}^{-1} \mathcal{D}_{-v} |+v\rangle = \mathcal{D}_{-v} E |+v\rangle = E |g_1\rangle. \tag{2.6}$$

2.1 Finding $|+v\rangle$

To set up our perturbative expansion, let us expand the Hamiltonian about a harmonic oscillator Hamiltonian H_0

$$H' = H_0 + \frac{x_0}{v} H_1 + \left(\frac{x_0}{v} \right)^2 H_2, \tag{2.7}$$

where

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x_0^2}{2} \frac{x^2}{x_0^2}, \quad H_1 = \frac{m\omega^2 x_0^2}{2} \frac{x^3}{x_0^3}, \quad H_2 = \frac{m\omega^2 x_0^2}{8} \frac{x^4}{x_0^4}, \quad \omega = \sqrt{\frac{8kv^2}{m}}. \tag{2.8}$$

Here x_0 is an arbitrary unit of length, which we introduced to make x_0/v dimensionless. We imagine that x_0/v is small while x_0, m and ω are fixed, and we will calculate states and energies as a power series in x_0/v. In this approximation the two wells are well-separated and are treated as two decoupled anharmonic oscillators. Here H_0, H_1 and H_2 are the leading order, 1st order and 2nd order terms in the expansion of the Hamiltonian, and there are no higher order terms.

Let $|0\rangle$ be the ground state of H_0

$$H_0 |0\rangle = E_0 |0\rangle. \tag{2.9}$$

We will expand g_1 as

$$|g_1\rangle = |0\rangle + \frac{x_0}{v} |1\rangle + \left(\frac{x_0}{v} \right)^2 |2\rangle + O\left(\left(\frac{x_0}{v} \right)^3 \right) \tag{2.10}$$

and its energy as

$$E = E_0 + \frac{x_0}{v} E_1 + \left(\frac{x_0}{v} \right)^2 E_2 + \mathcal{O}\left(\left(\frac{x_0}{v} \right)^3 \right). \tag{2.11}$$

Then to second order in perturbation theory, the Schrodinger equation is

$$\left(H_0 + \frac{x_0}{v} H_1 + \left(\frac{x_0}{v} \right)^2 H_2 \right) \left(|0\rangle + \frac{x_0}{v} |1\rangle + \left(\frac{x_0}{v} \right)^2 |2\rangle + \mathcal{O}\left(\left(\frac{x_0}{v} \right)^3 \right) \right)$$

$$= \left(E_0 + \frac{x_0}{v} E_1 + \left(\frac{x_0}{v} \right)^2 E_2 + \mathcal{O}\left(\left(\frac{x_0}{v} \right)^3 \right) \right) \left(|0\rangle + \frac{x_0}{v} |1\rangle + \left(\frac{x_0}{v} \right)^2 |2\rangle + \mathcal{O}\left(\left(\frac{x_0}{v} \right)^3 \right) \right).$$

(2.12)

To simplify calculations below, we fix unit normalization not on $|g_1\rangle$ but rather on $|0\rangle$, which fixes the normalization of $|g_1\rangle$ so long as each other term $|n\rangle$ is orthogonal to $|0\rangle$

$$\langle 0|0 \rangle = 1, \quad \langle 0|1 \rangle = \langle 0|2 \rangle = ... = \langle 0|n \rangle = 0. \tag{2.13}$$

Now our task is to express $|1\rangle$ and $|2\rangle$ in terms of operators acting on $|0\rangle$. Let us define the Heisenberg operators a and $a^\dagger$ as usual by

$$a = \sqrt{\frac{m\omega}{2}} \left(x + \frac{ip}{m\omega} \right), \quad a^\dagger = \sqrt{\frac{m\omega}{2}} \left(x - \frac{ip}{m\omega} \right), \tag{2.14}$$

then (2.8) becomes

$$H_0 = \omega \left(a^\dagger a + \tfrac{1}{2} \right), \quad H_1 = \sqrt{\frac{\omega}{32mx_0^2}} (a + a^\dagger)^3, \quad H_2 = \frac{1}{32mx_0^2} (a + a^\dagger)^4. \tag{2.15}$$

0th order

First, the terms of order unity in (2.12) reproduce Eq. (2.9) with the unperturbed energy

$$E_0 = \frac{1}{2}\omega. \tag{2.16}$$

1st order

At first order we consider the terms of $O\left(\frac{x_0}{v} \right)$ in (2.12),

$$H_0 |1\rangle + H_1 |0\rangle = E_1 |0\rangle + E_0 |1\rangle. \tag{2.17}$$

In terms of a and $a^\dagger$ in (2.15) this is,

$$\omega \left(a^\dagger a + \tfrac{1}{2} \right) |1\rangle + \frac{1}{4x_0} \sqrt{\frac{\omega}{2m}} (a + a^\dagger)^3 |0\rangle = E_1 |0\rangle + E_0 |1\rangle. \tag{2.18}$$

Solving (2.18), subject to the conditions (2.13), we find

$$E_1 = 0, \quad |1\rangle = -\frac{1}{12x_0\sqrt{2m\omega}} \left(9a^\dagger + a^\dagger a^\dagger a^\dagger \right) |0\rangle. \tag{2.19}$$

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2nd order

Finally, we consider the terms of $O\left(\left(\frac{x_0}{v}\right)^2\right)$ in (2.12),

$$H_0\,|2\rangle + H_1\,|1\rangle + H_2\,|0\rangle = E_0\,|2\rangle + E_1\,|1\rangle + E_2\,|0\rangle. \tag{2.20}$$

In terms of a and $a^\dagger$,

$$\omega\left(a^\dagger a + \tfrac{1}{2}\right)|2\rangle + \frac{1}{4x_0}\sqrt{\frac{\omega}{2m}}(a+a^\dagger)^3\,|1\rangle + \frac{1}{32mx_0^2}(a+a^\dagger)^4\,|0\rangle = E_0\,|2\rangle + E_1\,|1\rangle + E_2\,|0\rangle. \tag{2.21}$$

Solving (2.13) and (2.21) we obtain

$$E_2 = -\frac{1}{4mx_0^2}, \quad |2\rangle = \frac{1}{576m\omega x_0^2}\left(189a^{\dagger 2} + 27a^{\dagger 4} + a^{\dagger 6}\right)|0\rangle. \tag{2.22}$$

Putting It All Together

Thus, to the 2nd order, the energy of the ground state is

$$E = E_0 + \frac{x_0}{v}E_1 + \left(\frac{x_0}{v}\right)^2 E_2 = \frac{\omega}{2} - \frac{1}{4mv^2} \tag{2.23}$$

and the ground state $|g_1\rangle$ is

$$|g_1\rangle = \left[1 - \frac{1}{v}\frac{1}{\sqrt{2m\omega}}\left(\frac{3}{4}a^\dagger + \frac{1}{12}a^{\dagger 3}\right) + \frac{1}{v^2}\left(\frac{21}{64m\omega}a^{\dagger 2} + \frac{3}{64m\omega}a^{\dagger 4} + \frac{1}{576m\omega}a^{\dagger 6}\right)\right]|0\rangle.$$

We can represent this state by a wave function. Recall that the nth excited state of the harmonic oscillator is $\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$, and

$$\langle x|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi}\right)^{1/4}e^{-\frac{m\omega x^2}{2}}H_n(\sqrt{m\omega}x), \tag{2.24}$$

where H_n is the nth Hermite polynomial. The wave function of $|g_1\rangle$ is

$$
\begin{aligned}
\phi_{g_1}(x) &= \langle x|g_1\rangle \\
&= \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}m\omega x^2}\left[m^{\frac{1}{4}}\omega^{\frac{1}{4}} - \frac{1}{v}\left(\frac{1}{2}m^{\frac{1}{4}}\omega^{\frac{1}{4}}x + \frac{1}{6}m^{\frac{5}{4}}\omega^{\frac{5}{4}}x^3\right)\right. \\
&\quad \left. + \frac{1}{v^2}\left(-\frac{41}{192}m^{-\frac{3}{4}}\omega^{-\frac{3}{4}} + \frac{1}{4}m^{\frac{1}{4}}\omega^{\frac{1}{4}}x^2 + \frac{1}{12}m^{\frac{5}{4}}\omega^{\frac{5}{4}}x^4 + \frac{1}{72}m^{\frac{9}{4}}\omega^{\frac{9}{4}}x^6\right)\right]
\end{aligned} \tag{2.25}
$$

which is normalized such that

$$\int dx\,\phi_{g_1}^2(x) = 1 + \frac{29}{96m\omega v^2} + \frac{4981}{18432m^2\omega^2 v^4}. \tag{2.26}$$

As a check, one can substitute (2.25) and (2.23) into the full Schrodinger equation,

$$
\begin{aligned}
& H'\phi_{g_1}(x) - E\phi_{g_1}(x) \\
= \; & \frac{1}{m\pi^{1/4}}e^{-\frac{1}{2}m\omega x^2}\left[\frac{1}{v^3}\left(-\frac{m^{\frac{1}{4}}\omega^{\frac{1}{4}}x}{8} - \frac{19m^{\frac{5}{4}}\omega^{\frac{5}{4}}x^3}{128} + \frac{m^{\frac{9}{4}}\omega^{\frac{9}{4}}x^5}{16} + \frac{m^{\frac{13}{4}}\omega^{\frac{13}{4}}x^7}{48} + \frac{m^{\frac{17}{4}}\omega^{\frac{17}{4}}x^9}{144}\right)\right. \\
& \left.\frac{1}{v^4}\left(-\frac{41m^{-\frac{3}{4}}\omega^{-\frac{3}{4}}}{768} + \frac{m^{\frac{1}{4}}\omega^{\frac{1}{4}}x^2}{16} - \frac{3m^{\frac{5}{4}}\omega^{\frac{5}{4}}x^4}{512} + \frac{5m^{\frac{9}{4}}\omega^{\frac{9}{4}}x^6}{144} + \frac{m^{\frac{13}{4}}\omega^{\frac{13}{4}}x^8}{96} + \frac{m^{\frac{17}{4}}\omega^{\frac{17}{4}}x^{10}}{576}\right)\right] \\
= \; & \mathcal{O}\left(\left(\frac{1}{v}\right)^3\right). \tag{2.27}
\end{aligned}
$$

As expected, the three leading terms have all canceled, leaving only higher order terms. Of course these higher order terms are large when x is far from the bottom of the well centered at $x = 0$, as they will be at any order in perturbation theory.

One may obtain the ground state wave function $|{+}v\rangle$ of the original Hamiltonian H using the shift operator

$$
\begin{aligned}
\phi_{+v}(x) = \; & \mathcal{D}_{+v}\phi_{g_1}(x) \\
= \; & \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}m\omega(x-v)^2}\left[m^{\frac{1}{4}}\omega^{\frac{1}{4}} - \frac{1}{v}\left(\frac{1}{2}m^{\frac{1}{4}}\omega^{\frac{1}{4}}(x-v) + \frac{1}{6}m^{\frac{5}{4}}\omega^{\frac{5}{4}}(x-v)^3\right)\right. \\
& \left. + \frac{1}{v^2}\left(-\frac{41}{192}m^{-\frac{3}{4}}\omega^{-\frac{3}{4}} + \frac{1}{4}m^{\frac{1}{4}}\omega^{\frac{1}{4}}(x-v)^2 + \frac{1}{12}m^{\frac{5}{4}}\omega^{\frac{5}{4}}(x-v)^4 + \frac{1}{72}m^{\frac{9}{4}}\omega^{\frac{9}{4}}(x-v)^6\right)\right]. \tag{2.28}
\end{aligned}
$$

Note that this shift mixes the orders in our perturbative expansion, and so this wave function is not a solution to the original Schrodinger equation beyond the leading order. In fact, we have not defined a perturbative expansion of the original Hamiltonian H.

To obtain the corresponding quantities for the state $|{-}v\rangle$, one need only change the sign of v in the formulas above.

3 Constructing A Perturbative ϕ^4 Ground State

We are interested in the 1+1 dimensional quantum field theory of a real scalar field ϕ with a double-well ϕ^4 interaction

$$
H = \int dx : \frac{1}{2}\left(\partial_x\phi\right)^2 + \frac{1}{2}\pi^2 + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 : \tag{3.1}
$$

where the normal ordering is defined below and π is the momentum conjugate to ϕ. We will also define a mass

$$m = v\sqrt{2\lambda} \tag{3.2}$$

which corresponds to the mass of the field when expanded about the bottom of either well. In 1+1 dimensions ϕ is dimensionless, λ has dimensions of m^2 and v is dimensionless. We will consider a perturbative expansion in $1/v$. This theory has two degenerate ground states of energy E, centered near $\phi = v$ and $\phi = -v$, which we will call $|+v\rangle$ and $|-v\rangle$ respectively. In this section we will construct $|+v\rangle$.

Although ϕ is not a free field, in the Schrodinger picture we can Fourier transform ϕ and π to define a and $a^\dagger$

$$
\begin{aligned}
\phi(x) &= \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a^\dagger_{-p} \right) e^{ipx}, \\
\pi(x) &= \int \frac{dp}{2\pi} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a^\dagger_{-p} \right) e^{ipx}
\end{aligned}
\tag{3.3}
$$

where

$$\omega_p = \sqrt{m^2 + p^2}. \tag{3.4}$$

Now the canonical commutation relations

$$[\phi(x), \pi(y)] = i\delta(x - y) \tag{3.5}$$

imply that a and $a^\dagger$ obey the Heisenberg algebra

$$[a_p, a^\dagger_q] = 2\pi\delta(p - q) \tag{3.6}$$

even in the interacting theory. Our normal ordering prescription is that all $a^\dagger$ be placed to the left of all a.

3.1 The Displacement Operator $\mathcal{D}$

The displacement operator

$$\mathcal{D}_\alpha = e^{-i\alpha \int dx \pi(x)} = e^{\alpha\sqrt{m/2}\left(a^\dagger_0 - a_0\right)} \tag{3.7}$$

shifts the value of ϕ by α

$$[\phi, \mathcal{D}_\alpha] = \alpha \mathcal{D}_\alpha. \tag{3.8}$$

Also it commutes with normal ordering.

Applying it to the Hamiltonian H one finds the shifted Hamiltonian

$$H' = \mathcal{D}_{-v} H \mathcal{D}_{-v}^{-1} \tag{3.9}$$

given by

$$H' = \int dx : \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{v}\frac{m^2}{2}\phi^3 + \frac{1}{v^2}\frac{m^2}{8}\phi^4 : . \tag{3.10}$$

3.2   Finding the Operator Perturbatively

We denote the ground state of (3.10) as $|g_1\rangle$. So we have the relation

$$|+v\rangle = \mathcal{D}_{+v} |g_1\rangle \tag{3.11}$$

and Eq. (2.6) applies as written to the quantum field theory case, showing that $|g_1\rangle$ has energy E. Again our goal is to write $|g_1\rangle$ in terms of the free ground state.

First we perturbatively expand the Hamiltonian (3.10),

$$H' = H_0 + \frac{1}{v}H_1 + \frac{1}{v^2}H_2 \tag{3.12}$$

where

$$H_0 = \int dx : \frac{1}{2}\phi'^2 + \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 :, \qquad H_1 = \frac{1}{v}\int dx \frac{m^2}{2} : \phi^3 :, \qquad H_2 = \frac{1}{v^2}\int dx \frac{m^2}{8} : \phi^4 : . \tag{3.13}$$

H_0 is the Hamiltonian of the free, massive Klein-Gordon theory, and the next two terms are perturbations of order $O\left(\frac{1}{v}\right)$ and $O\left(\frac{1}{v^2}\right)$.

Expanding the ground state

$$|g_1\rangle = \left(|0\rangle + \frac{1}{v}|1\rangle + \frac{1}{v^2}|2\rangle\right) + \mathcal{O}\left(\frac{1}{v^3}\right) \tag{3.14}$$

subject to (2.13) and the energy as in Eq. (2.11) we obtain the eigenvalue equation for $|g_1\rangle$

$$\left(H_0 + \frac{1}{v}H_1 + \frac{1}{v^2}H_2\right)\left(|0\rangle + \frac{1}{v}|1\rangle + \frac{1}{v^2}|2\rangle + \mathcal{O}\left(\frac{1}{v^3}\right)\right)$$

$$= \left(E_0 + \frac{1}{v}E_1 + \frac{1}{v^2}E_2 + \mathcal{O}\left(\frac{1}{v^3}\right)\right)\left(|0\rangle + \frac{1}{v}|1\rangle + \frac{1}{v^2}|2\rangle + \mathcal{O}\left(\frac{1}{v^3}\right)\right). \tag{3.15}$$

0th order

The leading terms in (3.15) are

$$H_0 |0\rangle = E_0 |0\rangle \tag{3.16}$$

so that $|0\rangle$ is the free ground state and

$$E_0 = 0. \tag{3.17}$$

Note that this is superficially different from the energy in the quantum mechanical case, because we have normal ordered our Hamiltonian. Of course had we normal ordered in that case we would also have obtained a vanishing value of E_0.

1st order

The first order terms in (3.15) are

$$H_0 |1\rangle + H_1 |0\rangle = E_1 |0\rangle . \tag{3.18}$$

Using the normal ordering in H_1 and the fact that

$$a_p |0\rangle = 0 \tag{3.19}$$

as a result of (3.16) and (3.17), this can be written explicitly as

$$H_0 |1\rangle = E_1 |0\rangle - \frac{m^2}{2} \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}} |0\rangle . \tag{3.20}$$

Applying $\langle 0|$ to (3.20), together with the condition (2.13), we find

$$E_1 = 0 \tag{3.21}$$

and so then (3.20) is solved by inverting H_0, yielding a sum over frequencies in the denominator

$$|1\rangle = -\frac{m^2}{2} \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})} |0\rangle . \tag{3.22}$$

This looks similar to (2.19), except that the $a^\dagger$ term is missing. This again is the result of the normal ordering.

2nd order

Finally we arrive at the $O\left(\frac{1}{v^2}\right)$ terms in (3.15),

$$H_0 |2\rangle + H_1 |1\rangle + H_2 |0\rangle = E_2 |0\rangle . \tag{3.23}$$

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Applying $\langle 0|$ on the left, the first term vanishes because $E_0 = 0$ and the H_2 term vanishes because it is normal ordered. Using the normalization of $|0\rangle$ we obtain

$$E_2 = \langle 0|H_1|1\rangle. \tag{3.24}$$

Using (3.22) we find the ground state energy at this order

$$E_2 = \int dx \mathcal{E}_2(x) \tag{3.25}$$

$$\mathcal{E}_2(x) = -\frac{3}{16}m^4 \int \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \frac{1}{\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}} \frac{1}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})}. \tag{3.26}$$

The energy is infinite, however the energy density $\mathcal{E}_2$ is finite and constant. To the 2nd order, the energy density of the ϕ^4 vacuum state is

$$
\begin{aligned}
\mathcal{E} &= 0 + \frac{1}{v}\mathcal{E}_1 + \frac{1}{v^2}\mathcal{E}_2 \\
&= -\frac{1}{v^2}\frac{3}{16}m^4 \int \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \frac{1}{\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}} \frac{1}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})} \\
&\approx -0.022\lambda.
\end{aligned} \tag{3.27}
$$

Now we know everything in Eq. (3.23) except for $|2\rangle$. This can again be found by inverting H_0, providing another sum over frequencies in the denominator

$$
\begin{aligned}
|2\rangle &= \Bigg[\frac{9}{4}m^4 \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{8\omega_{p_1}^2 \omega_{p_2}\omega_{p_1+p_2}} \frac{a_{p_1}^\dagger a_{-p_1}^\dagger}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})} \\
&\quad + \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \frac{1}{\sqrt{16\omega_{p_1}\omega_{p_2}\omega_{p_3}\omega_{p_1+p_2+p_3}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{-p_1-p_2-p_3}^\dagger}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_3} + \omega_{p_1+p_2+p_3})} \\
&\quad \times \left(\frac{9}{4}m^4 \frac{1}{2\omega_{p_1+p_2}(\omega_{p_1+p_2} + \omega_{p_3} + \omega_{p_1+p_2+p_3})} - \frac{m^2}{8}\right) \\
&\quad + \int \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \frac{m^4}{4} \frac{1}{\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2} + \omega_{p_3} + \omega_{p_4} + \omega_{p_3+p_4}} \\
&\quad \times \frac{1}{\sqrt{64\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}\omega_{p_3}\omega_{p_4}\omega_{p_3+p_4}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger a_{-p_3-p_4}^\dagger}{(\omega_{p_3} + \omega_{p_4} + \omega_{p_3+p_4})}\Bigg] |0\rangle. \tag{3.28}
\end{aligned}
$$

Note that it resembles the corresponding expression in quantum mechanics (2.22). The two particle, four particle and six particle components have just the same origin in both cases.

Now we can write $|g_1\rangle$ to 2nd order

$$
\begin{aligned}
|g_1\rangle \;=\; & \Bigg\{ 1 - \frac{1}{v}\frac{m^2}{2} \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger}{(\omega_{p_1}+\omega_{p_2}+\omega_{p_1+p_2})} \\
& + \frac{1}{v^2}\Bigg[\frac{9}{4}m^4 \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{8\omega_{p_1}^2 \omega_{p_2}\omega_{p_1+p_2}} \frac{a_{p_1}^\dagger a_{-p_1}^\dagger}{(\omega_{p_1}+\omega_{p_2}+\omega_{p_1+p_2})} \\
& + \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \frac{1}{\sqrt{16\omega_{p_1}\omega_{p_2}\omega_{p_3}\omega_{p_1+p_2+p_3}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{-p_1-p_2-p_3}^\dagger}{(\omega_{p_1}+\omega_{p_2}+\omega_{p_3}+\omega_{p_1+p_2+p_3})} \\
& \times \left(\frac{9}{4}m^4 \frac{1}{2\omega_{p_1+p_2}(\omega_{p_1+p_2}+\omega_{p_3}+\omega_{p_1+p_2+p_3})} - \frac{m^2}{8}\right) \\
& + \int \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \frac{m^4}{4} \frac{1}{\omega_{p_1}+\omega_{p_2}+\omega_{p_1+p_2}+\omega_{p_3}+\omega_{p_4}+\omega_{p_3+p_4}} \\
& \times \frac{1}{\sqrt{64\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}\omega_{p_3}\omega_{p_4}\omega_{p_3+p_4}}} \frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger a_{-p_3-p_4}^\dagger}{(\omega_{p_3}+\omega_{p_4}+\omega_{p_3+p_4})} \Bigg] \Bigg\} |0\rangle . \quad (3.29)
\end{aligned}
$$

Let us denote the operator in the curly bracket by $\mathcal{O}'$

$$|g_1\rangle = \mathcal{O}' |0\rangle . \tag{3.30}$$

Let the operator which takes the free vacuum $|0\rangle$ to the ϕ^4 vacuum $|+v\rangle$ be $\mathcal{O}_1$

$$|+v\rangle = \mathcal{O}_1 |0\rangle \tag{3.31}$$

then

$$\mathcal{O}_1 = \mathcal{D}_{+v}\mathcal{O}'. \tag{3.32}$$

4 Interpolating Operator

To obtain $|-v\rangle$, the other ground state, we perform exactly the same manipulations but with the sign of v flipped. The resulting state $|g_1\rangle$ is identical to that in Eq. (3.29) except each v is replaced by $-v$. Let the corresponding operator, corresponding to $\mathcal{O}'$ with $v \to -v$, be $\mathcal{O}''$ and the operator that changes $|0\rangle$ to $|-v\rangle$ be $\mathcal{O}_2$

$$\mathcal{O}_2 = \mathcal{D}_{-v}\mathcal{O}''. \tag{4.1}$$

The operator $\mathcal{O}$ that changes $|+v\rangle$ to $|-v\rangle$

$$|-v\rangle = \mathcal{O} |+v\rangle \tag{4.2}$$

is then

$$
\begin{aligned}
\mathcal{O} &= \mathcal{O}_2 \mathcal{O}_1^{-1} \\
&= \mathcal{D}_{-v} \mathcal{O}'' \left(\mathcal{D}_{+v} \mathcal{O}' \right)^{-1} \\
&= \mathcal{D}_{-v} \mathcal{O}'' \mathcal{O}'^{-1} \mathcal{D}_{-v}.
\end{aligned}
\tag{4.3}
$$

Let us define the shorthand notation

$$
\mathcal{O}' = 1 - \frac{1}{v} \mathcal{A} + \frac{1}{v^2} \mathcal{B},
\tag{4.4}
$$

$$
\mathcal{O}'' = 1 + \frac{1}{v} \mathcal{A} + \frac{1}{v^2} \mathcal{B}
$$

where

$$
\mathcal{A} = \frac{m^2}{2} \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}} \frac{a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{-p_1-p_2}}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})},
\tag{4.5}
$$

$$
\begin{aligned}
\mathcal{B} &= \frac{9}{4} m^4 \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{8\omega_{p_1}^2 \omega_{p_2} \omega_{p_1+p_2}} \frac{a^\dagger_{p_1} a^\dagger_{-p_1}}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2})} \\
&+ \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \frac{1}{\sqrt{16\omega_{p_1}\omega_{p_2}\omega_{p_3}\omega_{p_1+p_2+p_3}}} \frac{a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{p_3} a^\dagger_{-p_1-p_2-p_3}}{(\omega_{p_1} + \omega_{p_2} + \omega_{p_3} + \omega_{p_1+p_2+p_3})} \\
&\times \left(\frac{9}{4} m^4 \frac{1}{2\omega_{p_1+p_2}(\omega_{p_1+p_2} + \omega_{p_3} + \omega_{p_1+p_2+p_3})} - \frac{m^2}{8} \right) \\
&+ \int \frac{dp_1 dp_2 dp_3 dp_4}{(2\pi)^4} \frac{m^4}{4} \frac{1}{\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2} + \omega_{p_3} + \omega_{p_4} + \omega_{p_3+p_4}} \\
&\times \frac{1}{\sqrt{64\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}\omega_{p_3}\omega_{p_4}\omega_{p_3+p_4}}} \frac{a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{-p_1-p_2} a^\dagger_{p_3} a^\dagger_{p_4} a^\dagger_{-p_3-p_4}}{(\omega_{p_3} + \omega_{p_4} + \omega_{p_3+p_4})}.
\end{aligned}
\tag{4.6}
$$

Then,

$$
\begin{aligned}
\mathcal{O}'^{-1} &= \left(1 - \frac{1}{v} \mathcal{A} + \frac{1}{v^2} \mathcal{B} \right)^{-1} \\
&= 1 + \frac{1}{v} \mathcal{A} + \frac{1}{v^2} \left(\mathcal{A}^2 - \mathcal{B} \right).
\end{aligned}
\tag{4.7}
$$

Finally,

$$
\begin{aligned}
\mathcal{O} &= \mathcal{D}_{-v}\left(1 + \frac{1}{v}\mathcal{A} + \frac{1}{v^2}\mathcal{B}\right)\left(1 + \frac{1}{v}\mathcal{A} + \frac{1}{v^2}\left(\mathcal{A}^2 - \mathcal{B}\right)\right)\mathcal{D}_{-v} \\
&= \mathcal{D}_{-v}\left[1 + \frac{2}{v}\mathcal{A} + \frac{2}{v^2}\mathcal{A}^2\right]\mathcal{D}_{-v} \\
&= e^{-v\sqrt{\frac{m}{2}}\left(a_0^\dagger - a_0\right)}\left[1 + \frac{m^2}{v}\int\frac{dp_1 dp_2}{(2\pi)^2}\frac{1}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}}\frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger}{\left(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2}\right)}\right. \\
&\quad + \left.\frac{m^4}{2v^2}\left(\int\frac{dp_1 dp_2}{(2\pi)^2}\frac{1}{\sqrt{8\omega_{p_1}\omega_{p_2}\omega_{p_1+p_2}}}\frac{a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_1-p_2}^\dagger}{\left(\omega_{p_1} + \omega_{p_2} + \omega_{p_1+p_2}\right)}\right)^2\right]e^{-v\sqrt{\frac{m}{2}}\left(a_0^\dagger - a_0\right)}. \quad (4.8)
\end{aligned}
$$

At this order, there is no dependence on $\mathcal{B}$.

The main motivation of the present work is to provide necessary formulas for the calculation of higher corrections to the ϕ^4 kink. If the center of the kink is at x_0 and the kink form factor, in the sense of Ref. [10], interpolates between $-v$ at $x \to -\infty$ and $+v$ at $x \to \infty$, then we expect that in the suitably defined limit $x_0 \to \infty$ the operator which creates the kink will reduce to $\mathcal{O}$.

5   Remarks

In this note we have performed a simple exercise in perturbative quantum field theory, finding the ground states of the ϕ^4 double-well theory and their energies. Armed with this perturbative expression for the ground state, we may now extend the calculation of kink state and mass in Ref. [5] beyond the leading quantum corrections. We wish to do this because our eventual goal, in the supersymmetric case, is to attempt to construct the kink at all orders but we would like the first terms in the perturbation theory as a starting point.

Another possible application of this ground state is in Higgs physics. The Higgs field is of this form, except that it is complex and also lives in 3+1 dimensions. However the radial mode, near the minimum, is governed by the same potential. A quantum description of this state may help one to study the following phenomena.

A superconductor can repel a magnetic field. It does this by letting its electrons travel in circles, which creates a counter magnetic field. This costs energy, and so changes the potential energy as a function of the Cooper pair condensate. The field equations imply that this energy must be minimized and so, in the presence of an external magnetic field, it is well known that the Cooper pair condensate is reduced.

Exactly the same argument may be applied to the Standard Model Higgs field. It creates mass for, for example, the top quark. But this mass costs the Higgs field energy, and so in the minimum energy field configuration, in the presence of a top quark, the Higgs condensate will be reduced. This reduction is quite localized about the top quark, which anyway does not live long. However, in a proposed 100 TeV collider, one expects that there will be a larger region where a number of particles are given masses by the Higgs field. If this energy density is of order the Higgs potential, then the backreaction on the Higgs field, reducing its value and the particle masses, may be considerable. This would be new physics, but simply calculated in perturbation theory with this potential coupled via a Yukawa coupling to fermions. This is the same coupling that we will need to add anyway to study the supersymmetric kink.

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