# Gyrokinetic theory with polynomial transforms: a model for ions and electrons in maximal ordering 

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#### Abstract

We propose a novel derivation of the gyrokinetic field-particle Lagrangian for non-collisional ion-electron plasmas in a magnetic background with strong variations (maximal ordering). Our approach follows the twostep reduction process, where the guiding-center coordinate transformation is followed by the gyrocenter coordinate transformation in the single-particle phase space. For the first time both steps are addressed within a unique methodology, based on near-identity coordinate transformations constructed as polynomial transforms. These are well-defined transformations composed of a finite number of terms that are linear and algebraic with respect to the generating functions. The derivation is carried out in a fully non-dimensional framework, based on parameters governing the magnetic fusion experiments ASDEX Upgrade and ITER. Our method leads to a gyrokinetic Vlasov-Maxwell model for ions and electrons, derived without the use of Lie perturbation methods. It is found that, based on the employed ordering, curvature terms such as the gyro-gauge term and the Baños drift appear at first order in the ion Hamiltonian, whereas ion polarization terms appear only at second order. By contrast, curvature terms are absent from the first-order electron Hamiltonian, where instead magnetic flutter plays a role.


## 1 Introduction

Gyrokinetics is one of the major frameworks used in theoretical and numerical studies of low-frequency turbulence in magnetized fusion plasmas [1,2]. Gyrokinetic models are based on a change of coordinates in particle phase space that separates fast from slow dynamics, namely cyclotron motion from drift motion. The idea is to derive a reduced set of dynamical equations that contain enough information for an adequate description of low-frequency phenomena in the plasma. As a result, the dimensionality of the problem is reduced and so is the computational cost in numerical simulations.

In mathematical terms, gyrokinetic theory can be considered as the asymptotic study of the Vlasov-Maxwell model with strong magnetic background in the quasi-neutral regime. This requires a suitable reformulation of the equations such that the corresponding asymptotic limit can be carried out in a meaningful way. The procedure can be understood in the context of averaging the characteristics (or Lagrangian paths) of the Vlasov equation, as described first in [3]. Rigorous mathematical accounts can be found, for example, in [4-6]. A first attempt to average the Vlasov-Maxwell system taking into account the self-consistent interaction between plasma particles and electromagnetic fields can be found in [7].

From another point of view, the studies of [8-11] laid the foundation of "structure-preserving" gyrokinetic theory. These works first showed that averaging could be carried out on the level of the variational principle, by transforming the particle Lagrangian (or its corresponding Poisson bracket structure). This strategy has the advantage of preserving symmetries of the plasma equations during the process of averaging, in particular their Hamiltonian structure: the averaged equations exactly conserve averaged versions of the true constants of the motion, such as energy or momentum. Most of the gyrokinetic simulation codes [12-15] are based on structure-preserving gyrokinetic models and often show improved stability and accuracy. The derivation of gyrokinetic models has been discussed and reviewed extensively in $[2,16]$, and references therein. The prevalent methodology is based on Lie transform perturbation theory, as presented for example in [17, 18], and most of the recent derivations have been carried out in this framework, as in [19-23].

This work is motivated by the need for an easier access to gyrokinetic theory, without having to rely extensively on Lie perturbation methods. Such methods, despite being mathematically elegant, are formulated in the language of differential geometry and may thus prevent readers from focusing on the essence of the gyrokinetic reduction. Here, we propose a different method which is inspired by the guiding-center theory of Littlejohn
[24]. A similar approach has been suggested recently in [25] and worked out in the long-wavelength regime of gyrokinetics in [26]. In this work we extend the methodology presented in [26] to address the description of turbulence on the microscopic scale of plasmas, such as the ion Larmor radius. As in Lie transform perturbation theory, our method is also based on near-identity phase-space coordinate transformations. However, we propose to construct such transformations in a different way. More precisely, our phase-space coordinate transformations are defined as polynomials of finite degree in powers of a given perturbation parameter, hence the name "polynomial transforms". The coefficients of such polynomials are the so-called generating functions (or generators) and represent the degrees of freedom that allow us to separate fast and slow scales. This ansatz is conceptually simpler than working with Lie transforms, which are asymptotic series (thus not necessarily convergent) constructed as products of operator exponentials which feature Lie derivatives along the generating vector fields. In this work we show that structure-preserving gyrokinetic equations can be derived without the use of such a complex mathematical machinery.

We follow the strategy of the two-step gyrokinetic reduction, where the guiding-center coordinate transformation is followed by the gyrocenter coordinate transformation [23]. To derive the reduced equations we apply polynomial transforms in each of the two steps, leading to a unified methodology for the complete reduction process. Moreover, in the spirit of asymptotic analysis our derivation is carried out in a fully non-dimensional framework. The perturbation parameter $\varepsilon \ll 1$ in our near-identity coordinate transformations is identified by a rigorous normalization of the Vlasov-Maxwell model. Our ordering in powers of $\varepsilon$ is then based on assumptions derived from realistic physical scenarios relevant for existing and future fusion experiments, such as the Tokamaks ASDEX Upgrade and ITER. In this way we clearly separate the physical assumptions (ordering) from the mathematical model reduction (averaging with polynomial transforms). Our methodology based on a priori normalization of the physical equations allows us to formulate a gyrokinetic theory for both ions and electrons within the same physical scenario and assumptions.

This paper is organized as follows. Section 2 introduces the basic equations of the Vlasov-Maxwell model for a non-collisional magnetized plasma, including a field-theoretic Lagrangian formulation. Section 3 defines the normalization scheme used for the purpose of non-dimensionalization and derives an ordering pattern based on physical considerations. Section 4 describes in detail the gyrokinetic reduction procedure, leading from physical phase-space coordinates to gyrocenter coordinates, and outlines the main results of this work, namely gyrokinetic Lagrangians for both ions and electrons in maximal ordering, together with the corresponding particle equations of motion and Maxwell's equations. Appendix A reviews briefly the field-theoretic Lagrangian formulation of the Vlasov-Maxwell model. Appendix B repeats some calculations pertaining to the guiding-center transformation, presented in detail in [26]. Appendix C contains the calculations pertaining to the derivation and proof of our main results for the gyrocenter transformation, namely Propositions 1-3 for ions and Propositions 4-6 for electrons.

## 2 The Vlasov-Maxwell model

We consider a non-collisional plasma composed of ions and electrons described in terms of the distribution functions $f_{\mathrm{s}}(t, \boldsymbol{x}, \boldsymbol{v})$, where s denotes the particle species, $t \in \mathbb{R}^{+}$denotes the time coordinate, and $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{3} \times \mathbb{R}^{3}$ are position and velocity coordinates in phase space. All equations are written in SI units in the following. The distribution functions $f_{\mathrm{s}}$ obey the non-collisional Vlasov equation

$$
\begin{equation*}
\frac{\partial f_{\mathrm{s}}}{\partial t}+\boldsymbol{v} \cdot \boldsymbol{\nabla} f_{\mathrm{s}}+\frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}(\boldsymbol{E}+\boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_{\mathrm{s}}}{\partial \boldsymbol{v}}=0 \tag{1}
\end{equation*}
$$

where $q_{\mathrm{s}}$ and $m_{\mathrm{s}}$ denote the particle charge and mass, respectively. The electromagnetic fields $\boldsymbol{E}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$ satisfy Maxwell's equations

$$
\begin{align*}
& \boldsymbol{\nabla} \cdot \boldsymbol{E}=\frac{\rho}{\varepsilon_{0}}  \tag{2a}\\
& \boldsymbol{\nabla} \cdot \boldsymbol{B}=0  \tag{2b}\\
& \boldsymbol{\nabla} \times \boldsymbol{E}=-\frac{\partial \boldsymbol{B}}{\partial t}  \tag{2c}\\
& \boldsymbol{\nabla} \times \boldsymbol{B}=\mu_{0} \boldsymbol{J}+\varepsilon_{0} \mu_{0} \frac{\partial \boldsymbol{E}}{\partial t} \tag{2d}
\end{align*}
$$

(Coulomb's law),
(absence of free magnetic poles),
(Faraday's law),
(Ampère-Maxwell's law),
where $\varepsilon_{0}$ and $\mu_{0}$ denote the vacuum electric permittivity and the vacuum magnetic permeability, respectively. The sources $\rho(t, \boldsymbol{x})$ and $\boldsymbol{J}(t, \boldsymbol{x})$ are expressed in terms of the distribution functions as

$$
\begin{align*}
& \rho=\sum_{\mathrm{s}} q_{\mathrm{s}} n_{\mathrm{s}}=\sum_{\mathrm{s}} q_{\mathrm{s}} \int \mathrm{~d}^{3} \boldsymbol{v} f_{\mathrm{s}}(t, \boldsymbol{x}, \boldsymbol{v}),  \tag{3a}\\
& \boldsymbol{J}=\sum_{\mathrm{s}} q_{\mathrm{s}} n_{\mathrm{s}} \boldsymbol{u}_{\mathrm{s}}=\sum_{\mathrm{s}} q_{\mathrm{s}} \int \mathrm{~d}^{3} \boldsymbol{v} \boldsymbol{v} f_{\mathrm{s}}(t, \boldsymbol{x}, \boldsymbol{v}) . \tag{3b}
\end{align*}
$$

The derivation of the Vlasov-Maxwell system (1)-(3) from an action principle was recognized first by [27]. Denoting by $\phi(t, \boldsymbol{x})$ and $\boldsymbol{A}(t, \boldsymbol{x})$ the electric scalar potential and the magnetic vector potential associated to the electric and magnetic fields via $\boldsymbol{E}=-\boldsymbol{\nabla} \phi-\partial \boldsymbol{A} / \partial t$ and $\boldsymbol{B}=\boldsymbol{\nabla} \times \boldsymbol{A}$, Low's action principle reads

$$
\begin{equation*}
\delta \int_{t_{0}}^{t_{1}} \mathrm{~d} t\left(L_{\mathrm{p}}+L_{\mathrm{EM}}\right)=0 \tag{4}
\end{equation*}
$$

where $\delta$ denotes the functional derivative and the Lagrangian is the sum of the electromagnetic free-field Lagrangian

$$
\begin{equation*}
L_{\mathrm{EM}}=\frac{\varepsilon_{0}}{2} \int \mathrm{~d}^{3} \boldsymbol{x}\left|\boldsymbol{\nabla} \phi+\frac{\partial \boldsymbol{A}}{\partial t}\right|^{2}-\frac{1}{2 \mu_{0}} \int \mathrm{~d}^{3} \boldsymbol{x}|\boldsymbol{\nabla} \times \boldsymbol{A}|^{2}, \tag{5}
\end{equation*}
$$

and the particle Lagrangian

$$
\begin{equation*}
L_{\mathrm{p}}=\sum_{\mathrm{s}} \int \mathrm{~d}^{3} \boldsymbol{x}_{0} \mathrm{~d}^{3} \boldsymbol{v}_{0} f_{\mathrm{s}}\left(t_{0}, \boldsymbol{x}_{0}, \boldsymbol{v}_{0}\right) L_{\mathrm{s}} \tag{6}
\end{equation*}
$$

Here, $L_{\mathrm{s}}$ denotes the single-particle Lagrangian for the respective particle species, defined on the tangent bundle of the single-particle phase space. In the phase-space coordinates $(\boldsymbol{x}, \boldsymbol{v})$, the single-particle Lagrangian $L_{\mathrm{s}}$ reads

$$
\begin{equation*}
L_{\mathrm{s}}=\left(m_{\mathrm{s}} \boldsymbol{v}+q_{\mathrm{s}} \boldsymbol{A}\right) \cdot \dot{\boldsymbol{x}}-H_{\mathrm{s}} \quad \text { with } \quad H_{s}:=m_{\mathrm{s}}|\boldsymbol{v}|^{2} / 2+q_{\mathrm{s}} \phi . \tag{7}
\end{equation*}
$$

Hence, $L_{\mathrm{s}}$ depends implicitly on the potentials $\phi$ and $\boldsymbol{A}$ and describes the self-consistent interaction between plasma particles and electromagnetic fields.

The variational principle (4) leads to the characteristics of the Vlasov equation by computing variations of $L_{\mathrm{s}}$ with respect to single-particle trajectories $(\boldsymbol{x}(t), \boldsymbol{v}(t))$, to Coulomb's law by computing variations with respect to $\phi$, and to Ampère-Maxwell's law by computing variations with respect to $\boldsymbol{A}$. We refer to Appendix A for more details. Only the non-homogeneous Maxwell's equations, featuring source terms coupling to the plasma particles, can be derived from the variational principle. With appropriate initial/boundary conditions, this results in a well-posed system for $\left(f_{\mathrm{s}}, \phi, \boldsymbol{A}\right)$, which describes the self-consistent interaction between plasma particles and electromagnetic fields.

An important aspect in gyrokinetic theory is the separation of the electromagnetic fields into background and fluctuating parts. In this work we assume that the magnetic field consists of dynamic fluctuations added to a static background, while the electric field consists only of dynamic fluctuations (without a static background):

$$
\begin{equation*}
\boldsymbol{B}(t, \boldsymbol{x})=\boldsymbol{B}_{0}(\boldsymbol{x})+\boldsymbol{B}_{1}(t, \boldsymbol{x}), \quad \boldsymbol{E}(t, \boldsymbol{x})=\boldsymbol{E}_{1}(t, \boldsymbol{x}) . \tag{8}
\end{equation*}
$$

A dynamic background electric field $\boldsymbol{E}_{0}(t, \boldsymbol{x})$ can be studied within the drift-kinetic framework, as in [26], but is neglected here in order to focus on aspects specific to the gyrokinetic regime. Similarly, the electromagnetic potentials are written as

$$
\begin{equation*}
\boldsymbol{A}(t, \boldsymbol{x})=\boldsymbol{A}_{0}(\boldsymbol{x})+\boldsymbol{A}_{1}(t, \boldsymbol{x}), \quad \phi(t, \boldsymbol{x})=\phi_{1}(t, \boldsymbol{x}), \tag{9}
\end{equation*}
$$

so that $\boldsymbol{B}_{0}=\boldsymbol{\nabla} \times \boldsymbol{A}_{0}$. Therefore, in the variational principle (4) variations have to be computed with respect to the dynamic potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$.

## 3 Normalization and ordering

The formulation of the Vlasov-Maxwell system as a perturbation problem requires the non-dimensionalization (or scaling) of the physical equations and the subsequent application of an ordering scheme that allows for a comparison of terms in relation to an asymptotic parameter $\varepsilon \ll 1$. Since gyrokinetics is ultimately the theory of low-frequency dynamics in strongly-magnetized plasmas, the perturbation parameter is typically defined as the ratio between a characteristic ion turbulence frequency $\widehat{\omega}_{\mathrm{i}}$ and the ion cyclotron frequency $\omega_{\mathrm{ci}}$ :

$$
\begin{equation*}
\varepsilon=\frac{\widehat{\omega}_{\mathrm{i}}}{\omega_{\mathrm{ci}}} \tag{10}
\end{equation*}
$$

Gyrokinetic theory can then be considered formally as the asymptotic analysis of the Vlasov-Maxwell model in the limit $\varepsilon \rightarrow 0$. The procedure of non-dimensionalization and the introduction of the scaling parameter $\varepsilon$ in the equations are fundamental. Therefore, we give here a detailed description of both steps, which typically are not treated extensively in the gyrokinetic literature. We first introduce the generic normalization of the Low action principle (4) and then suggest an ordering scheme that corresponds to a realistic physical scenario relevant for existing and future fusion experiments, such as the Tokamaks ASDEX Upgrade and ITER.

### 3.1 Normalization of the Vlasov-Maxwell model

In order to write the Vlasov-Maxwell model in non-dimensional form, we introduce reference scales (denoted by a hat) for time, length, and ion and electron velocities:

$$
t=\hat{t} t^{\prime}, \quad \boldsymbol{x}=\hat{x} \boldsymbol{x}^{\prime}, \quad \boldsymbol{v}= \begin{cases}\widehat{v}_{\mathrm{i}} \boldsymbol{v}^{\prime} & \text { ions }  \tag{11}\\ \widehat{v}_{\mathrm{e}} \boldsymbol{v}^{\prime \prime} & \text { electrons }\end{cases}
$$

Here, the primed quantities $t^{\prime}, x^{\prime}, \boldsymbol{v}^{\prime}$ and $\boldsymbol{v}^{\prime \prime}$ are non-dimensional and the characteristic velocities for ions and electrons, $\widehat{v}_{\mathrm{i}}$ and $\widehat{v}_{\mathrm{e}}$, can be chosen differently, which means that $\boldsymbol{v}^{\prime}$ and $\boldsymbol{v}^{\prime \prime}$ are different velocity coordinates. We then write the background magnetic field and its corresponding magnetic vector potential as

$$
\begin{equation*}
\boldsymbol{B}_{0}(\boldsymbol{x})=\widehat{B}_{0} \boldsymbol{B}_{0}^{\prime}\left(\frac{\hat{x}}{\ell_{0}} \boldsymbol{x}^{\prime}\right), \quad \boldsymbol{A}_{0}(\boldsymbol{x})=\hat{A}_{0} \boldsymbol{A}_{0}^{\prime}\left(\frac{\widehat{x}}{\ell_{0}} \boldsymbol{x}^{\prime}\right) \tag{12}
\end{equation*}
$$

where $\ell_{0}=\left\|\boldsymbol{\nabla} \boldsymbol{B}_{0} / B_{0}\right\|^{-1}$ denotes the length scale of the background magnetic field. Here, the primed functions $\boldsymbol{B}_{0}^{\prime}$ and $\boldsymbol{A}_{0}^{\prime}$ are dimensionless, of order $O(1)$ and with variations of order $O(1)$ in the limit $\varepsilon \rightarrow 0$. In particular, they only take non-dimensional (scaled) arguments. Choosing $\widehat{A}_{0}:=\widehat{B}_{0} \ell_{0}$, we obtain $\boldsymbol{B}_{0}^{\prime}=\boldsymbol{\nabla}^{\prime} \times \boldsymbol{A}_{0}^{\prime}$ in the scaled variables. We remark that for a uniform background magnetic field the considerations for the vector potential $\boldsymbol{A}_{0}$ are still valid $\left(\boldsymbol{A}_{0}=\left(\boldsymbol{B}_{0} \times \boldsymbol{x}\right) / 2\right.$ in this case $)$. With regard to the dynamic fields $\boldsymbol{B}_{1}$ and $\boldsymbol{E}_{1}$, we denote their length and time scales by $\ell_{1}$ and $\tau_{1}$, respectively. For the magnetic and electric fluctuations (and their corresponding potentials) we write

$$
\begin{array}{ll}
\boldsymbol{B}_{1}(t, \boldsymbol{x})=\widehat{B}_{1} \boldsymbol{B}_{1}^{\prime}\left(\frac{\hat{t}}{\tau_{1}} t^{\prime}, \frac{\hat{x}}{\ell_{1}} \boldsymbol{x}^{\prime}\right), & \boldsymbol{A}_{1}(t, \boldsymbol{x})=\widehat{A}_{1} \boldsymbol{A}_{1}^{\prime}\left(\frac{\hat{t}}{\tau_{1}} t^{\prime}, \frac{\hat{x}}{\ell_{1}} \boldsymbol{x}^{\prime}\right) \\
\boldsymbol{E}_{1}(t, \boldsymbol{x})=\widehat{E}_{1} \boldsymbol{E}_{1}^{\prime}\left(\frac{\hat{t}}{\tau_{1}} t^{\prime}, \frac{\hat{x}}{\ell_{1}} \boldsymbol{x}^{\prime}\right), & \phi_{1}(t, \boldsymbol{x})=\hat{\phi}_{1} \phi_{1}^{\prime}\left(\frac{\hat{t}}{\tau_{1}} t^{\prime}, \frac{\hat{x}}{\ell_{1}} \boldsymbol{x}^{\prime}\right) \tag{13b}
\end{array}
$$

Choosing $\widehat{A}_{1}:=\widehat{B}_{1} \ell_{1}$ and $\widehat{\phi}_{1}:=\widehat{E}_{1} \ell_{1}$, we obtain

$$
\begin{equation*}
\boldsymbol{B}_{1}^{\prime}=\boldsymbol{\nabla}^{\prime} \times \boldsymbol{A}_{1}^{\prime}, \quad \boldsymbol{E}_{1}^{\prime}=-\boldsymbol{\nabla}^{\prime} \phi_{1}^{\prime}-\frac{\widehat{B}_{1} \ell_{1}}{\widehat{E}_{1} \tau_{1}} \frac{\partial \boldsymbol{A}_{1}^{\prime}}{\partial t^{\prime}} \tag{14}
\end{equation*}
$$

in the scaled variables. We remark that the amplitudes $\widehat{A}_{1}$ and $\widehat{\phi}_{1}$ depend on the length scale $\ell_{1}$ of the fluctuations. Therefore, if the sizes of the field fluctuations $\widehat{B}_{1}$ and $\hat{E}_{1}$ are fixed, small-scale fluctuations are associated to small potentials, while large-scale fluctuations are associated to large potentials. The size of the potentials, in turn, plays a role in the ordering of terms in the particle Lagrangian, and thus in the overall asymptotic expansion.

In the Low action (4) we normalize the electromagnetic free-field Lagrangian (5) as $L_{\mathrm{EM}}=\widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}} \widehat{x}^{3} L_{\mathrm{EM}}^{\prime}$, where $\widehat{n}_{\mathrm{i}}$ denotes a reference ion density, $\widehat{T}_{\mathrm{i}}$ a reference ion temperature, and $k_{\mathrm{B}}$ is the Boltzmann constant. Therefore, we obtain the non-dimensional electromagnetic free-field Lagrangian

$$
\begin{equation*}
L_{\mathrm{EM}}^{\prime}=\frac{\varepsilon_{0} \widehat{E}_{1}^{2}}{\widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}}} \frac{1}{2} \int \mathrm{~d}^{3} \boldsymbol{x}^{\prime}\left|\boldsymbol{\nabla}^{\prime} \phi_{1}^{\prime}+\frac{\widehat{B}_{1} \ell_{1}}{\widehat{E}_{1} \tau_{1}} \frac{\partial \boldsymbol{A}_{1}^{\prime}}{\partial t^{\prime}}\right|^{2}-\frac{1}{\beta_{\mathrm{i}}} \int \mathrm{~d}^{3} \boldsymbol{x}^{\prime}\left|\boldsymbol{\nabla}^{\prime} \times \boldsymbol{A}_{0}^{\prime}+\frac{\widehat{B}_{1}}{\widehat{B}_{0}} \boldsymbol{\nabla}^{\prime} \times \boldsymbol{A}_{1}^{\prime}\right|^{2} \tag{15}
\end{equation*}
$$

where $\beta_{\mathrm{i}}=2 \mu_{0} \widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}} / \widehat{B}_{0}^{2}$ is the ion plasma beta. In order to normalize the single-particle Lagrangian (7), we note that $\dot{\boldsymbol{x}}$ represents three components of an element $(\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}}) \in \mathbb{R}^{3} \times \mathbb{R}^{3}$ of the tangent space at $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{3} \times \mathbb{R}^{3}$. Tangents $\dot{\boldsymbol{x}}$ have units of velocity and are therefore normalized as $\dot{\boldsymbol{x}}=(\widehat{x} / \widehat{t}) \dot{\boldsymbol{x}}^{\prime}$. We then normalize the single-particle Lagrangian (7) as $L_{\mathrm{s}}=m_{\mathrm{s}} \widehat{v}_{\mathrm{s}}^{2} L_{\mathrm{s}}^{\prime}$, obtaining the non-dimensional single-particle Lagrangian

$$
\begin{equation*}
L_{\mathrm{s}}^{\prime}=\frac{\widehat{x}}{\widehat{v}_{\mathrm{s}} \hat{t}}\left[\boldsymbol{v}^{\prime}+\frac{q_{\mathrm{s}} \widehat{B}_{0} \ell_{0}}{m_{\mathrm{s}} \widehat{v}_{\mathrm{s}}}\left(\boldsymbol{A}_{0}^{\prime}+\frac{\widehat{B}_{1} \ell_{1}}{\widehat{B}_{0} \ell_{0}} \boldsymbol{A}_{1}^{\prime}\right)\right] \cdot \dot{\boldsymbol{x}}^{\prime}-\frac{\left|\boldsymbol{v}^{\prime}\right|^{2}}{2}-\frac{q_{\mathrm{s}} \widehat{E}_{1} \ell_{1}}{m_{\mathrm{s}} \widehat{v}_{\mathrm{s}}^{2}} \phi_{1}^{\prime} \tag{16}
\end{equation*}
$$

|  |  |  | ASDEX | Upgrade |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | ions | electrons | ions | electrons |
| major radius | $R_{0}$ | [m] |  |  |  |  |
| minor radius | $a$ | [m] |  |  |  |  |
| toroidal magnetic field | $B_{\mathrm{T}}$ | [T] |  |  |  |  |
| average particle density | $\left\langle n_{\text {s }}\right\rangle$ | $\left[\frac{10^{20}}{\mathrm{~m}^{3}}\right]$ | 2.0 | 2.0 | 1.0 | 1.0 |
| average thermal energy | $\left\langle T_{\mathrm{s}}\right\rangle$ | $\left[\frac{\mathrm{keV}}{k_{\mathrm{B}}}\right]$ | 8.7 | 8.7 | 8.0 | 8.8 |
| cyclotron frequency | $\omega_{\text {cs }}$ | [Hz] | $1.9 \times 10^{8}$ | $6.9 \times 10^{11}$ | $2.5 \times 10^{8}$ | $9.3 \times 10^{11}$ |
| thermal velocity | $\widehat{v}_{\text {s }}$ | $\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$ | $6.4 \times 10^{5}$ | $3.9 \times 10^{7}$ | $6.2 \times 10^{5}$ | $3.9 \times 10^{7}$ |
| thermal frequency | $\widehat{\omega}_{\text {s }}$ | [ Hz ] | $8.0 \times 10^{5}$ | $4.9 \times 10^{7}$ | $3.1 \times 10^{5}$ | $2.0 \times 10^{7}$ |
| Larmor radius | $\rho_{\text {s }}$ | [m] | $3.4 \times 10^{-3}$ | $5.7 \times 10^{-5}$ | $2.4 \times 10^{-3}$ | $4.2 \times 10^{-5}$ |
| Debye length | $\lambda_{\text {s }}$ | [m] | $4.9 \times 10^{-5}$ | $4.9 \times 10^{-5}$ | $6.7 \times 10^{-5}$ | $7.0 \times 10^{-5}$ |
| $\widehat{\omega}_{\mathrm{s}} / \omega_{\mathrm{cs}}$ |  |  | $3.4 \times 10^{-3}$ | $5.7 \times 10^{-5}$ | $1.2 \times 10^{-3}$ | $2.1 \times 10^{-5}$ |
| $v_{\mathrm{s}}^{2} / c^{2}$ |  |  | $4.6 \times 10^{-6}$ | $1.7 \times 10^{-2}$ | $4.3 \times 10^{-6}$ | $1.7 \times 10^{-2}$ |
| $\lambda_{\mathrm{s}}^{2} / a^{2}$ |  |  | $3.7 \times 10^{-9}$ | $3.7 \times 10^{-9}$ | $1.1 \times 10^{-9}$ | $1.2 \times 10^{-9}$ |
| $\beta_{\mathrm{s}}$ |  |  | $1.2 \times 10^{-2}$ | $1.2 \times 10^{-2}$ | $2.4 \times 10^{-3}$ | $3.2 \times 10^{-3}$ |

Table 1: Physical scenarios for magnetic confinement fusion experiments: parameters for the Tokamaks ASDEX Upgrade [28] and ITER [29]. Note that we choose $\widehat{B}_{0}=B_{\mathrm{T}}, \widehat{n}_{\mathrm{s}}=\left\langle n_{\mathrm{s}}\right\rangle$ and $\widehat{T}_{\mathrm{s}}=\left\langle T_{\mathrm{s}}\right\rangle$.

Finally, we normalize the particle Lagrangian of the Low action in the same way as the electromagnetic free-field Lagrangian, namely as $L_{\mathrm{p}}=\widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}} \widehat{x}^{3} L_{\mathrm{p}}^{\prime}$, obtaining the non-dimensional particle Lagrangian

$$
\begin{equation*}
L_{\mathrm{p}}^{\prime}=\sum_{\mathrm{s}} \frac{m_{\mathrm{s}} \widehat{v}_{\mathrm{s}}^{5} \hat{f}_{s}}{\widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}}} \int \mathrm{~d}^{3} \boldsymbol{x}_{0}^{\prime} \mathrm{d}^{3} \boldsymbol{v}_{0}^{\prime} f_{\mathrm{s}}^{\prime}\left(t_{0}^{\prime}, \boldsymbol{x}_{0}^{\prime}, \boldsymbol{v}_{0}^{\prime}\right) L_{\mathrm{s}}^{\prime} \tag{17}
\end{equation*}
$$

All dependent variables in the Low action are now expressed in terms of non-dimensional functions. Therefore, the size of each term is determined only by the size of the non-dimensional coefficients in front of it. Such coefficients are, in turn, determined by the physical scenario under consideration, as we discuss in the next section.

### 3.2 Physical scenario and ordering

The normalization of the Vlasov-Maxwell system described in the previous section is generic. For the purpose of deriving a set of gyrokinetic equations by asymptotic methods we must quantify the size of the non-dimensional coefficients appearing in the physical quantities of interest in powers of the perturbation parameter $\varepsilon$. This process is usually referred to as ordering. Different orderings lead to different perturbation theories and to reduced models with different physical content. An ordering is thus the mathematical expression of a specific physical scenario. Two such scenarios for magnetic confinement fusion experiments are listed in Table 1. We choose as the characteristic length and time scales of observation the minor radius $a$ and the inverse ion thermal frequency:

$$
\begin{equation*}
\widehat{x}:=a, \quad \hat{t}^{-1}:=\widehat{\omega}_{\mathrm{i}} . \tag{18}
\end{equation*}
$$

The ion thermal frequency $\widehat{\omega}_{i}$, that is, the inverse of the time required for an ion to travel the distance $a$, is close to the frequency of micro-turbulence observed in Tokamaks [2, 30, 31]. By substituting in (10) the values shown in Table 1, we then obtain

$$
\begin{equation*}
\varepsilon \approx 10^{-3} \tag{19}
\end{equation*}
$$

Measurements in Tokamaks have shown that fluctuation levels in turbulent plasmas satisfy [16, 30, 31]

$$
\begin{equation*}
\frac{\widehat{B}_{1}}{\widehat{B}_{0}} \sim \frac{\widehat{E}_{1}}{\widehat{B}_{0} \widehat{v}_{\mathrm{i}}} \approx 10^{-3} \approx \varepsilon \tag{20}
\end{equation*}
$$

Moreover, if we consider electrons and deuterium ions we have $q_{\mathrm{e}} / q_{\mathrm{i}}=-1$ and

$$
\begin{equation*}
\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}} \approx 2.7 \times 10^{-3} \approx \varepsilon \tag{21}
\end{equation*}
$$

We also notice an ordering pattern in powers of $\varepsilon$ in the normalized plasma parameters of Table 1 (last four non-dimensional parameters). We then apply this ordering to the normalized Low action. By choosing

$$
\begin{equation*}
\ell_{0}:=a, \quad \ell_{1}:=\rho_{\mathrm{i}}, \quad \tau_{1}^{-1}:=\widehat{\omega}_{\mathrm{i}} \tag{22}
\end{equation*}
$$

we satisfy the maximal ordering $\ell_{1} / \ell_{0}=O(\varepsilon)$ in the limit $\varepsilon \rightarrow 0$. More precisely, what we mean here is that the ratio $\ell_{1} / \ell_{0}$ is approximately $\varepsilon$ and therefore it can be considered as a function of order $O(\varepsilon)$ if we were taking the formal limit $\varepsilon \rightarrow 0$ (which we actually never take, as $\varepsilon$ is a fixed number and cannot vanish). The normalized background magnetic field (12) becomes a function of $\boldsymbol{x}^{\prime}$, namely $\boldsymbol{B}_{0}^{\prime}\left(\boldsymbol{x}^{\prime}\right)$. On the other hand, since $\ell_{1} / \widehat{x}=O(\varepsilon)$, the fluctuating electric and magnetic fields and their corresponding potentials become stronglyvarying functions of $\boldsymbol{x}^{\prime}$, namely $\boldsymbol{E}_{1}^{\prime}\left(t^{\prime}, \boldsymbol{x}^{\prime} / \varepsilon\right)$ and $\boldsymbol{B}_{1}^{\prime}\left(t^{\prime}, \boldsymbol{x}^{\prime} / \varepsilon\right)$. More precisely, we assume the conventional gyrokinetic ordering for the fluctuations, namely $\left|\boldsymbol{k}_{\perp}\right| \rho_{\mathrm{i}}=O(1)$ and $k_{\|} \rho_{\mathrm{i}}=O(\varepsilon)$ in the limit $\varepsilon \rightarrow 0$, where $\boldsymbol{k}:=\left(k_{\|}, \boldsymbol{k}_{\perp}\right)$ denotes characteristic wave vectors parallel and perpendicular to $\boldsymbol{B}_{0}$, respectively. Moreover, for the coefficient appearing in (14) we have

$$
\begin{equation*}
\frac{\widehat{B}_{1} \ell_{1}}{\widehat{E}_{1} \tau_{1}}=\frac{\widehat{B}_{1} \rho_{\mathrm{i}} \widehat{\omega}_{\mathrm{i}}}{\widehat{E}_{1}}=\frac{\widehat{\omega}_{\mathrm{i}}}{\omega_{\mathrm{ci}}} \frac{\widehat{B}_{1}}{\widehat{E}_{\mathrm{i}}}=\varepsilon \frac{\widehat{B}_{1}}{\widehat{B}_{0}} \frac{\widehat{B}_{0} \widehat{v}_{\mathrm{i}}}{\widehat{E}_{1}} \approx \varepsilon \tag{23}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\boldsymbol{E}_{1}^{\prime}=-\nabla^{\prime} \phi_{1}^{\prime}-\varepsilon \frac{\partial \boldsymbol{A}_{1}^{\prime}}{\partial t^{\prime}} \tag{24}
\end{equation*}
$$

Moreover, the non-dimensional coefficients in the normalized electromagnetic free-field Lagrangian (15) have the following sizes:

$$
\begin{equation*}
\frac{\varepsilon_{0} \widehat{E}_{1}^{2}}{\widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \widehat{T}_{\mathrm{i}}}=\frac{\widehat{v}_{\mathrm{i}}^{2}}{c^{2}} \frac{\widehat{E}_{1}^{2}}{\widehat{B}_{0}^{2} \widehat{v}_{\mathrm{i}}^{2}} \frac{\widehat{B}_{0}^{2}}{\mu_{0} \widehat{n}_{\mathrm{i}} k_{\mathrm{B}} \hat{T}_{\mathrm{i}}} \approx \varepsilon^{3}, \quad \frac{1}{\beta_{\mathrm{i}}} \approx \frac{1}{\varepsilon} . \tag{25}
\end{equation*}
$$

In the normalized particle Lagrangian (17) we set $k_{\mathrm{B}} \widehat{T}_{\mathrm{e}}=k_{\mathrm{B}} \widehat{T}_{\mathrm{i}}$ and $\widehat{v}_{\mathrm{s}}^{3} \widehat{f}_{\mathrm{s}}=\hat{n}_{\mathrm{i}}$ to fix the characteristic size of the distribution function. Therefore, in our ordering the normalized ion and electron single-particle Lagrangians (16) read

$$
\begin{align*}
& L_{\mathrm{i}}^{\prime}=\left(\boldsymbol{v}^{\prime}+\frac{\boldsymbol{A}_{0}^{\prime}}{\varepsilon}+\varepsilon \boldsymbol{A}_{1}^{\prime}\right) \cdot \dot{\boldsymbol{x}}^{\prime}-\frac{\left|\boldsymbol{v}^{\prime}\right|^{2}}{2}-\varepsilon \phi_{1}^{\prime}  \tag{26a}\\
& L_{\mathrm{e}}^{\prime}=\left(\sqrt{\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}} \boldsymbol{v}^{\prime}-\frac{\boldsymbol{A}_{0}^{\prime}}{\varepsilon}-\varepsilon \boldsymbol{A}_{1}^{\prime}\right) \cdot \dot{\boldsymbol{x}}^{\prime}-\frac{\left|\boldsymbol{v}^{\prime}\right|^{2}}{2}+\varepsilon \phi_{1}^{\prime} \tag{26b}
\end{align*}
$$

The only difference between ions and electrons, besides the sign in front of the electromagnetic potentials due to the negative electron charge, is the factor $\sqrt{m_{\mathrm{e}} / m_{\mathrm{i}}} \approx \sqrt{\varepsilon}$ multiplying $\boldsymbol{v}^{\prime} \cdot \dot{\boldsymbol{x}}^{\prime}$, which defines an intermediate scale that is not an integer power of $\varepsilon$. As the final result of the ordering procedure, we obtain the normalized Low action principle

$$
\begin{equation*}
\delta \int_{t_{0}^{\prime}}^{t_{1}^{\prime}}\left(L_{\mathrm{p}}^{\prime}+L_{\mathrm{EM}}^{\prime}\right) \mathrm{d} t^{\prime}=0 \tag{27}
\end{equation*}
$$

with the Lagrangians given by (omitting the primes for a simpler notation)

$$
\begin{align*}
L_{\mathrm{p}}^{\prime} & =\int \mathrm{d}^{6} \boldsymbol{z}_{0} f_{\mathrm{i}}\left(t_{0}, \boldsymbol{z}_{0}\right)\left[\left(\boldsymbol{v}+\frac{\boldsymbol{A}_{0}}{\varepsilon}+\varepsilon \boldsymbol{A}_{1}\right) \cdot \dot{\boldsymbol{x}}-\frac{|\boldsymbol{v}|^{2}}{2}-\varepsilon \phi_{1}\right]  \tag{28a}\\
& +\int \mathrm{d}^{6} \boldsymbol{z}_{0} f_{\mathrm{e}}\left(t_{0}, \boldsymbol{z}_{0}\right)\left[\left(\sqrt{\varepsilon} \boldsymbol{v}-\frac{\boldsymbol{A}_{0}}{\varepsilon}-\varepsilon \boldsymbol{A}_{1}\right) \cdot \dot{\boldsymbol{x}}-\frac{|\boldsymbol{v}|^{2}}{2}+\varepsilon \phi_{1}\right], \\
L_{\mathrm{EM}}^{\prime} & =\frac{\varepsilon^{3}}{2} \int \mathrm{~d}^{3} \boldsymbol{x}\left|\boldsymbol{\nabla} \phi_{1}+\varepsilon \frac{\partial \boldsymbol{A}_{1}}{\partial t}\right|^{2}-\frac{1}{2 \varepsilon} \int \mathrm{~d}^{3} \boldsymbol{x}\left|\boldsymbol{\nabla} \times \boldsymbol{A}_{0}+\varepsilon \boldsymbol{\nabla} \times \boldsymbol{A}_{1}\right|^{2} . \tag{28b}
\end{align*}
$$

Taking variations with respect to particle trajectories, $\phi_{1}$ and $\boldsymbol{A}_{1}$ in (27) leads to the following normalized Vlasov-Maxwell equations (again omitting the primes):

$$
\begin{align*}
& \quad \frac{\partial f_{\mathrm{i}}}{\partial t}+\boldsymbol{v} \cdot \boldsymbol{\nabla} f_{\mathrm{i}}+\left[\boldsymbol{E}_{1}+\boldsymbol{v} \times\left(\frac{\boldsymbol{B}_{0}}{\varepsilon}+\boldsymbol{B}_{1}\right)\right] \cdot \frac{\partial f_{\mathrm{i}}}{\partial \boldsymbol{v}}=0  \tag{29a}\\
& \sqrt{\varepsilon} \frac{\partial f_{\mathrm{e}}}{\partial t}+\boldsymbol{v} \cdot \boldsymbol{\nabla} f_{\mathrm{e}}-\left[\boldsymbol{E}_{1}+\boldsymbol{v} \times \frac{1}{\sqrt{\varepsilon}}\left(\frac{\boldsymbol{B}_{0}}{\varepsilon}+\boldsymbol{B}_{1}\right)\right] \cdot \frac{\partial f_{\mathrm{e}}}{\partial \boldsymbol{v}}=0  \tag{29b}\\
& \quad \varepsilon^{2} \boldsymbol{\nabla} \cdot \boldsymbol{E}_{1}=\rho  \tag{29c}\\
& \quad \nabla \times\left(\frac{\boldsymbol{B}_{0}}{\varepsilon}+\boldsymbol{B}_{1}\right)=\boldsymbol{J}+\varepsilon^{3} \frac{\partial \boldsymbol{E}_{1}}{\partial t} \tag{29d}
\end{align*}
$$

where the normalized charge and current densities are given by

$$
\begin{equation*}
\rho=\int \mathrm{d}^{3} \boldsymbol{v} f_{\mathrm{i}}-\int \mathrm{d}^{3} \boldsymbol{v} f_{\mathrm{e}}, \quad \boldsymbol{J}=\int \mathrm{d}^{3} \boldsymbol{v} \boldsymbol{v} f_{\mathrm{i}}-\frac{1}{\sqrt{\varepsilon}} \int \mathrm{~d}^{3} \boldsymbol{v} \boldsymbol{v} f_{\mathrm{e}} . \tag{30}
\end{equation*}
$$

The factor $1 / \sqrt{\varepsilon}$ in front of the electron current density comes from the different choice of scales for the ion and electron thermal velocities. From the two Vlasov equations in (29) we deduce the charge conservation law

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\nabla \cdot \boldsymbol{J}=0 \tag{31}
\end{equation*}
$$

This is also a solvability condition for Maxwell's equations. Indeed, taking the divergence of Ampère-Maxwell's law in (29d), recalling that $\boldsymbol{E}_{1}(t, \boldsymbol{x} / \varepsilon)$ is strongly-varying in space, and inserting Coulomb's law from (29c) yields (31).

The normalized variational principle (27), or the normalized set of equations (29), are a suitable starting point for our perturbation analysis of the Vlasov-Maxwell system. Let us remark that the majority of gyrokinetic theories for micro-turbulence have been developed in a homogeneous background for the sake of conceptual clarity. However, because curvature is important in magnetically confined fusion plasmas ("neo-classical transport"), many of the state-of-the-art numerical codes feature a model with slowly-varying magnetic background, corresponding to $\boldsymbol{B}_{0}(\varepsilon \boldsymbol{x})$ in normalized variables. Curvature terms then appear only at the second order of the perturbation theory, and are often neglected for simplicity. In this work, we develop a consistent theory in the maximal ordering, corresponding to $\boldsymbol{B}_{0}(\boldsymbol{x})$ in normalized variables. This seems to be the scenario for current Tokamak and Stellarator experiments, but is also interesting for Spheromaks and even for the future large-scale Tokamak ITER. Reduced equations for smaller background curvature can easily be deduced from our more general results in maximal ordering.

## 4 Gyrokinetic reduction

The basic idea of gyrokinetic theory is to replace the exact trajectories of the plasma particles by the trajectories of their gyrocenters, which move on the time scale of the thermal frequency $\widehat{\omega}_{i}$ or slower. The dynamics occurring at scales faster than the cyclotron frequency $\omega_{\text {ci }}$ are "averaged out" in the gyrocenter picture. However, some effects of the fast motion of gyration are still present in form of drifts of the gyrocenters. In this section we make these concepts more precise by analyzing the formal asymptotic limit $\varepsilon \rightarrow 0$ in the normalized single-particle Lagrangians (26a)-(26b). From the reduced Lagrangians we then derive the gyrokinetic Vlasov equation for ions and electrons and define gyrocenter charge and current densities with polarization corrections, thus coupling plasma particles and electromagnetic fields from the normalized gyrocenter action principle. Primes are omitted from now on, in order to increase readability.

Following [32], we intend to replace the particle Lagrangian $L_{\mathrm{p}}$ in the Low action principle (27) by its gyrocenter representation $\mathcal{L}_{\mathrm{p}}$ :

$$
\int \mathrm{d}^{6} \boldsymbol{z}_{0}\left[f_{\mathrm{i}}\left(0, \boldsymbol{z}_{0}\right) L_{\mathrm{i}}+f_{\mathrm{e}}\left(0, \boldsymbol{z}_{0}\right) L_{\mathrm{e}}\right] \approx \int \mathrm{d}^{6} \mathcal{Z}_{0}\left[B_{\| \mathrm{i}}^{*} F_{\mathrm{i}}\left(0, \mathcal{Z}_{0}\right) \mathcal{L}_{\mathrm{i}}+B_{\| \mathrm{e}}^{*} F_{\mathrm{e}}\left(0, \mathcal{Z}_{0}\right) \mathcal{L}_{\mathrm{e}}\right]
$$

where $\mathrm{d}^{6} \mathcal{Z}_{0}:=\mathrm{d}^{3} \boldsymbol{X}_{0} \mathrm{~d} P_{\| 0} \mathrm{~d} \mu_{0} \mathrm{~d} \Theta_{0}$ denotes the measure in gyrocenter phase space, $F_{\mathrm{s}}$ denotes the gyrocenter distribution function, $\mathcal{L}_{\mathrm{s}}$ is the corresponding gyrocenter single-particle Lagrangian, to be derived below, and $B_{\| \mathrm{s}}^{*}$ is the Jacobian determinant. The gyrocenter coordinates are the gyrocenter position $\boldsymbol{X}$, the gyrocenter parallel momentum $P_{\|}$, the gyrocenter magnetic moment $\mu$ and the gyro-angle $\Theta$. The single-particle dynamics in the new coordinates is such that the time evolution of the gyro-angle $\Theta$ is decoupled from the rest of the coordinates, leading to a closed system of equations for the "slow" variables $\left(\boldsymbol{X}, P_{\|}\right)$, where $\mu$ is a constant of the motion. The slow system represents the averaged dynamics (see [26] for details).

The phase-space coordinate transformation $\mathcal{Z}:=\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right) \mapsto(\boldsymbol{x}, \boldsymbol{v})$ with Jacobian determinant denoted by $B_{\|}^{*}$ is the central object of gyrokinetic theory. It is usually derived in terms of (canonical) Lie transforms of the fundamental one-form associated to the single-particle Lagrangian $L_{\mathrm{s}}[10,11,23]$. Despite being an elegant mathematical framework, Lie transform perturbation theory introduces many formal complications, which seem not to be strictly necessary for averaging. In this work we replace Lie transforms with polynomials of finite degree in $\varepsilon$, algebraic in the generating functions. We show that also with this different ansatz for the phase-space nearidentity coordinate transformation it is possible to remove the gyro-angle dependence from the single-particle Lagrangian up to the desired order in $\varepsilon$, without changing its symplectic part (and thus the Jacobian $B_{\|}^{*}$ of the coordinate transformation). Our polynomial transforms are well-defined coordinate transformations (locally invertible), in contrast to the asymptotic series in Lie transform perturbation theory, where it is difficult to prove convergence and existence of the transforms. It is our hope that the simpler derivation based on polynomial transforms will enable more rigorous mathematical studies of gyrokinetic theory in the future.

### 4.1 Preliminary transformations

The phase-space coordinate transformation $\mathcal{Z} \mapsto(\boldsymbol{x}, \boldsymbol{v})$ is a composition of several coordinate changes, which are summarized in Tables 2 and 3 for ions and electrons, respectively. The first transformation moves the magnetic vector potential $\boldsymbol{A}_{1}$ from the symplectic part of the single-particle Lagrangian to the Hamiltonian, by defining the "momentum"

$$
\boldsymbol{p}:= \begin{cases}\boldsymbol{v}+\varepsilon \boldsymbol{A}_{1} & \text { ions }  \tag{32}\\ \boldsymbol{v}-\sqrt{\varepsilon} \boldsymbol{A}_{1} & \text { electrons }\end{cases}
$$

This is a near-identity transformation in $\boldsymbol{v}$ in the limit $\varepsilon \rightarrow 0$, with unit Jacobian determinant. It resembles the usual transformation to canonical coordinates, but it does not contain the background magnetic vector potential $\boldsymbol{A}_{0}$. The new single-particle Lagrangians read

$$
\begin{align*}
& L_{\mathrm{i}}=\left(\boldsymbol{p}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{x}}-\left(\frac{|\boldsymbol{p}|^{2}}{2}+\varepsilon \Psi_{1 \mathrm{i}}+\varepsilon^{2} \frac{\left|\boldsymbol{A}_{1}\right|^{2}}{2}\right)  \tag{33a}\\
& L_{\mathrm{e}}=\left(\sqrt{\varepsilon} \boldsymbol{p}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{x}}-\left(\frac{|\boldsymbol{p}|^{2}}{2}-\sqrt{\varepsilon} \Psi_{1 \mathrm{e}}+\varepsilon \frac{\left|\boldsymbol{A}_{1}\right|^{2}}{2}\right), \tag{33b}
\end{align*}
$$

where we introduced the generalized potentials

$$
\Psi_{1 \mathrm{~s}}:= \begin{cases}\phi_{1}-\boldsymbol{p} \cdot \boldsymbol{A}_{1} & \text { ions }  \tag{34}\\ \sqrt{\varepsilon} \phi_{1}-\boldsymbol{p} \cdot \boldsymbol{A}_{1} & \text { electrons }\end{cases}
$$

This first preliminary transformation is not necessary for performing the gyrokinetic reduction. However, it leads to simpler calculations in the following. Since all dynamic potentials now occur in the Hamiltonian, the $(\boldsymbol{x}, \boldsymbol{p})$-coordinate representation is also called the Hamiltonian picture. We also remark that electromagnetic gauge invariance has been broken by this preliminary coordinate transformation. We then introduce local cylindrical coordinates in $\boldsymbol{p}$-space, namely

$$
\begin{equation*}
p_{\|}:=\boldsymbol{p} \cdot \boldsymbol{b}_{0}, \quad \mu:=\frac{\left|\boldsymbol{b}_{0} \times \boldsymbol{p} \times \boldsymbol{b}_{0}\right|^{2}}{2 B_{0}}, \quad \theta:=\arctan \left(\frac{\boldsymbol{p} \cdot \boldsymbol{e}_{1}}{\boldsymbol{p} \cdot \boldsymbol{e}_{2}}\right) \tag{35}
\end{equation*}
$$

where $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{b}_{0}\right)$ represents a local static orthonormal basis of $\mathbb{R}^{3}$, given an arbitrary unit vector $\boldsymbol{e}_{1}$ perpendicular to $\boldsymbol{b}_{0}$. Denoting by $\boldsymbol{p}_{\perp}:=\boldsymbol{b}_{0} \times \boldsymbol{p} \times \boldsymbol{b}_{0}$ the component of $\boldsymbol{p}$ perpendicular to the local background

| physical coordinates $(\boldsymbol{x}, \boldsymbol{v})$ |  |
| :---: | :---: |
| Hamiltonian picture $(\boldsymbol{x}, \boldsymbol{p})$ | $\boldsymbol{p}:=\boldsymbol{v}+\varepsilon \boldsymbol{A}_{1}$ |
| angle coordinates $\left(\boldsymbol{x}, p_{\\|}, \mu, \theta\right)$ | $\begin{aligned} p_{\\|} & :=\boldsymbol{p} \cdot \boldsymbol{b}_{0} \\ \mu & :=\frac{1}{2 B_{0}}\left\|\boldsymbol{b}_{0} \times \boldsymbol{p} \times \boldsymbol{b}_{0}\right\|^{2} \\ \theta & :=\arctan \left(\frac{\boldsymbol{p} \cdot \boldsymbol{e}_{1}}{\boldsymbol{p} \cdot \boldsymbol{e}_{2}}\right) \end{aligned}$ |
| guiding-center coordinates $\overline{\boldsymbol{Z}}:=\left(\overline{\boldsymbol{X}}, \bar{p}_{\\|}, \bar{\mu}, \bar{\theta}\right)$ | $\begin{aligned} \boldsymbol{x} & :=\overline{\boldsymbol{X}}+\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{i}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{i}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{i}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{i}} \\ p_{\\|} & :=\bar{p}_{\\|}+\varepsilon \bar{G}_{1 \mathrm{i}}^{\\|}+\varepsilon^{2} \bar{G}_{2 \mathrm{i}}^{\\|}+\varepsilon^{3} \bar{G}_{3 \mathrm{i}}^{\\|} \\ \mu & :=\bar{\mu}+\varepsilon \bar{G}_{1 \mathrm{i}}^{\mu}+\varepsilon^{2} \bar{G}_{2 \mathrm{i}}^{\mu}+\varepsilon^{3} \bar{G}_{3 \mathrm{i}}^{\mu} \\ \theta & :=\bar{\theta}+\varepsilon \bar{G}_{1 \mathrm{i}}^{\Theta}+\varepsilon^{2} \bar{G}_{2 \mathrm{i}}^{\Theta}+\varepsilon^{3} \bar{G}_{3 \mathrm{i}}^{\Theta} \end{aligned}$ |
| preliminary gyrocenter coordinates $\boldsymbol{Z}:=\left(\boldsymbol{X}, P_{\\|}, \widehat{\mu}, \Theta\right)$ | $\begin{aligned} \overline{\boldsymbol{X}} & :=\boldsymbol{X} \quad+\varepsilon^{2} \boldsymbol{\rho}_{2 \mathrm{i}}+\varepsilon^{3} \boldsymbol{\rho}_{3 \mathrm{i}}+\varepsilon^{4} \boldsymbol{\rho}_{4 \mathrm{i}} \\ \bar{p}_{\\|} & :=P_{\\|}+\varepsilon G_{1 \mathrm{i}}^{\\|}+\varepsilon^{2} G_{2 \mathrm{i}}^{\\|}+\varepsilon^{3} G_{3 \mathrm{i}}^{\\|} \\ \bar{\mu} & :=\widehat{\mu}+\varepsilon G_{1 \mathrm{i}}^{\mu}+\varepsilon^{2} G_{2 \mathrm{i}}^{\mu}+\varepsilon^{3} G_{3 \mathrm{i}}^{\mu} \\ \bar{\theta} & :=\Theta+\varepsilon G_{1 \mathrm{i}}^{\Theta}+\varepsilon^{2} G_{2 \mathrm{i}}^{\Theta}+\varepsilon^{3} G_{3 \mathrm{i}}^{\Theta} \end{aligned}$ |
| gyrocenter coordinates $\mathcal{Z}:=\left(\boldsymbol{X}, P_{\\|}, \mu, \Theta\right)$ | $\mu:=\widehat{\mu}+\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle+\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle$ |

Table 2: Coordinate changes for ions involved in the phase-space coordinate transformation $\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right) \mapsto$ $(\boldsymbol{x}, \boldsymbol{v})$ relating physical coordinates and gyrocenter coordinates.

| physical coordinates $(\boldsymbol{x}, \boldsymbol{v})$ |  |
| :---: | :---: |
| Hamiltonian picture $(\boldsymbol{x}, \boldsymbol{p})$ | $\boldsymbol{p}:=\boldsymbol{v}-\sqrt{\varepsilon} \boldsymbol{A}_{1}$ |
| angle coordinates $\left(\boldsymbol{x}, p_{\\|}, \mu, \theta\right)$ | $\begin{aligned} p_{\\|} & :=\boldsymbol{p} \cdot \boldsymbol{b}_{0} \\ \mu & :=\frac{1}{2 B_{0}}\left\|\boldsymbol{b}_{0} \times \boldsymbol{p} \times \boldsymbol{b}_{0}\right\|^{2} \\ \theta & :=\arctan \left(\frac{\boldsymbol{p} \cdot \boldsymbol{e}_{1}}{\boldsymbol{p} \cdot \boldsymbol{e}_{2}}\right) \end{aligned}$ |
| guiding-center coordinates $\overline{\boldsymbol{Z}}:=\left(\overline{\boldsymbol{X}}, \bar{p}_{\\|}, \bar{\mu}, \bar{\theta}\right)$ | $\begin{aligned} \boldsymbol{x} & :=\overline{\boldsymbol{X}}+\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{e}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{e}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{e}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{e}} \\ p_{\\|} & :=\bar{p}_{\\|}+\varepsilon \bar{G}_{1 \mathrm{e}}^{\\|}+\varepsilon^{2} \bar{G}_{2 \mathrm{e}}^{\\|}+\varepsilon^{3} \bar{G}_{3 \mathrm{e}}^{\\|} \\ \mu & :=\bar{\mu}+\varepsilon \bar{G}_{1 \mathrm{e}}^{\mu}+\varepsilon^{2} \bar{G}_{2 \mathrm{e}}^{\mu}+\varepsilon^{3} \bar{G}_{3 \mathrm{e}}^{\mu} \\ \theta & :=\bar{\theta}+\varepsilon \bar{G}_{1 \mathrm{e}}^{\Theta}+\varepsilon^{2} \bar{G}_{2 \mathrm{e}}^{\Theta}+\varepsilon^{3} \bar{G}_{3 \mathrm{e}}^{\Theta} \end{aligned}$ |
| preliminary gyrocenter coordinates $\boldsymbol{Z}:=\left(\boldsymbol{X}, P_{\\|}, \widehat{\mu}, \Theta\right)$ | $\begin{aligned} & \overline{\boldsymbol{X}}:=\boldsymbol{X}+\varepsilon^{2} \boldsymbol{\rho}_{2 \mathrm{e}} \quad+\varepsilon^{\frac{5}{2}} \boldsymbol{\rho}_{\frac{5}{2} \mathrm{e}}+\varepsilon^{3} \boldsymbol{\rho}_{3 \mathrm{e}} \\ & \bar{p}_{\\|}:=P_{\\|}+\varepsilon G_{1 \mathrm{e}}^{\\|} \quad+\varepsilon^{2} G_{2 \mathrm{e}}^{\\|} \\ & \bar{\mu}:=\widehat{\mu}+\sqrt{\varepsilon} G_{\frac{1}{2} \mathrm{e}}^{\mu}+\varepsilon G_{1 \mathrm{e}}^{\mu} \\ & \bar{\theta}:=\Theta+\sqrt{\varepsilon} G_{\frac{1}{2} \mathrm{e}}^{\Theta}+\varepsilon G_{1 \mathrm{e}}^{\Theta} . \end{aligned}$ |
| gyrocenter coordinates $\mathcal{Z}:=\left(\boldsymbol{X}, P_{\\|}, \mu, \Theta\right)$ | $\mu:=\widehat{\mu}+\sqrt{\varepsilon}\left\langle G_{\frac{1}{2} \mathrm{e}}^{\mu}\right\rangle-\varepsilon\left\langle\gamma_{3}^{\Theta}\right\rangle$ |

Table 3: Coordinate changes for electrons involved in the phase-space coordinate transformation $\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right) \mapsto$ $(\boldsymbol{x}, \boldsymbol{v})$ relating physical coordinates and gyrocenter coordinates.
magnetic field, we have $\boldsymbol{p}=p_{\|} \boldsymbol{b}_{0}+\boldsymbol{p}_{\perp}$, with $\boldsymbol{p}_{\perp}=\left(\boldsymbol{p} \cdot \boldsymbol{e}_{1}\right) \boldsymbol{e}_{1}+\left(\boldsymbol{p} \cdot \boldsymbol{e}_{2}\right) \boldsymbol{e}_{2}$. From the definition of $\theta$ we can write $\left(\boldsymbol{p} \cdot \boldsymbol{e}_{1}\right)=-\sqrt{2 \mu B_{0}} \sin \theta$ and $\left(\boldsymbol{p} \cdot \boldsymbol{e}_{2}\right)=-\sqrt{2 \mu B_{0}} \cos \theta$ and thus define a second $\theta$-dependent orthonormal basis $\left(\boldsymbol{a}_{0}, \boldsymbol{b}_{0}, \boldsymbol{c}_{0}\right)$, with $\boldsymbol{a}_{0}:=\boldsymbol{e}_{1} \cos \theta-\boldsymbol{e}_{2} \sin \theta$ and $\boldsymbol{c}_{0}:=-\boldsymbol{e}_{1} \sin \theta-\boldsymbol{e}_{2} \cos \theta$. We note that $\boldsymbol{b}_{0} \times \boldsymbol{c}_{0}=\boldsymbol{a}_{0}, \partial \boldsymbol{a}_{0} / \partial \theta=\boldsymbol{c}_{0}$ and $\partial c_{0} / \partial \theta=-\boldsymbol{a}_{0}$, which will be used in later calculations. The transformation to angle coordinates thus reads

$$
\begin{equation*}
\boldsymbol{p}=p_{\|} \boldsymbol{b}_{0}+\sqrt{2 \mu B_{0}} \boldsymbol{c}_{0} \tag{36}
\end{equation*}
$$

with Jacobian determinant $B_{0}$. This leads to the single-particle Lagrangians

$$
\begin{align*}
& L_{\mathrm{i}}=\left(p_{\|} \boldsymbol{b}_{0}+\sqrt{2 \mu B_{0}} \boldsymbol{c}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \boldsymbol{\boldsymbol { x }}-\left(\frac{p_{\|}^{2}}{2}+\mu B_{0}+\varepsilon \Psi_{1 \mathrm{i}}+\varepsilon^{2} \frac{\left|\boldsymbol{A}_{1}\right|^{2}}{2}\right)  \tag{37a}\\
& L_{\mathrm{e}}=\left(\sqrt{\varepsilon} p_{\|} \boldsymbol{b}_{0}+\sqrt{\varepsilon} \sqrt{2 \mu B_{0}} \boldsymbol{c}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{x}}-\left(\frac{p_{\|}^{2}}{2}+\mu B_{0}-\sqrt{\varepsilon} \Psi_{1 \mathrm{e}}+\varepsilon \frac{\left|\boldsymbol{A}_{1}\right|^{2}}{2}\right) \tag{37b}
\end{align*}
$$

### 4.2 Guiding-center coordinates $\left(\overline{\boldsymbol{X}}, \bar{p}_{\|}, \bar{\mu}, \bar{\theta}\right)$

The guiding-center phase-space coordinate transformation dates back to the pioneering work of [24] and has the purpose of removing the gyro-angle dependence from those parts of the Lagrangians (37a)-(37b) that do not depend on the fluctuating potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$. For this reason, the gyrokinetic literature often describes the guiding-center phase-space coordinate transformation as a transformation acting on a single-particle Lagrangian that involves only quantities related to the background magnetic field $\boldsymbol{B}_{0}$ and does not feature any fluctuating fields. These are said to be added at a later stage, after the guiding-center coordinate transformation has been performed. We believe that this description is slightly misleading, as it seems to suggest the idea that the single-particle Lagrangian is modified by adding terms related to the fluctuating fields during the process of transforming the phase-space coordinates. In fact, the fluctuating fields are present in the single-particle Lagrangian since the beginning of the derivation (as it should be, once we identify the physical system that we want to describe), but their gyro-angle dependence is simply treated at a later stage, after the guidingcenter coordinate transformation has been performed. Following [26], we define the guiding-center coordinate transformation as a polynomial transform of the form

$$
\begin{align*}
\boldsymbol{x} & :=\overline{\boldsymbol{X}}+\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{~s}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{~s}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{~s}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{~s}},  \tag{38a}\\
p_{\|} & :=\bar{p}_{\|}+\varepsilon \bar{G}_{1 \mathrm{~s}}^{\|}+\varepsilon^{2} \bar{G}_{2 \mathrm{~s}}^{\|}+\varepsilon^{3} \bar{G}_{3 \mathrm{~s}}^{\|},  \tag{38b}\\
\mu & :=\bar{\mu}+\varepsilon \bar{G}_{1 \mathrm{~s}}^{\mu}+\varepsilon^{2} \bar{G}_{2 \mathrm{~s}}^{\mu}+\varepsilon^{3} \bar{G}_{3 \mathrm{~s}}^{\mu},  \tag{38c}\\
\theta & :=\bar{\theta}+\varepsilon \bar{G}_{1 \mathrm{~s}}^{\Theta}+\varepsilon^{2} \bar{G}_{2 \mathrm{~s}}^{\Theta}+\varepsilon^{3} \bar{G}_{3 \mathrm{~s}}^{\Theta}, \tag{38~d}
\end{align*}
$$

where $\overline{\boldsymbol{\rho}}_{n \mathrm{~s}}, \bar{G}_{n \mathrm{~s}}^{\|}, \bar{G}_{n \mathrm{~s}}^{\mu}$ and $\bar{G}_{n \mathrm{~s}}^{\Theta}$ denote the generators of the coordinate transformation for the respective particle species. The generators are functions of the guiding-center coordinates $\overline{\boldsymbol{Z}}:=\left(\overline{\boldsymbol{X}}, \overline{p_{\|}}, \bar{\mu}, \bar{\theta}\right)$ and may additionally depend on time. The guiding-center coordinates are the guiding-center position $\overline{\boldsymbol{X}}$, the guiding-center parallel momentum $\bar{p}_{\|}$, the guiding-center magnetic moment $\bar{\mu}$ and the guiding-center angle variable $\bar{\theta}$. The idea is then to substitute the coordinate transformation (38) in the single-particle Lagrangians (37a)-(37b), by using the transformation law of vector fields

$$
\begin{equation*}
\dot{\boldsymbol{x}}=\dot{\overline{\boldsymbol{X}}}+\sum_{n=1}^{4} \varepsilon^{n} \dot{\overline{\boldsymbol{\rho}}}_{n \mathrm{~s}}=\dot{\overline{\boldsymbol{X}}}+\sum_{n=1}^{4} \varepsilon^{n}\left(\frac{\partial \overline{\boldsymbol{\rho}}_{n \mathrm{~s}}}{\partial \overline{\boldsymbol{Z}}} \cdot \dot{\overline{\boldsymbol{Z}}}+\frac{\partial \overline{\boldsymbol{\rho}}_{n \mathrm{~s}}}{\partial t}\right) \tag{39}
\end{equation*}
$$

to obtain the corresponding guiding-center Lagrangians. The work of [26] showed that the gyro-angle dependence due to the term $\sqrt{2 \mu B_{0}} c_{0}$ can be indeed removed via polynomial transforms in maximal ordering. We repeat these calculations in Appendix B and arrive at

$$
\begin{align*}
L_{\mathrm{i}} \sim & \left(\bar{p}_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\overline{\boldsymbol{X}}}+\varepsilon \bar{\mu} \dot{\bar{\theta}}-\left[\bar{H}_{0}+\varepsilon \bar{H}_{1 \mathrm{i}}+\varepsilon^{2} \bar{H}_{2 \mathrm{i}}+\bar{O}\left(\varepsilon^{3}\right)\right]+O\left(\varepsilon^{4}\right)  \tag{40a}\\
L_{\mathrm{e}} \sim & \left(\sqrt{\varepsilon} \bar{p}_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\overline{\boldsymbol{X}}}-\varepsilon^{2} \bar{\mu} \dot{\bar{\theta}}  \tag{40b}\\
& -\left[\bar{H}_{0}+\sqrt{\varepsilon} \bar{H}_{\frac{1}{2} \mathrm{e}}+\varepsilon \bar{H}_{1 \mathrm{e}}+\varepsilon^{\frac{3}{2}} \bar{H}_{\frac{3}{2} \mathrm{e}}+\bar{O}\left(\varepsilon^{2}\right)\right]+O\left(\varepsilon^{4}\right)
\end{align*}
$$

where the symbol $\sim$ denotes the equivalence between Lagrangians, namely the fact that two Lagrangians differ only by the total differential of some scalar function. Moreover, the symbol $\bar{O}\left(\varepsilon^{n}\right)$ denotes corrections to the Hamiltonians of order $O\left(\varepsilon^{n}\right)$ that are independent of the guiding-center angle $\bar{\theta}$. The ion and electron guidingcenter Hamiltonians in (40a)-(40b) read

$$
\begin{array}{ll}
\bar{H}_{0}:=\frac{\bar{p}_{\|}^{2}}{2}+\bar{\mu} B_{0}, & \bar{H}_{0}:=\frac{\bar{p}_{\|}^{2}}{2}+\bar{\mu} B_{0} \\
\bar{H}_{1 \mathrm{i}}:=\Psi_{\mathrm{i}}+\delta H_{1}, & \bar{H}_{\frac{1}{2} \mathrm{e}}:=\boldsymbol{p} \cdot \boldsymbol{A}_{1} \\
\bar{H}_{2 \mathrm{i}}:=\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2}+\delta H_{2}, & \bar{H}_{1 \mathrm{e}}:=-\phi_{1}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2} \\
& \bar{H}_{\frac{3}{2} \mathrm{e}}:=-\delta H_{1}
\end{array}
$$

We remark the following comments about the guiding-center single-particle Lagrangians (40a)-(40b):

- The dynamic potentials in the Hamiltonians (41) are evaluated at the physical particle position $\boldsymbol{x} / \varepsilon$ :

$$
\begin{align*}
& \phi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right)=\phi_{1}\left(t, \frac{\overline{\boldsymbol{X}}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1 \mathrm{~s}}+\varepsilon \overline{\boldsymbol{\rho}}_{2 \mathrm{~s}}+O\left(\varepsilon^{2}\right)\right)  \tag{42a}\\
& \boldsymbol{A}_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right)=\boldsymbol{A}_{1}\left(t, \frac{\overline{\boldsymbol{X}}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1 \mathrm{~s}}+\varepsilon \overline{\boldsymbol{\rho}}_{2 \mathrm{~s}}+O\left(\varepsilon^{2}\right)\right) \tag{42~b}
\end{align*}
$$

The gyro-angle dependence in the generators $\overline{\boldsymbol{\rho}}_{\text {ns }}$ occurring in the arguments of the fluctuating potentials will be removed eventually by the transformation from guiding-center to gyrocenter coordinates, as discussed in detail in the next section.

- Due to our assumption of maximal ordering, the guiding-center Hamiltonians feature geometric terms related to the curvature of the background magnetic field, in particular

$$
\begin{equation*}
\delta H_{1}=\bar{\mu}\left[\frac{\bar{p}_{\|}}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}-\bar{p}_{\|}\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right], \tag{43}
\end{equation*}
$$

and a cumbersome term $\delta H_{2}$, which can be deduced from (103) in the appendix. The two terms in (43) are usually referred to as Baños drift [33] and gyro-gauge term, respectively. The curvature terms are less important for the electrons, where $\delta H_{1}$ appears at order $O\left(\varepsilon^{\frac{3}{2}}\right)$, because of the mass ratio between ions and electrons of order $O(\sqrt{\varepsilon})$.

- For electrons, the magnetic perturbations $\boldsymbol{A}_{1}$ are $O(\sqrt{\varepsilon})$ larger than the electric perturbations $\phi_{1}$. This can be already foreseen in the normalized Vlasov-Maxwell equations (29) and is due to the mass ratio between ions and electrons. Moreover, it shows the importance of electron dynamics in electromagnetic gyrokinetic simulations of fusion plasmas.
- Due to the error term $\bar{O}\left(\varepsilon^{2}\right)$ in the electron Hamiltonian, the electron guiding-center single-particle Lagrangian (40b) is less accurate than the ion guiding-center single-particle Lagrangian (40a). This is due to the fact that the guiding-center magnetic moment $\bar{\mu}$ has been computed with less precision for electrons than for ions. We could easily improve the accuracy of $\bar{\mu}$ for electrons, but, as we can see from (40b), the dynamic potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$ play a more prominent role than any curvature terms. In the Hamiltonian, the term $\boldsymbol{p} \cdot \boldsymbol{A}_{1}$ of $\Psi_{1}$ appears at order $O(\sqrt{\varepsilon})$ and the term $\left|\boldsymbol{A}_{1}\right|^{2} / 2$ appears at order $O(\varepsilon)$, whereas the first curvature terms appear at $O\left(\varepsilon^{\frac{3}{2}}\right)$. This is in contrast to the ions, where the first curvature term $\delta H_{1}$ appears already at order $O(\varepsilon)$, which is the same order as $\Psi_{1}$ and one order lower than the quadratic term $\left|\boldsymbol{A}_{1}\right|^{2} / 2$. In order to achieve an equally accurate description for the electrons, we should truncate the electron single-particle Lagrangian at order $O\left(\varepsilon^{5}\right)$ : this is beyond the scope of the work presented here, but does not represent a limitation of the method in general.
- The Jacobian determinants $J_{\mathrm{s}}$ of the guiding-center transformation $\overline{\boldsymbol{Z}} \mapsto(\boldsymbol{x}, \boldsymbol{v})$ can be computed directly from the symplectic part of the guiding-center single-particle Lagrangians (40a)-(40b):

$$
\begin{align*}
& J_{\mathrm{i}}=B_{\| \mathrm{i}}^{*}=B_{0}+\varepsilon \bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0},  \tag{44a}\\
& J_{\mathrm{e}}=B_{\| \mathrm{e}}^{*}=B_{0}-\varepsilon^{\frac{3}{2}} \bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0} . \tag{44b}
\end{align*}
$$

Such Jacobian determinants are exact because the symplectic forms in (40a)-(40b) remain the same at any order of the guiding-center expansion, as only the guiding-center Hamiltonians change with increased order of accuracy (see, for example, [26] for a proof of this statement). The Jacobian determinants confirm that geometric terms related to the curvature of the background magnetic field appear at order $O(\varepsilon)$ for the ions and at order $O\left(\varepsilon^{\frac{3}{2}}\right)$ for the electrons, in accordance with the guiding-center Hamiltonians (41).

### 4.3 Gyrocenter coordinates $\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right)$

The guiding-center single-particle Lagrangians (40a)-(40b) obtained from the guiding-center coordinate transformation still carry a dependence on the guiding-center angle $\bar{\theta}$ (the fast variable) in the arguments of the dynamic potentials (42). Consequently, the guiding-center magnetic moment $\bar{\mu}$ is not a constant of the motion and the dynamics of slow and fast variables are still coupled in the guiding-center phase space. The purpose of the gyrocenter phase-space coordinate transformation is to remove this residual dependence on the angle variable $\bar{\theta}$ from the Lagrangians, thus from the Hamiltonians (41). As for the guiding-center coordinate transformation (38), we define the gyrocenter coordinate transformation for ions as polynomial transforms of the form

$$
\begin{align*}
\overline{\boldsymbol{X}} & =\boldsymbol{X}+\varepsilon^{2} \boldsymbol{\rho}_{2 \mathrm{i}}+\varepsilon^{3} \boldsymbol{\rho}_{3 \mathrm{i}}+\varepsilon^{4} \boldsymbol{\rho}_{4 \mathrm{i}},  \tag{45a}\\
\bar{p}_{\|} & =P_{\|}+\varepsilon G_{1 \mathrm{i}}^{\|}+\varepsilon^{2} G_{2 \mathrm{i}}^{\|}+\varepsilon^{3} G_{3 \mathrm{i}}^{\|},  \tag{45b}\\
\bar{\mu} & =\widehat{\mu}+\varepsilon G_{1 \mathrm{i}}^{\mu}+\varepsilon^{2} G_{2 \mathrm{i}}^{\mu}+\varepsilon^{3} G_{3 \mathrm{i}}^{\mu},  \tag{45c}\\
\bar{\theta} & =\Theta+\varepsilon G_{1 \mathrm{i}}^{\Theta}+\varepsilon^{2} G_{2 \mathrm{i}}^{\Theta}+\varepsilon^{3} G_{3 \mathrm{i}}^{\Theta}, \tag{45~d}
\end{align*}
$$

and the gyrocenter coordinate transformation for electrons as polynomial transforms of the form

$$
\begin{align*}
\overline{\boldsymbol{X}} & =\boldsymbol{X}+\varepsilon^{2} \boldsymbol{\rho}_{2 \mathrm{e}}+\varepsilon^{\frac{5}{2}} \boldsymbol{\rho}_{\frac{5}{2} \mathrm{e}}+\varepsilon^{3} \boldsymbol{\rho}_{3 \mathrm{e}}  \tag{46a}\\
\bar{p}_{\|} & =P_{\|}+\varepsilon G_{1 \mathrm{e}}^{\|}+\varepsilon^{2} G_{2 \mathrm{e}}^{\|}  \tag{46b}\\
\bar{\mu} & =\widehat{\mu}+\sqrt{\varepsilon} G_{\frac{1}{2} \mathrm{e}}^{\mu}+\varepsilon G_{1 \mathrm{e}}^{\mu}  \tag{46c}\\
\bar{\theta} & =\Theta+\sqrt{\varepsilon} G_{\frac{1}{2} \mathrm{e}}^{\Theta}+\varepsilon G_{1 \mathrm{e}}^{\Theta} \tag{46~d}
\end{align*}
$$

Here, $\boldsymbol{Z}:=\left(\boldsymbol{X}, P_{\|}, \widehat{\mu}, \Theta\right)$ denote preliminary gyrocenter coordinates, and $\boldsymbol{\rho}_{n \mathrm{~s}}, G_{n \mathrm{~s}}^{\|}, G_{n \mathrm{~s}}^{\mu}$ and $G_{n \mathrm{~s}}^{\Theta}$ (with $n$ integer or half-integer) denote the generators of the coordinate transformation for the respective particle species. Our preliminary gyrocenter coordinates are the gyrocenter position $\boldsymbol{X}$, the gyrocenter parallel momentum $P_{\|}$, the preliminary gyrocenter magnetic moment $\hat{\mu}$ and the gyrocenter angle variable $\Theta$, also called gyro-angle. The polynomial transform for the electrons is defined by polynomials in powers of $\sqrt{\varepsilon}$ because of the mass ratio between ions and electrons. Moreover, it consists of fewer terms than the ion coordinate transformation because of the lower accuracy of the electron guiding-center single-particle Lagrangian. If more accuracy is desired, the number of terms in the polynomial transform can be increased, but this goes beyond the scope of this work. We also set $\rho_{1 \mathrm{~s}}=0$ a priori: it is, in principle, possible to keep these first-order generators in the calculations and then find out that they can be set to zero without loss of generality.

We remark the conceptual simplicity of the polynomial transform $\boldsymbol{Z} \mapsto \overline{\boldsymbol{Z}}$ compared to Lie transforms [16]: for each coordinate, the transformation is a polynomial of finite degree in $\varepsilon$ (the degree being adapted to the desired accuracy of the transformation) and it is moreover linear and algebraic in the generators. By substituting (45)-(46) in the Lagrangians (40a)-(40b), the gyrocenter generators can be chosen in order to eliminate the residual dependence on the gyro-angle $\Theta$. The method is analogous to the guiding-center transformation and it is discussed in detail in [26] for the long-wavelength regime, that is, the case of dynamic potentials with spatial variations on the macroscopic length scale $\widehat{x}$. In what follows we apply the same methodology to the short-wavelength (strongly-varying) regime expressed in (42).

The exact same ideas and computations of the guiding-center transformation can be applied also for the gyrocenter transformation. In particular, we make use of the equivalence of Lagrangians under the addition of
the total differential $\dot{S}$ of arbitrary scalar functions $S(t, \boldsymbol{Z})$ and write

$$
\begin{equation*}
L_{\mathrm{i}} \sim L_{\mathrm{i}}+\varepsilon^{2} \dot{S}_{2 \mathrm{i}}+\varepsilon^{3} \dot{S}_{3 \mathrm{i}}, \quad L_{\mathrm{e}} \sim L_{\mathrm{e}}+\varepsilon^{\frac{5}{2}} \dot{S}_{\frac{5}{2} \mathrm{e}}+\varepsilon^{3} \dot{S}_{3 \mathrm{e}} \tag{47}
\end{equation*}
$$

where the total differential $\dot{S}_{n s}$ reads

$$
\begin{equation*}
\dot{S}_{n \mathrm{~s}}=\frac{1}{\varepsilon} \nabla_{\perp} S_{n \mathrm{~s}} \cdot \dot{\boldsymbol{X}}+\nabla_{\|} S_{n \mathrm{~s}} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}+\frac{\partial S_{n \mathrm{~s}}}{\partial P_{\|}} \dot{P}_{\|}+\frac{\partial S_{n \mathrm{~s}}}{\partial \widehat{\mu}} \dot{\hat{\mu}}+\frac{\partial S_{n \mathrm{~s}}}{\partial \Theta} \dot{\Theta}+\frac{\partial S_{n \mathrm{~s}}}{\partial t} \tag{48}
\end{equation*}
$$

Here, $\nabla_{\perp}:=-\boldsymbol{b}_{0} \times \boldsymbol{\nabla} \times \boldsymbol{b}_{0}$ and $\nabla_{\|}:=\boldsymbol{b}_{0} \cdot \nabla$ denote the gradients with respect to the direction perpendicular and parallel to the background magnetic field, respectively. We remark that in the derivation of the gyrokinetic Lagrangians, the scalar functions $S_{n s}$ turn out to be functions of the fluctuating potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$ and thus have strong variations in the perpendicular directions, which has to be expressed in (48) by means of the factor $1 / \varepsilon$ in front of $\nabla_{\perp}$.

We summarize our results for ions in Propositions 1-3 and our results for electrons in Propositions 4-6. Proofs of these propositions are given in Appendix C. The species index is mostly omitted for more readability.

Proposition 1 (ion polynomial transform) The ion guiding-center single-particle Lagrangian (40a), expressed in the preliminary gyrocenter coordinates $\left(\boldsymbol{X}, P_{\|}, \widehat{\mu}, \Theta\right)$ via the polynomial transform (45), is equivalent to

$$
\begin{equation*}
L_{\mathrm{i}} \sim\left(P_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}-H_{0}+\sum_{n=1}^{3} \varepsilon^{n} L_{n}+O\left(\varepsilon^{4}\right) \tag{49}
\end{equation*}
$$

where $H_{0}=P_{\|}^{2} / 2+\widehat{\mu} B_{0}$ is the lowest-order Hamiltonian and the Lagrangians $L_{n}$ read

$$
\begin{equation*}
L_{n}=\gamma_{n}^{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}}+\gamma_{n}^{\|} \dot{P}_{\|}+\gamma_{n}^{\mu} \dot{\hat{\mu}}+\gamma_{n}^{\Theta} \dot{\Theta}-H_{n} \tag{50}
\end{equation*}
$$

where the components $\gamma_{n}^{\boldsymbol{X}}, \gamma_{n}^{\|}, \gamma_{n}^{\mu}, \gamma_{n}^{\Theta}$ and the Hamiltonians $H_{n}$ depend on the generators of the transformation (45) and are given in Appendix C.

Proposition 2 (preliminary ion gyrocenter Lagrangian) In the Lagrangians $L_{n}$ of (50) the generators of the polynomial transform (45) can be chosen such that

$$
\begin{equation*}
L_{\mathrm{i}} \sim\left(P_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}+\varepsilon\left(\widehat{\mu}+\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle+\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle\right) \dot{\Theta}-H_{0}+O\left(\varepsilon^{4}\right) \tag{51}
\end{equation*}
$$

where $\left\langle\gamma_{2}^{\Theta}\right\rangle$ and $\left\langle\gamma_{3}^{\Theta}\right\rangle$ are given in (143) and (154) in Appendix $C$ and, for a given function $g(\Theta),\langle g\rangle$ denotes its gyro-average and is defined as

$$
\begin{equation*}
\langle g\rangle:=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \Theta g(\Theta) . \tag{52}
\end{equation*}
$$

Proposition 3 (ion gyrocenter Lagrangian) The generalized gyrocenter magnetic moment

$$
\begin{equation*}
\mu:=\widehat{\mu}+\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle+\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle \tag{53}
\end{equation*}
$$

is a constant of the motion, accurate up to order $O\left(\varepsilon^{2}\right)$, with respect to the dynamics induced by the preliminary ion gyrocenter single-particle Lagrangian (51). Moreover, there is a one-to-one correspondence $\mu \mapsto \hat{\mu}$, which implies that

$$
\begin{equation*}
\mathcal{H}_{\mathrm{i}}:=\frac{P_{\|}^{2}}{2}+\widehat{\mu}(\mu) B_{0} \tag{54}
\end{equation*}
$$

is the ion gyrocenter Hamiltonian. In other words, $\mathcal{H}_{\mathrm{i}}$ is obtained from $H_{0}$ by inverting the transformation $\widehat{\mu} \mapsto \mu$ defined in (53). By expressing (51) in terms of the new gyrocenter coordinates $\mathcal{Z}:=\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right)$, we obtain the ion gyrocenter single-particle Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{i}} \sim\left(P_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}+\varepsilon \mu \dot{\Theta}-\left[\mathcal{H}_{\mathrm{i}}+\bar{O}\left(\varepsilon^{3}\right)\right]+O\left(\varepsilon^{4}\right) \tag{55}
\end{equation*}
$$

where the symbol $\bar{O}\left(\varepsilon^{3}\right)$ denotes corrections to the Hamiltonian of order $O\left(\varepsilon^{3}\right)$ that are independent of the gyro-angle $\Theta$ and $\mathcal{H}_{\mathrm{i}}=\mathcal{H}_{0 \mathrm{i}}+\varepsilon \mathcal{H}_{1 \mathrm{i}}+\varepsilon^{2} \mathcal{H}_{2 \mathrm{i}}$, with

$$
\begin{align*}
\mathcal{H}_{0 \mathrm{i}}= & \frac{P_{\|}^{2}}{2}+\mu B_{0}  \tag{56a}\\
\mathcal{H}_{1 \mathrm{i}}= & \left\langle\Psi_{1}\right\rangle+\delta H_{1},  \tag{56b}\\
\mathcal{H}_{2 \mathrm{i}}= & \left.\left.\frac{1}{2}\langle | \boldsymbol{A}_{1}\right|^{2}\right\rangle-\frac{1}{2 B_{0}} \frac{\mathrm{~d}}{\mathrm{~d} \mu}\left\langle{\widetilde{\Psi_{1}}}^{2}\right\rangle \\
& -\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp} \widetilde{\Psi_{1}}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{\Psi_{1}}\right\rangle  \tag{56c}\\
& -\frac{1}{B_{0}}\left\langle\nabla_{\|} \widetilde{\Psi_{1}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle+\delta \mathcal{G}_{2}+\delta H_{2}
\end{align*}
$$

Here,

$$
\begin{equation*}
\Psi_{1}=\Psi_{1}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1 \mathrm{i}}\right) \quad \text { with } \quad \overline{\boldsymbol{\rho}}_{1 \mathrm{i}}=\sqrt{\frac{2 \mu}{B_{0}(\boldsymbol{X})}} \boldsymbol{a}_{0}(\boldsymbol{X}, \Theta) \tag{57}
\end{equation*}
$$

and $\widetilde{\Psi_{1}}:=\Psi_{1}-\left\langle\Psi_{1}\right\rangle$ (and the same for $\boldsymbol{A}_{1}$ ). Moreover, $\delta \mathcal{G}_{2}$ is a term related to the curvature of the background magnetic field, besides $\delta H_{1}$ and $\delta H_{2}$ :

$$
\begin{align*}
\delta \mathcal{G}_{2}:=\delta G_{2} & -\frac{1}{2}\left\langle\widetilde{\widetilde{\Psi_{1}}}\left(\frac{\boldsymbol{b}_{0}}{B_{0}} \times \nabla_{\perp} B_{0}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{\Psi_{1}}}{B_{0}}\right\rangle \\
& +\frac{1}{2}\left\langle\left(\frac{\boldsymbol{b}_{0}}{B_{0}^{3}} \times \nabla_{\perp} \widetilde{\Psi_{1}}\right) \cdot \nabla_{\perp} B_{0} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{\Psi_{1}}\right\rangle \tag{58}
\end{align*}
$$

with $\delta G_{2}$ given in (141) in Appendix $C$.
Proposition 4 (electron polynomial transform) The electron guiding-center single-particle Lagrangian (40b), expressed in the preliminary gyrocenter coordinates $\left(\boldsymbol{X}, P_{\|}, \widehat{\mu}, \Theta\right)$ via the polynomial transform (46), is equivalent to

$$
\begin{equation*}
L_{\mathrm{e}} \sim\left(\sqrt{\varepsilon} P_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}-H_{0}-\sqrt{\varepsilon} H_{\frac{1}{2}}+\sum_{n=2}^{6} \varepsilon^{\frac{n}{2}} L_{\frac{n}{2}}+O\left(\varepsilon^{\frac{7}{2}}\right) \tag{59}
\end{equation*}
$$

where $H_{0}=P_{\|}^{2} / 2+\widehat{\mu} B_{0}$ is the lowest-order Hamiltonian, the Hamiltonian $H_{\frac{1}{2}}$ reads

$$
\begin{equation*}
H_{\frac{1}{2}}=G_{\frac{1}{2}}^{\mu} B_{0}+P_{\|} A_{1 \|}+\sqrt{2 \widehat{\mu} B_{0}} c_{0} \cdot \boldsymbol{A}_{1 \perp} \tag{60}
\end{equation*}
$$

and the Lagrangians $L_{\frac{n}{2}}$ read

$$
\begin{equation*}
L_{\frac{n}{2}}=\gamma_{\frac{n}{2}}^{X} \cdot \dot{\boldsymbol{X}}+\gamma_{\frac{n}{2}}^{\|} \dot{P}_{\|}+\gamma_{\frac{n}{2}}^{\mu} \dot{\hat{\mu}}+\gamma_{\frac{n}{2}}^{\Theta} \dot{\Theta}-H_{\frac{n}{2}} \tag{61}
\end{equation*}
$$

where the components $\gamma_{\frac{n}{2}}^{X}, \gamma_{\frac{n}{2}}^{\|}, \gamma_{\frac{n}{2}}^{\mu}, \gamma_{\frac{n}{2}}^{\Theta}$ and the Hamiltonians $H_{\frac{n}{2}}$ depend on the generators of the transformation (46) and are given in Appendix ${ }^{2}$.

Proposition 5 (preliminary electron gyrocenter Lagrangian) In the Hamiltonian $H_{\frac{1}{2}}$ and in the Lagrangians $L_{\frac{n}{2}}$ of (61) the generators of the polynomial transform (46) can be chosen such that

$$
\begin{equation*}
L_{\mathrm{e}} \sim\left(\sqrt{\varepsilon} P_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}-\varepsilon^{2}\left(\widehat{\mu}+\sqrt{\varepsilon}\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle-\varepsilon\left\langle\gamma_{3}^{\Theta}\right\rangle\right) \dot{\Theta}-H_{0}+O\left(\varepsilon^{\frac{7}{2}}\right) \tag{62}
\end{equation*}
$$

where $\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle$ and $\left\langle\gamma_{3}^{\Theta}\right\rangle$ are given in (180) and (186) in Appendix $C$ and, for a given function $g(\Theta)$, $\langle g\rangle$ denotes its gyro-average as defined in (52).

Proposition 6 (electron gyrocenter Lagrangian) The generalized gyrocenter magnetic moment

$$
\begin{equation*}
\mu:=\widehat{\mu}+\sqrt{\varepsilon}\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle-\varepsilon\left\langle\gamma_{3}^{\Theta}\right\rangle \tag{63}
\end{equation*}
$$

is a constant of the motion, accurate up to order $O(\varepsilon)$, with respect to the dynamics induced by the preliminary electron gyrocenter single-particle Lagrangian (62). Moreover, there is a one-to-one correspondence $\mu \mapsto \widehat{\mu}$, which implies that

$$
\begin{equation*}
\mathcal{H}_{\mathrm{e}}:=\frac{P_{\|}^{2}}{2}+\widehat{\mu}(\mu) B_{0} \tag{64}
\end{equation*}
$$

is the electron gyrocenter Hamiltonian. In other words, $\mathcal{H}_{\mathrm{e}}$ is obtained from $H_{0}$ by inverting the transformation $\widehat{\mu} \mapsto \mu$ defined in (63). By expressing (62) in terms of the new gyrocenter coordinates $\mathcal{Z}:=\left(\boldsymbol{X}, P_{\|}, \mu, \Theta\right)$, we obtain the electron gyrocenter single-particle Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{e}} \sim\left(\sqrt{\varepsilon} P_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}-\varepsilon^{2} \mu \dot{\Theta}-\left[\mathcal{H}_{\mathrm{e}}+\bar{O}\left(\varepsilon^{\frac{3}{2}}\right)\right]+O\left(\varepsilon^{\frac{7}{2}}\right), \tag{65}
\end{equation*}
$$

where the symbol $\bar{O}\left(\varepsilon^{\frac{3}{2}}\right)$ denotes corrections to the Hamiltonian of order $O\left(\varepsilon^{\frac{3}{2}}\right)$ that are independent of the gyro-angle $\Theta$ and $\mathcal{H}_{\mathrm{e}}=\mathcal{H}_{0 \mathrm{e}}+\sqrt{\varepsilon} \mathcal{H}_{\frac{1}{2} \mathrm{e}}+\varepsilon \mathcal{H}_{1 \mathrm{e}}$, with

$$
\begin{align*}
& \mathcal{H}_{0 \mathrm{e}}=\frac{P_{\|}^{2}}{2}+\mu B_{0}  \tag{66a}\\
& \mathcal{H}_{\frac{1}{2} \mathrm{e}}=P_{\|} A_{1 \|}  \tag{66b}\\
& \mathcal{H}_{1 \mathrm{e}}=-\phi_{1}+\frac{A_{1 \|}^{2}}{2}+\mu\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{1}\right) \cdot \boldsymbol{b}_{0} . \tag{66c}
\end{align*}
$$

Here, $\phi_{1}=\phi_{1}(t, \boldsymbol{X} / \varepsilon)$ and the same for $\boldsymbol{A}_{1}$.

## 5 Gyrokinetic Vlasov-Maxwell model

We first remark that the Jacobian determinants $\mathcal{J}_{\mathrm{s}}$ of the gyrocenter coordinate transformation $\mathcal{Z} \mapsto(\boldsymbol{x}, \boldsymbol{v})$ can be computed directly from the symplectic part of the gyrocenter single-particle Lagrangians (55) and (65) for ions and electrons, respectively:

$$
\begin{align*}
& \mathcal{J}_{\mathrm{i}}=B_{\| \mathrm{i}}^{*}=B_{0}+\varepsilon P_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0},  \tag{67a}\\
& \mathcal{J}_{\mathrm{e}}=B_{\| \mathrm{e}}^{*}=B_{0}-\varepsilon^{\frac{3}{2}} P_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0} \tag{67~b}
\end{align*}
$$

Such Jacobian determinants are exact and have the same form as the corresponding guiding-center Jacobian determinants in (44) because the symplectic forms in (55) and (65) remain the same at any order of the gyrocenter expansion, as only the gyrocenter Hamiltonians change with increased order of accuracy (as for the guiding-center coordinate transformation). The Jacobian determinants (67) confirm that geometric terms related to the curvature of the background magnetic field appear again at order $O(\varepsilon)$ for the ions and at order $O\left(\varepsilon^{\frac{3}{2}}\right)$ for the electrons.

The ion gyrokinetic equations of motion for the slow phase-space variables $\left(\boldsymbol{X}, P_{\|}\right)$derived from the ion gyrocenter Lagrangian (55) read

$$
\begin{align*}
\dot{\boldsymbol{X}} & =\frac{1}{B_{\| \mathrm{i}}^{*}}\left(\varepsilon \boldsymbol{b}_{0} \times \nabla \mathcal{H}_{\mathrm{i}}+\frac{\partial \mathcal{H}_{\mathrm{i}}}{\partial P_{\|}} \boldsymbol{B}_{\mathrm{i}}^{*}\right)+O\left(\varepsilon^{3}\right),  \tag{68a}\\
\dot{P}_{\|} & =-\frac{\boldsymbol{B}_{\mathrm{i}}^{*}}{B_{\| \mathrm{i}}^{*}} \cdot \nabla \mathcal{H}_{\mathrm{i}}+O\left(\varepsilon^{3}\right) \tag{68b}
\end{align*}
$$

where the modified magnetic field $\boldsymbol{B}_{\mathrm{i}}^{*}$ is defined as $\boldsymbol{B}_{\mathrm{i}}^{*}:=\boldsymbol{B}_{0}+\varepsilon P_{\|} \boldsymbol{\nabla} \times \boldsymbol{b}_{0}$ and its parallel component is defined as $B_{\| \mathrm{i}}^{*}:=\boldsymbol{B}_{\mathrm{i}}^{*} \cdot \boldsymbol{b}_{0}$, which is the ion Jacobian (67a). The gyrokinetic magnetic moment $\mu$ is a constant of the motion accurate up to order $O\left(\varepsilon^{3}\right): \dot{\mu}=O\left(\varepsilon^{3}\right)$. Moreover, the dynamics of the gyro-angle $\Theta$ is decoupled from the slow dynamics of $\left(\boldsymbol{X}, P_{\|}\right)$and described by $\dot{\Theta}=(1 / \varepsilon) \partial \mathcal{H}_{\mathrm{i}} / \partial \mu+O\left(\varepsilon^{2}\right)$, with the factor $1 / \varepsilon$ signifying that this dynamics is the fastest among all phase-space variables and with larger error terms of order $O\left(\varepsilon^{2}\right)$. The ion
gyrocenter distribution function $F_{\mathrm{i}}\left(t, \boldsymbol{X}, P_{\|}, \mu\right)$ is constant along solutions of (68), where $\mu$ is a time-independent parameter. The electron gyrokinetic equations of motion for the slow phase-space variables $\left(\boldsymbol{X}, P_{\|}\right)$derived from the electron gyrocenter Lagrangian (65) read

$$
\begin{align*}
\dot{\boldsymbol{X}} & =-\frac{1}{\sqrt{\varepsilon}} \frac{1}{B_{\| \mathrm{e}}^{*}}\left(\varepsilon^{\frac{3}{2}} \boldsymbol{b}_{0} \times \nabla \mathcal{H}_{\mathrm{e}}-\frac{\partial \mathcal{H}_{\mathrm{e}}}{\partial P_{\|}} \boldsymbol{B}_{\mathrm{e}}^{*}\right)+O(\varepsilon),  \tag{69a}\\
\dot{P}_{\|} & =-\frac{1}{\sqrt{\varepsilon}} \frac{\boldsymbol{B}_{\mathrm{e}}^{*}}{B_{\| \mathrm{e}}^{*}} \cdot \nabla \mathcal{H}_{\mathrm{e}}+O(\varepsilon), \tag{69b}
\end{align*}
$$

where the modified magnetic field $\boldsymbol{B}_{\mathrm{e}}^{*}$ is defined as $\boldsymbol{B}_{\mathrm{e}}^{*}:=\boldsymbol{B}_{0}-\varepsilon^{\frac{3}{2}} P_{\|} \boldsymbol{\nabla} \times \boldsymbol{b}_{0}$ and its parallel component is defined again as $B_{\| \mathrm{e}}^{*}:=\boldsymbol{B}_{\mathrm{e}}^{*} \cdot \boldsymbol{b}_{0}$, which is the electron Jacobian (67b). The gyrokinetic magnetic moment $\mu$ is a constant of the motion accurate up to order $O\left(\varepsilon^{\frac{3}{2}}\right): \dot{\mu}=O\left(\varepsilon^{\frac{3}{2}}\right)$. Moreover, the dynamics of the gyro-angle $\Theta$ is decoupled from the slow dynamics of $\left(\boldsymbol{X}, P_{\|}\right)$and described by $\dot{\Theta}=-\left(1 / \varepsilon^{2}\right) \partial \mathcal{H}_{\mathrm{e}} / \partial \mu+O\left(\varepsilon^{-\frac{1}{2}}\right)$, with the factor $1 / \varepsilon^{2}$ signifying again that this dynamics is the fastest among all phase-space variables and with larger error terms of order $O\left(\varepsilon^{-\frac{1}{2}}\right)$. The electron gyrocenter distribution function $F_{\mathrm{e}}\left(t, \boldsymbol{X}, P_{\|}, \mu\right)$ is constant along solutions of (69), where $\mu$ is a time-independent parameter.

The non-homogeneous gyrokinetic Maxwell's equations can be derived from the variational principle by taking variations of the Low Lagrangian with respect to the electromagnetic fluctuating potentials $\Phi_{1}$ and $\boldsymbol{A}_{1}$. After applying identity (79), the weak form of gyrokinetic Coulomb's law is obtained by taking variations with respect to $\Phi_{1}$ and reads

$$
\begin{aligned}
0= & \int \mathrm{d}^{6} \mathcal{Z}\left(B_{\| \mathrm{e}}^{*} F_{\mathrm{e}}\left\langle\delta \Phi_{1}\right\rangle-B_{\| \mathrm{i}}^{*} F_{\mathrm{i}}\left\langle\delta \Phi_{1}\right\rangle\right) \\
& +\varepsilon \int \mathrm{d}^{6} \mathcal{Z} B_{\| \mathrm{i}}^{*} F_{\mathrm{i}}\left(\frac{1}{B_{0}} \frac{\mathrm{~d}}{\mathrm{~d} \mu}\left\langle\widetilde{\Psi_{1 \mathrm{i}}} \widetilde{\delta \Phi_{1}}\right\rangle\right. \\
& +\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp} \widetilde{\delta \Phi_{1}}\right) \cdot \int^{\Theta} \mathrm{d} \Theta^{\prime} \nabla_{\perp} \widetilde{\Psi_{1 \mathrm{i}}}\right\rangle \\
& +\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp} \widetilde{\Psi_{1 \mathrm{i}}}\right) \cdot \int^{\Theta} \mathrm{d} \Theta^{\prime} \nabla_{\perp} \widetilde{\delta \Phi_{1}}\right\rangle \\
& \left.+\frac{1}{B_{0}}\left\langle\nabla_{\|} \delta \Phi_{1} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle+\frac{\delta\left(\delta \mathcal{G}_{2}\right)}{\delta \Phi_{1}}\left(\delta \Phi_{1}\right)\right)
\end{aligned}
$$

where $\delta \Phi_{1}$ denotes an arbitrary test function. Here we neglected all terms of order higher than $O\left(\varepsilon^{2}\right)$ and $O(\varepsilon)$ from the ion and electron Hamiltonians, respectively. The terms of order $O(1)$ and $O(\varepsilon)$ represent the gyrocenter charge density and the gyrocenter polarization density, respectively. Similarly, after applying again identity (79), the weak form of Ampère-Maxwell's law is obtained by taking variations with respect to $\boldsymbol{A}_{1}$ and
reads

$$
\begin{aligned}
0= & -\int \mathrm{d}^{3} \boldsymbol{x}\left(\nabla \times \boldsymbol{B}_{0}+\varepsilon \nabla \times\left(\nabla \times \boldsymbol{A}_{1}\right)\right) \cdot \delta \boldsymbol{A}_{1} \\
& -\sqrt{\varepsilon} \int \mathrm{d}^{6} \boldsymbol{\mathcal { Z }} B_{\| \mathrm{e}}^{*} F_{\mathrm{e}} P_{\|} \delta A_{1 \|}+\varepsilon \int \mathrm{d}^{6} \boldsymbol{\mathcal { Z }}\left[B_{\| \mathrm{i}}^{*} F_{\mathrm{i}}\left\langle\boldsymbol{P} \cdot \delta \boldsymbol{A}_{1}\right\rangle\right. \\
& \left.-B_{\| \mathrm{e}}^{*} F_{\mathrm{e}}\left(A_{1 \|} \delta A_{1 \|}+\mu\left(\boldsymbol{\nabla} \times \delta \boldsymbol{A}_{1}\right) \cdot \boldsymbol{b}_{0}\right)\right] \\
& -\varepsilon^{2} \int \mathrm{~d}^{6} \boldsymbol{\mathcal { Z }} B_{\| \mathrm{i}}^{*} F_{\mathrm{i}}\left(\frac{1}{B_{0}} \frac{\mathrm{~d}}{\mathrm{~d} \mu}\left\langle\widetilde{\Psi_{1 \mathrm{i}}} \boldsymbol{P} \cdot \widetilde{\delta \boldsymbol{A}_{1}}\right\rangle+\left\langle\boldsymbol{A}_{1} \cdot \delta \boldsymbol{A}_{1}\right\rangle\right. \\
& +\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp}\left(\boldsymbol{P} \cdot \widetilde{\delta \boldsymbol{A}_{1}}\right)\right) \cdot \int^{\Theta} \mathrm{d} \Theta^{\prime} \nabla_{\perp} \widetilde{\Psi_{1 \mathrm{i}}}\right\rangle \\
& +\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp} \widetilde{\Psi_{1 \mathrm{i}}}\right) \cdot \int^{\Theta} \mathrm{d} \Theta^{\prime} \nabla_{\perp}\left(\boldsymbol{P} \cdot \widetilde{\delta \boldsymbol{A}_{1}}\right)\right\rangle \\
& +\frac{1}{B_{0}}\left\langle\nabla_{\|}\left(\boldsymbol{P} \cdot \delta \boldsymbol{A}_{1}\right) \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle+\frac{1}{B_{0}}\left\langle\nabla_{\|} \Psi_{1 \mathrm{i}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{\delta A_{1 \|}}\right\rangle \\
& \left.+\frac{\delta\left(\delta \mathcal{G}_{2}\right)}{\delta \boldsymbol{A}_{1}}\left(\delta \boldsymbol{A}_{1}\right)\right),
\end{aligned}
$$

where $\delta \boldsymbol{A}_{1}$ denotes an arbitrary test function and $\boldsymbol{P}:=P_{\|} \boldsymbol{b}_{0}+\sqrt{2 \mu B_{0}} \boldsymbol{c}_{0}$. As before, we neglected all terms of order higher than $O\left(\varepsilon^{2}\right)$ and $O(\varepsilon)$ from the ion and electron Hamiltonians, respectively.

## 6 Conclusions and outlook

The main results of this work are summarized in Proposition 3 for ions and in Proposition 6 for electrons. We state the final ion and electron single-particle gyrocenter Lagrangians (55) and (65) in normalized form, obtained from the consecutive coordinate transformations listed in Tables 2 and 3, respectively. The results obtained in [23] and in the previous works of [10] and [11] for ions are recovered and augmented by terms related to the assumption of maximal ordering. In particular, novel terms are the geometric first-order and second-order corrections $\delta H_{1}, \delta H_{2}$ and $\delta \mathcal{G}_{2}$, appearing in the first-order and second-order ion gyrocenter Hamiltonians $\mathcal{H}_{1 \mathrm{i}}$ and $\mathcal{H}_{2 \mathrm{i}}$ in (56), respectively. In particular, the first-order curvature term

$$
\begin{equation*}
\delta H_{1}=\mu\left[\frac{P_{\|}}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}-P_{\|}\left(\nabla \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right] \tag{70}
\end{equation*}
$$

should not be neglected when computing ion trajectories in maximal ordering, since it is of the same size as the curvature term in $\boldsymbol{B}_{\mathrm{i}}^{*}$. Moreover, the term

$$
\begin{equation*}
-\frac{1}{B_{0}}\left\langle\nabla_{\|} \widetilde{\Psi_{1}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle \tag{71}
\end{equation*}
$$

appearing in the second-order ion gyrocenter Hamiltonian $\mathcal{H}_{2 \mathrm{i}}$, is also new. It is important to note that this term arises under the conventional gyrokinetic ordering $k_{\|} /\left|\boldsymbol{k}_{\perp}\right|=O(\varepsilon)$ (see discussions in section 3.2) and it is related to fluctuations with non-zero toroidal mode number.

Concerning the electrons, they turn out to be insensitive to magnetic background curvature effects up to second order in $\varepsilon$, even in maximal ordering. However, terms related to the parallel component of the fluctuating magnetic vector potential appear in the electron gyrocenter Hamiltonian already at order $O(\sqrt{\varepsilon})$. Moreover, the first-order term $\mu\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{1}\right) \cdot \boldsymbol{b}_{0}=\mu B_{1 \|}$ represents a correction to the perpendicular electron kinetic energy, which now reads $\mu\left(B_{0}+\varepsilon B_{1 \|}\right)$. This correction is absent for the ions at first order.

The ion gyrokinetic equations of motion (68) for the slow variables ( $\boldsymbol{X}, P_{\|}$) are accurate up to order $O\left(\varepsilon^{3}\right)$. The same accuracy is achieved for the conservation of the gyrocenter magnetic moment $\mu$ : the corrections $\bar{O}\left(\varepsilon^{3}\right)$ to the ion gyrocenter Hamiltonian in (55) are indeed independent of $\Theta$ and do not play a role in the Euler-Lagrange equation $\mathrm{d} / \mathrm{d} t\left(\partial \mathcal{L}_{\mathrm{i}} / \partial \dot{\Theta}\right)=\partial \mathcal{L}_{\mathrm{i}} / \partial \Theta$.

The electron gyrokinetic equations of motion (69) for the slow variables ( $\boldsymbol{X}, P_{\|}$) are accurate up to order $O(\varepsilon)$. By contrast, the conservation of the gyrocenter magnetic moment $\mu$ is accurate up to order $O\left(\varepsilon^{\frac{3}{2}}\right)$ : the corrections $\bar{O}\left(\varepsilon^{\frac{3}{2}}\right)$ to the electron gyrocenter Hamiltonian in (65) are indeed independent of $\Theta$ and do not play a role in the Euler-Lagrange equation $\left.\mathrm{d} / \mathrm{d} t\left(\partial \mathcal{L}_{\mathrm{e}} / \partial \dot{\Theta}\right)=\partial \mathcal{L}_{\mathrm{e}} / \partial \Theta\right)$.

The electron gyrokinetic Lagrangian, and the corresponding gyrokinetic equations of motion, have been derived within an ordering consistent with the ions, despite the order of accuracy of the results being different for the two species (due to the fact that the gyrocenter magnetic moment $\mu$ has been computed with less precision for electrons than for ions). We conclude that it is possible to derive a set of gyrokinetic Vlasov-Maxwell equations for ions and electrons within our unique methodology based on guiding-center and gyrocenter polynomial transforms and within unique ordering assumptions relevant for realistic fusion scenarios (maximal ordering). Our technique is alternative to the use of Lie transforms and, combined with our rigorous normalization procedure, can provide useful insights into the derivation of gyrokinetic models and a solid starting point for their further rigorous mathematical investigation.

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## A Variational principle

In (6) the single-particle Lagrangian $L_{\mathrm{s}}$ is evaluated at the $t$-family of maps $\Psi_{t}: \mathbb{R}^{6} \rightarrow \mathbb{R}^{6}$, parametrically depending on time. The map $\Psi_{t}$ is the flow map of the characteristics of the Vlasov equation (1),

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{x}}{\mathrm{~d} t}=\boldsymbol{v}, \quad \frac{\mathrm{d} \boldsymbol{v}}{\mathrm{~d} t}=\frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}[\boldsymbol{E}(t, \boldsymbol{x})+\boldsymbol{v} \times \boldsymbol{B}(t, \boldsymbol{x})] . \tag{72}
\end{equation*}
$$

The flow transports a particle that is at $\boldsymbol{z}_{0}=\left(\boldsymbol{x}_{0}, \boldsymbol{v}_{0}\right)$ at the initial time $t_{0}$ to the phase space position $\Psi_{t}\left(\boldsymbol{z}_{0}\right)$ at time $t$; it is a volume-preserving diffeomorphism, $\Psi_{t} \in \operatorname{Diff}{ }_{\text {vol }}\left(\mathbb{R}^{6}\right)$. The description of an ensemble of particles via the particle Lagrangian $L_{\mathrm{p}}$ arises from the single picture in the following way: Newton's equation of motion for a charged particle can be deduced from the variational principle

$$
\begin{equation*}
\delta \int_{t_{0}}^{t_{1}} L_{\mathrm{s}}\left(\boldsymbol{z}(t), \frac{\mathrm{d} \boldsymbol{z}(t)}{\mathrm{d} t}\right) \mathrm{d} t=0 \tag{73}
\end{equation*}
$$

The extremum defined by (73) is denoted by $\boldsymbol{z}(t)=(\boldsymbol{x}(t), \boldsymbol{v}(t))$ and is the solution of the Euler-Lagrange equations, given by (72). Hence, $\boldsymbol{z}(t)=\Psi_{t}\left(\boldsymbol{z}_{0}\right)$ and (73) can be written as

$$
\begin{equation*}
\delta \int_{t_{0}}^{t_{1}} \int \delta^{6}\left(\boldsymbol{z}_{1}-\boldsymbol{z}_{0}\right) L_{\mathrm{s}}\left(\Psi_{t}\left(\boldsymbol{z}_{1}\right), \frac{\mathrm{d} \Psi_{t}\left(\boldsymbol{z}_{1}\right)}{\mathrm{d} t}\right) \mathrm{d}^{6} \boldsymbol{z}_{1} \mathrm{~d} t=0 \tag{74}
\end{equation*}
$$

Formally, the ensemble description is obtained by replacing the delta function with the initial particle distribution, $\delta^{6}\left(\boldsymbol{z}_{1}-\boldsymbol{z}_{0}\right) \rightarrow f_{\mathrm{s}, 0}\left(\boldsymbol{z}_{1}\right)$, which yields

$$
\begin{equation*}
\delta \int_{t_{0}}^{t_{1}} \int f_{\mathrm{s}, 0}\left(\boldsymbol{z}_{1}\right) L_{\mathrm{s}}\left(\Psi_{t}\left(\boldsymbol{z}_{1}\right), \frac{\mathrm{d} \Psi_{t}\left(\boldsymbol{z}_{1}\right)}{\mathrm{d} t}\right) \mathrm{d}^{6} \boldsymbol{z}_{1} \mathrm{~d} t=0 \tag{75}
\end{equation*}
$$

The particle Lagrangian $L_{\mathrm{p}}$ from (6) is obtained by taking the sum over the species and by relabeling the variables of integration $\boldsymbol{z}_{1} \rightarrow \boldsymbol{z}_{0}$. By construction, the variation (75) with respect to $\Psi_{t}$ yields the characteristics of the Vlasov equation. The Vlasov equation itself enters the picture via the definition

$$
\begin{equation*}
f_{\mathrm{s}}(t, \boldsymbol{z}):=\int \delta^{6}\left(\boldsymbol{z}-\Psi_{t}\left(\boldsymbol{z}_{0}\right)\right) f_{\mathrm{s}, 0}\left(\boldsymbol{z}_{0}\right) \mathrm{d}^{6} \boldsymbol{z}_{0} \tag{76}
\end{equation*}
$$

of the particle distribution functions. Remark that $\boldsymbol{z} \in \mathbb{R}^{6}$ denote coordinates here and not a path in phase space. Hence (76) makes the link between Lagrangian paths $\Psi_{t}$ and the distribution function $f_{\mathrm{s}}$ via $f_{\mathrm{s}}(t, \boldsymbol{z})=$ $f_{\mathrm{s}, 0}\left(\left(\Psi_{t}\right)^{-1}(\boldsymbol{z})\right)$, or simply $f_{\mathrm{s}}=f_{\mathrm{s}, 0} \circ\left(\Psi_{t}\right)^{-1}$. This implies in particular that $f_{s}$ is constant along the Lagrangian paths,

$$
\begin{equation*}
f_{\mathrm{s}}\left(t, \Psi_{t}\left(\boldsymbol{z}_{0}\right)\right)=f_{\mathrm{s}, 0}\left(\boldsymbol{z}_{0}\right) \tag{77}
\end{equation*}
$$

which is the statement of the Vlasov equation. Let us now write the characteristic equations in the form

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{z}}{\mathrm{~d} t}=\boldsymbol{F}(\boldsymbol{z}), \quad \boldsymbol{z}(t=0)=\boldsymbol{z}_{0} \tag{78}
\end{equation*}
$$

with the vector field $\boldsymbol{F}=(\boldsymbol{v}, \boldsymbol{a})$, where $\boldsymbol{a}=\left(q_{\mathrm{s}} / m_{\mathrm{s}}\right)(\boldsymbol{E}+\boldsymbol{v} \times \boldsymbol{B})$. From the definition of the distribution function it follows that

$$
\begin{equation*}
\int \mathrm{d}^{6} \boldsymbol{z}_{0} f_{\mathrm{s}, 0}\left(\boldsymbol{z}_{0}\right) L_{\mathrm{s}}\left(\Psi_{t}, \frac{\mathrm{~d} \Psi_{t}}{\mathrm{~d} t}\right)=\int \mathrm{d}^{6} \boldsymbol{z} f_{\mathrm{s}}(t, \boldsymbol{z}) L_{\mathrm{s}}(\boldsymbol{z}, \boldsymbol{F}(\boldsymbol{z})) \tag{79}
\end{equation*}
$$

This is easily verified by substituting the definition (76) into the right-hand side and preforming the integration over $\boldsymbol{z}$.

## B Guiding-center transformation

Here we review how the ion and electron guiding-center Lagrangians (40a)-(40b) are obtained from the polynomial transforms (38), following [26]. In order to shorten the calculations, we use the coordinate $p_{\perp}:=\sqrt{2 \mu B_{0}}$ instead of the magnetic moment $\mu$ introduced in (35). Hence, we have the momentum transformation

$$
\begin{equation*}
p_{\perp}:=\bar{p}_{\perp}+\varepsilon \bar{G}_{1 \mathrm{~s}}^{\perp}+\varepsilon^{2} \bar{G}_{2 \mathrm{~s}}^{\perp}+\varepsilon^{3} \bar{G}_{3 \mathrm{~s}}^{\perp} \tag{80}
\end{equation*}
$$

and denote by $\overline{\boldsymbol{Z}}_{\perp}:=\left(\overline{\boldsymbol{X}}, \bar{p}_{\|}, \bar{p}_{\perp}, \bar{\theta}\right)$ the guiding-center coordinates. The final result is then transformed back to the representation in terms of the magnetic moment. Let us start with the ions and consider the starting Lagrangian (37a). We substitute the polynomial transform (38), with $p_{\perp}$ from (80) instead of $\mu$, and expand in Taylor series around $\overline{\boldsymbol{Z}}$. For instance,

$$
\begin{align*}
\boldsymbol{A}_{0}(\boldsymbol{x})= & \boldsymbol{A}_{0}(\overline{\boldsymbol{X}} \\
= & \left.+\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{i}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{i}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{i}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{i}}\right)  \tag{81}\\
=\boldsymbol{A}_{0}(\overline{\boldsymbol{X}}) & +\left[\left(\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{i}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{i}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{i}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{i}}\right) \cdot \boldsymbol{\nabla}\right] \boldsymbol{A}_{0}(\overline{\boldsymbol{X}}) \\
& +\frac{1}{2}\left[\left(\varepsilon \overline{\boldsymbol{\rho}}_{1 \mathrm{i}}+\varepsilon^{2} \overline{\boldsymbol{\rho}}_{2 \mathrm{i}}+\varepsilon^{3} \overline{\boldsymbol{\rho}}_{3 \mathrm{i}}+\varepsilon^{4} \overline{\boldsymbol{\rho}}_{4 \mathrm{i}}\right) \cdot \boldsymbol{\nabla}\right]^{2} \boldsymbol{A}_{0}(\overline{\boldsymbol{X}})+\ldots
\end{align*}
$$

We also use the equivalence of Lagrangians under the addition of total differentials of arbitrary scalar functions and the transformation law of tangents (39) (here applied to $\dot{\boldsymbol{A}}_{0}(\overline{\boldsymbol{X}})$ ) to write

$$
\begin{align*}
\boldsymbol{A}_{0}(\overline{\boldsymbol{X}}) \cdot \dot{\overline{\boldsymbol{\rho}}}_{n \mathrm{i}}\left(\overline{\boldsymbol{X}}, \bar{p}_{\|}, \bar{p}_{\perp}, \bar{\theta}, t\right) & \sim-\dot{\boldsymbol{A}}_{0}(\overline{\boldsymbol{X}}) \cdot \overline{\boldsymbol{\rho}}_{n \mathrm{i}}\left(\overline{\boldsymbol{X}}, \bar{p}_{\|}, \bar{p}_{\perp}, \bar{\theta}, t\right) \\
& =-\dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla} \boldsymbol{A}_{0}(\overline{\boldsymbol{X}}) \cdot \overline{\boldsymbol{\rho}}_{n \mathrm{i}}\left(\overline{\boldsymbol{X}}, \bar{p}_{\|}, \bar{p}_{\perp}, \bar{\theta}, t\right) \tag{82}
\end{align*}
$$

We then encounter terms like

$$
\begin{align*}
{\left[-\nabla \boldsymbol{A}_{0}(\overline{\boldsymbol{X}}) \cdot \overline{\boldsymbol{\rho}}_{n \mathrm{i}}+\overline{\boldsymbol{\rho}}_{n \mathrm{i}} \cdot \nabla \boldsymbol{A}_{0}(\overline{\boldsymbol{X}})\right] \cdot \dot{\overline{\boldsymbol{X}}} } & =-\left[\boldsymbol{\rho}_{n \mathrm{i}} \times\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{0}(\overline{\boldsymbol{X}})\right)\right] \cdot \dot{\boldsymbol{X}}  \tag{83}\\
& =-\left[\boldsymbol{\rho}_{n \mathrm{i}} \times \boldsymbol{B}_{0}(\overline{\boldsymbol{X}})\right] \cdot \dot{\boldsymbol{X}}
\end{align*}
$$

in the transformed Lagrangian. The computations for polynomial transforms of arbitrary order have been carried out in [26]. We repeat here in particular the results from Proposition 1 on page 12 of this work to write the series expansion of the Lagrangian, up to order $N=3$ (omitting the species index for more readability). The new Lagrangian written without the dynamical potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$, which are not transformed in the guiding-center step, reads

$$
\begin{equation*}
L_{\mathrm{i}} \sim \frac{\bar{L}_{-1}}{\varepsilon}+\bar{L}_{0}+\varepsilon \bar{L}_{1}+\varepsilon^{2} \bar{L}_{2}+\varepsilon^{3} \bar{L}_{3}+O\left(\varepsilon^{4}\right) \tag{84}
\end{equation*}
$$

where $\bar{L}_{-1}=\boldsymbol{A}_{0} \cdot \dot{\overline{\boldsymbol{X}}}$ and

$$
\begin{align*}
& \bar{L}_{0}=\left(\bar{p}_{\|} \boldsymbol{b}_{0}+\bar{p}_{\perp} \boldsymbol{c}_{0}-\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}\right) \cdot \dot{\overline{\boldsymbol{X}}}-\frac{\bar{p}_{\|}^{2}}{2}-\frac{\bar{p}_{\perp}^{2}}{2}  \tag{85}\\
& \bar{L}_{1}=\left(\bar{G}_{1}^{\|} \boldsymbol{b}_{0}+\bar{G}_{1}^{\perp} \boldsymbol{c}_{0}-\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{1}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{1}^{\|}-\bar{p}_{\perp} \bar{G}_{1}^{\perp}+\boldsymbol{\mathcal { L }}_{1}  \tag{86}\\
& \bar{L}_{2}=\left(\bar{G}_{2}^{\|} \boldsymbol{b}_{0}+\bar{G}_{2}^{\perp} \boldsymbol{c}_{0}-\overline{\boldsymbol{\rho}}_{3} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{2}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{2}^{\|}-\bar{p}_{\perp} \bar{G}_{2}^{\perp}+\boldsymbol{\mathcal { L }}_{2}  \tag{87}\\
& \bar{L}_{3}=\left(\bar{G}_{3}^{\|} \boldsymbol{b}_{0}+\bar{G}_{3}^{\perp} \boldsymbol{c}_{0}-\overline{\boldsymbol{\rho}}_{4} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{3}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{3}^{\|}-\bar{p}_{\perp} \bar{G}_{3}^{\perp}+\boldsymbol{\mathcal { L }}_{3} \tag{88}
\end{align*}
$$

In order to determine the guiding-center Lagrangians we need the explicit expressions of

$$
\begin{equation*}
\mathcal{Q}_{1}:=\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{B}_{0}\right) \times \overline{\boldsymbol{\rho}}_{1}-\bar{p}_{\|} \overline{\boldsymbol{\rho}}_{1} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right)-\bar{p}_{\perp} \overline{\boldsymbol{\rho}}_{1} \times\left(\boldsymbol{\nabla} \times \boldsymbol{c}_{0}\right)-\bar{p}_{\perp} \bar{G}_{1}^{\Theta} \boldsymbol{a}_{0} \tag{89}
\end{equation*}
$$

and of

$$
\begin{aligned}
\mathcal{L}_{1}:= & -\frac{1}{2} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left(\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}\right)-\dot{\bar{p}}_{\|}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{p}}_{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}\right)+\dot{\bar{\theta}} \bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{a}_{0}\right), \\
\mathcal{L}_{2}:= & -\dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)-\dot{\bar{G}}_{1}^{\|}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{G}}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}\right)+\dot{\bar{\theta}} \bar{G}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{a}_{0}\right) \\
& -\dot{\bar{p}}_{\|}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{p}}_{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{c}_{0}\right)+\dot{\theta}_{\bar{p}}^{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right) \\
& +\bar{p}_{\|} \overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \dot{\overline{\boldsymbol{\rho}}}_{1}+\bar{p}_{\perp} \overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{c}_{0} \cdot \dot{\overline{\boldsymbol{\rho}}}_{1}-\bar{p}_{\perp} \bar{G}_{1}^{\Theta} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot \boldsymbol{a}_{0} \\
& -\frac{1}{3} \dot{\boldsymbol{\rho}}_{1} \cdot\left[\overline{\boldsymbol{\rho}}_{1} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]-\frac{1}{6} \overline{\boldsymbol{\rho}}_{1} \times\left[\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{B}_{0}\right] \cdot \dot{\overline{\boldsymbol{X}}} \\
\boldsymbol{\mathcal { L }}_{3}:= & -\frac{1}{2} \dot{\overline{\boldsymbol{\rho}}}_{2} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)-\dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left(\overline{\boldsymbol{\rho}}_{3} \times \boldsymbol{B}_{0}\right) \\
& -\dot{\bar{G}}_{2}^{\|}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{G}}_{2}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}\right)+\dot{\bar{\theta}} \bar{G}_{2}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{a}_{0}\right) \\
& -\dot{\bar{G}}_{1}^{\|}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{G}}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{c}_{0}\right)+\dot{\bar{\theta}} \bar{G}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right) \\
& -\dot{\bar{p}}_{\|( }\left(\overline{\boldsymbol{\rho}}_{3} \cdot \boldsymbol{b}_{0}\right)-\dot{\bar{p}}_{\perp}\left(\overline{\boldsymbol{\rho}}_{3} \cdot \boldsymbol{c}_{0}\right)+\dot{\bar{\theta}} \bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{3} \cdot \boldsymbol{a}_{0}\right) \\
+ & \sum_{m=1}^{2} \bar{G}_{2-m}^{\|} \overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}}_{m}+\bar{p}_{\|}\left[\overline{\boldsymbol{\rho}}_{2} \cdot \nabla \boldsymbol{b}_{0}+\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{b}_{0}\right] \cdot \dot{\overline{\boldsymbol{\rho}}}_{1} \\
& +\sum_{m=1}^{2} \bar{G}_{2-m}^{\perp}\left[\overline{\boldsymbol{\rho}}_{1} \cdot \nabla \boldsymbol{c}_{0}-\bar{G}_{1}^{\Theta} \boldsymbol{a}_{0}\right] \cdot \dot{\overline{\boldsymbol{\rho}}}_{m} \\
& +\bar{p}_{\perp}\left[\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{\nabla} \boldsymbol{c}_{0}-\bar{G}_{2}^{\Theta} \boldsymbol{a}_{0}+\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}+\bar{G}_{1}^{\Theta} \frac{\partial}{\partial \bar{\theta}}\right)^{2} \boldsymbol{c}_{0}\right] \cdot \dot{\overline{\boldsymbol{\rho}}_{1}} \\
& -\dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left[\overline{\boldsymbol{\rho}}_{2} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]-\frac{3}{24} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left[\overline{\boldsymbol{\rho}}_{1} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{B}_{0}\right] \\
& -\frac{1}{2} \overline{\boldsymbol{\rho}}_{2} \times\left[\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{B}_{0}\right] \cdot \dot{\boldsymbol{\boldsymbol { X }}}-\frac{1}{\boldsymbol{\rho}_{1} \times\left[\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{3} \boldsymbol{B}_{0}\right] \cdot \dot{\boldsymbol{X}}}
\end{aligned}
$$

We remark that the above expansions are straightforward but cumbersome, in particular at higher orders, as in $\mathcal{L}_{3}$ for example. One could automatize these expansions in a symbolic computer program, similar to the ideas in [34], where a different approach of setting up the guiding-center transform has been implemented.

In the following we choose the generators $\overline{\boldsymbol{\rho}}_{n}, \bar{G}_{n}^{\|}, \bar{G}_{n}^{\perp}$ and $\bar{G}_{n}^{\Theta}$ in order to cancel as many terms as possible from the Lagrangians. In $\bar{L}_{0}$ from (85) we require $\bar{p}_{\perp} \boldsymbol{c}_{0}=\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}$, which can be obtained by setting

$$
\begin{equation*}
\overline{\boldsymbol{\rho}}_{1}=\frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{a}_{0} \tag{90}
\end{equation*}
$$

Moreover, it has been shown in [26, Theorem 1] that the remaining generators in (86)-(88) can be chosen such that

$$
\begin{equation*}
\bar{L}_{1} \sim \frac{\bar{p}_{\perp}^{2}}{2 B_{0}} \dot{\bar{\theta}}, \quad \bar{L}_{2} \sim-\frac{\delta H_{1}}{B_{0}} \dot{\bar{\theta}}, \quad \bar{L}_{3} \sim-\frac{\delta H_{2}}{B_{0}} \dot{\bar{\theta}} \tag{91}
\end{equation*}
$$

where $-\delta H_{1} / B_{0}$ and $-\delta H_{2} / B_{0}$ are the $\bar{\theta}$-averages of the terms multiplying $\dot{\bar{\theta}}$ in $\mathcal{L}_{2}$ and $\mathcal{L}_{3}$ of (87) and (88), respectively. It is mandatory to keep these terms in the Lagrangian to avoid secularities in the averaged equations of motion. Substituting (90) and (91) into (84) yields

$$
\begin{equation*}
L_{\mathrm{i}} \sim\left(\bar{p}_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\overline{\boldsymbol{X}}}+\varepsilon\left(\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}-\varepsilon \frac{\delta H_{1}}{B_{0}}-\varepsilon^{2} \frac{\delta H_{2}}{B_{0}}\right) \dot{\bar{\theta}}-\frac{\bar{p}_{\|}^{2}}{2}-\frac{\bar{p}_{\perp}^{2}}{2}+O\left(\varepsilon^{4}\right) \tag{92}
\end{equation*}
$$

without the dynamical potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$. Computing the Euler-Lagrange equation $\partial L_{\mathrm{i}} / \partial \bar{\theta}-\frac{\mathrm{d}}{\mathrm{d} t} \partial L_{\mathrm{i}} / \partial \dot{\bar{\theta}}=0$ and noting that (92) is independent of $\bar{\theta}$ up to third order, we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}-\varepsilon \frac{\delta H_{1}}{B_{0}}-\varepsilon^{2} \frac{\delta H_{2}}{B_{0}}\right)=O\left(\varepsilon^{3}\right) \tag{93}
\end{equation*}
$$

Hence, the guiding-center generalized magnetic moment

$$
\begin{equation*}
\bar{\mu}:=\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}-\varepsilon \frac{\delta H_{1}}{B_{0}}-\varepsilon^{2} \frac{\delta H_{2}}{B_{0}} \tag{94}
\end{equation*}
$$

is a constant of the motion accurate up to order $O\left(\varepsilon^{3}\right)$ and should be adopted as one of the coordinates. As indicated by the notation $\delta H_{1}$ and $\delta H_{2}$, there is a one-to-one correspondence between the guiding-center generalized magnetic moment $\bar{\mu}$ and the guiding-center Hamiltonian $\bar{H}$ :

$$
\begin{equation*}
\bar{H}:=\frac{\bar{p}_{\|}^{2}}{2}+\frac{\bar{p}_{\perp}^{2}(\bar{\mu})}{2} \tag{95}
\end{equation*}
$$

In other words, the guiding-center Hamiltonian is obtained by expressing $\bar{p}_{\perp}^{2} / 2$ in terms of $\bar{\mu}$ by inverting the transformation $\bar{p}_{\perp} \mapsto \bar{\mu}$ defined in (94). This one-to-one correspondence is typical for polynomial transforms and occurs also in the gyrocenter transformation. It has an important consequence for the accuracy of the derived guiding-center Lagrangian, namely it yields

$$
\begin{equation*}
L_{\mathrm{i}} \sim\left(\bar{p}_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}+\varepsilon \bar{\mu} \dot{\bar{\theta}}-\left[\bar{H}+\bar{O}\left(\varepsilon^{3}\right)\right]+O\left(\varepsilon^{4}\right) \tag{96}
\end{equation*}
$$

Here, the notation $\bar{O}\left(\varepsilon^{3}\right)$ denotes corrections to the Hamiltonian $\bar{H}$ that are of order $O\left(\varepsilon^{3}\right)$ and independent of $\bar{\theta}$, originating from the inversion of (94). The loss of one order of accuracy in the Hamiltonian from (92) to (96) occurs because the generalized magnetic moment $\bar{\mu}$ has been determined only up to $O\left(\varepsilon^{2}\right)$ in (94), implying that the guiding-center Hamiltonian $\bar{H}$ can only be second-order accurate. Is is important to note that neglecting the second-order term $\delta \mathrm{H}_{2}$ in (92) leads to a guiding-center Hamiltonian that is only first-order accurate. We arrive to a similar conclusion for the electrons below.

Let us now identify the higher-order terms $\delta H_{1}$ and $\delta H_{2}$ in the guiding-center generalized magnetic moment (94). As mentioned earlier, they are the $\bar{\theta}$-averages of the terms multiplying $\dot{\bar{\theta}}$ in (87) and (88), respectively. From (87) and (88) we observe that all $\dot{\bar{\theta}}$-terms are in $\mathcal{L}_{2}$ and $\mathcal{L}_{3}$. From (90) we deduce that

$$
\begin{equation*}
\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}=\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}=0, \quad \dot{\overline{\boldsymbol{\rho}}}_{1}=\frac{\partial \overline{\boldsymbol{\rho}}_{1}}{\partial \bar{\theta}} \dot{\bar{\theta}}+\text { rest }=\frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{c}_{0} \dot{\bar{\theta}}+\text { rest } . \tag{97}
\end{equation*}
$$

Therefore, the term multiplying $\dot{\bar{\theta}}$ in $\mathcal{L}_{2}$, denoted by $\mathcal{L}_{2}^{\theta}$, reads

$$
\begin{align*}
\mathcal{L}_{2}= & \mathcal{L}_{2}^{\theta} \dot{\bar{\theta}}+\text { rest } \\
= & {\left[-\frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{c}_{0} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)+\bar{G}_{1}^{\perp} \frac{\bar{p}_{\perp}}{B_{0}}+\bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right)\right.}  \tag{98}\\
& \left.+\bar{p}_{\|} \frac{\bar{p}_{\perp}^{2}}{B_{0}^{2}} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}-\frac{1}{3} \frac{\bar{p}_{\perp}^{3}}{B_{0}^{3}} \boldsymbol{c}_{0} \cdot\left[\boldsymbol{a}_{0} \times\left(\boldsymbol{a}_{0} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]\right] \dot{\bar{\theta}}+\text { rest } .
\end{align*}
$$

From $\boldsymbol{b}_{0} \times \boldsymbol{c}_{0}=\boldsymbol{a}_{0}$ it follows that the first and the third term in the above bracket cancel each other. In (86) the generator $\bar{G}_{1}^{\perp}$ is chosen to remove the Hamiltonian and the generator $\bar{G}_{1}^{\|}$is chosen to remove the parallel
component of the vector multiplying $\dot{\overline{\boldsymbol{X}}}$. This leads to

$$
\begin{align*}
\bar{G}_{1}^{\perp} & =-\frac{\bar{p}_{\|}}{\bar{p}_{\perp}} \bar{G}_{1}^{\|}=\frac{\bar{p}_{\|}}{\bar{p}_{\perp}}\left(\boldsymbol{\mathcal { Q }}_{1} \cdot \boldsymbol{b}_{0}-\frac{\bar{p}_{\perp}^{2}}{2 B_{0}} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right)  \tag{99}\\
& =\frac{\bar{p}_{\|}}{B_{0}}\left[\frac{1}{2} \bar{p}_{\perp} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}+\bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{c}_{0}+\bar{p}_{\perp}\left(\boldsymbol{\nabla} \times \boldsymbol{c}_{0}\right) \cdot \boldsymbol{c}_{0}-\frac{\bar{p}_{\perp}}{2} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right] \\
& =\frac{\bar{p}_{\|}}{B_{0}}\left[-\frac{1}{2} \bar{p}_{\perp} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}+\bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{c}_{0}+\frac{\bar{p}_{\perp}}{2} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right],
\end{align*}
$$

Here, we used $\left(\boldsymbol{\nabla} \times \boldsymbol{c}_{0}\right) \cdot \boldsymbol{c}_{0}=\boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}-\boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}$ to arrive at the last line. The first correction to the magnetic moment in (94) can now be given as the $\bar{\theta}$-average of $\mathcal{L}_{2}^{\theta}$, defined in (98),

$$
\begin{equation*}
-\frac{\delta H_{1}}{B_{0}}:=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \bar{\theta} \boldsymbol{\mathcal { L }}_{2}^{\theta}=\frac{\bar{p}_{\|}}{B_{0}} \frac{\bar{p}_{\perp}^{2}}{2 B_{0}}\left(-\frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}+\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right) . \tag{100}
\end{equation*}
$$

Here we used that $\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}=\boldsymbol{\nabla} \boldsymbol{e}_{2} \cdot \boldsymbol{e}_{1}$ is independent of $\bar{\theta}$ as well as the average

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \bar{\theta}\left(\boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}\right)=-\frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0} \tag{101}
\end{equation*}
$$

The result (100) shows that it is straightforward to compute the inverse of (94) up to first order:

$$
\begin{equation*}
\frac{\bar{p}_{\perp}^{2}}{2}=\bar{\mu} B_{0}\left[1+\varepsilon\left(\frac{\bar{p}_{\|}}{B_{0}} \frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}-\frac{\bar{p}_{\|}}{B_{0}}\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right)\right]+O\left(\varepsilon^{2}\right) \tag{102}
\end{equation*}
$$

which corresponds to the term $\delta H_{1}$ stated in (43). Let us move on to the computation of $\delta H_{2}$. The term multiplying $\dot{\bar{\theta}}$ in $\mathcal{L}_{3}$, denoted by $\mathcal{L}_{3}^{\theta}$, reads

$$
\begin{aligned}
\mathcal{L}_{3}= & \boldsymbol{L}_{3}^{\theta} \dot{\bar{\theta}}+\text { rest } \\
= & \left(-\frac{1}{2} \frac{\partial \overline{\boldsymbol{\rho}}_{2}}{\partial \bar{\theta}} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)+\bar{G}_{2}^{\perp} \frac{\bar{p}_{\perp}}{B_{0}}-\frac{\partial \bar{G}_{1}^{\|}}{\partial \bar{\theta}}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{b}_{0}\right)-\frac{\partial \bar{G}_{1}^{\perp}}{\partial \bar{\theta}}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{c}_{0}\right)+\bar{G}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right)\right. \\
& +\sum_{m=1}^{2} \bar{G}_{2-m}^{\|} \overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \frac{\partial \overline{\boldsymbol{\rho}}_{m}}{\partial \bar{\theta}}+\bar{p}_{\|}\left[\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0}+\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{b}_{0}\right] \cdot \frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{c}_{0} \\
& +\bar{p}_{\perp}\left[\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{c}_{0}-\bar{G}_{1}^{\Theta} \boldsymbol{a}_{0}\right] \cdot \frac{\partial \overline{\boldsymbol{\rho}}_{2}}{\partial \bar{\theta}}+\bar{p}_{\perp}\left[\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}+\bar{G}_{1}^{\Theta} \frac{\partial}{\partial \bar{\theta}}\right)^{2} \boldsymbol{c}_{0}\right] \cdot \frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{c}_{0} \\
& \left.-\frac{\bar{p}}{B_{0}} \boldsymbol{c}_{0} \cdot\left[\overline{\boldsymbol{\rho}}_{2} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]-\frac{3}{24} \frac{\bar{p}}{B_{0}} \boldsymbol{c}_{0} \cdot\left[\overline{\boldsymbol{\rho}}_{1} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{B}_{0}\right]\right) \dot{\bar{\theta}}+\text { rest } .
\end{aligned}
$$

The second correction to the magnetic moment in (94) can then be defined as the $\bar{\theta}$-average of $\mathcal{L}_{3}^{\theta}$ :

$$
\begin{equation*}
-\frac{\delta H_{2}}{B_{0}}:=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \bar{\theta} \mathcal{L}_{3}^{\theta} \tag{103}
\end{equation*}
$$

The second-order Hamiltonian correction due to the magnetic curvature would then be the sum of (103) and the second-order term in (118). We see however that such a term is too cumbersome for practical applications, for instance the implementation in a gyrokinetic simulation code.

We now proceed in the same fashion for the electrons and substitute the polynomial transform (38), with $p_{\perp}$ from (80) instead of $\mu$, in the electron starting Lagrangian (37b). We omit again the species index for more readability. After expanding in Taylor series the static background field, we find the following Lagrangians at
the respective orders $\varepsilon^{n}, 0 \leqslant n \leqslant 3$ (and $\bar{L}_{-1}=-\boldsymbol{A}_{0} \cdot \dot{\overline{\boldsymbol{X}}}$ ):

$$
\begin{align*}
& \bar{L}_{0}=\left(\sqrt{\varepsilon} \bar{p}_{\|} \boldsymbol{b}_{0}+\sqrt{\varepsilon} \bar{p}_{\perp} \boldsymbol{c}_{0}+\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}\right) \cdot \dot{\overline{\boldsymbol{X}}}-\frac{\bar{p}_{\|}^{2}}{2}-\frac{\bar{p}_{\perp}^{2}}{2}  \tag{104}\\
& \bar{L}_{1}=\left(\sqrt{\varepsilon} \bar{G}_{1}^{\|} \boldsymbol{b}_{0}+\sqrt{\varepsilon} \bar{G}_{1}^{\perp} \boldsymbol{c}_{0}+\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{1}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{1}^{\|}-\bar{p}_{\perp} \bar{G}_{1}^{\perp}+\mathcal{L}_{1}  \tag{105}\\
& \bar{L}_{2}=\left(\sqrt{\varepsilon} \bar{G}_{2}^{\|} \boldsymbol{b}_{0}+\sqrt{\varepsilon} \bar{G}_{2}^{\perp} \boldsymbol{c}_{0}+\overline{\boldsymbol{\rho}}_{3} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{2}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{2}^{\|}-\bar{p}_{\perp} \bar{G}_{2}^{\perp}+\mathcal{L}_{2}  \tag{106}\\
& \bar{L}_{3}=\left(\sqrt{\varepsilon} \bar{G}_{3}^{\|} \boldsymbol{b}_{0}+\sqrt{\varepsilon} \bar{G}_{3}^{\perp} \boldsymbol{c}_{0}+\overline{\boldsymbol{\rho}}_{4} \times \boldsymbol{B}_{0}+\boldsymbol{\mathcal { Q }}_{3}\right) \cdot \dot{\boldsymbol{X}}-\bar{p}_{\|} \bar{G}_{3}^{\|}-\bar{p}_{\perp} \bar{G}_{3}^{\perp}+\mathcal{L}_{3} \tag{107}
\end{align*}
$$

We will then need the explicit expressions of

$$
\mathcal{Q}_{1}:=-\frac{1}{2}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{B}_{0}\right) \times \overline{\boldsymbol{\rho}}_{1}-\sqrt{\varepsilon} \bar{p}_{\|} \overline{\boldsymbol{\rho}}_{1} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right)-\sqrt{\varepsilon} \bar{p}_{\perp} \overline{\boldsymbol{\rho}}_{1} \times\left(\boldsymbol{\nabla} \times \boldsymbol{c}_{0}\right)-\sqrt{\varepsilon} \bar{p}_{\perp} \bar{G}_{1}^{\Theta} \boldsymbol{a}_{0}
$$

and of

$$
\begin{aligned}
\mathcal{L}_{1}:= & \frac{1}{2} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left(\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}\right)-\sqrt{\varepsilon} \dot{\bar{p}}_{\|}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}\right)-\sqrt{\varepsilon} \dot{\bar{p}}_{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}\right)+\sqrt{\varepsilon} \dot{\bar{\theta}} \bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{a}_{0}\right), \\
\mathcal{L}_{2}:= & \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)-\sqrt{\varepsilon} \dot{\bar{G}}_{1}^{\|}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}\right)-\sqrt{\varepsilon} \dot{\bar{G}}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{c}_{0}\right)+\sqrt{\varepsilon} \dot{\bar{\theta}} \bar{G}_{1}^{\perp}\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{a}_{0}\right) \\
& -\sqrt{\varepsilon} \dot{\bar{p}}_{\|}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{b}_{0}\right)-\sqrt{\varepsilon} \dot{\bar{p}}_{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{c}_{0}\right)+\sqrt{\varepsilon} \dot{\bar{\theta}} \bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right) \\
& +\sqrt{\varepsilon} \bar{p}_{\|} \overline{\boldsymbol{\rho}}_{1} \cdot \nabla \boldsymbol{b}_{0} \cdot \dot{\overline{\boldsymbol{\rho}}}_{1}+\sqrt{\varepsilon} \bar{p}_{\perp} \overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla} \boldsymbol{c}_{0} \cdot \dot{\overline{\boldsymbol{\rho}}}_{1}-\sqrt{\varepsilon} \bar{p}_{\perp} \bar{G}_{1}^{\Theta} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot \boldsymbol{a}_{0} \\
& +\frac{1}{3} \dot{\overline{\boldsymbol{\rho}}}_{1} \cdot\left[\overline{\boldsymbol{\rho}}_{1} \times\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]+\frac{1}{6} \overline{\boldsymbol{\rho}}_{1} \times\left[\left(\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{B}_{0}\right] \cdot \dot{\overline{\boldsymbol{X}}}
\end{aligned}
$$

Here again we choose the electron generators $\boldsymbol{\rho}_{n}, \bar{G}_{n}^{\|}, \bar{G}_{n}^{\perp}$, and $\bar{G}_{n}^{\Theta}$ in order to cancel as many terms as possible from the Lagrangians. In $\bar{L}_{0}$ from (104) we require $-\sqrt{\varepsilon} \bar{p}_{\perp} \boldsymbol{c}_{0}=\overline{\boldsymbol{\rho}}_{1} \times \boldsymbol{B}_{0}$, which can be obtained by setting

$$
\begin{equation*}
\bar{\rho}_{1}:=-\sqrt{\varepsilon} \frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{a}_{0} \tag{108}
\end{equation*}
$$

The factor $\sqrt{\varepsilon}$ reflects the smallness of the electron Larmor radius compared to the ion Larmor radius. The methodology is now the same as for ions. However, the ordering and the signs of the various terms has been regarded with care. In (105) we require $\overline{\boldsymbol{\rho}}_{1} \cdot \boldsymbol{b}_{0}=0$ and

$$
\begin{align*}
\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0} & =-\sqrt{\varepsilon} \bar{G}_{1}^{\perp} \boldsymbol{c}_{0}-\boldsymbol{\mathcal { Q }}_{1}+\varepsilon \frac{p_{\perp}^{2}}{2 B_{0}} \nabla \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}  \tag{109}\\
G_{1}^{\|} & =-\frac{1}{\sqrt{\varepsilon}} \boldsymbol{\mathcal { Q }}_{1} \cdot \boldsymbol{b}_{0}-\sqrt{\varepsilon} \frac{\bar{p}_{\perp}^{2}}{2 B_{0}} \boldsymbol{b}_{0} \cdot \nabla \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}  \tag{110}\\
\bar{G}_{1}^{\perp} & =-\frac{\bar{p}_{\|}}{\bar{p}_{\perp}} \bar{G}_{1}^{\|} \tag{111}
\end{align*}
$$

which, in contrast to (91), leads to

$$
\begin{equation*}
\bar{L}_{1} \sim-\varepsilon \frac{p_{\perp}^{2}}{2 B_{0}} \dot{\bar{\theta}} \tag{112}
\end{equation*}
$$

At second order in (106) we can remove all terms except the gyro-averages of the terms multiplying $\dot{\bar{\theta}}$, which in analogy to (98) are given by

$$
\begin{align*}
\mathcal{L}_{2}= & \mathcal{L}_{2}^{\theta} \dot{\bar{\theta}}+\text { rest } \\
= & {\left[-\sqrt{\varepsilon} \frac{\bar{p}_{\perp}}{B_{0}} \boldsymbol{c}_{0} \cdot\left(\overline{\boldsymbol{\rho}}_{2} \times \boldsymbol{B}_{0}\right)-\varepsilon \bar{G}_{1}^{\perp} \frac{\bar{p}_{\perp}}{B_{0}}+\sqrt{\varepsilon} \bar{p}_{\perp}\left(\overline{\boldsymbol{\rho}}_{2} \cdot \boldsymbol{a}_{0}\right)\right.}  \tag{113}\\
& \left.+\varepsilon^{3 / 2} \bar{p}_{\|} \frac{\bar{p}_{\perp}^{2}}{B_{0}^{2}} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}-\frac{\varepsilon^{3 / 2}}{3} \frac{\bar{p}_{\perp}^{3}}{B_{0}^{3}} \boldsymbol{c}_{0} \cdot\left[\boldsymbol{a}_{0} \times\left(\boldsymbol{a}_{0} \cdot \boldsymbol{\nabla}\right) \boldsymbol{B}_{0}\right]\right] \dot{\bar{\theta}}+\text { rest } .
\end{align*}
$$

Again due to $\boldsymbol{b}_{0} \times \boldsymbol{c}_{0}=\boldsymbol{a}_{0}$, the first and the third term in the above bracket cancel each other. Moreover, from (110) and (111) we have

$$
\begin{align*}
\bar{G}_{1}^{\perp} & =-\frac{\bar{p}_{\|}}{\bar{p}_{\perp}} \bar{G}_{1}^{\|}=\frac{\bar{p}_{\|}}{\bar{p}_{\perp}}\left(\frac{1}{\sqrt{\varepsilon}} \boldsymbol{\mathcal { Q }}_{1} \cdot \boldsymbol{b}_{0}+\sqrt{\varepsilon} \frac{\bar{p}_{\perp}^{2}}{2 B_{0}} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \\
& =\sqrt{\varepsilon} \frac{\bar{p}_{\|}}{B_{0}}\left[-\frac{1}{2} \bar{p}_{\perp} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}-\bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{c}_{0}-\bar{p}_{\perp}\left(\boldsymbol{\nabla} \times \boldsymbol{c}_{0}\right) \cdot \boldsymbol{c}_{0}+\frac{\bar{p}_{\perp}}{2} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right]  \tag{114}\\
& =\sqrt{\varepsilon} \frac{\bar{p}_{\|}}{B_{0}}\left[\frac{1}{2} \bar{p}_{\perp} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{c}_{0}-\bar{p}_{\|}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{c}_{0}-\frac{\bar{p}_{\perp}}{2} \boldsymbol{b}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right]
\end{align*}
$$

Since the gyro-average of $\mathcal{L}_{2}^{\theta}$ is the only term that cannot be removed via the generators, we obtain

$$
\begin{equation*}
\bar{L}_{2} \sim\left\langle\mathcal{L}_{2}^{\theta}\right\rangle \dot{\bar{\theta}} \tag{115}
\end{equation*}
$$

where

$$
\left\langle\mathcal{L}_{2}^{\theta}\right\rangle=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \bar{\theta} \boldsymbol{\mathcal { L }}_{2}^{\theta}=\varepsilon^{\frac{3}{2}} \frac{\bar{p}_{\|}}{B_{0}} \frac{\bar{p}_{\perp}^{2}}{2 B_{0}}\left[-\frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}+\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right]=-\varepsilon^{\frac{3}{2}} \frac{\delta H_{1}}{B_{0}}
$$

This is the same result as for the ions in (100) but with a factor $\varepsilon^{\frac{3}{2}}$. Carrying out the analogous computations for $\bar{L}_{3}$, we find that generators can be chosen such that $\bar{L}_{3}=O(\varepsilon)$. Therefore, from (112) and (115) we obtain the electron Lagrangian

$$
\begin{equation*}
L_{\mathrm{e}} \sim\left(\sqrt{\varepsilon} \bar{p}_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}-\varepsilon^{2}\left(\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}+\varepsilon^{\frac{3}{2}} \frac{\delta H_{1}}{B_{0}}\right) \dot{\bar{\theta}}-\frac{\bar{p}_{\|}^{2}}{2}-\frac{\bar{p}_{\perp}^{2}}{2}+O\left(\varepsilon^{4}\right) \tag{116}
\end{equation*}
$$

In contrast to (142) and (94) for the ions, for the electrons we have

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}+\varepsilon^{\frac{3}{2}} \frac{\delta H_{1}}{B_{0}}\right)=O\left(\varepsilon^{2}\right) \quad \Rightarrow \quad \bar{\mu}:=\frac{\bar{p}_{\perp}^{2}}{2 B_{0}}+\varepsilon^{\frac{3}{2}} \frac{\delta H_{1}}{B_{0}} \tag{117}
\end{equation*}
$$

which is however less accurate than for ions, namely only up to order $O\left(\varepsilon^{2}\right)$. The electron guiding-center Hamiltonian is obtained from the inverse of the mapping $p_{\perp} \mapsto \bar{\mu}$ :

$$
\begin{align*}
& \bar{\mu} B_{0}= \\
&=\frac{\bar{p}_{\perp}^{2}}{2}\left[1+\varepsilon^{3 / 2}\left(\frac{\bar{p}_{\|}}{B_{0}} \frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}-\frac{\bar{p}_{\|}}{B_{0}}\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right)\right]  \tag{118}\\
& \Leftrightarrow \quad \frac{\bar{p}_{\perp}^{2}}{2}=\bar{\mu} B_{0}\left[1-\varepsilon^{3 / 2}\left(\frac{\bar{p}_{\|}}{B_{0}} \frac{1}{2}\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right) \cdot \boldsymbol{b}_{0}-\frac{\bar{p}_{\|}}{B_{0}}\left(\boldsymbol{\nabla} \boldsymbol{a}_{0} \cdot \boldsymbol{c}_{0}\right) \cdot \boldsymbol{b}_{0}\right)\right]+O\left(\varepsilon^{2}\right),
\end{align*}
$$

which proves the result (40b) for the electron guiding-center Hamiltonian.

## C Gyrocenter transformation: proofs

We collect here the proofs of Propositions 1-3 for ions and of Propositions 4-6 for electrons. The species index is mostly omitted for more readability.

Proof 1 (ion polynomial transform) Before proving Proposition 1, we remark that the Lagrangians $L_{n}$ in (49), for $n=1,2,3$, read

$$
\begin{equation*}
L_{n}:=\gamma_{n}^{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}}+\gamma_{n}^{\|} \dot{P}_{\|}+\gamma_{n}^{\mu} \dot{\hat{\mu}}+\gamma_{n}^{\Theta} \dot{\Theta}-H_{n} \tag{119}
\end{equation*}
$$

The components $\boldsymbol{\gamma}_{n}^{\boldsymbol{X}}$, for $n=1,2,3$, are given by

$$
\begin{align*}
\gamma_{1}^{\boldsymbol{X}} & =G_{1}^{\|} \boldsymbol{b}_{0}-\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}+\nabla_{\perp} S_{2}  \tag{120a}\\
\gamma_{2}^{\boldsymbol{X}} & =G_{2}^{\|} \boldsymbol{b}_{0}-\boldsymbol{\rho}_{3} \times \boldsymbol{B}_{0}+\nabla_{\perp} S_{3}+\nabla_{\|} S_{2} \boldsymbol{b}_{0}+\boldsymbol{F}_{\perp}+\delta \boldsymbol{\gamma}_{2}^{\boldsymbol{X}}  \tag{120b}\\
\boldsymbol{\gamma}_{3}^{\boldsymbol{X}} & =G_{3}^{\|} \boldsymbol{b}_{0}-\boldsymbol{\rho}_{4} \times \boldsymbol{B}_{0}+\nabla_{\|} S_{3} \boldsymbol{b}_{0}+F_{\|} \boldsymbol{b}_{0}+\delta \boldsymbol{\gamma}_{3}^{\boldsymbol{X}} \tag{120c}
\end{align*}
$$

where the terms $\boldsymbol{F}_{\perp}$ and $F_{\|}$are defined as

$$
\begin{gather*}
\boldsymbol{F}_{\perp}=G_{1}^{\|} \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{2} \cdot \boldsymbol{b}_{0}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \nabla_{\perp} \boldsymbol{\rho}_{2}+G_{1}^{\mu} \nabla_{\perp} G_{1}^{\Theta}  \tag{121a}\\
F_{\|}=G_{1}^{\|} \nabla_{\|} \boldsymbol{\rho}_{2} \cdot \boldsymbol{b}_{0}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \nabla_{\|} \boldsymbol{\rho}_{2}+G_{1}^{\mu} \nabla_{\|} G_{1}^{\Theta} \tag{121b}
\end{gather*}
$$

and the terms $\delta \gamma_{n}^{\boldsymbol{X}}$, for $n=2,3$, contain terms related to the curvature of the background magnetic field and are given by

$$
\begin{align*}
\delta \gamma_{2}^{\boldsymbol{X}}= & -P_{\|} \boldsymbol{\rho}_{2} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right),  \tag{122a}\\
\delta \gamma_{3}^{\boldsymbol{X}}= & -P_{\|} \boldsymbol{\rho}_{3} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right)+P_{\|}\left(\boldsymbol{\rho}_{3} \cdot \boldsymbol{\nabla}\right) \boldsymbol{b}_{0} \\
& +G_{1}^{\|}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{b}_{0}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla} \boldsymbol{B}_{0}\right) \times \boldsymbol{\rho}_{2} . \tag{122b}
\end{align*}
$$

The components $\gamma_{n}^{\|}$, for $n=1,2,3$, are given by

$$
\begin{align*}
& \gamma_{1}^{\|}=0  \tag{123a}\\
& \gamma_{2}^{\|}=-\boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{2}+\frac{\partial S_{2}}{\partial P_{\|}}  \tag{123b}\\
& \gamma_{3}^{\|}=-\boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{3}+\frac{\partial S_{3}}{\partial P_{\|}}+\left(G_{1}^{\|} \boldsymbol{b}_{0}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial P_{\|}}+G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial P_{\|}} . \tag{123c}
\end{align*}
$$

The components $\gamma_{n}^{\mu}$, for $n=1,2,3$, are given by

$$
\begin{align*}
& \gamma_{1}^{\mu}=0  \tag{124a}\\
& \gamma_{2}^{\mu}=-G_{1}^{\Theta}+\frac{\partial S_{2}}{\partial \hat{\mu}}  \tag{124b}\\
& \gamma_{3}^{\mu}=-G_{2}^{\Theta}+\frac{\partial S_{3}}{\partial \hat{\mu}}+\left(G_{1}^{\|} \boldsymbol{b}_{0}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \hat{\mu}}+G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial \hat{\mu}} \tag{124c}
\end{align*}
$$

The components $\gamma_{n}^{\Theta}$, for $n=1,2,3$, are given by

$$
\begin{align*}
\gamma_{1}^{\Theta} & =\hat{\mu}  \tag{125a}\\
\gamma_{2}^{\Theta} & =G_{1}^{\mu}+\frac{\partial S_{2}}{\partial \Theta}  \tag{125b}\\
\gamma_{3}^{\Theta} & =G_{2}^{\mu}+\frac{\partial S_{3}}{\partial \Theta}+\left(G_{1}^{\|} \boldsymbol{b}_{0}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}+G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial \Theta} . \tag{125c}
\end{align*}
$$

Finally, the Hamiltonians $H_{n}$, for $n=1,2,3$, are given by

$$
\begin{align*}
H_{1}= & G_{1}^{\mu} B_{0}+P_{\|} G_{1}^{\|}+\Psi_{1}+\delta H_{1}  \tag{126a}\\
H_{2}= & G_{2}^{\mu} B_{0}+\hat{\mu} \boldsymbol{\rho}_{2} \cdot \nabla B_{0}+P_{\|} G_{2}^{\|}+\frac{1}{2}\left(G_{1}^{\|}\right)^{2}+\left[\left(\overline{\boldsymbol{\rho}}_{2}+\boldsymbol{\rho}_{2}\right) \cdot \boldsymbol{\nabla}\right. \\
& \left.+\left(\bar{G}_{1}^{\|}+G_{1}^{\|}\right) \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+\left(\bar{G}_{1}^{\mu}+G_{1}^{\mu}\right) \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}+\left(\bar{G}_{1}^{\Theta}+G_{1}^{\Theta}\right) \frac{\mathrm{d}}{\mathrm{~d} \Theta}\right] \Psi_{1}  \tag{126b}\\
& +\left(G_{1}^{\|} \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+G_{1}^{\mu} \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}\right) \delta H_{1}+\delta H_{2}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2}+\frac{\partial S_{2}}{\partial t}, \\
H_{3}= & G_{3}^{\mu} B_{0}+\delta H_{3}, \tag{126c}
\end{align*}
$$

where the generalized potential reads

$$
\begin{align*}
\Psi_{\mathrm{i}}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\boldsymbol{Z}), P_{\|}, \widehat{\mu}, \Theta\right)= & \phi_{1}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\boldsymbol{Z})\right)-P_{\|} A_{1 \|}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\boldsymbol{Z})\right)  \tag{127}\\
& -\sqrt{2 \widehat{\mu} B_{0}(\boldsymbol{X})} \boldsymbol{c}_{0}(\boldsymbol{X}, \Theta) \cdot \boldsymbol{A}_{1 \perp}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\boldsymbol{Z})\right)
\end{align*}
$$

with $A_{1 \|}:=\boldsymbol{A}_{1} \cdot \boldsymbol{b}_{0}$ and $\boldsymbol{A}_{1 \perp}:=\boldsymbol{b}_{0} \times \boldsymbol{A}_{1} \times \boldsymbol{b}_{0}$. Moreover, $\delta H_{1}$ is the curvature term introduced in (43), we do not write the explicit expression of $\delta H_{2}$, and the explicit expression of $\delta H_{3}$ is not relevant for our order of accuracy.

The results stated in Proposition 1 are obtained by substituting the gyrocenter coordinate transformation (45) into the guiding-center single-particle Lagrangian (40a) and computing its Taylor expansion in powers of $\varepsilon$ up to order $\varepsilon^{3}$ (starting from $1 / \varepsilon$ ). We first denote by $\Gamma$ the symplectic part of the guiding-center Lagrangian (40a):

$$
\begin{equation*}
\Gamma:=\left(\bar{p}_{\|} \boldsymbol{b}_{0}+\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\boldsymbol{X}}+\varepsilon \bar{\mu} \dot{\bar{\theta}} \tag{128}
\end{equation*}
$$

The coefficients $\Gamma_{n}$, for $n=-1,0,1,2,3$, of the Taylor expansion of $\Gamma$ read

$$
\begin{align*}
\Gamma_{-1}= & A_{0} \cdot \dot{\boldsymbol{X}},  \tag{129a}\\
\Gamma_{0}= & P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}},  \tag{129b}\\
\Gamma_{1}= & G_{1}^{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}+\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}+\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{2}+\widehat{\mu} \dot{\Theta},  \tag{129c}\\
\Gamma_{2}= & {\left[G_{2}^{\|}+P_{\|}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right)\right] \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}+\left(\boldsymbol{\rho}_{3} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}+\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{3} }  \tag{129d}\\
& +P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}}_{2}+G_{1}^{\mu} \dot{\Theta}+\widehat{\mu} \dot{G}_{1}^{\Theta} \\
\Gamma_{3}= & {\left[G_{3}^{\|}+P_{\|}\left(\boldsymbol{\rho}_{3} \cdot \boldsymbol{\nabla}\right)+G_{1}^{\|}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right)\right] \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}+\left(\boldsymbol{\rho}_{4} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}} } \\
& +\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{4}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}+\left[G_{1}^{\|} \boldsymbol{b}_{0}+\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0}\right] \cdot \dot{\boldsymbol{\rho}}_{2}  \tag{129e}\\
& +P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}}_{3}+G_{2}^{\mu} \dot{\Theta}+\widehat{\mu} \dot{G}_{2}^{\Theta}+G_{1}^{\mu} \dot{G}_{1}^{\Theta} .
\end{align*}
$$

The results for the symplectic part follow by using the equivalence relations (for generic generators $\boldsymbol{\rho}$ and $G^{\Theta}$ )

$$
\begin{align*}
(\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}+\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}} & \sim-\left(\boldsymbol{\rho} \times \boldsymbol{B}_{0}\right) \cdot \dot{\boldsymbol{X}},  \tag{130a}\\
P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}} & \sim-P_{\|}\left(\boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}\right) \cdot \dot{\boldsymbol{X}}-\left(\boldsymbol{b}_{0} \cdot \boldsymbol{\rho}\right) \dot{P}_{\|}  \tag{130b}\\
(\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}+\frac{1}{2}(\boldsymbol{\rho} \cdot \boldsymbol{\nabla})^{2} \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}} & \sim-\frac{1}{2}\left(\boldsymbol{\rho} \times \boldsymbol{B}_{0}\right) \cdot \dot{\boldsymbol{\rho}} \\
& +\frac{1}{2}\left[\left(\boldsymbol{\rho} \cdot \nabla \boldsymbol{B}_{0}\right) \times \boldsymbol{\rho}\right] \cdot \dot{\boldsymbol{X}},  \tag{130c}\\
\hat{\mu} \dot{G}^{\Theta} & \sim-G^{\Theta} \dot{\hat{\mu}} \tag{130d}
\end{align*}
$$

together with the vector identity $(\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{b}_{0}-\boldsymbol{\nabla} \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}=-\boldsymbol{\rho} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right)$, and by adding the terms corresponding to the total differentials $\dot{S}_{2}$ and $\dot{S}_{3}$. For the Hamiltonian part, we note that the fluctuating potential $\Psi_{1}$ must be first transformed to the guiding-center coordinates $\overline{\boldsymbol{Z}}$ and then to the preliminary gyrocenter coordinates $\boldsymbol{Z}$. We first recall that in physical coordinates we have

$$
\begin{equation*}
\Psi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}, p_{\|}, \mu, \theta\right)=\phi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right)-p_{\|} A_{1 \|}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right)-\sqrt{2 \mu B_{0}(\boldsymbol{x})} \boldsymbol{c}_{0}(\boldsymbol{x}, \theta) \cdot \boldsymbol{A}_{1 \perp}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right) \tag{131}
\end{equation*}
$$

We shall first substitute the guiding-center coordinate transformation (38). Using that $\phi_{1}$ and $\boldsymbol{A}_{1}$ in (131) are normalized functions with size and variations of order $O(1)$ in the limit $\varepsilon \rightarrow 0$, we can safely expand in a Taylor series around $\left(\overline{\boldsymbol{X}} / \varepsilon+\overline{\boldsymbol{\rho}}_{1}, \bar{p}_{\|}, \bar{\mu}, \bar{\theta}\right)$ and obtain

$$
\begin{align*}
\Psi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}, p_{\|}, \mu, \theta\right)= & \Psi_{1}\left(t, \frac{\overline{\boldsymbol{X}}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\bar{\mu}, \bar{\theta}), \bar{p}_{\|}, \bar{\mu}, \bar{\theta}\right)  \tag{132}\\
& +\varepsilon\left(\overline{\boldsymbol{\rho}}_{2} \cdot \nabla+\bar{G}_{1}^{\|} \frac{\mathrm{d}}{\mathrm{~d} \bar{p}_{\|}}+\bar{G}_{1}^{\mu} \frac{\mathrm{d}}{\mathrm{~d} \bar{\mu}}+\bar{G}_{1}^{\Theta} \frac{\mathrm{d}}{\mathrm{~d} \bar{\theta}}\right) \Psi_{1}+O\left(\varepsilon^{2}\right)
\end{align*}
$$

The same reasoning applies when we substitute the gyrocenter coordinate transformation (45) into (132), yielding

$$
\begin{align*}
\Psi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}, p_{\|}, \mu, \theta\right)= & \Psi_{1}\left(t, \frac{\boldsymbol{X}}{\varepsilon}+\overline{\boldsymbol{\rho}}_{1}(\hat{\mu}, \Theta), P_{\|}, \hat{\mu}, \Theta\right) \\
+ & \varepsilon\left[\left(\overline{\boldsymbol{\rho}}_{2}+\boldsymbol{\rho}_{2}\right) \cdot \boldsymbol{\nabla}+\left(\bar{G}_{1}^{\|}+G_{1}^{\mathrm{U}}\right) \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+\left(\bar{G}_{1}^{\mu}+G_{1}^{\mathrm{\mu}}\right) \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}\right.  \tag{133}\\
& \left.+\left(\bar{G}_{1}^{\Theta}+G_{1}^{\Theta}\right) \frac{\mathrm{d}}{\mathrm{~d} \Theta}\right] \Psi_{1}+O\left(\varepsilon^{2}\right)
\end{align*}
$$

This explains the second line of the second-order Hamiltonian $H_{2}$ in (126b) as well as the expression for $\Psi_{1}$ given in (127).

Proof 2 (preliminary ion gyrocenter Lagrangian) The components $\boldsymbol{\gamma}_{n}^{\boldsymbol{X}}$, for $n=1,2,3$, in (120a)-(120c) vanish if and only if we set

$$
\begin{array}{ll}
G_{1}^{\|}=0, & \boldsymbol{\rho}_{2 \perp}=\frac{\boldsymbol{b}_{0}}{B_{0}} \times \nabla_{\perp} S_{2} \\
G_{2}^{\|}=-\nabla_{\|} S_{2}-\delta \gamma_{2 \|}^{\boldsymbol{X}}, & \boldsymbol{\rho}_{3 \perp}=\frac{\boldsymbol{b}_{0}}{B_{0}} \times\left(\nabla_{\perp} S_{3}+\boldsymbol{F}_{\perp}+\delta \gamma_{2 \perp}^{\boldsymbol{X}}\right) \\
G_{3}^{\|}=-\nabla_{\|} S_{3}-F_{\|}-\delta \gamma_{3 \|}^{\boldsymbol{X}}, & \boldsymbol{\rho}_{4 \perp}=\frac{\boldsymbol{b}_{0}}{B_{0}} \times \delta \gamma_{3 \perp}^{\boldsymbol{X}} \tag{134c}
\end{array}
$$

The components $\gamma_{n}^{\|}$, for $n=2,3$, in (123b)-(123c) vanish if and only if we set

$$
\begin{align*}
& \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{2}=\frac{\partial S_{2}}{\partial P_{\|}}  \tag{135a}\\
& \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{3}=\frac{\partial S_{3}}{\partial P_{\|}}+\left(G_{1}^{\|} \boldsymbol{b}_{0}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial P_{\|}}+G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial P_{\|}} \tag{135b}
\end{align*}
$$

The components $\gamma_{n}^{\mu}$, for $n=2,3$, in (124b)-(124c) vanish if and only if we set

$$
\begin{align*}
& G_{1}^{\Theta}=\frac{\partial S_{2}}{\partial \widehat{\mu}}  \tag{136a}\\
& G_{2}^{\Theta}=\frac{\partial S_{3}}{\partial \widehat{\mu}}+\left(G_{1}^{\|} \boldsymbol{b}_{0}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \widehat{\mu}}+G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial \widehat{\mu}} \tag{136b}
\end{align*}
$$

The Hamiltonians $H_{n}$, for $n=1,2,3$, in (126a)-(126c) vanish if and only if we set

$$
\begin{align*}
G_{1}^{\mu}= & -\frac{1}{B_{0}}\left(P_{\|} G_{1}^{\|}+\Psi_{i}+\delta H_{1}\right)  \tag{137a}\\
G_{2}^{\mu}= & -\frac{1}{B_{0}}\left\{\widehat{\mu} \rho_{2} \cdot \nabla B_{0}+P_{\|} G_{2}^{\|}+\frac{1}{2}\left(G_{1}^{\|}\right)^{2}+\left[\left(\overline{\boldsymbol{\rho}}_{2}+\boldsymbol{\rho}_{2}\right) \cdot \nabla\right.\right. \\
& \left.+\left(\bar{G}_{1}^{\|}+G_{1}^{\|}\right) \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+\left(\bar{G}_{1}^{\mu}+G_{1}^{\mu}\right) \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}+\left(\bar{G}_{1}^{\theta}+G_{1}^{\Theta}\right) \frac{\mathrm{d}}{\mathrm{~d} \Theta}\right] \Psi_{1}  \tag{137b}\\
& \left.+\left(G_{1}^{\|} \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+G_{1}^{\mu} \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}\right) \delta H_{1}+\delta H_{2}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2}+\frac{\partial S_{2}}{\partial t}\right\} \\
G_{3}^{\mu}= & -\frac{\delta H_{3}}{B_{0}} \tag{137c}
\end{align*}
$$

The only degrees of freedom left are the arbitrary scalar functions $S_{2}$ and $S_{3}$. Since these functions must be $2 \pi$-periodic in the gyro-angle $\Theta$, we cannot eliminate $\gamma_{2}^{\Theta}$ and $\gamma_{3}^{\Theta}$, given by (125b)-(125c), entirely from the Lagrangian. The reason is that the equation $\partial S_{n} / \partial \Theta=g$, for a given function $g$, has $2 \pi$-periodic solutions $S_{n}$ if and only if $\langle g\rangle=0$, where $\langle g\rangle$ denotes the gyro-average of $g$ defined in (52). Denoting by $\widetilde{g}:=g-\langle g\rangle$ the fluctuating part of $g$ (with zero gyro-average), the dependence on the gyro-angle $\Theta$ can be removed from (125b)-(125c) by setting, for $n=2,3$,

$$
\begin{equation*}
\gamma_{n}^{\Theta}=\left\langle\gamma_{n}^{\Theta}\right\rangle \tag{138}
\end{equation*}
$$

or, equivalently, by requiring that $S_{n}$, for $n=2,3$, satisfy the differential equations

$$
\begin{align*}
& \frac{\partial S_{2}}{\partial \Theta}=-\widetilde{G_{1}^{\mu}}  \tag{139a}\\
& \frac{\partial S_{3}}{\partial \Theta}=-\widetilde{G_{2}^{\mu}}-\widetilde{G_{1}^{\|} \boldsymbol{b}_{0} \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}}+\frac{1}{2} \boldsymbol{\rho}_{2} \times \overline{\boldsymbol{B}_{0}} \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}-\widetilde{G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial \Theta}} . \tag{139b}
\end{align*}
$$

The solutions of (139) read (with arbitrary lower bound $\Theta_{0}$ of integration)

$$
\begin{align*}
S_{2}(\Theta)= & S_{2}\left(\Theta_{0}\right)-\int_{\Theta_{0}}^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{G_{1}^{\mu}}  \tag{140a}\\
S_{3}(\Theta)= & S_{3}\left(\Theta_{0}\right) \\
& -\int_{\Theta_{0}}^{\Theta} \mathrm{d} \Theta^{\prime}\left(\widetilde{G_{2}^{\mu}}+\widetilde{G_{1}^{\|} \boldsymbol{b}_{0} \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}-\frac{1}{2} \boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}} \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}+\widetilde{G_{1}^{\mu} \frac{\partial G_{1}^{\Theta}}{\partial \Theta}}\right) . \tag{140b}
\end{align*}
$$

Proof 3 (ion gyrocenter Lagrangian) Before proving Proposition 3, we remark that the $\delta G_{2}$ in (58) reads

$$
\begin{equation*}
\delta G_{2}:=\left\langle\left(\bar{\rho}_{2} \cdot \nabla+\bar{G}_{1}^{\|} \frac{\mathrm{d}}{\mathrm{~d} P_{\|}}+\bar{G}_{1}^{\mu} \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}+\bar{G}_{1}^{\Theta} \frac{\mathrm{d}}{\mathrm{~d} \Theta}\right) \Psi_{1}\right\rangle . \tag{141}
\end{equation*}
$$

The term $\delta G_{2}$ is linear in the fluctuating potential $\Psi_{1}$ and couples to higher-order generators of the guidingcenter transformation.

In order to prove our results, we first note that computing the Euler-Lagrange equation $\partial L_{\mathrm{i}} / \partial \Theta-\frac{\mathrm{d}}{\mathrm{d} t} \partial L_{\mathrm{i}} / \partial \dot{\Theta}=$ 0 for the preliminary gyrocenter single-particle Lagrangian (51), and noting that $\bar{O}\left(\varepsilon^{3}\right)$ is independent of $\Theta$, we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\hat{\mu}+\varepsilon \gamma_{2}^{\Theta}+\varepsilon^{2} \gamma_{3}^{\Theta}\right)=O\left(\varepsilon^{3}\right) \tag{142}
\end{equation*}
$$

Hence, the gyrocenter magnetic moment $\mu$ defined in (53) is conserved with second-order accuracy in $\varepsilon$. Let us now compute the terms $\left\langle\gamma_{2}^{\Theta}\right\rangle$ and $\left\langle\gamma_{3}^{\Theta}\right\rangle$ that define the transformation $\widehat{\mu} \mapsto \mu$ in (53). From (125b) and (137a) we have

$$
\begin{equation*}
\left\langle\gamma_{2}^{\Theta}\right\rangle=\left\langle G_{1}^{\mu}\right\rangle=-\frac{\left\langle\Psi_{1}\right\rangle}{B_{0}}-\frac{\delta H_{1}}{B_{0}}, \tag{143}
\end{equation*}
$$

where we used the result $G_{1}^{\|}=0$ from (134a) and the fact that the geometric term $\delta H_{1}$ does not depend on the gyro-angle. Moreover, from (125c) we have

$$
\begin{equation*}
\left\langle\gamma_{3}^{\Theta}\right\rangle=\left\langle G_{2}^{\mu}\right\rangle-\frac{1}{2}\left\langle\frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta} \cdot\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right)\right\rangle+\left\langle\frac{\partial G_{1}^{\Theta}}{\partial \Theta} G_{1}^{\mu}\right\rangle . \tag{144}
\end{equation*}
$$

The gyro-average of $G_{2}^{\mu}$ can be computed from (137b), obtaining

$$
\begin{align*}
\left\langle G_{2}^{\mu}\right\rangle= & \left.-\frac{1}{B_{0}}\left\langle P_{\|} G_{2}^{\|}+\left(\boldsymbol{\rho}_{2} \cdot \nabla+G_{1}^{\mu} \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}}+G_{1}^{\Theta} \frac{\mathrm{d}}{\mathrm{~d} \Theta}\right) \Psi_{1}+\left.\frac{1}{2}\langle | \boldsymbol{A}_{1}\right|^{2}\right\rangle\right\rangle  \tag{145}\\
& -\frac{\left\langle G_{1}^{\mu}\right\rangle}{B_{0}} \frac{\mathrm{~d}}{\mathrm{~d} \hat{\mu}} \delta H_{1}-\frac{\delta H_{2}}{B_{0}}-\frac{\delta G_{2}}{B_{0}} \tag{146}
\end{align*}
$$

where we used $\left\langle\boldsymbol{\rho}_{2}\right\rangle=0$. In order to compute the second term on the right-hand side of (144), the generator $\rho_{2}$ is determined by the function $S_{2}$ via (134a) and (135a). Omitting the arbitrary lower bound of integration $\Theta_{0}$, we have

$$
\begin{equation*}
S_{2}=-\int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{G_{1}^{\mu}}=\int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{\Psi_{1}}}{B_{0}} \tag{147}
\end{equation*}
$$

and, recalling the functional form of $\Psi_{1}$ in (131), we obtain

$$
\begin{equation*}
\frac{\partial S_{2}}{\partial P_{\|}}=-\int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{A_{1 \|}}}{B_{0}} \tag{148}
\end{equation*}
$$

Therefore, the generator $\rho_{2}$ reads

$$
\begin{equation*}
\boldsymbol{\rho}_{2}=\frac{\boldsymbol{b}_{0}}{B_{0}} \times \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{\Psi_{1}}}{B_{0}}-\boldsymbol{b}_{0} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{A_{1 \|}}}{B_{0}}, \tag{149}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
-\frac{1}{2}\left\langle\frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta} \cdot\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right)\right\rangle=-\frac{1}{2}\left\langle\left(\frac{\boldsymbol{b}_{0}}{B_{0}} \times \nabla_{\perp} \frac{\widetilde{\Psi_{1}}}{B_{0}}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{\Psi_{1}}}{B_{0}}\right\rangle \tag{150}
\end{equation*}
$$

In order to compute the last term in (144), we get from (136a)

$$
\begin{equation*}
G_{1}^{\Theta}=\frac{\partial S_{2}}{\partial \hat{\mu}}=\int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\mathrm{d}}{\mathrm{~d} \hat{\mu}} \frac{\widetilde{\Psi_{1}}}{B_{0}} \tag{151}
\end{equation*}
$$

yielding

$$
\begin{equation*}
\left\langle\frac{\partial G_{1}^{\Theta}}{\partial \Theta} G_{1}^{\mu}\right\rangle=-\frac{1}{2 B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \widehat{\mu}}\left\langle\widetilde{\Psi}_{1}^{2}\right\rangle \tag{152}
\end{equation*}
$$

In order to get an explicit expression for $\left\langle\gamma_{3}^{\Theta}\right\rangle$, we need to compute the right-hand side of (146) term by term. Using (134b) and the fact that $\left\langle S_{2}\right\rangle=0$ and $\left\langle\delta \gamma_{2 \|}^{\boldsymbol{X}}\right\rangle=0$ from (122), we find $\left\langle G_{2}^{\|}\right\rangle=0$. The second to fourth terms in (146) read

$$
\begin{align*}
-\frac{1}{B_{0}}\left\langle\boldsymbol{\rho}_{2} \cdot \nabla \Psi_{1}\right\rangle= & \left\langle\left(\frac{b_{0}}{B_{0}^{2}} \times \nabla_{\perp} \widetilde{\Psi_{1}}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{\Psi_{1}}}{B_{0}}\right\rangle \\
& +\frac{1}{B_{0}}\left\langle\nabla_{\|} \widetilde{\Psi_{1}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \frac{\widetilde{A_{1 \|}}}{B_{0}}\right\rangle  \tag{153a}\\
-\frac{1}{B_{0}}\left\langle G_{1}^{\mathrm{\mu}} \frac{\mathrm{~d}}{\mathrm{~d} \hat{\mu}} \Psi_{\mathrm{i}}\right\rangle= & \frac{1}{2 B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \widehat{\mu}}\left\langle\Psi_{\mathrm{i}}^{2}\right\rangle+\frac{1}{B_{0}^{2}} \delta H_{1} \frac{\mathrm{~d}}{\mathrm{~d} \hat{\mu}}\left\langle\Psi_{\mathrm{i}}\right\rangle,  \tag{153b}\\
-\frac{1}{B_{0}}\left\langle G_{1}^{\Theta} \frac{\mathrm{d}}{\mathrm{~d} \Theta} \Psi_{\mathrm{i}}\right\rangle= & \frac{1}{2 B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \hat{\mu}}\left\langle\widetilde{\Psi}_{\mathrm{i}}^{2}\right\rangle \tag{153c}
\end{align*}
$$

where we integrated by parts in order to obtain the last equality. Substitution of (150) and (152)-(153b) into (144) yields

$$
\begin{align*}
\left\langle\gamma_{3}^{\Theta}\right\rangle= & \left.-\left.\frac{1}{2 B_{0}}\langle | \boldsymbol{A}_{1}\right|^{2}\right\rangle+\frac{1}{2 B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \widehat{\mu}}\left\langle\Psi_{1}^{2}\right\rangle \\
& +\frac{1}{2}\left\langle\left(\frac{\boldsymbol{b}_{0}}{B_{0}^{3}} \times \nabla_{\perp} \widetilde{\Psi_{1}}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{\Psi_{1}}\right\rangle \\
& +\frac{1}{B_{0}^{2}}\left\langle\nabla_{\|} \widetilde{\Psi_{1}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle-\frac{\delta \mathcal{G}_{2}}{B_{0}}  \tag{154}\\
& +\frac{1}{B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \widehat{\mu}}\left(\left\langle\Psi_{1}\right\rangle \delta H_{1}+\frac{1}{2} \delta H_{1}^{2}\right)-\frac{\delta H_{2}}{B_{0}}
\end{align*}
$$

The generalized magnetic moment $\mu$ can now be computed explicitly as a function of the fluctuating potentials from (143) and (154). It remains to identify the gyrocenter Hamiltonian. For this purpose, we need to invert (53) and substitute the result into $H_{0}$. From

$$
\begin{equation*}
\mu=\widehat{\mu}+\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle(\widehat{\mu})+\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle(\widehat{\mu}), \tag{155}
\end{equation*}
$$

we obtain

$$
\begin{align*}
\hat{\mu} & =\mu-\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle\left(\mu-\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle(\mu)\right)-\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle(\mu)+\bar{O}\left(\varepsilon^{3}\right) \\
& =\mu-\varepsilon\left\langle\gamma_{2}^{\Theta}\right\rangle(\mu)+\varepsilon^{2}\left\langle\gamma_{2}^{\Theta}\right\rangle(\mu) \frac{\mathrm{d}}{\mathrm{~d} \mu}\left\langle\gamma_{2}^{\Theta}\right\rangle(\mu)-\varepsilon^{2}\left\langle\gamma_{3}^{\Theta}\right\rangle(\mu)+\bar{O}\left(\varepsilon^{3}\right) \tag{156}
\end{align*}
$$

At order $O\left(\varepsilon^{2}\right)$ we compute the difference

$$
\begin{equation*}
\left\langle\gamma_{3}^{\Theta}\right\rangle-\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} \mu}\left\langle\gamma_{2}^{\Theta}\right\rangle^{2}=\left\langle\gamma_{3}^{\Theta}\right\rangle-\frac{1}{2 B_{0}^{2}} \frac{\mathrm{~d}}{\mathrm{~d} \mu}\left(\left\langle\Psi_{1}\right\rangle+\delta H_{1}\right)^{2}, \tag{157}
\end{equation*}
$$

and obtain, using the explicit formulas (143) and (154) for $\left\langle\gamma_{2}^{\Theta}\right\rangle$ and $\left\langle\gamma_{3}^{\Theta}\right\rangle$,

$$
\begin{align*}
\widehat{\mu}= & \mu+\frac{\varepsilon}{B_{0}}\left(\left\langle\Psi_{1}\right\rangle+\delta H_{1}\right)+\frac{\varepsilon^{2}}{B_{0}}\left(\left.\frac{1}{2}\langle | \boldsymbol{A}_{1}\right|^{2}\right\rangle-\frac{1}{2 B_{0}} \frac{\mathrm{~d}}{\mathrm{~d} \hat{\mu}}\left\langle{\widetilde{\Psi_{1}}}^{2}\right\rangle \\
& -\frac{1}{2 B_{0}^{2}}\left\langle\left(\boldsymbol{b}_{0} \times \nabla_{\perp} \widetilde{\Psi_{1}}\right) \cdot \nabla_{\perp} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{\Psi_{1}}\right\rangle  \tag{158}\\
& \left.-\frac{1}{B_{0}}\left\langle\nabla_{\|} \widetilde{\Psi_{1}} \int^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{A_{1 \|}}\right\rangle+\delta \mathcal{G}_{2}+\delta H_{2}\right)+\bar{O}\left(\varepsilon^{3}\right),
\end{align*}
$$

where we used the fact that $\langle g\rangle^{2}-\left\langle g^{2}\right\rangle=-\left\langle\widetilde{g}^{2}\right\rangle$, for a given function $g(\Theta)$. Substituting this into the Hamiltonian $H_{0}$ completes the proof.

Proof 4 (electron polynomial transform) Before proving Proposition 4, we remark that the Lagrangians $L_{\frac{n}{2}}$, for $n=2, \ldots, 6$, read

$$
\begin{equation*}
L_{\frac{n}{2}}=\gamma_{\frac{n}{2}}^{X} \cdot \dot{\boldsymbol{X}}+\gamma_{\frac{n}{2}}^{\|} \dot{P}_{\|}+\gamma_{\frac{n}{2}}^{\mu} \dot{\hat{\mu}}+\gamma_{\frac{n}{2}}^{\Theta} \dot{\Theta}-H_{\frac{n}{2}} . \tag{159}
\end{equation*}
$$

The components $\gamma_{\frac{n}{2}}^{X}$, for $n=2, \ldots, 6$, are given by

$$
\begin{align*}
& \boldsymbol{\gamma}_{1}^{\boldsymbol{X}}=\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0},  \tag{160a}\\
& \boldsymbol{\gamma}_{\frac{3}{2}}^{\boldsymbol{X}}:=G_{1}^{\|} \boldsymbol{b}_{0}+\boldsymbol{\rho}_{\frac{5}{2}} \times \boldsymbol{B}_{0}+\nabla_{\perp} S_{\frac{5}{2}},  \tag{160b}\\
& \boldsymbol{\gamma}_{2}^{\boldsymbol{X}}=\boldsymbol{\rho}_{3} \times \boldsymbol{B}_{0}+\nabla_{\perp} S_{3}+\boldsymbol{F}_{2 \perp},  \tag{160c}\\
& \boldsymbol{\gamma}_{\frac{5}{2}}^{\boldsymbol{X}}:=G_{2} \| \boldsymbol{b}_{0}+\nabla_{\|} S_{\frac{5}{2}} \boldsymbol{b}_{0}+\boldsymbol{F}_{\frac{5}{2} \perp}+\delta \boldsymbol{\gamma}_{\frac{5}{2}}^{\boldsymbol{X}},  \tag{160d}\\
& \boldsymbol{\gamma}_{3}^{\boldsymbol{X}}=\nabla_{\|} S_{3} \boldsymbol{b}_{0}+\boldsymbol{F}_{3 \perp}+F_{3 \|} \boldsymbol{b}_{0}+\delta \boldsymbol{\gamma}_{3}^{\boldsymbol{X}}, \tag{160e}
\end{align*}
$$

where the terms $\boldsymbol{F}_{\frac{n}{2} \perp}$, for $n=4,5,6$, and $F_{3 \|}$ are defined as

$$
\begin{align*}
\boldsymbol{F}_{2 \perp}= & \frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{2}-G_{\frac{1}{2}}^{\mu} \boldsymbol{\nabla}_{\perp} G_{\frac{1}{2}}^{\Theta}  \tag{161a}\\
\boldsymbol{F}_{\frac{5}{2} \perp}= & G_{1}^{\|} \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{2} \cdot \boldsymbol{b}_{0}-G_{\frac{1}{2}}^{\mu} \boldsymbol{\nabla}_{\perp} G_{1}^{\Theta}-G_{1}^{\mu} \boldsymbol{\nabla}_{\perp} G_{\frac{1}{2}}^{\Theta}  \tag{161b}\\
& -\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{\frac{5}{2}}-\left(\boldsymbol{\rho}_{\frac{5}{2}} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{2}, \\
\boldsymbol{F}_{3 \perp}= & G_{1}^{\|} \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{\frac{5}{2}} \cdot \boldsymbol{b}_{0}+\frac{1}{2}\left(\boldsymbol{\rho}_{\frac{5}{2}} \times \boldsymbol{B}_{0}\right) \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{\frac{5}{2}}-G_{1}^{\mu} \boldsymbol{\nabla}_{\perp} G_{1}^{\Theta}  \tag{161c}\\
& -\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{3}-\left(\boldsymbol{\rho}_{3} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \boldsymbol{\nabla}_{\perp} \boldsymbol{\rho}_{2}, \\
F_{3 \|}= & \frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \nabla_{\|} \boldsymbol{\rho}_{2}-G_{\frac{1}{2}}^{\mu} \nabla_{\|} G_{\frac{1}{2}}^{\Theta}, \tag{161d}
\end{align*}
$$

and the terms $\delta \gamma_{\frac{n}{2}}^{X}$, for $n=5,6$, contain terms related to the curvature of the background magnetic field and are given by

$$
\begin{align*}
& \delta \gamma_{\frac{5}{2}}^{\boldsymbol{X}}=-P_{\|} \boldsymbol{\rho}_{2} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right),  \tag{162a}\\
& \delta \gamma_{3}^{\boldsymbol{X}}=-P_{\|} \boldsymbol{\rho}_{\frac{5}{2}} \times\left(\boldsymbol{\nabla} \times \boldsymbol{b}_{0}\right)-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla} \boldsymbol{B}_{0}\right) \times \boldsymbol{\rho}_{2} . \tag{162b}
\end{align*}
$$

The components $\gamma_{\frac{n}{2}}^{\|}$, for $n=2, \ldots, 6$, are given by

$$
\begin{align*}
& \gamma_{1}^{\|}=\gamma_{\frac{3}{2}}^{\|}=\gamma_{2}^{\|}=0  \tag{163a}\\
& \gamma_{\frac{5}{2}}^{\|}=-\boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{2}+\frac{\partial S_{\frac{5}{2}}}{\partial P_{\|}},  \tag{163b}\\
& \gamma_{3}^{\|}=-\boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{\frac{5}{2}}+\frac{\partial S_{3}}{\partial P_{\|}}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial P_{\|}}-G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial P_{\|}} . \tag{163c}
\end{align*}
$$

The components $\gamma_{\frac{n}{2}}^{\mu}$, for $n=2, \ldots, 6$, are given by

$$
\begin{align*}
& \gamma_{1}^{\mu}=\gamma_{\frac{3}{2}}^{\mu}=\gamma_{2}^{\mu}=0  \tag{164a}\\
& \gamma_{\frac{5}{2}}^{\mu}=G_{\frac{1}{2}}^{\Theta}+\frac{\partial S_{\frac{5}{2}}}{\partial \hat{\mu}},  \tag{164b}\\
& \gamma_{3}^{\mu}=G_{1}^{\Theta}+\frac{\partial S_{3}}{\partial \widehat{\mu}}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \hat{\mu}}-G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \hat{\mu}} . \tag{164c}
\end{align*}
$$

The components $\gamma_{\frac{n}{2}}^{\Theta}$, for $n=2, \ldots, 6$, are given by

$$
\begin{align*}
& \gamma_{1}^{\Theta}=\gamma_{\frac{3}{2}}^{\Theta}=0  \tag{165a}\\
& \gamma_{2}^{\Theta}=-\hat{\mu}  \tag{165b}\\
& \gamma_{\frac{5}{2}}^{\Theta}=-G_{\frac{1}{2}}^{\mu}+\frac{\partial S_{\frac{5}{2}}}{\partial \Theta},  \tag{165c}\\
& \gamma_{3}^{\Theta}=-G_{1}^{\mu}+\frac{\partial S_{3}}{\partial \Theta}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}-G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \Theta} . \tag{165d}
\end{align*}
$$

Finally, the Hamiltonian $H_{1}$ is given by

$$
\begin{align*}
H_{1}= & G_{1}^{\mu} B_{0}+P_{\|} G_{1}^{\|}-\Phi_{1}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2} \\
& +\sqrt{\frac{B_{0}}{2 \hat{\mu}}} G_{\frac{1}{2}}^{\mu} \boldsymbol{c}_{0} \cdot \boldsymbol{A}_{1 \perp}-\sqrt{2 \hat{\mu} B_{0}} G_{\frac{1}{2}}^{\Theta} \boldsymbol{a}_{0} \cdot \boldsymbol{A}_{1 \perp}  \tag{166}\\
& -\sqrt{\frac{2 \hat{\mu}}{B_{0}}} \boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{A}_{1} \cdot\left(P_{\|} \boldsymbol{b}_{0}+\sqrt{2 \hat{\mu} B_{0}} \boldsymbol{c}_{0}\right)+O(\sqrt{\varepsilon})
\end{align*}
$$

where $\boldsymbol{A}_{1 \perp}:=\boldsymbol{b}_{0} \times \boldsymbol{A}_{1} \times \boldsymbol{b}_{0}$, as before, and the fluctuating potentials $\Phi_{1}$ and $\boldsymbol{A}_{1}$ are evaluated at the position $\boldsymbol{X} / \varepsilon$. We remark that the higher-order Hamiltonians $H_{2}$ and $H_{3}$ are not relevant for our order of accuracy.

The result is obtained by substituting the gyrocenter coordinate transformation (45) into the guiding-center single-particle Lagrangian (40b) and computing its Taylor expansion in powers of $\sqrt{\varepsilon}$ up to order $\varepsilon^{3}$ (starting from $1 / \varepsilon)$. We denote again by $\Gamma$ the symplectic part of the guiding-center Lagrangian (40b),

$$
\begin{equation*}
\Gamma:=\left(\sqrt{\varepsilon} \bar{p}_{\|} \boldsymbol{b}_{0}-\frac{\boldsymbol{A}_{0}}{\varepsilon}\right) \cdot \dot{\overline{\boldsymbol{X}}}-\varepsilon^{2} \overline{\bar{\mu}} \dot{\bar{\theta}} . \tag{167}
\end{equation*}
$$

The coefficients $\Gamma_{\frac{n}{2}}$, for $n=-2, \ldots, 6$, of the Taylor expansion of $\Gamma$ read

$$
\begin{align*}
\Gamma_{-1}= & -A_{0} \cdot \dot{\boldsymbol{X}},  \tag{168a}\\
\Gamma_{-\frac{1}{2}}= & \Gamma_{0}=0  \tag{168b}\\
\Gamma_{\frac{1}{2}}= & P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}},  \tag{168c}\\
\Gamma_{1}= & -\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}-\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{2},  \tag{168d}\\
\Gamma_{\frac{3}{2}}= & G_{1}^{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}-\left(\boldsymbol{\rho}_{\frac{5}{2}} \cdot \nabla\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}-\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{\frac{5}{2}},  \tag{168e}\\
\Gamma_{2}= & -\left(\boldsymbol{\rho}_{3} \cdot \nabla\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}-\boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{3}-\widehat{\mu} \dot{\Theta},  \tag{168f}\\
\Gamma_{\frac{5}{2}}= & \left(G_{2}^{\|}+P_{\|}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right)\right) \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}+P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}}_{2}-G_{\frac{1}{2}}^{\mu} \dot{\Theta}-\hat{\mu} \dot{G}_{\frac{1}{2}}^{\Theta}  \tag{168g}\\
\Gamma_{3}= & P_{\|}\left(\boldsymbol{\rho}_{\frac{5}{2}} \cdot \nabla\right) \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{X}}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \cdot \boldsymbol{\nabla}\right)^{2} \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{X}}-\left(\boldsymbol{\rho}_{2} \cdot \nabla\right) \boldsymbol{A}_{0} \cdot \dot{\boldsymbol{\rho}}_{2}  \tag{168h}\\
& +P_{\|} \boldsymbol{b}_{0} \cdot \dot{\boldsymbol{\rho}}_{\frac{5}{2}}-G_{1}^{\mu} \dot{\Theta}-\widehat{\mu} \dot{G}_{1}^{\Theta}-G_{\frac{1}{2}}^{\mu} \dot{G}_{\frac{1}{2}}^{\Theta} .
\end{align*}
$$

The results for the symplectic part follow by using the same equivalence relations used for ions and by adding the terms corresponding to the total differentials $\dot{S}_{\frac{5}{2}}$ and $\dot{S}_{3}$. For the Hamiltonian part, we note that, because of the guiding-center generator (obtained in Appendix B) (108),

$$
\begin{equation*}
\bar{\rho}_{1 \mathrm{e}}=-\sqrt{\varepsilon} \sqrt{\frac{2 \bar{\mu}}{B_{0}}} \boldsymbol{a}_{0} \tag{169}
\end{equation*}
$$

the fluctuating potentials $\phi_{1}$ and $\boldsymbol{A}_{1}$ can be expanded in a Taylor series around $\boldsymbol{X} / \varepsilon$, yielding

$$
\begin{align*}
\Phi_{1}\left(t, \frac{\boldsymbol{x}}{\varepsilon}\right) & =\Phi_{1}\left(t, \frac{\overline{\boldsymbol{X}}}{\varepsilon}-\sqrt{\varepsilon} \sqrt{\frac{2 \bar{\mu}}{B_{0}}} \boldsymbol{a}_{0}(\overline{\boldsymbol{X}}, \bar{\theta})+O(\varepsilon)\right) \\
& =\Phi_{1}\left(t, \frac{\boldsymbol{X}}{\varepsilon}\right)-\sqrt{\varepsilon} \sqrt{\frac{2 \hat{\mu}}{B_{0}}} \boldsymbol{a}_{0}(\boldsymbol{X}, \Theta) \cdot \nabla \Phi_{1}\left(t, \frac{\boldsymbol{X}}{\varepsilon}\right)+O(\varepsilon) \tag{170}
\end{align*}
$$

and the same for $\boldsymbol{A}_{1}$. This explains the particular form of the Hamiltonian (166).
Proof 5 (preliminary electron gyrocenter Lagrangian) The components $\gamma_{\frac{n}{2}}^{X}$, for $n=2, \ldots, 6$, in (160) vanish if and only if we set

$$
\begin{align*}
& G_{1}^{\|}:=0  \tag{171a}\\
& G_{2}^{\|}:=-\nabla_{\|} S_{\frac{5}{2}}-\delta \gamma_{\frac{5}{2} \|}^{\boldsymbol{X}}-\sqrt{\varepsilon}\left(\nabla_{\|} S_{3}+F_{3 \|}+\delta \gamma_{3 \|}^{\boldsymbol{X}}\right), \tag{171b}
\end{align*}
$$

as well as

$$
\begin{align*}
& \rho_{2 \perp}:=0  \tag{172a}\\
& \rho_{\frac{5}{2} \perp}:=-\frac{\boldsymbol{b}_{0}}{B_{0}} \times \nabla_{\perp} S_{\frac{5}{2}}  \tag{172b}\\
& \rho_{3 \perp}:=-\frac{\boldsymbol{b}_{0}}{B_{0}} \times\left[\nabla_{\perp} S_{3}+\boldsymbol{F}_{2 \perp}+\sqrt{\varepsilon}\left(\boldsymbol{F}_{\frac{5}{2} \perp}+\delta \boldsymbol{\gamma}_{\frac{5}{2} \perp}^{\boldsymbol{X}}\right)+\varepsilon\left(\boldsymbol{F}_{3 \perp}+\delta \boldsymbol{\gamma}_{3 \perp}^{\boldsymbol{X}}\right)\right] . \tag{172c}
\end{align*}
$$

The components $\gamma_{\frac{n}{2}}^{\|}$, for $n=5,6$, in (163) vanish if and only if we set

$$
\begin{align*}
& \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{2}=\frac{\partial S_{\frac{5}{2}}}{\partial P_{\|}}  \tag{173a}\\
& \boldsymbol{b}_{0} \cdot \boldsymbol{\rho}_{\frac{5}{2}}=\frac{\partial S_{3}}{\partial P_{\|}}+\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial P_{\|}}-G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial P_{\|}} \tag{173b}
\end{align*}
$$

The components $\gamma_{\frac{n}{2}}^{\mu}$, for $n=5,6$, in (164) vanish if and only if we set

$$
\begin{align*}
G_{\frac{1}{2}}^{\Theta} & =-\frac{\partial S_{\frac{5}{2}}}{\partial \hat{\mu}}  \tag{174a}\\
G_{1}^{\Theta} & =-\frac{\partial S_{3}}{\partial \hat{\mu}}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \hat{\mu}}+G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \hat{\mu}} \tag{174b}
\end{align*}
$$

The Hamiltonians $H_{\frac{n}{2}}$, for $n=1,2$, in (166) vanish if and only if we set

$$
\begin{align*}
G_{\frac{1}{2}}^{\mu}= & -\frac{1}{B_{0}}\left(P_{\|} A_{1 \|}+\sqrt{2 \widehat{\mu} B_{0}} \boldsymbol{c}_{0} \cdot \boldsymbol{A}_{1 \perp}\right)  \tag{175a}\\
G_{1}^{\mu}= & -\frac{1}{B_{0}}\left\{P_{\|} G_{1}^{\|}-\Phi_{1}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2}\right. \\
& +\sqrt{\frac{B_{0}}{2 \widehat{\mu}}} G_{\frac{1}{2}}^{\mu} \boldsymbol{c}_{0} \cdot \boldsymbol{A}_{1 \perp}-\sqrt{2 \widehat{\mu} B_{0}} G_{\frac{1}{2}}^{\Theta} \boldsymbol{a}_{0} \cdot \boldsymbol{A}_{1 \perp}  \tag{175b}\\
& \left.-\sqrt{\frac{2 \hat{\mu}}{B_{0}}} \boldsymbol{a}_{0} \cdot \nabla \boldsymbol{A}_{1} \cdot\left(P_{\|} \boldsymbol{b}_{0}+\sqrt{2 \widehat{\mu} B_{0}} \boldsymbol{c}_{0}\right)+O(\sqrt{\varepsilon})\right\} .
\end{align*}
$$

The only degrees of freedom left are the arbitrary functions $S_{\frac{5}{2}}$ and $S_{3}$. As for ions, the dependence on the gyro-angle $\Theta$ can be removed from (165) by setting, for $n=5,6$,

$$
\begin{equation*}
\gamma_{\frac{n}{2}}^{\Theta}=\left\langle\gamma_{\frac{n}{2}}^{\Theta}\right\rangle \tag{176}
\end{equation*}
$$

or, equivalently, by requiring that $S_{\frac{n}{2}}$, for $n=5,6$, satisfy the differential equations

$$
\begin{align*}
& \frac{\partial S_{\frac{5}{2}}}{\partial \Theta}=\widetilde{G_{\frac{1}{2}}^{\mu}}  \tag{177a}\\
& \frac{\partial S_{3}}{\partial \Theta}=\widetilde{G_{1}^{\mu}}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}+\widetilde{G_{\frac{1}{2}}^{\mu}} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \Theta} \tag{177b}
\end{align*}
$$

The solutions of (177) read (with arbitrary lower bound $\Theta_{0}$ of integration)

$$
\begin{align*}
& S_{\frac{5}{2}}(\Theta)=S_{\frac{5}{2}}\left(\Theta_{0}\right)+\int_{\Theta_{0}}^{\Theta} \mathrm{d} \Theta^{\prime} \widetilde{G_{\frac{1}{2}}^{\mu}}  \tag{178a}\\
& S_{3}(\Theta)=S_{3}\left(\Theta_{0}\right)+\int_{\Theta_{0}}^{\Theta} \mathrm{d} \Theta^{\prime}\left(\widetilde{G_{1}^{\mu}}-\frac{1}{2}\left(\boldsymbol{\rho}_{2} \times \boldsymbol{B}_{0}\right) \cdot \frac{\partial \boldsymbol{\rho}_{2}}{\partial \Theta}+\widetilde{\left.G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \Theta}\right)}\right. \tag{178b}
\end{align*}
$$

Proof 6 (electron gyrocenter Lagrangian) Using that $\left\langle\gamma_{2}^{\Theta}\right\rangle=-\hat{\mu}-\sqrt{\varepsilon}\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle$, the Euler-Lagrange equation $\partial L_{\mathrm{e}} / \partial \Theta-\frac{\mathrm{d}}{\mathrm{d} t} \partial L_{\mathrm{e}} / \partial \dot{\Theta}=0$ for the preliminary gyrocenter single-particle Lagrangian (62) yields

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\widehat{\mu}+\sqrt{\varepsilon}\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle-\varepsilon\left\langle\gamma_{3}^{\Theta}\right\rangle\right)=O\left(\varepsilon^{\frac{3}{2}}\right) \tag{179}
\end{equation*}
$$

Hence, the gyrocenter magnetic moment $\mu$ defined in (63) is conserved with first-order accuracy in $\varepsilon$. Let us now compute the terms $\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle$ and $\left\langle\gamma_{3}^{\Theta}\right\rangle$ that define the transformation $\widehat{\mu} \mapsto \mu$ in (63). From (175) we have

$$
\begin{align*}
\left\langle G_{\frac{1}{2}}^{\mu}\right\rangle= & -\frac{1}{B_{0}} P_{\|} A_{1 \|}  \tag{180}\\
\left\langle G_{1}^{\mu}\right\rangle= & -\frac{1}{B_{0}}\left(-\phi_{1}+\frac{1}{2}\left|\boldsymbol{A}_{1}\right|^{2}-\left\langle\left(\boldsymbol{c}_{0} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}\right\rangle\right.  \tag{181}\\
& \left.-\left\langle\left(\boldsymbol{a}_{0} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}\right\rangle-2 \widehat{\mu}\left\langle\boldsymbol{a}_{0} \cdot \boldsymbol{\nabla} \boldsymbol{A}_{1} \cdot \boldsymbol{c}_{0}\right\rangle\right)+\bar{O}(\sqrt{\varepsilon}) . \tag{182}
\end{align*}
$$

Moreover, from (174) we have

$$
\begin{equation*}
\left\langle G_{\frac{1}{2}}^{\mu} \frac{\partial G_{\frac{1}{2}}^{\Theta}}{\partial \Theta}\right\rangle=-\frac{1}{B_{0}}\left\langle\left(\boldsymbol{c}_{0} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}\right\rangle \tag{183}
\end{equation*}
$$

By computing the additional terms

$$
\begin{equation*}
\left\langle\left(\boldsymbol{a}_{0} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}\right\rangle=\frac{1}{2}\left[\left(\boldsymbol{e}_{1} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}+\left(\boldsymbol{e}_{2} \cdot \boldsymbol{A}_{1 \perp}\right)^{2}\right]=\frac{1}{2}\left|\boldsymbol{A}_{1 \perp}\right|^{2}, \tag{184}
\end{equation*}
$$

and

$$
\begin{align*}
\left\langle a_{0} \cdot \nabla A_{1} \cdot c_{0}\right\rangle & =\frac{1}{2}\left(e_{2} \cdot \nabla A_{1} \cdot e_{1}-e_{1} \cdot \nabla A_{1} \cdot e_{2}\right) \\
& =\frac{1}{2} e_{2} \cdot\left[e_{1} \times\left(\nabla \times A_{1}\right)\right]=-\frac{1}{2}\left(\nabla \times \boldsymbol{A}_{1}\right) \cdot \boldsymbol{b}_{0}, \tag{185}
\end{align*}
$$

we finally obtain

$$
\begin{equation*}
\left\langle\gamma_{3}^{\Theta}\right\rangle=\frac{1}{B_{0}}\left[-\phi_{1}+\frac{1}{2} A_{1 \|}^{2}+\widehat{\mu}\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{1}\right) \cdot \boldsymbol{b}_{0}\right]+\bar{O}(\sqrt{\varepsilon}), \tag{186}
\end{equation*}
$$

where we used the results $G_{1}^{\|}=0$ and $\boldsymbol{\rho}_{2 \perp}=0$ from (171) and (172), respectively. It remains to identify the gyrocenter Hamiltonian. For this purpose, we need to invert (63) and substitute the result into $H_{0}$. Thanks to (180), the inversion is trivial and yields

$$
\begin{equation*}
\widehat{\mu}=\mu+\sqrt{\varepsilon} \frac{1}{B_{0}} P_{\|} A_{1 \|}+\varepsilon\left\langle\gamma_{3}^{\Theta}\right\rangle(\mu)+\bar{O}\left(\varepsilon^{3 / 2}\right) . \tag{187}
\end{equation*}
$$

Substituting this into the Hamiltonian $H_{0}$ completes the proof.

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