Basic Queueing Theory

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This book is dedicated to my wife without whom this work could have been finished much earlier.

- If anything can go wrong, it will.
- If you change queues, the one you have left will start to move faster than the one you are in now.
- Your queue always goes the slowest.
- Whatever queue you join, no matter how short it looks, it will always take the longest for you to get served.

(Murphy' Laws on reliability and queueing)

Contents

Preface

Modern information technologies require innovations that are based on modeling, analyzing, designing and finally implementing new systems. The whole developing process assumes a well-organized team work of experts including engineers, computer scientists, mathematicians, physicist just to mention some of them. Modern infocommunication networks are one of the most complex systems where the reliability and efficiency of the components play a very important role. For the better understanding of the dynamic behavior of the involved processes one have to deal with constructions of mathematical models which describe the stochastic service of randomly arriving requests. Queueing Theory is one of the most commonly used mathematical tool for the performance evaluation of such systems.

The aim of the book is to present the basic methods, approaches in a Markovian level for the analysis of not too complicated systems. The main purpose is to understand how models could be constructed and how to analyze them. It is assumed the reader has been exposed to a first course in probability theory, however in the text I give a refresher and state the most important principles I need later on. My intention is to show what is behind the formulas and how we can derive formulas. It is also essential to know which kind of questions are reasonable and then how to answer them.

My experience and advice are that if it is possible solve the same problem in different ways and compare the results. Sometimes very nice closed-form, analytic solutions are obtained but the main problem is that we cannot compute them for higher values of the involved variables. In this case the algorithmic or asymptotic approaches could be very useful. My intention is to find the balance between the mathematical and practitioner needs. I feel that a satisfactory middle ground has been established for understanding and applying these tools to practical systems. I hope that after understanding this book the reader will be able to create his owns formulas if needed.

It should be underlined that most of the models are based on the assumption that the involved random variables are exponentially distributed and independent of each other. We must confess that this assumption is artificial since in practice the exponential distribution is not so frequent. However, the mathematical models based on the memoryless property of the exponential distribution greatly simplifies the solution methods resulting in computable formulas. By using these relatively simple formulas one can easily foresee the effect of a given parameter on the performance measure and hence the trends can be forecast. Clearly, instead of the exponential distribution one can use other distributions but in that case the mathematical models will be much more complicated. The analytic

results can help us in validating the results obtained by stochastic simulation. This approach is quite general when analytic expressions cannot be expected. In this case not only the model construction but also the statistical analysis of the output is important.

The primary purpose of the book is to show how to create simple models for practical problems that is why the general theory of stochastic processes is omitted. It uses only the most important concepts and sometimes states theorem without proofs, but each time the related references are cited.

I must confess that the style of the following books greatly influenced me, even if they are in different level and more comprehensive than this material: Allen [2], Jain [41], Kleinrock [48], Kobayashi and Mark [51], Stewart [74], Tijms [91], Trivedi [94].

This book is intended not only for students of computer science, engineering, operation research, mathematics but also those who study at business, management and planning departments, too. It covers more than one semester and has been tested by graduate students at Debrecen University over the years. It gives a very detailed analysis of the involved queueing systems by giving density function, distribution function, generating function, Laplace-transform, respectively. Furthermore, Java-applets are provided to calculate the main performance measures immediately by using the pdf version of the book in a WWW environment. Of course these applets can be run if one reads the printed version.

I have attempted to provide examples for the better understanding and a collection of exercises with detailed solution helps the reader in deepening her/his knowledge. I am convinced that the book covers the basic topics in stochastic modeling of practical problems and it supports students in all over the world.

I am indebted to Professors József Bíró and Zalán Heszberger for their review, comments and suggestions which greatly improved the quality of the book. I am also very grateful to Tamás Török, Zoltán Nagy and Ferenc Veres for their help in editing. .

All comments and suggestions are welcome at:

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Debrecen, 2012.

János Sztrik

Part I

Basic Queueing Theory

Chapter 1

Fundamental Concepts of Queueing Theory

Queueing theory deals with one of the most unpleasant experiences of life, waiting. Queueing is quite common in many fields, for example, in telephone exchange, in a supermarket, at a petrol station, at computer systems, etc. I have mentioned the telephone exchange first because the first problems of queueing theory was raised by calls and Erlang was the first who treated congestion problems in the beginning of 20th century, see Erlang [21, 22].

His works inspired engineers, mathematicians to deal with queueing problems using probabilistic methods. Queueing theory became a field of applied probability and many of its results have been used in operations research, computer science, telecommunication, traffic engineering, reliability theory, just to mention some. It should be emphasized that is a living branch of science where the experts publish a lot of papers and books. The easiest way is to verify this statement one should use the Google Scholar for queueing related items. A Queueing Theory Homepage has been created where readers are informed about relevant sources, for example books, softwares, conferences, journals, etc. I highly recommend to visit it at

`http://web2.uwindsor.ca/math/hlynka/queue.html`

There is only a few books and lectures notes published in Hungarian language, I would mention the work of Györfi and Páli [33], Jereb and Telek [43], Kleinrock [48], Lakatos and Szeidl , Telek [55] and Sztrik [84, 83, 82, 81]. However, it should be noted that the Hungarian engineers and mathematicians have effectively contributed to the research and applications. First of all we have to mention Lajos Takács who wrote his pioneer and famous book about queueing theory [88]. Other researchers are J. Tomkó, M. Arató, L. Györfi, A. Benczúr, L. Lakatos, L. Szeidl, L. Jereb, M. Telek, J. Bíró, T. Do, and J. Sztrik. The Library of Faculty of Informatics, University of Debrecen, Hungary offer a valuable collection of queueing and performance modeling related books in English, and Russian, too. Please visit:

`http://irh.inf.unideb.hu/user/jsztrik/education/05/3f.html`

I may draw your attention to the books of Takagi [85, 86, 87] where a rich collection of references is provided.

1.1 Performance Measures of Queueing Systems

To characterize a queueing system we have to identify the probabilistic properties of the incoming flow of requests, service times and service disciplines. The arrival process can be characterized by the distribution of the **interarrival times** of the customers, denoted by $A(t)$, that is

$$A(t) = P(\text{ interarrival time} < t).$$

In queueing theory these interarrival times are usually assumed to be independent and identically distributed random variables. The other random variable is the **service time**, sometimes it is called service request, work. Its distribution function is denoted by $B(x)$, that is

$$B(x) = P(\text{ service time} < x).$$

The service times, and interarrival times are commonly supposed to be independent random variables.

The **structure of service and service discipline** tell us the **number of servers**, the **capacity of the system**, that is the maximum number of customers staying in the system including the ones being under service. The **service discipline** determines the rule according to the next customer is selected. The most commonly used laws are

- FIFO - First In First Out: who comes earlier leaves earlier
- LIFO - Last Come First Out: who comes later leaves earlier
- RS - Random Service: the customer is selected randomly
- Priority.

The aim of all investigations in queueing theory is to get the main performance measures of the system which are the probabilistic properties (distribution function, density function, mean, variance) of the following random variables: number of customers in the system, number of waiting customers, utilization of the server/s, response time of a customer, waiting time of a customer, idle time of the server, busy time of a server. Of course, the answers heavily depends on the assumptions concerning the distribution of interarrival times, service times, number of servers, capacity and service discipline. It is quite rare, except for elementary or Markovian systems, that the distributions can be computed. Usually their mean or transforms can be calculated.

For simplicity consider first a single-server system Let ϱ, called **traffic intensity**, be defined as

$$\varrho = \frac{\text{mean service time}}{\text{mean interarrival time}}.$$

Assuming an infinity population system with arrival intensity λ, which is reciprocal of the mean interarrival time, and let the mean service denote by $1/\mu$. Then we have

$$\varrho = \text{arrival intensity} * \text{mean service time } = \frac{\lambda}{\mu}.$$

If $\varrho > 1$ then the systems is overloaded since the requests arrive faster than as the are served. It shows that more server are needed.

Let $\chi(A)$ denote the characteristic function of event A, that is

$$\chi(A) = \begin{cases} 1 & \text{, if } A \text{ occurs,} \\ 0 & \text{, if } A \text{ does not ,} \end{cases}$$

furthermore let $N(t) = 0$ denote the event that at time T the server is idle, that is no customer in the system. Then the **utilization of the server during time** T is defined by

$$\frac{1}{T} \int_0^T \chi\left(N(t) \neq 0\right) dt \ ,$$

where T is a long interval of time. As $T \to \infty$ we get the **utilization of the server** denoted by U_s and the following relations holds with probability 1

$$U_s = \lim_{T\to\infty} \frac{1}{T} \int_0^T \chi\left(N(t) \neq 0\right) dt = 1 - P_0 = \frac{E\delta}{E\delta + Ei},$$

where P_0 is the steady-state probability that the server is idle $E\delta$, Ei denote the mean busy period, mean idle period of the server, respectively.

This formula is a special case of the relationship valid for continuous-time Markov chains and proved in Tomkó [93].

Theorem 1 *Let $X(t)$ be an ergodic Markov chain, and A is a subset of its state space. Then with probability* 1

$$\lim_{T\to\infty} \frac{1}{T} \left(\int_0^T \chi(X(t) \in A) dt \right) = \sum_{i\in A} P_i = \frac{m(A)}{m(A) + m(\overline{A})},$$

where $m(A)$ and $m(\overline{A})$ denote the mean sojourn time of the chain in A and $\overline{A}$ during a cycle,respectively. The ergodic (stationary, steady-state) distribution of $X(t)$ is denoted by P_i.

In an m-server system the mean number of arrivals to a given server during time T is $\lambda T/m$ given that the arrivals are uniformly distributed over the servers. Thus the utilization of a given server is

$$U_s = \frac{\lambda}{m\mu}.$$

The other important measure of the system is the **throughput of the system** which is defined as the mean number of requests serviced during a time unit. In an m-server system the mean number of completed services is $m\varrho\mu$ and thus

$$\text{throughput } = mU_s\mu = .$$

However, if we consider now the customers for a tagged customer the **waiting and response times** are more important than the measures defined above. Let us define by W_j, T_j the waiting, response time of the jth customer, respectively. Clearly the waiting time is the time a customer spends in the queue waiting for service, and response time is the time a customer spends in the system, that is

$$T_j = W_j + S_j,$$

where S_j denotes its service time. Of course, W_j and T_j are random variables and their mean, denoted by $\overline{W_j}$ and $\overline{T_j}$, are appropriate for measuring the efficiency of the system. It is not easy in general to obtain their distribution function.

Other characteristic of the system is the **queue length, and the number of customers in the system**. Let the random variables $Q(t)$, $N(t)$ denote the number of customers in the queue, in the system at time t, respectively. Clearly, in an m-server system we have

$$Q(t) = \max\{0, N(t) - m\}.$$

The primary aim is to get their distributions, but it is not always possible, many times we have only their mean values or their generating function.

1.2 Kendall's Notation

Before starting the investigations of elementary queueing systems let us introduce a notation originated by **Kendall** to describe a queueing system.
Let us denote a system by

$$A \ / \ B \ / \ m \ / \ K \ / \ n / \ D,$$

where

A: distribution function of the interarrival times,

B: distribution function of the service times,

m: number of servers,

K: capacity of the system, the maximum number of customers in the system including the one being serviced,

n: population size, number of sources of customers,

D: service discipline.

Exponentially distributed random variables are notated by M, meaning Markovain or memoryless.
Furthermore, if the population size and the capacity is infinite, the service discipline is

FIFO, then they are omitted.

Hence $M/M/1$ denotes a system with Poisson arrivals, exponentially distributed service times and a single server. $M/G/m$ denotes an m-server system with Poisson arrivals and generally distributed service times. $M/M/r/K/n$ stands for a system where the customers arrive from a finite-source with n elements where they stay for an exponentially distributed time, the service times are exponentially distributed, the service is carried out according to the request's arrival by r severs, and the system capacity is K.

1.3 Basic Relations for Birth-Death Processes

Since birth-death processes play a very important role in modeling elementary queueing systems let us consider some useful relationships for them. Clearly, arrivals mean birth and services mean death.

As we have seen earlier the steady-state distribution for birth-death processes can be obtained in a very nice closed-form, that is

$$P_i = \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i} P_0, \quad i = 1, 2, \cdots, \quad {P_0}^{-1} = 1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i}. \tag{1.1}$$

Let us consider the distributions at the moments of arrivals, departures, respectively, because we shall use them later on.

Let N_a, N_d denote the state of the process at the instant of births, deaths, respectively, and let $\Pi_k = P(N_a = k), D_k = P(N_d = k), \quad k = 0, 1, 2, \ldots$ stand for their distributions.

By applying the Bayes's theorem it is easy to see that

$$\Pi_k = \lim_{h \to 0} \frac{(\lambda_k h + o(h))P_k}{\sum_{j=0}^{\infty} (\lambda_j h + o(h))P_j} = \frac{\lambda_k P_k}{\sum_{j=0}^{\infty} \lambda_j P_j}. \tag{1.2}$$

Similarly

$$D_k = \lim_{h \to 0} \frac{(\mu_{k+1} h + o(h))P_{k+1}}{\sum_{j=1}^{\infty} (\mu_j h + o(h))P_j} = \frac{\mu_{k+1} P_{k+1}}{\sum_{j=1}^{\infty} \mu_j P_j}. \tag{1.3}$$

Since $P_{k+1} = \dfrac{\lambda_k}{\mu_{k+1}} P_k, \quad k = 0, 1, \ldots,$ thus

$$D_k = \frac{\lambda_k P_k}{\sum_{i=0}^{\infty} \lambda_i P_i} = \Pi_k, \quad k = 0, 1, \ldots. \tag{1.4}$$

In words, the above relation states that the steady-state distributions at the moments of births and deaths are the same. It should be underlined, that it does not mean that it is equal to the steady-state distribution at a random point as we will see later on.

Further essential observation is that in steady-state the mean birth rate is equal to the mean death rate. This can be seen as follows

$$\overline{\lambda} = \sum_{i=0}^{\infty} \lambda_i P_i = \sum_{i=0}^{\infty} \mu_{i+1} P_{i+1} = \sum_{k=1}^{\infty} \mu_k P_k = \overline{\mu}. \tag{1.5}$$

1.4 Queueing Softwares

To solve practical problems the first step is to identify the appropriate queueing system and then to calculate the performance measures. Of course the level of modeling heavily depends on the assumptions. It is recommended to start with a simple system and then if the results do not fit to the problem continue with a more complicated one. Various software packages help the interested readers in different level. The following links worths a visit

`http://web2.uwindsor.ca/math/hlynka/qsoft.html`

For practical oriented teaching courses we also have developed a collection of Java-applets calculating the performance measures not only for elementary but for more advanced queueing systems. It is available at

`http://irh.inf.unideb.hu/user/jsztrik/education/09/english/index.html`

For simulation purposes I recommend

`http://www.win.tue.nl/cow/Q2/`

If the preprepared systems are not suitable for your problem then you have to create your queueing system and then the **creation** starts and the primary aim of the present book is to help this process.

For further readings the interested reader is referred to the following books: Allen [2], Bose [9], Daigle [18], Gnedenko and Kovalenko [31], Gnedenko, Belyayev and Solovyev [29], Gross and Harris [32], Jain [41], Jereb and Telek [43], Kleinrock [48], Kobayashi [50, 51], Kulkarni [54], Nelson [59], Stewart [74], Sztrik [81], Tijms [91], Trivedi [94].
The present book has used some parts of Allen [2], Gross and Harris [32], Kleinrock [48], Kobayashi [50], Sztrik [81], Tijms [91], Trivedi [94].

Chapter 2

Infinite-Source Queueing Systems

Queueing systems can be classified according to the cardinality of their sources, namely finite-source and infinite-source models. In finite-source models the arrival intensity of the request depends on the state of the system which makes the calculations more complicated. In the case of infinite-source models, the arrivals are independent of the number of customers in the system resulting a mathematically tractable model. In queueing networks each node is a queueing system which can be connected to each other in various way. The main aim of this chapter is to know how these nodes operate.

2.1 The $M/M/1$ Queue

An $M/M/1$ queueing system is the simplest non-trivial queue where the requests arrive according to a Poisson process with rate λ, that is the interarrival times are independent, exponentially distributed random variables with parameter λ. The service times are also assumed to be independent and exponentially distributed with parameter μ. Furthermore, all the involved random variables are supposed to be independent of each other.

Let $N(t)$ denote the number of customers in the system at time t and we shall say that the system is at state k if $N(t) = k$. Since all the involved random variables are exponentially distributed, consequently they have the memoryless property, $N(t)$ is a continuous-time Markov chain with state space $0, 1, \cdots$.

In the next step let us investigate the transition probabilities during time h. It is easy to see that

$$\begin{aligned} P_{k,k+1}(h) = &(\lambda h + o(h))\,(1 - (\mu h + o(h)) + \\ &+ \sum_{k=2}^{\infty} (\lambda h + o(h))^k\,(\mu h + o(h))^{k-1}\,, \\ &k = 0, 1, 2, \ldots\ . \end{aligned}$$

By using the independence assumption the first term is the probability that during h one customer has arrived and no service has been finished. The summation term is the probability that during h at least 2 customers has arrived and at the same time at least 1

has been serviced. It is not difficult to verify the second term is $o(h)$ due to the property of the Poisson process. Thus

$$P_{k,k+1}(h) = \lambda h + o(h).$$

Similarly, the transition probability from state k into state $k-1$ during h can be written as

$$\begin{aligned} P_{k,k-1}(h) &= (\mu h + o(h))\,(1 - (\lambda h + o(h)) + \\ &+ \sum_{k=2}^{\infty} (\lambda h + o(h))^{k-1}\,(\mu h + o(h))^{k} \\ &= \mu h + o(h). \end{aligned}$$

Furthermore, for non-neighboring states we have

$$P_{k,j} = o(h), \qquad \mid k - j \mid \geq 2.$$

In summary, the introduced random process $N(t)$ is a birth-death process with rates

$$\lambda_k = \lambda, \qquad k = 0, 1, 2, ..., \quad \mu_k = \mu, \qquad k = 1, 2, 3....$$

That is all the birth rates are λ, and all the death rates are μ.
As we notated the system capacity is infinite and the service discipline is FIFO.

To get the steady-state distribution let us substitute these rates into formula (1.1) obtained for general birth-death processes. Thus we obtain

$$P_k = P_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu} = P_0 \left(\frac{\lambda}{\mu}\right)^k, \qquad k \geq 0.$$

By using the normalization condition we can see that this geometric sum is convergent iff $\lambda/\mu < 1$ and

$$P_0 = \left(1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k\right)^{-1} = 1 - \frac{\lambda}{\mu} = 1 - \varrho$$

where $\varrho = \frac{\lambda}{\mu}$. Thus

$$P_k = (1 - \varrho)\varrho^k, \qquad k = 0, 1, 2, ...,$$

which is a modified geometric distribution with success parameter $1 - \varrho$.

In the following we calculate the *the main performance measures of the system*

- *Mean number of customers in the system*

$$\overline{N} = \sum_{k=0}^{\infty} k P_k = (1 - \varrho)\varrho \sum_{k=1}^{\infty} k \varrho^{k-1} =$$

$$= (1-\varrho)\varrho \sum_{k=1}^{\infty} \frac{d\varrho^k}{d\varrho} = (1-\varrho)\varrho \frac{d}{d\varrho}\left(\frac{1}{1-\varrho}\right) = \frac{\varrho}{1-\varrho}.$$

Variance

$$\begin{aligned} Var(N) &= \sum_{k=0}^{\infty}(k-\overline{N})^2 P_k = \sum_{k=0}^{\infty}\left(k - \frac{\varrho}{1-\varrho}\right)^2 P_k \\ &= \sum_{k=0}^{\infty} k^2 P_k + \left(\frac{\varrho}{1-\varrho}\right)^2 - \sum_{k=0}^{\infty} 2k\frac{\varrho}{1-\varrho}P_k \\ &= \sum_{k=0}^{\infty} k(k-1)P_k + \frac{\varrho^2}{(1-\varrho)^2} + \frac{\varrho}{1-\varrho} - 2\left(\frac{\varrho}{1-\varrho}\right)^2 \\ &= (1-\varrho)\varrho^2 \frac{d^2}{d\varrho^2}\sum_{k=0}^{\infty}\varrho^k + \frac{\varrho}{1-\varrho} - \left(\frac{\varrho}{1-\varrho}\right)^2 \\ &= \frac{2\varrho^2}{(1-\varrho)^2} + \frac{\varrho}{1-\varrho} - \left(\frac{\varrho}{1-\varrho}\right)^2 = \frac{\varrho}{(1-\varrho)^2}. \end{aligned}$$

- *Mean number of waiting customers, mean queue length*

$$\overline{Q} = \sum_{k=1}^{\infty}(k-1)P_k = \sum_{k=1}^{\infty} kP_k - \sum_{k=1}^{\infty} P_k = \overline{N} - (1-P_0) = \overline{N} - \varrho = \frac{\varrho^2}{1-\varrho}.$$

Variance

$$Var(Q) = \sum_{k=1}^{\infty}(k-1)^2 P_k - \overline{Q}^2 = \frac{\varrho^2(1+\varrho-\varrho^2)}{(1-\varrho)^2}.$$

- *Server utilization*

$$U_s = 1 - P_0 = \frac{\lambda}{\mu} = \varrho.$$

By using Theorem 1 it is easy to see that

$$P_0 = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + E\delta},$$

where $E\delta$ a *is the mean busy period length of the server*, $\frac{1}{\lambda}$ is the mean idle time of the server. Since the server is idle until a new request arrives which is exponentially distributed with parameter λ. Hence

$$1 - \varrho = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + E\delta},$$

and thus

$$E\delta = \frac{1}{\lambda}\frac{\varrho}{1-\varrho} = \frac{1}{\lambda}\overline{N} = \frac{1}{\mu-\lambda}.$$

In the next few lines we show how this performance measure can be obtained in a different way.
To do so we need the following notations.
Let $\mathbb{E}(\nu_A)$, $\mathbb{E}(\nu_D)$ denote the mean number of customers that have arrived, departed during the mean busy period of the server, respectively. Furthermore, let $\mathbb{E}(\nu_S)$ denote the mean number of customers that have arrived during a mean service time. Clearly

$$\begin{aligned}\mathbb{E}(\nu_D) &= \mathbb{E}(\delta)\mu,\\ \mathbb{E}(\nu_S) &= \frac{\lambda}{\mu},\\ \mathbb{E}(\nu_A) &= \mathbb{E}(\delta)\lambda,\\ \mathbb{E}(\nu_A)+1 &= \mathbb{E}(\nu_D),\end{aligned}$$

and thus after substitution we get

$$\mathbb{E}(\delta) = \frac{1}{\mu-\lambda}.$$

Consequently

$$\begin{aligned}\mathbb{E}(\nu_D) &= \mathbb{E}(\delta)\mu = \frac{1}{1-\varrho}\\ \mathbb{E}(\nu_A) &= \mathbb{E}(\nu_S)\mathbb{E}(\nu_D) = \frac{\lambda}{\mu}\frac{1}{1-\varrho} = \frac{\varrho}{1-\varrho}\\ \mathbb{E}(\nu_A) &= \mathbb{E}(\delta)\lambda = \frac{\varrho}{1-\varrho}.\end{aligned}$$

- *Distribution of the response time of a customer*

  Before investigating the response we show that in any queueing system where the arrivals are Poisson distributed

  $$P_k(t) = \Pi_k(t),$$

  where $P_k(t)$ denotes the probability that at time t the system is a in state k, and $\Pi_k(t)$ denotes the probability that an arriving customers find the system in state k at time t. Let

  $$A(t, t+\Delta t)$$

  denote the event that an arrival occurs in the interval $(t, t+\Delta t)$. Then

  $$\Pi_k(t) := \lim_{\Delta t\to 0} P\left(N(t)=k | A(t,t+\Delta t)\right),$$

  Applying the definition of the conditional probability we have

  $$\Pi_k(t) = \lim_{\Delta t\to 0} \frac{P\left(N(t)=k\ ,\ A(t,t+\Delta t)\right)}{P\left(A(t,t+\Delta t)\right)} =$$

$$= \lim_{\Delta t \to 0} \frac{P\left(A(t, t+\Delta t) | N(t) = k\right) P\left(N(t) = k\right)}{P\left(A(t, t+\Delta t)\right)}.$$

However, in the case of a Poisson process event $A(t, t+\Delta t)$ does not depends on the number of customers in the system at time t and even the time t is irrespective thus we obtain

$$P\left(A(t, t+\Delta t) | N(t) = k\right) = P\left(A(t, t+\Delta t)\right),$$

hence for birth-death processes we have

$$\Pi_k(t) = P\left(N(t) = k\right).$$

That is the probability that an arriving customer find the system in state k is equal to the probability that the system is in state k.

In stationary case applying formula (1.2) with substitutions $\lambda_i = \lambda, \quad i = 0, 1, \ldots$ we have the same result.

If a customer arrives it finds the server idle with probability P_0 hence the waiting time is 0. Assume, upon arrival a tagged customer, the system is in state n. This means that the request has to wait until the residual service time of the customer being serviced plus the service times of the customers in the queue. As we assumed the service is carried out according to the arrivals of the requests. Since the service times are exponentially distributed the remaining service time has the same distribution as the original service time. Hence the waiting time of the tagged customer is Erlang distributed with parameters (n, μ) and the response time is Erlang distributed with $(n+1, \mu)$. Just to remind you the density function of an Erlang distribution with parameters (n, μ) is

$$f_n(x) = \frac{\mu(\mu x)^{n-1}}{(n-1)!} e^{-\mu x}, \qquad x \geq 0.$$

Hence applying the theorem of total probability for the density function of the response time we have

$$f_T(x) = \sum_{n=0}^{\infty} (1-\varrho)\varrho^n \frac{(\mu x)^n}{n!} \mu e^{-\mu x} = \mu(1-\varrho)e^{-\mu x} \sum_{n=0}^{\infty} \frac{(\varrho \mu x)^n}{n!} =$$

$$= \mu(1-\varrho)e^{-\mu(1-\varrho)x}.$$

Its distribution function is

$$F_T(x) = 1 - e^{-\mu(1-\varrho)x}.$$

That is the response time is exponentially distributed with parameter
$\mu(1-\varrho) = \mu - \lambda$.
Hence the expectation and variance of the response time are

$$\overline{T} = \frac{1}{\mu(1-\varrho)}, \quad Var(T) = (\frac{1}{\mu(1-\varrho)})^2.$$

Furthermore

$$\overline{T} = \frac{1}{\mu(1-\varrho)} = \frac{1}{\mu-\lambda} = E\delta.$$

- *Distribution of the waiting time*

Let $f_W(x)$ denote the density function of the waiting time. Similarly to the above considerations for $x > 0$ we have

$$f_W(x) = \sum_{n=1}^{\infty} \frac{(\mu x)^{n-1}}{(n-1)!} \mu e^{-\mu x} \varrho^n (1-\varrho) = (1-\varrho)\varrho\mu \sum_{k=0}^{\infty} \frac{(\mu x \varrho)^k}{k!} e^{-\mu x} =$$

$$= (1-\varrho)\varrho\mu e^{-\mu(1-\varrho)x}.$$

Thus

$$\begin{array}{ll} f_W(0) = 1-\varrho, & \text{if } x = 0, \\ f_W(x) = \varrho(1-\varrho)\mu e^{-\mu(1-\varrho)x}, & \text{if } x > 0. \end{array}$$

Hence

$$F_W(x) = 1 - \varrho + \varrho\left(1 - e^{-\mu(1-\varrho)x}\right) = 1 - \varrho e^{-\mu(1-\varrho)x}.$$

The mean waiting time is

$$\overline{W} = \int_0^{\infty} x f_W(x) dx = \frac{\varrho}{\mu(1-\varrho)} = \varrho E\delta = \overline{N}\frac{1}{\mu}.$$

Since $T = W + S$, in addition W and S are independent we get

$$Var(T) = \frac{1}{(\mu(1-\rho))^2} = Var(W) + \frac{1}{\mu^2},$$

thus

$$Var(W) = \frac{1}{(\mu(1-\rho))^2} - \frac{1}{\mu^2} = \frac{2\rho - \rho^2}{(\mu(1-\rho))^2} = \rho\frac{2}{(\mu(1-\rho))^2} - \frac{\rho^2}{(\mu(1-\rho))^2},$$

that is exactly $\mathbb{E}(W^2) - (\mathbb{E}W)^2$.

Notice that

$$\lambda\overline{T} = \lambda\frac{1}{\mu(1-\varrho)} = \frac{\varrho}{1-\varrho} = \overline{N}. \tag{2.1}$$

Furthermore

$$\lambda\overline{W} = \lambda\frac{\varrho}{\mu(1-\varrho)} = \frac{\varrho^2}{1-\varrho} = \overline{Q}. \tag{2.2}$$

Relations (2.1), (2.2) are called **Little formulas or Little theorem, or Little law** which remain valid under more general conditions.

Let us examine the states of an $M/M/1$ system at the departure instants of the customers. Our aim is to calculate the distribution of the departure times of the customers. As it was proved in (1.3) at departures the distribution is

$$D_k = \frac{\lambda_k P_k}{\sum_{i=0}^{\infty} \lambda_i P_i}.$$

In the case of Poisson arrivals $\lambda_k = \lambda, k = 0, 1, \ldots$, hence $D_k = P_k$.
Now we are able to calculate the Laplace-transform of the interdeparture time d. Conditioning on the state of the server at the departure instants, by using the theorem of total Laplace-transform we have

$$L_d(s) = \varrho \frac{\mu}{\mu + s} + (1 - \varrho) \frac{\lambda}{\lambda + s} \frac{\mu}{\mu + s},$$

since if the server is idle for the next departure a request should arrive first. Hence

$$L_d(s) = \frac{\mu\varrho(\lambda + s) + (1 - \varrho)\lambda\mu}{(\lambda + s)(\mu + s)} = \frac{\lambda\mu\varrho + \lambda s + \lambda\mu - \lambda\mu\varrho}{(\lambda + s)(\mu + s)}$$

$$= \frac{\lambda(s + \mu)}{(\lambda + s)(\mu + s)} = \frac{\lambda}{\lambda + s},$$

which shows that the distribution is exponential with parameter λ and not with μ as one might expect. The independence follows from the memoryless property of the exponential distributions and from their independence. This means that the departure process is a Poisson process with rate λ.

This observation is very important to investigate tandem queues, that is when several simple $M/M/1$ queueing systems as nodes are connected in serial to each other. Thus at each node the arrival process is a Poisson process with parameter λ and the nodes operate independently of each other. Hence if the service times have parameter μ_i at the ith node then introducing traffic intensity $\varrho_i = \dfrac{\lambda}{\mu_i}$ all the performance measures for a given node could be calculated. Consequently, the mean number of customers in tha network is the sum of the mean number of customers in the nodes. Similarly, the mean waiting and response times for the network can be calculated as the sum of the related measures in the nodes.

Now, let us show how the density function d can be obtained directly without using tha Laplace-transforms. By applying the theorem of total probability we have

$$\begin{aligned} f_d(x) &= \varrho\mu e^{-\mu x} + (1 - \varrho)\left(\frac{\lambda\mu}{\lambda - \mu} e^{-\mu x} + \frac{\lambda\mu}{\mu - \lambda} e^{-\lambda x}\right) \\ &= \lambda e^{-\mu x} + \frac{\mu - \lambda}{\mu}\left(\frac{\lambda\mu}{\mu - \lambda} e^{-\lambda x} - \frac{\lambda\mu}{\mu - \lambda} e^{-\mu x}\right) \\ &= \lambda e^{-\mu x} + \lambda e^{-\lambda x} - \lambda e^{-\mu x} = \lambda e^{-\lambda x}. \end{aligned}$$

Now let us consider an $M/G/1$ system and we are interested in under which service time distribution the interdeparture time is exponentially distributed with parameterλ. First prove that the utilization of the system is $U_S = \varrho = \lambda\mathbb{E}(S)$. As it is understandable for any stationary stable $G/G/1$ queueing system the mean number of departures during the mean busy period length of the server is one more than the mean number of arrivals during the mean busy period length of the server. That is

$$\frac{\mathbb{E}(\delta)}{\mathbb{E}(S)} = 1 + \frac{\mathbb{E}(\delta)}{\mathbb{E}(\tau)},$$

where $\mathbb{E}(\tau)$ denotes the mean interarrival times. Hence

$$\mathbb{E}(\tau) + \mathbb{E}(\delta) = \mathbb{E}(\delta)\frac{\mathbb{E}(\tau)}{\mathbb{E}(S)}$$
$$\mathbb{E}(\delta) = \frac{\mathbb{E}(\tau)\mathbb{E}(S)}{\mathbb{E}(\tau) - \mathbb{E}(S)} = \mathbb{E}(S)\frac{1}{1-\varrho},$$

where $\varrho = \frac{\mathbb{E}(S)}{\mathbb{E}(\tau)}$. Clearly

$$U_S = \frac{\mathbb{E}(\delta)}{\mathbb{E}(\tau) + \mathbb{E}(\delta)} = \frac{\mathbb{E}(S)\frac{1}{1-\varrho}}{\mathbb{E}(\tau) + \frac{\mathbb{E}(S)}{1-\varrho}} = \frac{\frac{\varrho}{1-\varrho}}{1 + \frac{\varrho}{1-\varrho}} = \varrho < 1.$$

Thus the utilization for an $M/G/1$ system is ϱ. It should be noted that an $M/G/1$ system $D_k = P_k$, that is why our question can be formulated as

$$\frac{\lambda}{\lambda+s} = \varrho L_S(s) + (1-\varrho)\frac{\lambda}{\lambda+s}L_S(s) = L_S(s)\left(\varrho + \frac{\lambda(1-\varrho)}{\lambda+s}\right)$$
$$= L_S(s)\frac{\lambda^2\mathbb{E}(S) + s\lambda\mathbb{E}(S) + \lambda - \lambda^2\mathbb{E}(S)}{\lambda+s} = L_S(s)\frac{\lambda(1+s\mathbb{E}(S))}{\lambda+s},$$

thus

$$L_S(s) = \frac{1}{1+s\mathbb{E}(S)},$$

which is the Laplace-transform of an exponential distribution with mean $\mathbb{E}(S)$. In summary, only exponentially distributed service times assures that Poisson arrivals involves Poisson departures with the same parameters.

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1/MM1.html`

Example 1 *Let us consider a small post office in a village where on the average 70 customers arrive according to a Poisson process during a day. Let us assume that the service times are exponentially distributed with rate 10 clients per hour and the office operates 10 hours daily. Find the mean queue length, and the probability that the number of waiting customer is greater than 2. What is the mean waiting time and the probability that the waiting time is greater than 20 minutes ?*

Solution:
Let the time unit be an hour. Then $\lambda = 7$, $\mu = 10$, $\rho = \frac{7}{10}$

$$\begin{aligned}
\overline{N} &= \frac{\rho}{1-\rho} = \frac{7}{3} \\
\overline{Q} &= \overline{N} - \rho = \frac{7}{3} - \frac{7}{10} = \frac{70-21}{30} = \frac{49}{30} \\
P(n>3) &= 1 - P(n \le 3) = 1 - P_0 - P_1 - P_2 - P_3 \\
&= 1 - 1 + \rho - (1-\rho)(\rho + \rho^2 + \rho^3) = \rho^4 = 0.343 \cdot 0.7 = 0.2401 \\
\overline{W} &= \frac{\overline{N}}{\mu} = \frac{7}{3 \cdot 10} = \frac{7}{30} \text{ hour } \approx 14 \text{ minutes} \\
P\left(W > \frac{1}{3}\right) &= 1 - F_W\left(\frac{1}{3}\right) = 0.7 \cdot e^{-10 \cdot \frac{1}{3} \cdot 0.3} = 0.7 \cdot e^{-1} = 0.257
\end{aligned}$$

■

2.2 The $M/M/1$ Queue with Balking Customers

Let us consider a modification of an $M/M/1$ system in which customers are discouraged when more and more requests are present at their arrivals. Let us denote by b_k the probability that a customers joints to the systems provided there are k customers in the system at the moment of his arrival.

It is easy to see, that the number of customers in the system is a birth-death process with birth rates

$$\lambda_k = \lambda \cdot b_k, \quad k = 0, 1, \ldots$$

Clearly, there are various candidates for b_k but we have to find such probabilities which result not too complicated formulas for the main performance measures. Keeping in mind this criteria let us consider the following

$$b_k = \frac{1}{k+1}, \quad k = 0, 1, \ldots$$

Thus

$$P_k = \frac{\rho^k}{k!} P_0, \qquad k = 0, 1, \ldots,$$

and then using the normalization condition we get

$$P_k = \frac{\rho^k}{k!} e^{-\rho}, \qquad k = 0, 1, \ldots$$

The stability condition is E $\rho < \infty$, that is we do not need the condition $\rho < 1$ as in an $M/M/1$ system.
Notice that the number of customers follows a Poisson law with parameter ρ and we can expect that the performnace measures can be obtained in a simple way.

Performance measures

- $$U_S = 1 - P_0 = 1 - e^{-\rho},$$
$$U_S = \frac{\mathbb{E}(\delta)}{\frac{1}{\lambda} + \mathbb{E}(\delta)},$$

  hence

$$\mathbb{E}(\delta) = \frac{1}{\lambda} \cdot \frac{U_S}{1 - U_S} = \frac{1}{\lambda} \cdot \frac{1 - e^{-\rho}}{e^{-\rho}}.$$

- $$\overline{N} = \rho,$$
$$Var(N) = \rho$$

- $$\overline{Q} = \overline{N} - U_S = \rho - (1 - e^{-\rho}) = \rho + e^{-\rho} - 1.$$
$$\begin{aligned}\mathbb{E}(Q^2) &= \sum_{k=1}^{\infty} (k-1)^2 P_k = \sum_{k=1}^{\infty} k^2 P_k - 2\sum_{k=1}^{\infty} k P_k + \sum_{k=1}^{\infty} P_k \\ &= \mathbb{E}(N^2) - 2\overline{N} + U_S = \rho + \rho^2 - 2\rho + U_S = \rho^2 - \rho + 1 - e^{-\rho}.\end{aligned}$$

  Thus

$$\begin{aligned}Var(Q) &= \mathbb{E}(Q^2) - (\mathbb{E}(Q))^2 = \rho^2 - \rho + 1 - e^{-\rho} - (\rho + e^{-\rho} - 1)^2 \\ &= \rho^2 - \rho + 1 - e^{-\rho} - \rho^2 - e^{-2\rho} - 1 - 2\rho e^{-\rho} + 2\rho + 2e^{-\rho} \\ &= \rho - e^{-2\rho} + e^{-\rho} - 2\rho e^{-\rho} = \rho - e^{-\rho}(e^{-\rho} + 2\rho - 1).\end{aligned}$$

- To get the distribution of the response and waiting times we have to know the distribution of the system at the instant when an arriving customer joins to the system.

  By applying the Bayes's rule it is not difficult to see that

$$\Pi_k = \frac{\frac{\lambda}{k+1} \cdot P_k}{\sum_{i=0}^{\infty} \frac{\lambda}{i+1} \cdot P_i} = \frac{\frac{\rho^{k+1}}{(k+1)!} \cdot e^{-\rho}}{\sum_{i=0}^{\infty} \frac{\rho^{i+1}}{(i+1)!} e^{-\rho}} = \frac{P_{k+1}}{1 - e^{-\rho}}.$$

  Notice, that this time

$$\Pi_k \neq P_k.$$

  Let us first determine $\overline{T}$ and then $\overline{W}$.

  By the law of total expectations we have

$$\overline{T} = \sum_{k=0}^{\infty} \frac{k+1}{\mu} \Pi_k = \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{1 - e^{-\rho}} = \frac{1}{\mu(1 - e^{-\rho})} \cdot \overline{N} = \frac{\rho}{\mu(1 - e^{-\rho})}.$$
$$\overline{W} = \overline{T} - \frac{1}{\mu} = \frac{1}{\mu} \left(\frac{\rho + e^{-\rho} - 1}{1 - e^{-\rho}} \right).$$

As we have proved in formula (1.5)

$$\overline{\lambda} = \sum_{k=0}^{\infty} \lambda_k P_k = \sum_{k=1}^{\infty} \mu_k P_k = \sum_{k=1}^{\infty} \mu P_k = \mu(1 - e^{-\rho}),$$

thus

$$\overline{\lambda} \cdot \overline{T} = \mu(1 - e^{-\rho}) \cdot \frac{\rho}{\mu(1 - e^{-\rho})} = \rho = \overline{N},$$
$$\overline{\lambda} \cdot \overline{W} = \mu(1 - e^{-\rho}) \cdot \frac{\rho + e^{-\rho} - 1}{\mu(1 - e^{-\rho})} = \rho + e^{-\rho} - 1 = \overline{Q}$$

which is the **Little formula** for this system.

- To find the distribution of T and W we have to use the same approach as we did earlier, namely

$$f_T(x) = \sum_{k=0}^{\infty} f_T(x|k) \cdot \Pi_k = \sum_{k=0}^{\infty} \frac{\mu(\mu x)^k e^{-\mu x}}{k!} \cdot \frac{\rho^{k+1}}{(k+1)!} \frac{e^{-\rho}}{1 - e^{-\rho}}$$
$$= \frac{\lambda e^{-(\rho+\mu x)}}{1 - e^{-\rho}} \sum_{k=0}^{\infty} \frac{(\mu x \rho)^k}{k!(k+1)!},$$

which is difficult to calculate. We have the same problems with $f_W(x)$, too.

However, the Laplace-transforms $L_T(s)$ and $L_W(s)$ can be obtained and the hence the higher moments can be derived.
Namely

$$L_T(s) = \sum_{k=0}^{\infty} L_T(s|k)\Pi_k = \sum_{k=0}^{\infty} \left(\frac{\mu}{\mu + s}\right)^{k+1} \frac{\frac{\rho^{k+1}}{(k+1)!} e^{-\rho}}{1 - e^{-\rho}}$$
$$= \frac{e^{-\rho}}{1 - e^{-\rho}} \sum_{k=0}^{\infty} \left(\frac{\mu\rho}{\mu + s}\right)^{k+1} \frac{1}{(k+1)!} = \frac{e^{-\rho}}{1 - e^{-\rho}} \left(e^{\frac{\mu\rho}{\mu+s}} - 1\right).$$
$$L_W(s) = L_T(s) \cdot \frac{\mu + s}{\mu}.$$

Find $\overline{T}$ by the help of $L_T(s)$ to check the formula. It is easy to see that

$$L_T'(s) = \frac{e^{-\rho}}{1 - e^{-\rho}} \cdot e^{\frac{\mu\rho}{\mu+s}} (-\mu\rho(\mu + s)^{-2})$$
$$L_T'(0) = -\frac{e^{-\rho}}{1 - e^{-\rho}} e^{\rho} \cdot \frac{\rho}{\mu} = -\frac{\rho}{\mu(1 - e^{-\rho})}.$$

Hence

$$\overline{T} = \frac{\rho}{\mu(1 - e^{-\rho})},$$

as we have obtained earlier. $\overline{W}$ can be verified similarly.

To get $Var(T)$ and $Var(W)$ we can use the Laplace-transform method. As we have seen

$$L_T(s) = \frac{e^{-\rho}}{1-e^{-\rho}}\left(e^{\frac{\lambda}{\mu+s}}-1\right).$$

Thus

$$L_T'(s) = \frac{e^{-\rho}}{1-e^{-\rho}} \cdot e^{\frac{\lambda}{\mu+s}}(-1)\lambda(\mu+s)^{-2},$$

therefore

$$L_T''(s) = \frac{e^{-\rho}}{1-e^{-\rho}} \cdot \left(e^{\frac{\lambda}{\mu+s}}\left((-1)\lambda(\mu+s)^{-2}\right)^2 + 2\lambda(\mu+s)^{-3} \cdot e^{\frac{\lambda}{\mu+s}}\right).$$

Hence

$$L_T''(0) = \frac{e^{-\rho}}{1-e^{-\rho}}\left(e^{\rho}\left(-\frac{\rho}{\mu}\right)^2 + \frac{2\rho}{\mu^2}e^{\rho}\right) = \frac{1}{\mu^2}\cdot\frac{\rho^2+2\rho}{1-e^{-\rho}}.$$

Consequently

$$\begin{aligned} Var(T) &= \frac{1}{\mu^2}\cdot\frac{\rho^2+2\rho}{1-e^{-\rho}} - \left(\frac{\rho}{\mu(1-e^{-\rho})}\right)^2 \\ &= \frac{(\rho^2+2\rho)\left(1-e^{-\rho}\right)-\rho^2}{\mu^2\left(1-e^{-\rho}\right)^2} = \frac{\rho^2+2\rho-\rho^2e^{-\rho}-2\rho e^{-\rho}-\rho^2}{\mu^2\left(1-e^{-\rho}\right)^2} \\ &= \frac{2\rho-\rho^2e^{-\rho}-2\rho e^{-\rho}}{\mu^2\left(1-e^{-\rho}\right)^2} = \frac{\rho(2-(\rho+2)e^{-\rho})}{\mu^2\left(1-e^{-\rho}\right)^2}. \end{aligned}$$

However, W and T can be considered as a random sum, too. That is

$$Var(W) = \mathbb{E}(N_a)\frac{1}{\mu^2} + Var(N_a)\left(\frac{1}{\mu}\right)^2 = \frac{1}{\mu^2}(\mathbb{E}(N_a)+Var(N_a)).$$

$$\begin{aligned} \mathbb{E}(N_a) &= \sum_{k=1}^{\infty} k\Pi_k = \sum_{k=1}^{\infty}\frac{kP_{k+1}}{1-e^{-\rho}} \\ &= \frac{1}{1-e^{-\rho}}\left(\sum_{k=0}^{\infty}(k+1)P_{k+1} - \sum_{k=0}^{\infty}P_{k+1}\right) \\ &= \frac{1}{1-e^{-\rho}}\left(\rho+e^{-\rho}-1\right). \end{aligned}$$

Since

$$Var(N_a) = \mathbb{E}(N_a^2) - (\mathbb{E}(N_a))^2$$

first we have to calculate $\mathbb{E}(N_a^2)$, that is

$$\begin{aligned}
\mathbb{E}(N_a^2) &= \sum_{k=1}^{\infty} k^2 \Pi_k = \sum_{k=1}^{\infty} k^2 \frac{P_{k+1}}{1-e^{-\rho}} \\
&= \frac{1}{1-e^{-\rho}} \sum_{k=0}^{\infty} \left((k+1)^2 - 2k - 1\right) P_{k+1} \\
&= \frac{1}{1-e^{-\rho}} \left(\sum_{k=0}^{\infty} (k+1)^2 P_{k+1} - 2 \sum_{k=0}^{\infty} k P_{k+1} - \sum_{k=0}^{\infty} P_{k+1} \right) \\
&= \frac{1}{1-e^{-\rho}} \left(\rho + \rho^2 - 2\left(\rho + e^{-\rho} - 1\right) - \left(1 - e^{-\rho}\right)\right) \\
&= \frac{1}{1-e^{-\rho}} \left(\rho^2 - \rho - e^{-\rho} + 1\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
Var(N_a) &= \frac{1}{1-e^{-\rho}} \left(\rho^2 - \rho - e^{-\rho} + 1\right) - \left(\frac{1}{1-e^{-\rho}} (\rho + e^{-\rho} - 1)\right)^2 \\
&= \left(\frac{1}{1-e^{-\rho}}\right)^2 \left((1-e^{-\rho}) \left(\rho^2 - \rho - e^{-\rho} + 1\right) - \left(\rho + e^{-\rho} - 1\right)^2 \right) \\
&= \left(\frac{1}{1-e^{-\rho}}\right)^2 (\rho^2 - \rho - e^{-\rho} + 1 - \rho^2 e^{-\rho} + \rho e^{-\rho} + e^{-2\rho} - e^{-\rho} \\
&- \rho^2 - e^{-2\rho} - 1 - 2\rho e^{-\rho} + 2\rho - 2e^{-\rho}) \\
&= \frac{\rho - e^{-\rho}(\rho^2 + \rho)}{(1-e^{-\rho})^2}.
\end{aligned}$$

Finally

$$\begin{aligned}
Var(W) &= \left(\frac{1}{\mu}\right)^2 \left(\frac{1}{1-e^{-\rho}} (\rho + e^{-\rho} - 1) + \frac{\rho - e^{-\rho}(\rho^2 + \rho)}{(1-e^{-\rho})^2} \right) \\
&= \frac{1}{(\mu(1-e^{-\rho}))^2} ((\rho + e^{-\rho} - 1)(1-e^{-\rho}) + \rho - e^{-\rho}(\rho^2 + \rho)).
\end{aligned}$$

Thus

$$\begin{aligned}
Var(T) &= Var(W) + \frac{1}{\mu^2} \\
Var(T) &= \left(\frac{1}{\mu(1-e^{-\rho})}\right)^2 (\rho + e^{-\rho} - 1)(1-e^{-\rho}) + \rho - e^{-\rho}(\rho^2 + \rho) + (1-e^{-\rho})^2) \\
&= \frac{(1-e^{-\rho})(\rho + e^{-\rho} - 1 + 1 - e^{-\rho}) + \rho - e^{-\rho}(\rho^2 + \rho)}{(\mu(1-e^{-\rho}))^2} \\
&= \frac{2\rho - 2\rho e^{-\rho} - \rho^2 e^{-\rho}}{(\mu(1-e^{-\rho})^2}
\end{aligned}$$

which is the same we have obtained earlier.

2.3 Priority $M/M/1$ Queues

In the following let us consider an $M/M/1$ systems with priorities. This means that we have two classes of customers. Each type of requests arrive according to a Poisson process with parameter λ_1, and λ_2, respectively and the processes are supposed to be independent of each other. The service times for each class are assumed to be exponentially distributed with parameter μ. The system is stable if

$$\rho_1 + \rho_2 < 1,$$

where $\rho_i = \lambda_i/\mu$, $i = 1, 2$.
Let us assume that class 1 has priority over class 2. This section is devoted to the investigation of preemptive and non-preemptive systems and some mean values are calculated.

Preemptive Priority

According to the discipline the service of a customer belonging to class 2 is never carried out if there is customer belonging to class 1 in the system. In other words it means that class 1 preempts class 2 that is if a class 2 customer is under service when a class 1 request arrives the service stops and the service of class 1 request starts. The interrupted service is continued only if there is no class 1 customer in the system.

Let N_i denote the number of class i customers in the system and let T_i stand for the response time of class i requests. Our aim is to calculate $\mathbb{E}(N_i)$ and $\mathbb{E}(T_i)$ for $i = 1, 2$. Since type 1 always preempts type 2 the service of class 1 customers is independent of the number of class 2 customers. Thus we have

$$\mathbb{E}(T_1) = \frac{1/\mu}{1-\rho_1}, \qquad \mathbb{E}(N_1) = \frac{\rho_1}{1-\rho_1}\,. \tag{2.3}$$

Since for all customers the service time is exponentially distributed with the same parameter, the number of customers does not depends on the order of service. Hence for the total number of customers in an $M/M/1$ we get

$$\mathbb{E}(N_1) + \mathbb{E}(N_2) = \frac{\rho_1 + \rho_2}{1-\rho_1-\rho_2}, \tag{2.4}$$

and then inserting (2.3) we obtain

$$\mathbb{E}(N_2) = \frac{\rho_1 + \rho_2}{1-\rho_1-\rho_2} - \frac{\rho_1}{1-\rho_1} = \frac{\rho_2}{(1-\rho_1)(1-\rho_1-\rho_2)}\,,$$

and using the Little's law we have

$$\mathbb{E}(T_2) = \frac{\mathbb{E}(N_2)}{\lambda_2} = \frac{1/\mu}{(1-\rho_1)(1-\rho_1-\rho_2)}\,.$$

Example 2 *Let us compare what is the difference if preemptive priority discipline is applied instead of FIFO.*

Let $\lambda_1 = 0.5$, $\lambda_2 = 0.25$ and $\mu = 1$. In FIFO case we get

$$\mathbb{E}(T) = 4.0, \quad \mathbb{E}(W) = 3.0, \quad \mathbb{E}(N) = 3.0$$

and in priority case we obtain

$$\mathbb{E}(T_1) = 2.0, \quad \mathbb{E}(W_1) = 1.0, \quad \mathbb{E}(N_1) = 1.0$$

$$\mathbb{E}(T_2) = 8.0, \quad \mathbb{E}(W_2) = 6.0, \quad \mathbb{E}(N_2) = 2.0$$

Non-preemptive Priority

The only difference between the two disciplines is that in the case the arrival of a class 1 customer does not interrupt the service of type 2 request. That is why sometimes this discipline is call HOL (Head Of the Line). Of course after finishing the service of class 1 starts.

By using the law of total expectations the mean response time for class 1 can be obtained as

$$\mathbb{E}(T_1) = \mathbb{E}(N_1)\frac{1}{\mu} + \frac{1}{\mu} + \rho_2\frac{1}{\mu}.$$

The last term shows the situation when an arriving class 1 customer find the server busy servicing a class 2 customer. Since the service time is exponentially distributed the residual service time has the same distribution as the original one. Furthermore, because of the Poisson arrivals the distribution at arrival moments is the same as at random moments, that is the probability that the server is busy with class 2 customer is ρ_2. By using the Little's law

$$\mathbb{E}(N_1) = \lambda_1\mathbb{E}(T_1),$$

after substitution we get

$$\mathbb{E}(T_1) = \frac{(1+\rho_2)/\mu}{1-\rho_1}, \qquad \mathbb{E}(N_1) = \frac{(1+\rho_2)\rho_1}{1-\rho_1}\,.$$

To get the means for class 2 the same procedure can be performed as in the previous case. That is using (2.4) after substitution we obtain

$$\mathbb{E}(N_2) = \frac{(1-\rho_1(1-\rho_1-\rho_2))\rho_2}{(1-\rho_1)(1-\rho_1-\rho_2)}\,,$$

and then applying the Little's law we have

$$\mathbb{E}(T_2) = \frac{(1-\rho_1(1-\rho_1-\rho_2))/\mu}{(1-\rho_1)(1-\rho_1-\rho_2)}\,.$$

Example 3 *Now let us compare the difference between the two priority disciplines.*
Let $\lambda_1 = 0.5$, $\lambda_2 = 0.25$ and $\mu = 1$, then

$$\mathbb{E}(T_1) = 2.5, \quad \mathbb{E}(W_1) = 1.5, \quad \mathbb{E}(N_1) = 1.25$$

$$\mathbb{E}(T_2) = 7.0, \quad \mathbb{E}(W_2) = 6.0, \quad \mathbb{E}(N_2) = 1.75$$

Of course knowing the mean response time and mean number of customers in the system the mean waiting time and the mean number of waiting customers can be obtained in the usual way.

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcPrio/MMcPrio.html`

`http://www.win.tue.nl/cow/Q2/`

2.4 The $M/M/1/K$ Queue, Systems with Finite Capacity

Let K be the capacity of an $M/M/1$ system, that is the maximum number of customers in the system including the one under service. It is easy to see that the nu,ber of customers in the systems is a birth-death process with rates $\lambda_k = \lambda$, $k = 0, \ldots, K-1$ és $\mu_k = \mu$, $k = 1, \ldots, K$. For the steady-state distribution we have

$$P_k = \frac{\rho^k}{\sum_{i=0}^{K} \rho^i}, \qquad k = 0, \ldots, K,$$

that is

$$P_0 = \frac{1}{\sum_{i=0}^{K} \rho^i} = \begin{cases} \frac{1}{K+1}, & \rho = 1 \\ \\ \frac{1-\rho}{1-\rho^{K+1}}, & \rho \neq 1. \end{cases}$$

It sholud be noted that the system is stable for any $\rho > 0$ when K is fixed. However, if $K \to \infty$ the the stability condition is $\rho < 1$ since the distribution of $M/M/1/K$ converges to the distribution of $M/M/1$.
It can be verified analytically since $\rho^K \to 0$ then $P_0 \to 1 - \rho$.

Similarly to an $M/M/1$ systems after reasonable modifications the *performance measures* can be computed as

- $$U_S = 1 - P_0,$$
$$\mathbb{E}(\delta) = \frac{1}{\lambda} \frac{U_S}{1 - U_S}$$

- $$\begin{aligned}\overline{N} &= \sum_{k=1}^{K} k\rho^k P_0 = \rho P_0 \sum_{k=1}^{K} k\rho^{k-1} \\ &= \rho P_0 \left(\sum_{k=1}^{K} \rho^k\right)' = \rho P_0 \left(\rho \frac{1-\rho^K}{1-\rho}\right)' = \rho P_0 \left(\frac{\rho-\rho^{K+1}}{1-\rho}\right)' \\ &= \left(\left(1-(K+1)\rho^K\right)(1-\rho)+\rho-\rho^{K+1}\right) \cdot \frac{\rho P_0}{(1-\rho)^2} \\ &= \frac{\rho P_0 \left(1-(K+1)\rho^K-\rho+(K+1)\rho^{K+1}+\rho-\rho^{K+1}\right)}{(1-\rho)^2} \\ &= \frac{\rho P_0 \left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)^2} \\ &= \frac{\rho \left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)(1-\rho^{K+1})}.\end{aligned}$$

- $$\overline{Q} = \sum_{k=1}^{K} (k-1)P_k = \sum_{k=1}^{K} kP_k - \sum_{k=1}^{K} P_k = \overline{N} - U_S$$

- To obtain the distribution of the response and waiting time we have to know the distribution of the system at the moment when the tagged customer enters into to system. It should be underlined that the customer should enter into the system and it is not the same as an arriving customer. An arriving customer can join the system or can be lost because the system is full. By using the Bayes' theorem it is easy to see that

$$\Pi_k = \frac{\lambda P_k}{\sum_{i=0}^{K-1} \lambda P_i} = \frac{P_k}{1-P_K}.$$

Similarly to the investigations we carried out in an $M/M/1$ system the mean and the density function of the response time can be obtained by the help of the law of total means and law of total probability, respectively.

For the expectation we have

$$\begin{aligned}\overline{T} &= \sum_{k=0}^{K-1} \frac{k+1}{\mu} \Pi_k = \sum_{k=0}^{K-1} \frac{k+1}{\mu} \frac{\rho^k P_0}{1-P_k} \\ &= \frac{1}{\lambda(1-P_K)} \sum_{k=0}^{K-1} (k+1)P_{k+1} = \frac{\overline{N}}{\lambda(1-P_K)}.\end{aligned}$$

Consequently

$$\overline{W} = \overline{T} - \frac{1}{\mu} = \frac{\overline{N}}{\lambda(1-P_K)} - \frac{1}{\mu}.$$

We would like to show that the Little's law is valid in this case and the same time we can check the correctness of the formula.
It can easily be seen that the average arrival rate into the system is $\overline{\lambda} = \lambda(1-P_K)$ and thus

$$\overline{\lambda} \cdot \overline{T} = \lambda(1-P_K)\frac{\overline{N}}{\lambda(1-P_K)} = \overline{N}.$$

Similarly

$$\begin{aligned}\overline{\lambda} \cdot \overline{W} &= \overline{\lambda}\left(\frac{\overline{N}}{\lambda(1-P_K)} - \frac{1}{\mu}\right) = \overline{N} - \frac{\overline{\lambda}}{\mu} \\ &= \overline{N} - \rho(1-P_K) = \overline{N} - U_S = \overline{Q},\end{aligned}$$

since

$$\overline{\lambda} = \overline{\mu} = \mu U_S.$$

Now let us find the density function of the response and waiting times
By using the theorem of total probability we have

$$f_T(x) = \sum_{k=0}^{K-1} \mu \frac{(\mu x)^k}{k!} e^{-\mu x} \frac{P_k}{1-P_K},$$

and thus for the distribution function we get

$$\begin{aligned}F_T(x) &= \sum_{k=0}^{K-1} \left(\int_0^x \mu \frac{(\mu t)^k}{k!} e^{-\mu t} \mathrm{d}t \right) \frac{P_k}{1-P_K} \\ &= \sum_{k=0}^{K-1} \left(1 - \sum_{i=0}^{k} \frac{(\mu x)^i}{i!} e^{-\mu x} \right) \frac{P_k}{1-P_K} \\ &= 1 - \sum_{k=0}^{K-1} \left(\sum_{i=0}^{k} \frac{(\mu x)^i}{i!} e^{-\mu x} \right) \frac{P_k}{1-P_K}.\end{aligned}$$

Thes formulas are more complicated due to the finite summation as in the case of an $M/M/1$ system, but it is not difficult to see that in the limiting case as $K \to \infty$ we have

$$f_T(x) = \mu(1-\rho)e^{-\mu(1-\rho)x}.$$

For the density and distribution function of the waiting time we obtain

$$
\begin{aligned}
f_W(0) &= \frac{P_0}{1-P_K} \\
f_W(x) &= \sum_{k=1}^{K-1} \mu \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} \frac{P_k}{1-P_K}, \qquad x > 0 \\
F_W(x) &= \frac{P_0}{1-P_K} + \sum_{k=1}^{K-1} \left(1 - \sum_{i=0}^{k-1} \frac{(\mu x)^i}{i!} e^{-\mu x} \right) \frac{P_k}{1-P_K} \\
&= 1 - \sum_{k=1}^{K-1} \left(\sum_{i=0}^{k-1} \frac{(\mu x)^i}{i!} e^{-\mu x} \right) \cdot \frac{P_k}{1-P_K}.
\end{aligned}
$$

These formulas can be calculated very easily by a computer.
As we can see the probability P_K plays an important role in the calculations. Notice that it is exactly the probability that an arriving customer find the system full that is it lost. It is called **blocking** or **lost probability** and denoted by P_B. Its correctness can be proved by the help of the Bayes's rule, namely

$$
P_B = \frac{\lambda P_K}{\sum_{k=0}^{K} \lambda P_k} = P_K.
$$

If we would like to show the dependence on K and ρ it can be denoted by

$$
P_B(K, \rho) = \frac{\rho^K}{\sum_{k=0}^{K} \rho^k}.
$$

Notice that

$$
P_B(K, \rho) = \frac{\rho \rho^{K-1}}{\sum_{k=0}^{K-1} \rho^k + \rho \rho^{K-1}} = \frac{\rho P_B(K-1, \rho)}{1 + \rho P_B(K-1, \rho)}.
$$

Starting with the initial value $P_B(1, \rho) = \dfrac{\rho}{1+\rho}$ the probability of loss can be computed recursively. It is obvious that this sequence tends to 0 as $\rho < 1$. Consequently by using the recursion we can always find an K-t, for which

$$
P_B(K, \rho) < P^*,
$$

where P^* is a predefined limit value for the probability of loss.

To find the value of K without recursion we have to solve the inequality

$$
\frac{\rho^K (1-\rho)}{1-\rho^{K+1}} < P^*
$$

which is more complicated task.

Alternatively can can find an approximation method, too. Use the distribution of an $M/M/1$ system and find the probability that in the system there are at least K customers. It is easy to see that

$$P_B(K,\rho) = \frac{\rho^K(1-\rho)}{1-\rho^{K+1}} < \sum_{k=K}^{\infty} \rho^k(1-\rho) = \rho^K,$$

and thus if

$$\rho^K < P^*,$$

then $P_B^*(K,\rho) < P^*$. That is

$$\begin{aligned} K\ln\rho &< \ln P^* \\ K &> \frac{\ln P^*}{\ln \rho}. \end{aligned}$$

Now let us turn our attention to the Laplace-transform of the response and waiting times. First let us compute it for the response time. Similarly to the previous arguments we have

$$\begin{aligned} L_T(s) &= \sum_{k=0}^{K-1} \left(\frac{\mu}{\mu+s}\right)^{k+1} \frac{\rho^k P_0}{1-P_K} \\ &= \frac{P_0}{\rho(1-P_K)} \sum_{l=1}^{K} \left(\frac{\mu\rho}{\mu+s}\right)^l \\ &= \frac{P_0}{\rho(1-P_K)} \frac{\lambda}{\mu+s} \frac{1-\left(\frac{\lambda}{\mu+s}\right)^K}{1-\frac{\lambda}{\mu+s}} \\ &= \frac{\mu P_0}{(1-P_K)} \frac{1-\left(\frac{\lambda}{\mu+s}\right)^K}{\mu-\lambda+s}. \end{aligned}$$

The Laplace-transform of the waiting time can be obtained as

$$\begin{aligned} L_W(s) &= \sum_{k=0}^{K-1} \left(\frac{\mu}{\mu+s}\right)^{k} \frac{\rho^k P_0}{1-P_K} \\ &= \frac{P_0}{1-P_K} \sum_{k=0}^{K-1} \left(\frac{\mu\rho}{\mu+s}\right)^{k} \\ &= \frac{P_0}{1-P_K} \frac{1-\left(\frac{\lambda}{\mu+s}\right)^K}{1-\frac{\lambda}{\mu+s}} \\ &= \frac{P_0}{1-P_K} \frac{(\mu+s)\left(1-\left(\frac{\lambda}{\mu+s}\right)^K\right)}{\mu-\lambda+s}, \end{aligned}$$

which also follows from relation

$$L_T(s) = L_W(s) \cdot \frac{\mu}{\mu + s}.$$

By the help of the Laplace-transforms the higher moments of the involved random variables can be computed, too.

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1K/MM1K.html`

2.5 The $M/M/\infty$ Queue

Similarly to the previous systems it is easy to see that the number of customers in the system, that is the process $(N(t), t \geq 0)$ is a birth-death process with rates

$$\lambda_k = \lambda, \quad k = 0, 1, \ldots$$
$$\mu_k = k\mu, \quad k = 1, 2, \ldots.$$

Hence the steady-state distribution can be obtained as

$$P_k = \frac{\varrho^k}{k!} P_0, \text{ where } P_0^{-1} = \sum_{k=0}^{\infty} \frac{\varrho^k}{k!} = e^{\varrho},$$

That is

$$P_k = \frac{\varrho^k}{k!} e^{-\varrho},$$

showing that N follows a Poisson law with parameter ϱ.

It is easy to see that the *performance measures* can be computed as

$$\overline{N} = \varrho, \quad \overline{\lambda} = \lambda, \quad \overline{T} = \frac{1}{\mu}, \quad \overline{W} = 0, \quad \overline{r} = \overline{N}, \quad \overline{\mu} = \overline{r}\mu$$
$$U_r = 1 - e^{-\varrho}, \quad \frac{\mathbb{E}(\delta_r)}{\frac{1}{\lambda}} = \frac{1 - e^{-\varrho}}{e^{-\varrho}}, \quad \mathbb{E}(\delta_r) = \frac{1}{\lambda} \frac{1 - e^{-\varrho}}{e^{-\varrho}}.$$

It can be proved that these formulas remain valid for an $M/G/\infty$ system as well where $\mathbb{E}(S) = \frac{1}{\mu}$.

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMinf/MMinf.html`

2.6 The $M/M/n/n$ Queue, Erlang-Loss System

This system is the oldest and thus the most famous system in queueing theory. The origin of the traffic theory or congestion theory started by the investigation of this system and Erlang was the first who obtained his well-reputed formulas, see for example Erlang [21, 22].
By assumptions customers arrive according to a Poisson process and the service times are exponentially distributed. However, if n servers all busy when a new customer arrives it will be lost because the system is full. The most important question is what proportion of the customers is lost.

The process $(N(t), t \geq 0)$ is said to be in state k if k servers are busy, which is the same as k customers are in the system. It is easy to see that $(N(t), t \geq 0)$is a birth-death process with rates

$$\lambda_k = \begin{cases} \lambda, & \text{if } k < n, \\ 0, & \text{if } k \geq n, \end{cases}$$

$$\mu_k = k\mu, \quad k = 1, 2, ..., n.$$

Clearly the steady-state distribution exists since the process has a finite state space. The stationary distribution can be obtained as

$$P_k = \begin{cases} P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} & \text{, if } k \leq n, \\ 0 & \text{, if } k > 0. \end{cases}$$

Due to the normalizing condition we have

$$P_0 = \left(\sum_{k=0}^{n} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} \right)^{-1},$$

and thus the distribution is

$$P_k = \frac{\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}{\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}} = \frac{\frac{\varrho^k}{k!}}{\sum_{i=0}^{n} \frac{\varrho^i}{i!}}, \qquad k \leq n.$$

The most important measure of the system is

$$P_n = \frac{\frac{\varrho^n}{n!}}{\sum_{k=0}^{n} \frac{\varrho^k}{k!}} = B(n, \rho)$$

which was introduced by Erlang and it is referred to as **Erlang's B-formula, or loss formula** and generally denoted by $B(n, \lambda/\mu)$.

By using the Bayes's rule it is easy to see that P_n is the probability that an arriving customer is lost. For moderate n the probability P_0 can easily be computed. For large n and small ϱ $P_0 \approx e^{-\varrho}$, and thus

$$P_k \approx \frac{\varrho^k}{k!} e^{-\varrho},$$

that is the Poisson distribution. For large n and large ϱ

$$\sum_{j=0}^{n} \frac{\varrho^j}{j!} \neq e^{\varrho}.$$

However, in this case the central limit theorem can be used, since the denominator is the sum of the first $(n+1)$ terms of a Poisson distribution with mean ϱ. Thus by the central limit theorem this Poisson distribution can be approximated by a normal law with mean ϱ and variance $\sqrt{\varrho}$ that is

$$P_n \approx \frac{\Phi(s) - \Phi(s - 1/\sqrt{\varrho})}{\Phi(s)} = 1 - \frac{\Phi(s - 1/\sqrt{\varrho})}{\Phi(s)},$$

where

$$\Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

and

$$s = \frac{n + \frac{1}{2} - \varrho}{\sqrt{\varrho}}.$$

Another way to calculate $B(n, \rho)$ is to find a recursion. This can be obtained as follows

$$\begin{aligned} B(n,p) &= \frac{\frac{\rho^n}{n!}}{\sum_{i=0}^{n} \frac{\rho^i}{i!}} = \frac{\frac{\rho}{n} \frac{\rho^{n-1}}{(n-1)!}}{\sum_{i=0}^{n} \frac{\rho^i}{i!} + \frac{\rho}{n} \frac{\rho^{n-1}}{(n-1)!}} \\ &= \frac{\frac{\rho}{n} B(n-1,\rho)}{1 + \frac{\rho}{n} B(n-1,\rho)} = \frac{\rho B(n-1,\rho)}{n + \rho B(n-1,\rho)}. \end{aligned}$$

Using $B(1,\rho) = \dfrac{\rho}{1+\rho}$ as an initial value the probabilities $B(n,\rho)$ can be computed for any n. It is important since the direct calculation can cause a problem due to the value of the factorial.
For example for $n = 1000, \rho = 1000$ the exact formula cannot be computed but the approximation and the recursion gives the value 0.024.

Due to the great importance of $B(n,\rho)$ in practical problems so-called calculators have been developed which can be found at

`http://www.erlang.com/calculator/`

To compare the approximations and the exact values we also have developed our own Java script which can be used at

`http://jani.uw.hu/erlang/erlang.html`

Now determine the *main performance measures* of this $M/M/n/n$ system

- *Mean number of customers in the systems, mean number of busy servers*

$$\overline{N} = \overline{n} = \sum_{j=0}^{n} jP_j = \sum_{j=0}^{n} j\frac{\varrho^j}{j!}P_0 = \varrho\sum_{j=0}^{n-1}\frac{\varrho^i}{i!}P_0 = \varrho(1-P_n),$$

  thus the mean number of requests for a given server is

$$\frac{\varrho}{n}(1-P_n).$$

- *Utilization of a server*

  As we have seen

$$U_s = \sum_{i=1}^{n}\frac{i}{n}P_i = \frac{\bar{n}}{n}.$$

  This case

$$U_s = \frac{\varrho}{n}(1-P_n).$$

- *The mean idle period for a given server*

  By applying the well-known relation

$$P(\text{the server is idle }) = \frac{1/\mu}{\overline{e}+1/\mu},$$

  where $\overline{e}$ is the mean idle time of the server. Thus

$$\frac{\varrho}{n}(1-P_n) = \frac{1/\mu}{\overline{e}+1/\mu},$$

  hence

$$\overline{e} = \frac{n}{\lambda(1-P_n)} - \frac{1}{\mu}.$$

- *The mean busy period of the system*

  Clearly

$$U_r = 1 - P_0 = \frac{E\delta_r}{\frac{1}{\lambda}+E\delta_r},$$

  thus

$$E\delta_r = \frac{1-P_0}{\lambda P_0} = \frac{\displaystyle\sum_{i=1}^{n}\frac{\varrho^i}{i!}}{\lambda\left(1+\displaystyle\sum_{i=1}^{n}\frac{\varrho^i}{i!}\right)}.$$

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcc/MMcc.html`

Example 4 *In busy parking lot cars arrive according to a Poisson process one in* 20 *seconds and stay there in the average of* 10 *minutes.*
How many parking places are required if the probability of a loss is no to exceed 1% ?

Solution:

$$\rho = \frac{\lambda}{\mu} = \frac{10}{\frac{1}{3}} = 30, P_n = 0.01.$$

Following a normal approximation

$$P_n = 0.01 = \frac{\frac{\rho^n}{n!}e^{-\rho}}{\Phi\left(\frac{n+\frac{1}{2}-\rho}{\sqrt{\rho}}\right)} = \frac{\Phi\left(\frac{n+\frac{1}{2}-\rho}{\sqrt{\rho}}\right) - \Phi\left(\frac{n-\frac{1}{2}-\rho}{\sqrt{\rho}}\right)}{\Phi\left(\frac{n+\frac{1}{2}-\rho}{\sqrt{\rho}}\right)}.$$

Thus

$$0.99\Phi\left(\frac{n+\frac{1}{2}-\rho}{\sqrt{\rho}}\right) = \Phi\left(\frac{n-\frac{1}{2}-\rho}{\sqrt{\rho}}\right).$$

It is not difficult to verify by using the Table for the standard normal distribution that $n = 41$.

Thus the approximation value of P_{41} is 0.009917321712214377,
and the exact value is 0.01043318100246811.
■

Example 5 *A telephone exchange consists of* 50 *lines and calls arrive according to a Poisson process, the mean interarrival time is* 10 *minutes. The mean service time is* 5 *minutes.*
Find the main performance measures.

Solution:
Using Poisson approximation where $\rho = \frac{\lambda}{\mu} = 0.5$
$P_{50} = 0.00000$, event for $n = 6$
$P_6 = 0,00001$. This means that a call is almost never lost.
Mean number of busy lines can be obtain as

$$\overline{n} = \rho(1 - P_n) = \rho = 0.5 \ ,$$

The utilization of a line is

$$\frac{0.5}{50} = \frac{5 \times 10^{-1}}{5 \times 10} = 10^{-2}$$

The utilization of the system is

$$U_r = 1 - 0.606 = 0.394$$

The mean busy period of the system can be obtained as

$$E\delta_r = \frac{(1-P_0)}{(\lambda P_0)} = \frac{0.394}{2 \times 0.606} = \frac{0.394}{1.212} = 0.32 \text{ minutes}$$

Mean idle period of a line is

$$\overline{e} = \frac{n}{\lambda(1-P_n)} - \frac{\rho}{\lambda} = \frac{50}{2(1-0)} - \frac{0,5}{2} = 25 - \frac{1}{4} = 24.75 \text{ minutes}$$

■

Heterogeneous Servers

In the case of an $M/\overrightarrow{M}/n/n$ system the service time distribution depends on the index of the server. That is the service time is exponentially distributed with parameter μ_i for server i. An arriving customer choose randomly among the idle servers, that is each idle server is chosen with the same probability. Since the servers are heterogeneous it is not enough to to the number of busy servers but we have to identify them by their index. It means that we have to deal with general Markov-processes.

Let $(i_1, \ldots, i_k)$ denote the indexes of the busy servers, which are the combinations of n objects taken k at a time without replacement. Thus the state space of the Markov-chain is the set of these combinations, that is $(0, (i_1, \ldots, i_k) \in C_k^n, k = 1, \ldots, n)$.

Let us denote by

$$\begin{aligned} P_0 &= P(0), \\ P(i_1, \ldots, i_k) &= P((i_1, \ldots, i_k)), (i_1, \ldots, i_k) \in C_k^n, \quad k = 1, \ldots, n \end{aligned}$$

the steady-state distribution of the chain which exists since the chain has a finite state space and it is irreducible. The set of steady-state balance equations can be written as

$$\lambda P_0 = \sum_{j=1}^{n} \mu_j P(j) \tag{2.5}$$

$$\begin{aligned} (\lambda + \sum_{j=1}^{k} \mu_{i_j}) P(i_1, \ldots, i_k) = \frac{\lambda}{n-k+1} \sum_{j=1}^{k} P(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k) \\ + \sum_{j \neq i_1, \ldots, i_k} \mu_j P(i'_1, \ldots, i'_k, j') \end{aligned} \tag{2.6}$$

$$\left(\sum_{j=1}^{n} \mu_j \right) P(1, \ldots, n) = \lambda \sum_{j=1}^{n} P(1, \ldots, j-1, j+1, \ldots, n) \tag{2.7}$$

where $(i'_1, \ldots, i'_k, j')$ denotes the ordered set $i_1, \ldots, i_k, j$, i_{-1} and i_{n+1} are not defined. Despite of the large number of unknowns, which is 2^n, the solution is quite simple, namely

$$P(i_1, \ldots, i_k) = (n-k)! \prod_{j=1}^{k} \varrho_{i_j} C, \tag{2.8}$$

where $\varrho_j = \dfrac{\lambda}{\mu_i}, \quad j = 1, \ldots, n, \quad P_0 = n!C$, which can be determined by the help of the normalizing condition

$$P_0 + \sum_{k=1}^{n} \sum_{(i_1,\ldots,i_k) \in C_k^n} P(i_1, \ldots, i_k) = 1.$$

Let us check the first equation (2.5). By substitution we have

$$\lambda n! C = \sum_{j=1}^{n} \mu_j \frac{\lambda}{\mu_j} (n-1)! C = n! \lambda C.$$

Lets us check now the third equation (2.7)

$$\left(\sum_{j=1}^{n} \mu_j \right) \frac{\lambda^n}{\mu_1 \cdots \mu_n} C = \lambda \sum_{j=1}^{n} \frac{\lambda^{n-1} C}{\mu_1 \cdots \mu_{j-1} \mu_{j+1} \cdots \mu_n} = \frac{\lambda^n}{\mu_1 \cdots \mu_n} \left(\sum_{j=1}^{n} \mu_j \right) C.$$

Finally let us check the most complicated one, the second set of equations (2.6), namely

$$\begin{aligned}
&(\lambda + \sum_{j=1}^{k} \mu_{i_j})(n-k)! \prod_{j=1}^{k} \varrho_{i_j} C \\
&= \frac{\lambda}{n-k+1} (n-k+1)! \sum_{j=1}^{k} \frac{\lambda^{k-1} C}{\mu_{i_1} \cdots \mu_{i_{j-1}} \mu_{i_{j+1}} \cdots \mu_{i_k}} \\
&+ \sum_{j \neq i_1, \ldots, i_k} (n-k-1)! \frac{\lambda^{k+1} \mu_j C}{\mu_{i_1} \cdots \mu_{i_k} \mu_j} \\
&= (n-k)! \sum_{j=1}^{k} \frac{\mu_{i_j} \lambda^k C}{\mu_{i_1} \cdots \mu_{i_k}} + \lambda \sum_{j \neq i_1, \ldots, i_k} (n-k-1)! \frac{\lambda^k C}{\mu_{i_1} \cdots \mu_{i_k}} \\
&= (n-k)! \left(\sum_{j=1}^{k} \mu_{i_j} \right) \frac{\lambda^k C}{\mu_{i_1} \cdots \mu_{i_k}} + \lambda (n-k)! \frac{\lambda^k C}{\mu_{i_1} \cdots \mu_{i_k}},
\end{aligned}$$

which shows the equality.

Thus the usual *performance measures* can be obtained as

- the utilization of the jth server U_j can be calculated as

$$U_j = \sum_{k=1}^{n} \sum_{j \in (i_1,\ldots,i_k)} P(i_1, \ldots, i_k),$$

and thus

$$U_j = \frac{\frac{1}{\mu_j}}{\frac{1}{\mu_j} + \mathbb{E}(e_j)},$$

where $\mathbb{E}(e_j)$ is the mean idle period of the jth server. Hence

$$\mathbb{E}(e_j) = \frac{1}{\mu_j} \frac{1 - U_j}{U_j}.$$

- $\overline{N} = \sum_{j=1}^{n} U_j$
- The probability of loss is $P_B = P(1, \ldots, n)$.

It should be noted that in this case the following relation also holds

$$\lambda(1 - P_B) = \sum_{j=1}^{n} U_j \mu_j.$$

In homogeneous case, that is when $\mu_j = \mu, j = 1, \ldots, n$, after substitution we have

$$P_k = \sum_{(i_1,\ldots,i_k) \in C_k^n} P(i_1, \ldots, i_k) = \binom{n}{k} (n-k)! \varrho^k C = \frac{\varrho^k}{k!} n! C = \frac{\varrho^k}{k!} P_0 = \frac{\frac{\varrho^k}{k!}}{\sum_{j=1}^{n} \frac{\varrho^j}{j!}},$$

that is it reduces to the Erlang's formula derived earlier.

It should be noted that these formulas remains valid under generally distributed service times with finite means with $\rho_i = \lambda E(S_i)$. In other words the Erlang's loss formula is robust to the distribution of the service time, it does not depend on the distribution itself but only on its mean.

2.7 The $M/M/n$ Queue

It is a variation of the classical queue assuming that the service is provided by n servers operating independently of each other. This modification is natural since if the mean arrival rate is greater than the service rate the system will not be stable, that is why the number of servers should be increased. However, in this situation we have parallel services and we are interested in the distribution of first service completion.
That is why we need the following observation.

Let X_i be exponentially distributed random variables with parameter μ_i, $(i = 1, 2, ..., r)$ and denote by Y their minimum. It is not difficult to see that Y is also exponentially distributed with parameter $\sum_{i=1}^{r} \mu_i$ since

$$P(Y < x) = 1 - P(Y \geq x) = 1 - P(X_i \geq x,\ i = 1, ..., r) =$$

$$= 1 - \prod_{i=1}^{r} P(X_i \geq x) = 1 - e^{-\left(\sum_{i=1}^{r} \mu_i\right)x}.$$

Similarly to the earlier investigations, it can easily be verified that the number of customers in the system is a birth-death process with the following transition probabilities

$$P_{k,k-1}(h) = (1 - (\lambda h + o(h)))\,(\mu_k h + o(h)) + o(h) = \mu_k h + o(h),$$
$$P_{k,k+1}(h) = (\lambda h + o(h))\,(1 - (\mu_k h + o(h))) + o(h) = \lambda h + o(h),$$

where

$$\mu_k = \min(k\mu, n\mu) = \begin{cases} k\mu & \text{, for } 0 \leq k \leq n, \\ \\ n\mu & \text{, for } n < k. \end{cases}$$

It is understandable that the stability condition is $\lambda/n\mu < 1$.

To obtain the distribution P_k we have to distinguish two cases according to as μ_k depends on k. Thus if $k < n$, then we get

$$P_k = P_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} = P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}.$$

Similarly, if $k \geq n$, then we have

$$P_k = P_0 \prod_{i=0}^{n-1} \frac{\lambda}{(i+1)\mu} \prod_{j=n}^{k-1} \frac{\lambda}{n\mu} = P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{n! n^{k-n}}.$$

In summary

$$P_k = \begin{cases} P_0 \dfrac{\rho^k}{k!} & \text{, for } k \leq n, \\ \\ P_0 \dfrac{a^k n^n}{n!} & \text{, for } k > n, \end{cases}$$

where

$$a = \frac{\lambda}{n\mu} = \frac{\rho}{n} < 1.$$

This a is exactly the $u$tilization of a given server . Furthermore

$$P_0 = \left(1 + \sum_{k=1}^{n-1} \frac{\rho^k}{k!} + \sum_{k=n}^{\infty} \frac{\rho^k}{n!} \frac{1}{n^{k-n}}\right)^{-1},$$

and thus

$$P_0 = \left(\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{\rho^n}{n!} \frac{1}{1-a} \right)^{-1}.$$

Since the arrivals follow a Poisson law the the distribution of the system at arrival instants equals to the distribution at random moments, hence the probability that an arriving customer has to wait is

$$P(\text{waiting}) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty} P_0 \frac{\rho^k}{n!} \frac{1}{n^{k-n}}.$$

that is it can be written as

$$P(\text{waiting}) = \frac{\frac{\rho^n}{n!} \frac{1}{1-a}}{\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{\rho^n}{n!} \frac{1}{1-a}} = \frac{\frac{\rho^n}{n!} \frac{n}{n-\rho}}{\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{\rho^n n}{n!(n-\rho)}} = C(n, \rho).$$

This probability is frequently used in different practical problems, for example in telephone systems, call centers, just to mention some of them. It is also a very famous formula which is referred to as **Erlang's C formula**,or **Erlang's delay formula** and it is denoted by $C(n, \lambda/\mu)$.

The main *performance measures* of the systems can be obtained as follows

- *For the mean queue length we have*

$$\overline{Q} = \sum_{k=n}^{\infty} (k-n) P_k = \sum_{j=0}^{\infty} j P_{n+j} = \sum_{j=0}^{\infty} \frac{j \left(\frac{\lambda}{\mu}\right)^{n+j}}{n! n^j} P_0 =$$

$$= \sum_{j=0}^{\infty} j \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} a^j P_0 = P_0 \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} a \sum_{j=0}^{\infty} \frac{da^j}{da} = P_0 \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} a \frac{d}{da} \sum_{j=0}^{\infty} a^j =$$

$$= P_0 \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} \frac{a}{(1-a)^2} = \frac{\rho}{n-\rho} C(n, \rho).$$

- For the *m*ean number of busy servers we obtain

$$\overline{n} = \sum_{k=0}^{n-1} k P_k + \sum_{k=n}^{\infty} n P_k = P_0 \left(\rho \sum_{k=0}^{n-2} \frac{\rho^k}{k!} + \frac{\rho^n}{(n-1)!} \frac{1}{1-a} \right) =$$

$$= \rho \left(\sum_{k=0}^{n-2} \frac{\rho^k}{k!} + \frac{\rho^{n-1}}{(n-1)!} + \frac{\rho^{n-1}}{(n-1)!} \left(\frac{1}{1-a} - 1 \right) \right) P_0 =$$

$$= \rho \left(\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{\rho^n}{n!} \frac{1}{1-a} \right) P_0 = \rho \frac{1}{p_0} P_0 = \rho.$$

- *For the mean number of customers in the system we get*

$$\begin{aligned}\overline{N} &= \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{n-1} kP_k + \sum_{k=n}^{\infty}(k-n)P_k + \sum_{k=n}^{\infty} nP_k = \overline{n} + \overline{Q} \\ &= \rho + \frac{\rho}{n-\rho}C(n,\rho),\end{aligned}$$

which is understandable since a customer is either in the queue or in service. Let us denote by $\overline{S}$-gal the mean number of idle servers. Then it is easy to see that

$$\begin{aligned}\overline{n} &= n - \overline{S}, \\ \overline{S} &= n - \frac{\lambda}{\mu},\end{aligned}$$

thus

$$\overline{N} = n - \overline{S} + \overline{Q},$$

hence

$$\overline{N} - n = \overline{Q} - \overline{S}.$$

- *Distribution of the waiting time*

An arriving customer has to wait if at his arrival the number of customers in the system is at least n. In this case the time while a customer is serviced is exponentially distributed with parameter $n\mu$, consequently if there $n+j$ customers in the system the waiting time is Erlang distributed with parameters $(j+1, n\mu)$. By applying the theorem of total probability for the density function of the waiting time we have

$$f_W(x) = \sum_{j=0}^{\infty} P_{n+j}(n\mu)^{j+1}\frac{x^j}{j!}e^{-n\mu x}.$$

Substituting the distribution we get

$$\begin{aligned}f_W(x) &= \sum_{j=0}^{\infty} P_0 \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} a^j (n\mu)^{j+1}\frac{x^j}{j!}e^{-n\mu x} \\ &= \frac{P_0\left(\frac{\lambda}{\mu}\right)^n}{n!} n\mu e^{-n\mu x}\sum_{j=0}^{\infty}\frac{(an\mu x)^j}{j!} \\ &= \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0 n\mu e^{-(n\mu-\lambda)x} \\ &= \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0 n\mu e^{-n\mu(1-a)x} \\ &= \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0 \frac{1}{1-a} n\mu(1-a)e^{-n\mu(1-a)x} \\ &= P(\text{waiting})n\mu(1-a)e^{-n\mu(1-a)x}.\end{aligned}$$

Hence for the complement of the distribution function we obtain

$$P(W > x) = \int_x^\infty f_W(u)du = P(\text{waiting})e^{-n\mu(1-a)x}$$
$$= C(n,\rho)\cdot e^{-\mu(n-\rho)x}.$$

Therefore the distribution function can be written as

$$F_W(x) = 1 - P(\text{waiting}) + P(\text{waiting})\left(1 - e^{-n\mu(1-a)x}\right)$$
$$= 1 - P(\text{waiting})e^{-n\mu(1-a)x} = 1 - C(n,\rho)\cdot e^{-\mu(n-\rho)x}.$$

Consequently the mean waiting time can be calculated as

$$\overline{W} = \int_0^\infty x f_W(x)dx = \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}P_0\frac{1}{(1-a)^2 n\mu} = \frac{1}{\mu(n-\rho)}C(n,\rho).$$

- *Distribution of the response time*

The service immediately starts if at arrival the number of customer in the system is than n. However, if the arriving customer has to wait then the response time is the sum of this waiting and service times. By applying the law of total probability for the density function of the response time we get

$$f_T(x) = P(\text{no waiting})\mu e^{-\mu x} + f_{W+S}(x)$$

As we have proved

$$f_W(x) = P(\text{waiting})e^{-n\mu(1-a)x}n\mu(1-a).$$

Thus

$$f_{W+S}(z) = \int_0^z f_W(x)\mu e^{-\mu(z-x)}dx =$$

$$= P(\text{waiting})n\mu(1-a)\mu\int_0^z e^{-n\mu(1-a)x}e^{-\mu(z-x)}dx =$$

$$= \frac{\rho^n}{n!}P_0\frac{1}{(1-a)}n\mu(1-a)\mu e^{-z\mu}\int_0^z e^{-\mu(n-1-\lambda/\mu)x}dx =$$

$$= \frac{\rho^n}{n!}P_0 n\mu\frac{1}{n-1-\lambda/\mu}e^{-\mu z}\left(1 - e^{-\mu(n-1-\lambda/\mu)z}\right).$$

Therefore

$$f_T(x) = \left(1 - \left(\frac{\lambda}{\mu}\right)^n\frac{P_0}{n!(1-a)}\right)\mu e^{-\mu x} +$$

$$+\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}n\mu P_0\frac{1}{n-1-\lambda/\mu}e^{-\mu x}\left(1-e^{-\mu(n-1-\lambda/\mu)x}\right)=$$

$$=\mu e^{-\mu x}\left(1-\frac{\left(\frac{\lambda}{\mu}\right)^n P_0}{n!(1-a)}+\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}nP_0\frac{1}{n-1-\lambda/\mu}\left(1-e^{-\mu(n-1-\lambda/\mu)x}\right)\right)=$$

$$=\mu e^{-\mu x}\left(1+\frac{\left(\frac{\lambda}{\mu}\right)^n P_0}{n!(1-a)}\frac{1-(n-\lambda/\mu)e^{-\mu(n-1-\lambda/\mu)x}}{n-1-\lambda/\mu}\right).$$

Consequently for the complement of the distribution function of the response time we have

$$P(T>x)=\int_x^\infty f_T(y)dy=$$

$$=\int_x^\infty \mu e^{-\mu y}+\frac{\left(\frac{\lambda}{\mu}\right)^n P_0}{n!(1-a)}\frac{1}{n-1-\lambda/\mu}\left(\mu e^{-\mu y}-\mu(n-\lambda/\mu)e^{-\mu(n-\lambda/\mu)y}\right)dy=$$

$$=e^{-\mu x}+\left(\frac{\lambda}{\mu}\right)^n P_0\frac{1}{n!(1-a)(n-1-\lambda/\mu)}\left(e^{-\mu x}-e^{-\mu(n-\lambda/\mu)x}\right)=$$

$$=e^{-\mu x}\left(1+\frac{\left(\frac{\lambda}{\mu}\right)^n P_0}{n!(1-a)}\frac{1-e^{-\mu(n-1-\lambda/\mu)x}}{n-1-\lambda/\mu}\right).$$

Thus the distribution function can be written as

$$F_T(x)=1-P(T>x).$$

In addition for the mean response time we obtain

$$\overline{T}=\int_0^\infty xf_T(x)dx=\frac{1}{\mu}+\frac{1}{n\mu}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}P_0\frac{1}{(1-a)^2}=\frac{1}{\mu}+\overline{W},$$

as it was expected.

In stationary case the mean number of arriving customer should be equal to the mean number of departing customers, so the mean number of customer in the system is equal to the mumber of customers arrived during a mean response time. That is

$$\lambda\overline{T}=\overline{N}=\overline{Q}+\overline{n},$$

in addition

$$\lambda\overline{W}=\overline{Q}.$$

These are the **Little's formulas**, that can be proved by simple calculations. As we have seen

$$\overline{N} = \rho + P_0 \frac{\rho^n}{n!(1-a)^2} a.$$

Since

$$\overline{T} = \frac{1}{\mu} + \frac{1}{n\mu} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} P_0 \frac{1}{(1-a)^2},$$

thus

$$\lambda \overline{T} = \frac{\lambda}{\mu} + \frac{\rho^n}{n!} P_0 \frac{a}{(1-a)^2},$$

that is

$$\overline{N} = \lambda \overline{T},$$

because $\dfrac{\lambda}{\mu} = \rho$.

Furthermore

$$\overline{Q} = \lambda \overline{W},$$

since

$$\overline{n} = \rho.$$

- *Overall utilization of the servers* can be obtained as

  The utilization of a single server is

$$U_s = \sum_{k=1}^{n-1} \frac{k}{n} P_k + \sum_{k=n}^{\infty} P_k = \frac{\bar{n}}{n} = a.$$

  Hence the overall utilization can be written as

$$U_n = nU_s = \bar{n}.$$

- *The mean busy period of the system* can be computed as

  The system is said to be idle if the is no customer in the system, otherwise the system is busy. Let $E\delta_r$ denote the mean busy period of the system. Then the utilization of the system is

$$U_r = 1 - P_0 = \frac{E\delta_r}{\frac{1}{\lambda} + E\delta_r},$$

  thus

$$E\delta_r = \frac{1 - P_0}{\lambda P_0}.$$

  If the individual servers are considered then we assume that a given server becomes busy earlier if it became idle earlier. Hence if $j < n$ customers are in the system then the number of idle servers is $n - j$.

Let as consider a given server. On the condition that at the instant when it became idle the number of customers in the system was j its mean idle time is

$$\overline{e}_j = \frac{n-j}{\lambda}.$$

The probability of this situation is

$$a_j = \frac{P_j}{\sum_{i=0}^{n-1} P_i}.$$

Then applying the law of total expectations for its mean idle period we have

$$\overline{e} = \sum_{j=0}^{n-1} a_j \overline{e}_j = \sum_{j=0}^{n-1} \frac{(n-j)P_j}{\lambda \sum_{i=0}^{n-1} P_i} = \frac{\overline{S}}{\lambda P(e)},$$

where $P(e)$ denotes the probability that an arriving customer find an idle server.

Since

$$U_s = a = \frac{E\delta}{\overline{e} + E\delta},$$

thus

$$a\overline{e} = (1-a)E\delta,$$

where $E\delta$ denotes it busy period.

Hence

$$E\delta = \frac{a}{1-a}\frac{\overline{S}}{\lambda P(e)}.$$

In the case of $n = 1$ it reduces to

$$\overline{S} = 1 - a, \qquad P(e) = P_0 = 1 - a, \qquad a = \frac{\lambda}{\mu},$$

thus

$$E\delta = \frac{1}{\mu - \lambda},$$

which was obtained earlier.

In the following we are going to show what is the connection between these two famous Erlang's formulas. Namely, first we prove how the delay formula can be expressed by the help of loss formula, that is

$$C\left(m, \frac{\lambda}{\mu}\right) = \frac{(\frac{\lambda}{\mu})^m}{m!} \frac{1}{1-\frac{\lambda}{m\mu}} \frac{1}{\sum_{k=0}^{m-1} \frac{(\frac{\lambda}{\mu})^k}{k!} + \frac{(\frac{\lambda}{\mu})^m}{m!} \frac{1}{1-\frac{\lambda}{m\mu}}} = \frac{\frac{(\frac{\lambda}{\mu})^m}{m!}}{\sum_{k=0}^{m-1} \frac{(\frac{\lambda}{\mu})^m}{m!}(1-\frac{\lambda}{m\mu}) + \frac{(\frac{\lambda}{\mu})^m}{m!}}$$

$$= \frac{B(m, \frac{\lambda}{\mu})}{(1 - B(m, \frac{\lambda}{\mu}))(1-\frac{\lambda}{m\mu}) + B(m, \frac{\lambda}{\mu})} = \frac{B(m, \frac{\lambda}{\mu})}{1 - \frac{\lambda}{m\mu}(1 - B(m, \frac{\lambda}{\mu}))}.$$

As we have seen in the previous investigations the delay probability $C(n,\rho)$, plays an important role in determining the main performance measures. Notice that the above formula can be rewritten as

$$C(n,\rho) = \frac{nB(n,\rho)}{n-\rho+\rho B(n,\rho)},$$

moreover it can be proved that there exists a recursion for it, namely

$$C(n,\rho) = \frac{\rho(n-1-\rho)\cdot C(n-1,\rho)}{(n-1)(n-\rho)-\rho C(n-1,\rho)},$$

starting with the value $C(1,\rho)=\rho$.

If the quality of service parameter is $C(n,\rho)$ then it is easy to see that there exists an olyan n^*_α, for which $C(n^*_\alpha,\rho)<\alpha$. This n^*_α can easily be calculated by a computer using the above recursion.

Let us show another method for calculating this value. As we have seen earlier the probability of loss can be approximated as

$$B(n,\rho) \approx \frac{\varphi\left(\frac{n-\rho}{\sqrt{\rho}}\right)}{\sqrt{\rho}\phi\left(\frac{n-\rho}{\sqrt{\rho}}\right)}.$$

Let $k=\frac{n-\rho}{\sqrt{\rho}}$, thus $n=\rho+\sqrt{\rho}k$. Hence

$$\begin{aligned} C(n,\rho) = \frac{nB(n,\rho)}{n-\rho+\rho B(n,\rho)} &\approx \frac{(\rho+k\sqrt{\rho})\frac{\varphi(k)}{\sqrt{\rho}\phi(k)}}{\rho+k\sqrt{k}-\rho+\rho\frac{\varphi(k)}{\sqrt{\rho}\phi(k)}} \\ &\approx \frac{\sqrt{\rho}\frac{\varphi(k)}{\phi(k)}}{\sqrt{\rho}\left(k+\frac{\varphi(k)}{\phi(k)}\right)} = \left(1+k\frac{\phi(k)}{\varphi(k)}\right)^{-1}. \end{aligned}$$

That is if we would like to find such an n^*_α for which $C(n^*_\alpha,\rho)<\alpha$, then we have to solve the following equation

$$\left(1+k_\alpha\frac{\phi(k_\alpha)}{\varphi(k_\alpha)}\right)^{-1} \approx \alpha$$

which can be rewritten as

$$k_\alpha\frac{\phi(k_\alpha)}{\varphi(k_\alpha)} = \frac{1-\alpha}{\alpha}$$

If k_α is given then

$$n^*_\alpha = \rho + k_\alpha\sqrt{\rho}.$$

It should be noted that the search for k_α is independent of the value of ρ and n thus it can be calculated for various values of α.

For example, if $\alpha = 0.8, 0.5, 0.2, 0.1$,
then the corresponding k_α-as are $0.1728, 0.5061, 1.062, 1.420$.

The formula $n^*_\alpha = \rho + k_\alpha\sqrt{\rho}$ is called as **square-root staffing rule**. As we can see in the following Table it gives a very good approximation, see Tijms [91].

Table 2.1: Exact and approximated values of n^*

	$\alpha = 0.5$		$\alpha = 0.2$		$\alpha = 0.1$	
	exact	approximation	exact	approximation	exact	approximation
$\rho = 1$	2	2	3	3	3	3
$\rho = 5$	7	7	8	8	9	9
$\rho = 10$	12	12	14	14	16	15
$\rho = 50$	54	54	58	58	61	61
$\rho = 100$	106	106	111	111	115	115
$\rho = 250$	259	259	268	267	274	273
$\rho = 500$	512	512	525	524	533	532
$\rho = 1000$	1017	1017	1034	1034	1046	1045

Let us see an example for illustration.
Let us consider two service centers which operate separately. Then using this rule overall we have to use $2(\rho + k_\alpha\sqrt{\rho})$ servers. However, if we have a joint queue to get the same service level we should use $2\rho + k_\alpha\sqrt{2\rho}$ servers. The reduction is

$$(2 - \sqrt{2})k_\alpha\sqrt{\rho}$$

, that is the reason that the joint queue is used in practice.

$C(n, \rho)$ is of great importance in practical problems hence so-called calculators have been developed and can be used at the link

`http://www.erlang.com/calculator/`

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMc/MMc.html`

Example 6 *Consider a service center with* 4 *servers where* $\lambda = 6$, $\mu = 2$.
Find the performance measures of the system.

Solution:

$$P_0 = 0.0377, \quad \overline{Q} = 1.528, \quad \overline{N} = 4.528, \quad \overline{S} = 1, \quad \overline{n} = 3,$$

$$P(W > 0) = P(n \geq 4) = C(4, 3) = 0.509, \overline{W} = 0.255 \text{ time unit}, \quad \overline{T} = 0.755 \text{ time unit},$$

$$U_n = \frac{3}{4}, \overline{e} = 0.35 \text{ time unit }, \quad E\delta = 1.05 \text{ time unit },$$

$$E\delta_r = 4.2 \text{ time unit }, \quad U_r = 0.9623.$$

■

Example 7 *Find the number of runways in an airport such a way the the probability of waiting of an airplane should not exceed* 0.1. *The arrivals are supposed to be Poisson distributed with rate* $\lambda = 27$ *per hour and the service times are exponentially distributed with a mean of* 2 *minutes.*

Solution:
First use the same time unit for the rates, let us compute in hours. Hence $\mu = 30$ and for stability we need $\frac{\lambda}{n\mu} < 1$ which results $n > 1$.

Denote by $P_i(W > 0)$ the probability of waiting for i runways. By applying the corresponding formulas we get

$$P_2(W > 0) = 0.278, \quad P_3(W > 0) = 0.070, \quad P_4(W > 0) = 0.014.$$

Hence the solution is $n = 3$. In this case $P_0 = 0.403$ and $\overline{W} = 0.0665 hour, \quad \overline{Q} = 0.03$.
■

Example 8 *Consider a fast food shop where to the customers arrive according to a Poisson law one customer in* 6 *seconds on the average. The service time is exponentially distributed with* 20 *seconds mean. Assuming that the maintenance cost of a server is* 100 *Hungarian Forint and the waiting cost is the same find the optimal value of the server which minimizes the mean cost per hour.*

Solution:

$$\overline{Q} = \lambda \overline{W} = 100 \times \frac{3600}{6} \overline{W}$$

$$E(TC) = 100 \times n + 100 \times 600 \times \overline{W}$$

$$\frac{\lambda}{\mu} = \frac{\frac{1}{6}}{\frac{1}{20}} = \frac{20}{6} \text{ thus } n \geq 4 \ .$$

Computing for the values $n = 4,\ 5,\ 6,\ 7,\ 8$ we have found that the minimum is achieved at $n = 5$. This case the performance measures are

$$\overline{W} = 3.9 second, \quad P(e) = 0.66, \quad P(W) = 0.34 \ ,$$

$$E\delta = 29.7 second, \quad \overline{e} = 14.9 second, \quad \overline{Q} = 0.65 \ ,$$

$$\overline{n} = 2.5, \quad \overline{N} = 3.15, \quad \overline{S} = 2.5, \quad E(TC) = 565 \text{ HUF/hour}.$$

■

2.8 The $M/M/c/K$ Queue - Multiserver, Finite-Capacity Systems

This queue is a variation of a multiserver system and only maximum K customers are allowed to stay in the system. As earlier the number of customers in the system is a birth-deat process with appropriate rates and for the steady-state distribution we have

$$P_n = \begin{cases} \frac{\lambda^n}{n!\mu^n} P_0, & \text{for } 0 \le n \le c \\ \\ \frac{\lambda^n}{c^{n-c}c!\mu^n} P_0, & \text{for } c \le n \le K. \end{cases}$$

From the normalizing condition for P_0 we have

$$P_0 = \left(\sum_{n=0}^{c-1} \frac{\lambda^n}{n!\mu^n} + \sum_{n=c}^{K} \frac{\lambda^n}{c^{n-c}c!\mu^n} \right)^{-1}.$$

To simplify this expression let $\rho = \dfrac{\lambda}{\mu}, \quad a = \dfrac{\rho}{c}.$

Then

$$\sum_{n=c}^{K} \frac{\rho^n}{c^{n-c}c!} = \frac{\rho^c}{c!} \sum_{n=c}^{K} a^{n-c} = \begin{cases} \frac{\rho^c}{c!} \frac{1-a^{K-c+1}}{1-a}, & \text{if } a \ne 1 \\ \\ \frac{\rho^c}{c!}(K-c+1), & \text{if } a = 1. \end{cases}$$

Thus

$$P_0 = \begin{cases} \left[\frac{\rho^c}{c!} \frac{1-a^{K-c+1}}{1-a} + \sum_{n=0}^{c-1} \frac{\rho^n}{n!} \right]^{-1}, & \text{if } a \ne 1 \\ \\ \left[\frac{\rho^c}{c!}(K-c+1) + \sum_{n=0}^{c-1} \frac{\rho^n}{n!} \right]^{-1}, & \text{if } a = 1. \end{cases}$$

The main performance measures can be obtained as follows

- Mean queue length

$$\begin{aligned} \overline{Q} &= \sum_{n=c+1}^{K} (n-c) P_n = \sum_{n=c+1}^{K} (n-c) \frac{\lambda^n}{c^{n-c}c!\mu^n} P_0 = \frac{P_0 \rho^c}{c!} \sum_{n=c+1}^{K} (n-c) \frac{\rho^{n-c}}{c^{n-c}} \\ &= \frac{P_0 \rho^c a}{c!} \sum_{n=c+1}^{K} (n-c) a^{n-c-1} = \frac{P_0 \rho^c a}{c!} \sum_{i=1}^{K-c} i a^{i-1} = \frac{P_0 \rho^c a}{c!} \frac{d}{da} \left(\sum_{i=0}^{K-c} a^i \right) \\ &= \frac{P_0 \rho^c a}{c!} \frac{d}{da} \left(\frac{1-a^{K-c+1}}{1-a} \right) \end{aligned}$$

  which results

$$\overline{Q} = \frac{P_0 \rho^c a}{c!(1-a)^2} [1 - a^{K-c+1} - (1-a)(K-c+1)a^{K-c}]$$

In particular, if $a = 1$ then the L'Hopital's rule should be applied twice.

- *Mean number of customers in the system*
  It is easy to see that

$$\overline{\lambda} = \lambda(1 - P_K) = \mu\overline{c}$$

  and since

$$\overline{N} = \overline{Q} + \overline{c}$$

  we get

$$\overline{N} = \overline{Q} + \rho(1 - P_K).$$

- *Mean response and waiting times*
  The mean times can be obtained by applying the **Little's law**, that is

$$\overline{T} = \frac{\overline{N}}{\lambda(1 - P_K)}$$
$$\overline{W} = \frac{\overline{Q}}{\lambda(1 - P_K)}$$

  In the case of an $M/M/1/K$ system these formulas are simplified to

$$P_0 = \begin{cases} \frac{1-a}{1-a^{K+1}}, & (a \neq 1) \\ \frac{1}{K+1}, & (a = 1) \end{cases}$$

$$P_n = \begin{cases} \frac{(1-a)a^n}{1-a^{K+1}}, & (a \neq 1) \\ \frac{1}{K+1}, & (a = 1) \end{cases}$$

$$\overline{Q} = \begin{cases} \frac{a}{1-a} - \frac{a(Ka^K+1)}{1-a^{K+1}}, & (a \neq 1) \\ \frac{K(K-1)}{2(K+1)}, & (a = 1) \end{cases}$$

$$\overline{N} = \overline{Q} + (1 - P_0).$$

- *Distribution at the arrival instants*
  By applying the Bayes's rule we have

$$\begin{aligned} \Pi_n &\equiv P(\text{ there are } n \text{ customers in the system} | \text{ a customer is about to enter into the system }) \\ &= \lim_{\Delta t \to 0} \left\{ \frac{[\lambda \Delta t + o(\Delta t)]P_n}{\sum_{n=0}^{K-1} [\lambda \Delta t + o(\Delta t)]P_n} \right\} \\ &= \lim_{\Delta t \to 0} \left\{ \frac{[\lambda + o(\Delta t)/\Delta t]P_n}{\sum_{n=0}^{K-1} [\lambda + o(\Delta t)/\Delta t]P_n} \right\} \\ &= \frac{\lambda P_n}{\lambda \sum_{n=0}^{K-1} P_n} = \frac{P_n}{1 - P_K}, \qquad (n \leq K - 1). \end{aligned}$$

  Obviously in the case of an $M/M/c/\infty$ system $\Pi_n = P_n$ since P_K tends to 0.

- *Distribution of the waiting time*
  As in the previous parts for $F_W(t)$ the theorem of total probability is applied resulting

$$F_W(t) = F_W(0) + \sum_{n=c}^{K-1} \Pi_n \int_0^t \frac{c\mu(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx$$
$$= F_W(0) + \sum_{n=c}^{K-1} \Pi_n \left(1 - \int_t^\infty \frac{c\mu(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx\right).$$

  Since

$$\int_t^\infty \frac{\lambda(\lambda x)^m}{m!} e^{-\lambda x}\, dx = \sum_{i=0}^{m} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

  applying substitutions $m = n - c$, $\lambda = c\mu$ we have

$$\int_t^\infty \frac{c\mu(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx = \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!},$$

  thus

$$F_W(t) = F_W(0) + \sum_{n=c}^{K-1} \Pi_n - \sum_{n=c}^{K-1} \Pi_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$$
$$= 1 - \sum_{n=c}^{K-1} \Pi_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}.$$

The Laplace-transform of the waiting and response times can be derived similarly, by using the law of total Laplace-transforms.

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcK/MMcK.html`

2.9 The $M/G/1$ Queue

So far systems with exponentially distributed serviced times have been treated. We must admit that it is a restriction since in many practical problems these times are not exponentially distributed. It means that the investigation of queueing systems with generally distributed service times is natural. It is not the aim of this book to give a detailed analysis of this important system I concentrate only on the mean value approach and some practice oriented theorems are stated without proofs. A simple proof for the Little's law is also given.

Little's Law

As a first step for the investigations let us give a simple proof for the **Little's theorem, Little's law, Little's formula**, which states a relation between the mean number of customers in the systems, mean arrival rate and the mean response time. Similar version can be stated for the mean queue length, mean arrival rate and mean waiting time.

Let $\alpha(t)$ denote the number of customers arrived into the system in a time interval $(0,t)$, and let $\delta(t)$ denote the number of departed customers in $(0,t)$. Supposing that $N(0)=0$, the number of customert in the system at time t is $N(t)=\alpha(t)-\delta(t)$.
Let the mean arrival rate into the system during $(0,t)$ be defined as

$$\bar{\lambda}_t := \frac{\alpha(t)}{t}.$$

Let $\gamma(t)$ denote the overall sojourn times of the customers until t and let $\overline{T}_t$ be defined as the mean sojourn time for a request. Clearly

$$\overline{T}_t = \frac{\gamma(t)}{\alpha(t)}.$$

Finally, let $\bar{N}_t$ denote the mean number of customers in the system during in the interval $(0,t)$, that is

$$\bar{N}_t = \frac{\gamma(t)}{t}.$$

From these relations we have

$$\bar{N}_t = \bar{\lambda}_t \bar{T}_t.$$

Supposing that the following limits exist

$$\bar{\lambda} = \lim_{t\to\infty} \bar{\lambda}_t, \qquad \bar{T} = \lim_{t\to\infty} \bar{T}_t.$$

we get

$$\bar{N} = \bar{\lambda}\bar{T},$$

which is called **Little's law** .

Similar version is

$$\bar{Q} = \bar{\lambda}\bar{W}.$$

The Embedded Markov Chain

As before let $N(t)$ denote the number of customers in the system at time t. As time evolves the state changes and we can see that changes to neighboring states occur, up and down, that is from state k either to $k+1$ or to $k-1$. Since we have a single server the number of $k \to k+1$ type transitions may differ by at most one from the number of $k+1 \to k$ type transitions. So if the system operate for a long time the relative frequencies should be the same. It means that in stationary case the distributions at the arrival instants and the departure instants should be the same. More formally, $\Pi_k = D_k$.

For further purposes we need the following statements
Statement 1 For Poisson arrivals

$$P\left(N\left(t\right)=k\right)=P\left(\text{an arrival at time } t \text{ finds } \ k \text{ customers in the system }\right).$$

Statement 2 If in any system $N\left(t\right)$ changes its states by one then if either one of the following limiting distribution exists, so does the other and they are equal.

$$\Pi_k := \lim_{t\to\infty}\left(\text{an arrival at time } t \text{ finds } \ k \text{ customers in the system }\right),$$

$$D_k := \lim_{t\to\infty}\left(\text{a departure at time } t \text{ leaves } \ k \text{ customers behind }\right),$$

$$\Pi_k = D_k.$$

Thus for an $M/G/1$ system

$$\Pi_k = P_k = D_k,$$

that is in stationary case these 3 types of distributions are the same.

Due to their importance we prove them. Les us consider first Statement 1.
Introduce the following notation

$$P_k(t) := P\left(N\left(t\right)=k\right),$$

$$\Pi_k(t) := P\left(\text{an arriving customer at instant } \ t \text{ finds } \ k \text{ customers in the system }\right).$$

Let $A(t,t+\Delta t)$ be the event that one arrival occurs in the interval $(t,t+\Delta t)$. Then

$$\Pi_k(t) = \lim_{\Delta t\to 0} P\left(N\left(t\right)=k \mid A(t,t+\Delta t)\right).$$

By the definition of the conditional probability we have

$$\Pi_k(t) = \lim_{\Delta t\to 0} \frac{P\left(N\left(t\right)=k, A(t,t+\Delta t)\right)}{P\left(A(t,t+\Delta t)\right)} =$$

$$= \lim_{\Delta t\to 0} \frac{P\left(A(t,t+\Delta t) \mid N\left(t\right)=k\right) P\left(N(t)=k\right)}{P\left(A(t,t+\Delta t)\right)}.$$

Due to the memoryless property of the exponential distribution event $A(t,t+\Delta t)$ does not depend on the number of customers in the systems and even on t itself thus

$$P\left(A(t,t+\Delta t) \mid N\left(t\right)=k\right) = P\left(A(t,t+\Delta t)\right),$$

hence

$$\Pi_k(t) = \lim_{\Delta t\to 0} P\left(N(t)=k\right),$$

that is

$$\Pi_k(t) = P_k(t).$$

This holds for the limiting distribution as well, namely

$$\Pi_k = \lim_{t\to\infty} \Pi_k(t) = \lim_{t\to\infty} P_k(t) = P_k.$$

Let us prove Statement 2 by the help of Statement 1 .

Let $\hat{R}_k(t)$ denote the number of arrivals into the system when it is in state k during the time interval $(0,t)$ and let $\hat{D}_k(t)$ denote the number of departures that leave the system behind in state k during $(0,t)$. Clearly

$$|\hat{R}_k(t) - \hat{D}_k(t)| \leq 1. \tag{2.9}$$

Furthermore if the total number of departures is denoted by $D(t)$, and the total number of arrivals is denoted by $R(t)$ then

$$D(t) = R(t) + N(0) - N(t).$$

The distribution at the departure instants can be written as

$$D_k = \lim_{t\to\infty} \frac{\hat{D}_k(t)}{D(t)}.$$

It is easy to see that the after simple algebra we have

$$\frac{\hat{D}_k(t)}{D(t)} = \frac{\hat{R}_k(t) + \hat{D}_k(t) - \hat{R}_k(t)}{R(t) + N(0) - N(t)}.$$

Since $N(0)$ is finite and $N(t)$ is also finite due to the stationarity from (2.9) and $\hat{R}(t) \to \infty$, with probability one follows that

$$D_k = \lim_{t\to\infty} \frac{\hat{D}_k(t)}{D(t)} = \lim_{t\to\infty} \frac{\hat{R}_k(t)}{R(t)} = \Pi_k.$$

Consequently, by using Statement 1 the equality of the three probabilities follows.

Mean Value Approach

Let S denote the service time and let R denote the residual (remaining) service time. Then it can easily be seen that

$$\begin{aligned}\mathbb{E}(W) &= \sum_{k=1}^{\infty} (\mathbb{E}(R) + (k-1)\mathbb{E}(S))\,\Pi_k \\ &= \sum_{k=1}^{\infty} \mathbb{E}(R)P_k + \left(\sum_{k=1}^{\infty} (k-1)P_k\right)\mathbb{E}(S) = \mathbb{E}(R)\rho + \mathbb{E}(Q)\mathbb{E}(S),\end{aligned}$$

where $\mathbb{E}(R)$ denotes the mean residual time.
By applying the Little's law we have

$$\mathbb{E}(Q) = \lambda\mathbb{E}(W),$$

and thus

$$\mathbb{E}(W) = \frac{\rho\mathbb{E}(R)}{1-\rho} \tag{2.10}$$

known as **Pollaczek-Khintchine mean value formula**.
In subsection 2.9 we will show that

$$\mathbb{E}(R) = \frac{\mathbb{E}(S^2)}{2\mathbb{E}(S)}\,, \tag{2.11}$$

which can be written as

$$\mathbb{E}(R) = \frac{\mathbb{E}(S^2)}{2\mathbb{E}(S)} = \frac{Var(S) + \mathbb{E}^2(S)}{2\mathbb{E}(S)} = \frac{1}{2}(C_S^2 + 1)\mathbb{E}(S), \tag{2.12}$$

where C_S^2 is the squared coefficient of the service time S. It should be noted that mean residual service time depends on the first two moments of the service time.

Thus for the mean waiting time we have

$$\mathbb{E}(W) = \frac{\rho\mathbb{E}(R)}{1-\rho} = \frac{\rho}{2(1-\rho)}(C_S^2 + 1)\mathbb{E}(S).$$

By using the Little's law for the mean queue length we get

$$\mathbb{E}(Q) = \frac{\rho^2}{1-\rho}\frac{C_S^2 + 1}{2}.$$

Clearly, the mean response time and the mean number of customers in the systems can be expressed as

$$\mathbb{E}(T) = \frac{\rho}{1-\rho}\frac{C_S^2 + 1}{2}\mathbb{E}(S) + \mathbb{E}(S),$$

$$\mathbb{E}(N) = \rho + \frac{\rho^2}{1-\rho}\frac{C_S^2 + 1}{2},$$

which are also referred to as **Pollaczek-Khintchine mean value formulas**.

Example 9 *For an exponential distribution $C_S^2 = 1$, and thus $\mathbb{E}(R) = \mathbb{E}(S)$ which is evident from the memoryless property of the exponential distribution. In this case we get*

$$\mathbb{E}(W) = \frac{\rho}{1-\rho}\mathbb{E}(S), \quad \mathbb{E}(Q) = \frac{\rho^2}{1-\rho}, \quad \mathbb{E}(T) = \frac{1}{1-\rho}\mathbb{E}(S), \quad \mathbb{E}(N) = \frac{\rho}{1-\rho}.$$

Example 10 *In the case of deterministic service time $C_S^2 = 0$, thus $\mathbb{E}(R) = \mathbb{E}(S)/2$. Consequently we have*

$$\mathbb{E}(W) = \frac{\rho}{1-\rho}\frac{\mathbb{E}(S)}{2}, \quad \mathbb{E}(Q) = \frac{\rho^2}{2(1-\rho)}$$

$$\mathbb{E}(T) = \frac{1}{1-\rho}\frac{\mathbb{E}(S)}{2} + \mathbb{E}(S), \quad \mathbb{E}(N) = \rho + \frac{\rho^2}{2(1-\rho)}.$$

For an $M/G/1$ system we have proved that

$$\Pi_k = D_k = P_k, \qquad k = 0, 1, \dots$$

therefore the generating function of the number of customers in the system is equal to the generating function of the number of customers at departure instant. Furthermore, it is clear that the number of customers at departure instants is equal the number customers arrived during the response time. In summary we have

$$D_k = P_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} f_T(x) \mathrm{d}x.$$

Thus the corresponding generating function can be obtained as

$$\begin{aligned} G_N(z) &= \sum_{k=0}^{\infty} z^k \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} f_T(x) \mathrm{d}x \\ &= \int_0^\infty \sum_{k=0}^{\infty} \frac{(\lambda x z)^k}{k!} e^{-\lambda x} f_T(x) \mathrm{d}x \\ &= \int_0^\infty e^{-\lambda(1-z)x} f_T(x) \mathrm{d}x = L_T(\lambda(1-z)), \end{aligned}$$

that is it can be expressed by the help of the Laplace-transform of the response time T. By applying the properties of the generating function and the Laplace-transform we have

$$G_N^{(k)}(1) = \mathbb{E}(N(N-1)\dots(N(-k+1))) = (-1)^k L_T^{(k)}(0)\lambda^k = \lambda^k \mathbb{E}(T^k).$$

In particular, the first derivative results to the Little's law, that is

$$\overline{N} = \lambda \overline{T},$$

and hence this formula can be considered as the generalization of the Little's law for an $M/G/1$ queueing systems.
By the help of this relation the higher moments of N can be obtained, thus the variance can be calculated if the second moment of T is known.

Residual Service Time

Let us suppose that the tagged customer arrives when the server is busy and denote the total service time of the request in service by X, that is a special interval. Let $f_X(x)$ denote the density function of X. The key observation to find $f_X(x)$ is that it is more likely that the tagged customer arrives in a longer service time than in a short one. Thus the probability that X is of length x should be proportional to the length x as well as the frequency of such service times, which is $f_S(x)\,dx$. Thus we may write

$$P(x \le X \le x + dx) = f_X(x)dx = C x f_S(x)dx,$$

where C is a constant to normalize this density. That is

$$C^{-1} = \int_{x=0}^{\infty} x f_S(x) dx = E(S),$$

thus

$$f_X(x) = \frac{x f_S(x)}{E(S)} \ .$$

$$\mathbb{E}(X) = \int_0^{\infty} x f_X(x)\, dx = \frac{1}{\mathbb{E}(S)} \int_0^{\infty} x^2 f_S(x)\, dx = \frac{\mathbb{E}(S^2)}{\mathbb{E}(S)}.$$

Since the tagged customers arrives randomly in service time S hence the mean residual can be obtained as

$$\mathbb{E}(R) = \frac{\mathbb{E}(X)}{2} = \frac{\mathbb{E}(S^2)}{2\mathbb{E}(S)}$$

Example 11 *Let the service time be Erlang distributed with parameters* (n, μ) *then*

$$\mathbb{E}(S) = \frac{n}{\mu}, \quad Var(S) = \frac{n}{\mu^2},$$

thus

$$\mathbb{E}(S^2) = Var(S) + \mathbb{E}^2(S) = \frac{n(1+n)}{\mu^2}$$

hence

$$\mathbb{E}(R) = \frac{1+n}{2\mu}.$$

It is easy to see that using this approach the the density function the residual service time can be calculated. Given that the tagged customer arrives in a service time of length x, the arrival moment will be a random point within this service time, that is it will be uniformly distributed within the service time interval $(0, x)$. Thus we have

$$P(x \leq X \leq x + dx, y \leq R \leq y + dy) = \frac{dy}{x} f_X(x) dx, qquad 0 \leq y \leq x.$$

After substitution for $f_X(x)$ and integrating over x we get the desired density function of the residual service time, that is

$$f_R(y) = \frac{1 - F_S(y)}{\mathbb{E}(S)}.$$

Hence

$$\mathbb{E}(R) = \int_0^{\infty} x f_R(x) dx = \int_0^{\infty} x \frac{1 - F_S(x)}{\mathbb{E}(S)} dx,$$

Thus

$$\mathbb{E}(R) = \frac{\mathbb{E}(S^2)}{2\mathbb{E}(S)}.$$

Now let us show how to calculate this type of integrals.
Let X be a nonnegative random variable with finite nth moment. Then

$$\int_0^\infty x^n f(x)dx = \int_0^y x^n f(x)dx + \int_y^\infty x^n f(x)dx,$$

thus

$$\int_y^\infty x^n f(x)dx = \int_0^\infty x^n f(x)dx - \int_0^y x^n f(x)dx.$$

Since

$$\int_y^\infty x^n f(x)dx \geq y^n \int_y^\infty f(x)dx = y^n \left(1 - F\left(y\right)\right),$$

hence

$$0 \leq y^n \left(1 - F\left(y\right)\right) \leq \int_0^\infty x^n f(x)dx - \int_0^y x^n f(x)dx,$$

therefore

$$0 \leq \lim_{y\to\infty} y^n \left(1 - F\left(y\right)\right) \leq \int_0^\infty x^n f(x)dx - \lim_{y\to\infty} \int_0^y x^n f(x)dx,$$

that is

$$\lim_{y\to\infty} y^n \left(1 - F\left(y\right)\right) = 0.$$

After these using integration by parts keeping in mind the above relation we get

$$\int_0^\infty x^{n-1}(1 - F(x))dx = \int_0^\infty \frac{x^n}{n} f(x)dx = \frac{\mathbb{E}(X^n)}{n}.$$

In particular, for $n = 2$ we obtain

$$\mathbb{E}(R) = \frac{\mathbb{E}(S^2)}{2\mathbb{E}(S)}$$

Pollaczek-Khintchine and Takács formulas

The following relations are commonly referred to as **Pollaczek-Khintchine transform equations**

$$G_N(z) = L_S(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{L_S(\lambda - \lambda z) - z},$$
$$L_T(t) = L_S(t)\frac{t(1-\rho)}{t - \lambda + \lambda L_S(t)},$$
$$L_W(t) = \frac{t(1-\rho)}{t - \lambda + \lambda L_S(t)},$$

by the help of which, in principle, the distribution of the number of customers in the system, the density function of the response and waiting times can be obtained. Of course this time we must be able to invert the involved Laplace-transforms.

Takács Recurrence Theorem

$$\mathbb{E}(W^k) = \frac{\lambda}{1-\rho} \sum_{i=1}^{k} \binom{k}{i} \frac{\mathbb{E}(S^{i+1})}{i+1} \mathbb{E}(W^{k-i})$$

that is moments of the waiting time can be obtained in terms of lower moments of the waiting time and moments of the service time. It should be noted to get the kth moment of W the $k+1$th moment of the service time should exist.

Since W,S are independent and $T = W + S$ the kth moment of the response time can also be computed by

$$\mathbb{E}(T^k) = \sum_{l=0}^{k} \binom{n}{l} \mathbb{E}(W^l) \cdot \mathbb{E}(S^{k-l}).$$

By using these formulas the following relations can be proved

$$\begin{aligned}
\mathbb{E}(W) &= \frac{\lambda \mathbb{E}(S^2)}{2(1-\rho)} = \frac{\rho \mathbb{E}(S)}{1-\rho} \left(\frac{1+C_S^2}{2} \right), \\
\mathbb{E}(T) &= \mathbb{E}(W) + \mathbb{E}(S), \\
\mathbb{E}(W^2) &= 2(\overline{W})^2 + \frac{\lambda \mathbb{E}(S^3)}{3(1-\rho)}, \\
\mathbb{E}(T^2) &= \mathbb{E}(W^2) + \frac{\mathbb{E}(S^2)}{1-\rho}, \\
Var(W) &= \mathbb{E}(W^2) - (\mathbb{E}(W))^2, \\
Var(T) &= Var(W+S) = Var(W) + Var(S).
\end{aligned}$$

Since

$$\mathbb{E}(N(N-1)) = \lambda^2 \mathbb{E}(T^2)$$

after elementary but lengthly calculation we have

$$Var(N) = \frac{\lambda \mathbb{E}(S^3)}{3(1-\rho)} + \left(\frac{\lambda \mathbb{E}(S^2)}{2(1-\rho)} \right)^2 + \frac{\lambda(3-2\rho)\mathbb{E}(S^2)}{2(1-\rho)} + \rho(1-\rho).$$

Since

$$\begin{aligned}
\mathbb{E}(Q^2) &= \sum_{k=1}^{\infty} (k-1)^2 P_k = \sum_{k=1}^{\infty} k^2 P_k - 2 \sum_{k=1}^{\infty} k P_k + \sum_{k=1}^{\infty} P_k \\
&= \mathbb{E}(N^2) - 2\overline{N} + \rho
\end{aligned}$$

by elementary computations we can prove that

$$Var(Q) = \frac{\lambda \mathbb{E}(S^3)}{3(1-\rho)} + \left(\frac{\lambda \mathbb{E}(S^2)}{2(1-\rho)}\right)^2 + \frac{\lambda \mathbb{E}(S^2)}{2(1-\rho)}.$$

Now let us turn our attention to the Laplace-transform of the busy period of the server.

Lajos Takács proved that

$$L_\delta(t) = L_S(t + \lambda - \lambda L_\delta(t)),$$

that is for the Laplace-transform $L_\delta(t)$ a function equation should be solved (which is usually impossible to invert).
However, by applying this equation the moments the busy period can be calculated.

First determine $\mathbb{E}(\delta)$. Using the properties of the Laplace-transform we have

$$\begin{aligned} L'_\delta(0) &= (1 - \lambda L'_\delta(0)) L'_S(0) \\ \mathbb{E}(\delta) &= (1 + \lambda \mathbb{E}(\delta)) \mathbb{E}(S) \\ \mathbb{E}(\delta) &= \frac{\mathbb{E}(S)}{1-\rho} = \frac{1}{\lambda} \frac{\rho}{1-\rho} \end{aligned}$$

which was obtained earlier by the well-known relation

$$\frac{\mathbb{E}(\delta)}{\frac{1}{\lambda} + \mathbb{E}(\delta)} = \rho.$$

After elementary but lengthly calculations it can be proved that

$$Var(\delta) = \frac{Var(S) + \rho(\mathbb{E}(S))^2}{(1-\rho)^3}.$$

Now let us consider the generating function of the customers served during a busy period. It can be proved that

$$G_{N_d(\delta)}(z) = z L_S(\lambda - \lambda G_{N_d(\delta)}(z))$$

which is again a functional equation but using derivations the higher moments can be computed.
Thus for the mean numbers we have

$$\begin{aligned} \mathbb{E}(N_d(\delta)) &= 1 + \lambda \mathbb{E}(S) \mathbb{E}(N_d(\delta)) \\ \mathbb{E}(N_d(\delta)) &= \frac{1}{1-\rho}, \end{aligned}$$

which can also be obtained by relation

$$\mathbb{E}(\delta) = \mathbb{E}(S)\mathbb{E}(N_d(\delta))$$

since

$$\frac{1}{\lambda}\frac{\rho}{1-\rho} = \mathbb{E}(S) \cdot \mathbb{E}(N_d(\delta))$$
$$\mathbb{E}(N_d(\delta)) = \frac{\rho}{\rho(1-\rho)} = \frac{1}{1-\rho}.$$

It can be proved that

$$Var(N_d(\delta)) = \frac{\rho(1-\rho) + \lambda^2\mathbb{E}(S^2)}{(1-\rho)^3}.$$

It is interesting to note that the computation of $Var(\delta)$, $Var(N_d(\delta))$ does not require the existence of $\mathbb{E}(S^3)$, as it in the case of $Var(N)$, $Var(Q)$, $Var(T)$, $Var(W)$.

As it is one of the most widely used queueing system the calculation of the main performance measure is of great importance. Tis can be done by the help of our Java applets

Java applets for direct calculations can be found at

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MH21/MH21.html`

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MGamma1/MGamma1.html`

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MEk1/MEk1.html`

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MD1/MD1.html`

Chapter 3

Finite-Source Systems

So far we have been dealing with such queueing systems where arrivals followed a Poisson process, that is the source of customers is infinite. In this chapter we are focusing on the *finite-source population* models. They are also very important from practical point of view since in many situation the source is finite. Let us investigate the example of the so-called *machine interference problem* treated by many experts.

Let us consider n machines that operates independently of each other. The operation times and service times are supposed to be independent random variables with given distribution function. After failure the broken machines are repaired by a single or multiple repairmen according to a certain discipline. Having been repaired the machine starts operating again and the whole process is repeated.

This simple model has many applications in various fields, for example in manufacturing, computer science, reliability theory, management science, just to mention some of them. For a detailed references on the finite-source models and their application the interested reader is recommended to visit the following link

`http://irh.inf.unideb.hu/user/jsztrik/research/fsqreview.pdf`

3.1 The $M/M/r/r/n$ Queue, Engset-Loss System

As we can see depending on the system capacity r in an $M/M/r/r/n$ a customer may find the system full. Despite of the infinite-source model where the customer is lost, in the finite-source model this request returns to the source and stay there for a exponentially distributed time. Since all the random variables are supposed to be exponentially distributed the number of customers in the system is a birth-death process with the following rates

$$\begin{aligned} \lambda_k &= (n-k)\lambda \quad , \quad 0 \le k < r, \\ \mu_k &= k\mu \quad , \quad 1 \le k \le r, \end{aligned}$$

hence the distribution can be obtained as

$$P_k = \binom{n}{k}\rho^k P_0, \qquad 0 \le k \le r,$$

$$P_k = \frac{\binom{n}{k}\rho^k}{\sum_{i=0}^{r}\binom{n}{i}\rho^i},$$

which is called a **truncated binomial or Engset distribution** .
This is the distribution of a **finite-source loss or Engset system** .
Specially, if $r = n$ that is no loss and each customer has its own server the distribution has a very nice form, namely

$$P_k = \frac{\binom{n}{k}\rho^k}{\sum_{i=0}^{n}\binom{n}{i}\rho^i} = \frac{\binom{n}{k}\rho^k}{(1+\rho)^n}$$

$$= \binom{n}{k}\left(\frac{\rho}{1+\rho}\right)^k\left(1-\frac{\rho}{1+\rho}\right)^{n-k},$$

that is we have a binomial distribution with success parameter i$p = \dfrac{\rho}{1+\rho}$.
That is p is the probability that a given request is in the system. It is easy to see that this distribution remains valid even for a $G/G/n/n/n$ system since

$$p = \frac{\mathbb{E}(S)}{\mathbb{E}(S)+\mathbb{E}(\tau)} = \frac{\rho}{1+\rho},$$

where $\rho = \frac{\mathbb{E}(S)}{\mathbb{E}(\tau)}$, and $\mathbb{E}(\tau)$ denotes the mean time a customer spends in the source.

As before it is easy to see that the *performance measures* are as follows

- *Mean number of customers in the system* $\overline{N}$

$$\overline{N} = \sum_{k=0}^{r} kP_k, \quad \overline{r} = \overline{N}, \quad U_S = \frac{\overline{r}}{r} = \frac{\overline{N}}{r},$$

- *Mean number of customers in the source* $\overline{m}$

$$\overline{m} = n - \overline{N}$$

- *Utilization of a source* U_t

$$U_t = \frac{\overline{m}}{n} = \frac{\mathbb{E}(\tau)}{\mathbb{E}(\tau)+\frac{1}{\mu}},$$

thus

$$\mathbb{E}(\tau) = \frac{1}{\mu}\frac{U_t}{1-U_t}.$$

This help us to calculate the mean number of retrials of a customer from the source to enter to the system. That it we have

$$\mathbb{E}(N_R) = \lambda \mathbb{E}(\tau),$$

hence the mean number of rejection is $E(N_R) - 1$.

The blocking probability, that is the probability that a customer find the system full at his arrival, by the help of the Bayes's theorem can be calculated as

$$P_B(n,r) = \frac{(n-r)P_r(n,r)}{\sum_{i=0}^{r}(n-i)P_k(n,r)} = P_r(n-1,r).$$

This can easily be verified by

$$\begin{aligned}
P_B(n,r) &= \lim_{h\to 0}\frac{((n-r)\lambda h + o(h))P_r(n,r)}{\sum_{i=0}^{r}((n-i)\lambda h + o(h))P_i(n,r)} = \frac{(n-r)P_r(n,r)}{\sum_{i=0}^{r}(n-i)P_i(n,r)} \\
&= \frac{(n-r)\binom{n}{r}\rho^r}{\sum_{i=0}^{r}(n-i)\binom{n}{i}\rho^i} = \frac{(n-r)\frac{n!}{r!(n-r)!}\rho^r}{\sum_{i=0}^{r}(n-i)\frac{n!}{i!(n-i)!}\rho^i} \\
&= \frac{\frac{(n-1)!}{r!(n-1-r)!}\rho^r}{\sum_{i=0}^{r}\frac{(n-1)!}{i!(n-1-i)!}\rho^i} = \frac{\binom{n-1}{r}\rho^r}{\sum_{i=0}^{r}\binom{n-1}{i}\rho^i} = P_r(n-1,r).
\end{aligned}$$

Let $E(n,r,\rho)$ denote the blocking probability, that is $E(n,r,\rho) = P_r(n-1,r)$, which is called **Engset's loss formula**.
In the following we show a recursion for this formula, namely

$$\begin{aligned}
E(n,r,\rho) &= \frac{\binom{n-1}{r}\rho^r}{\sum_{i=0}^{r}\binom{n-1}{i}\rho^i} = \frac{\binom{n-1}{r-1}\frac{n-r}{r}\rho^r}{\sum_{i=0}^{r-1}\binom{n-1}{i}\rho^i + \binom{n-1}{r-1}\frac{n-r}{r}\rho^r} \\
&= \frac{\frac{n-r}{r}\rho E(n,r-1,\rho)}{1+\frac{n-r}{r}\rho E(n,r-1,\rho)} = \frac{(n-r)\rho E(n,r-1,\rho)}{r+(n-r)\rho E(n,r-1,\rho)}.
\end{aligned}$$

The initial value is

$$E(n,1,\rho) = P_1(n-1,1) = \frac{(n-1)\rho}{1+(n-1)\rho}.$$

It is clear that

$$\lim_{n\to\infty,\quad \lambda\to 0,\quad n\lambda\to\lambda'} E(n,r,\rho) = B(n,\rho'),$$

where

$$\rho' = \frac{\lambda'}{\mu}$$

which van be seen formally, too. Moreover, as $(n-r)\rho \to \rho'$ the well-known recursion for $B(n,\rho')$ is obtained which also justifies the correctness of the recursion for $E(n,r,\rho)$.

In particular, if $r = n$ then it is easy to see that $\overline{N} = n\frac{\rho}{1+\rho}$ and thus

$$U_S = \frac{\rho}{1+\rho}, \quad \overline{m} = \frac{n}{1+\rho}, \quad U_t = \frac{1}{1+\rho}, \quad \mathbb{E}(\tau) = \frac{1}{\lambda}, \quad \mathbb{E}(N_R) = 1, \quad P_B = 0,$$

which was expected.

In general case

$$\overline{\mu} = \overline{r}\mu = \overline{\lambda} = \sum_{k=0}^{r-1} \lambda(n-k)P_k \neq \lambda(n-\overline{N}), \quad \overline{T} = \frac{1}{\mu}.$$

Let us consider the distribution of the system at the instant when an arriving customer enters into the system.
By using the Bayes's law we have

$$\Pi_k = \lim_{h\to 0} \frac{(\lambda_k h + o(h))P_k}{\sum_{i=0}^{r-1}(\lambda_i h + o(h))P_i} = \frac{\lambda_k P_k}{\sum_{i=0}^{r-1}\lambda_i P_i}, \qquad k = 0,\cdots,r-1$$

$$\overline{T} = \frac{1}{\mu}\sum_{k=0}^{r-1} \frac{\lambda_k P_k}{\sum_{i=0}^{r-1}\lambda_i P_i} = \frac{1}{\mu}$$

$$\overline{\lambda}\cdot\overline{T} = \mu\overline{r}\cdot\frac{1}{\mu} = \overline{r} = \overline{N}$$

which **Little's formula** for the finite-source loss system.

3.2 The $M/M/1/n/n$ Queue

It is the traditional machine interference problem, where the broken machines has to wait and the single repairman fixes the failed machine in FIFO order. Assume the the operating times are exponentially distributed with parameter λ and the repair rate is μ. All random variables are supposed to be independent of each other.

Let $N(t)$ denote the number of customers in the system at time t, which is a birth-death process with birth rates

$$\lambda_k = \begin{cases} (n-k)\lambda & \text{, ha } 0 \le k \le n, \\ \\ 0 & \text{, ha } k > n, \end{cases}$$

and with death rate

$$\mu_k = \mu, \qquad k \ge 1.$$

Thus for the distribution we have

$$P_k = \frac{n!}{(n-k)!}\varrho^k P_0 = (n-k+1)\varrho P_{k-1},$$

where

$$\varrho = \frac{\lambda}{\mu},$$

and

$$P_0 = \frac{1}{1 + \sum\limits_{k=1}^{n} \frac{n!}{(n-k)!}\varrho^k} = \frac{1}{\sum\limits_{k=0}^{n} \frac{n!}{(n-k)!}\varrho^k}.$$

Since the state space is finite the steady-state distribution always exists but if $\varrho > 1$ then more repairmen is needed.

For numerical calculation other forms are preferred that is why we introduce some notations.
Let $P(k;\lambda)$ a λ denote the Poisson distribution with parameter λ) and let $Q(k;\lambda)$ denote its cumulative distribution function, that is

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}, \qquad 0 \le k < \infty;$$

$$Q(k;\lambda) = \sum_{i=0}^{k} P(i;\lambda), \qquad 0 \le k < \infty.$$

First we show that

$$P_k = \frac{P(n-k;R)}{Q(n;R)}, \qquad 0 \le k \le n,$$

where

$$R = \frac{\mu}{\lambda} = \varrho^{-1}.$$

By elementary calculations we have

$$\frac{P(n-k;R)}{Q(n;R)} = \frac{\frac{n!}{(n-k)!}\left(\frac{\mu}{\lambda}\right)^{n-k} e^{-\frac{\mu}{\lambda}}}{\sum\limits_{i=0}^{n} \frac{n!}{i!}\left(\frac{\mu}{\lambda}\right)^{i} e^{-\frac{\mu}{\lambda}}} = \frac{\frac{n!}{(n-k)!}\left(\frac{\lambda}{\mu}\right)^{k}}{\sum\limits_{i=0}^{n} \frac{n!}{i!}\left(\frac{\lambda}{\mu}\right)^{n-i}} = \frac{\frac{n!}{(n-k)!}\left(\frac{\lambda}{\mu}\right)^{k}}{\sum_{k=0}^{n} \frac{n!}{(n-i)!}\left(\frac{\lambda}{\mu}\right)^{i}} = P_k.$$

Hence a very important consequence is

$$P_0 = B(n,R).$$

The *main performance measures* can be obtained as follows

- *Utilization of the server* and the *throughput of the system*
  For the utilization of the server we have

  $$U_s = 1 - P_0 = 1 - B(n,R).$$

  By using the cumulative distribution function this cab be written as

  $$U_s = \frac{Q(n-1;R)}{Q(n;R)}.$$

  For the throughput of the system we obtain

  $$\lambda_t = \mu U_s.$$

- *Mean number of customers in the system* $\overline{N}$ can be calculated as

  $$\overline{N} = \sum_{k=0}^{n} kP_k = n - \sum_{k=0}^{n}(n-k)P_k =$$

  $$= n - \frac{1}{\varrho}\sum_{k=0}^{n}(n-k)\varrho P_k = n - \frac{1}{\varrho}\sum_{k=0}^{n-1} P_{k+1} =$$

  $$= n - \frac{1}{\varrho}(1-P_0) = n - \frac{U_s}{\varrho}.$$

  In other form

  $$\overline{N} = n - \frac{RQ(n-1;R)}{Q(n;R)} = n - \frac{U_s}{\varrho}.$$

- *Mean queue length, mean number of customers waiting* can be derived as

$$\overline{Q} = \sum_{k=1}^{n}(k-1)P_k = \sum_{k=1}^{n} kP_k - \sum_{k=1}^{n} P_k = n - \frac{\mu}{\lambda}(1-P_0) - (1-P_0) =$$

$$= n - (1-P_0)(1+\frac{\mu}{\lambda}) = n - \left(1+\frac{1}{\varrho}\right)U_s.$$

- *Mean number of customers in the source* can be calculated as

$$\overline{m} = \sum_{k=0}^{n}(n-k)P_k = n - \overline{N} = \frac{\mu}{\lambda}(1-P_0) = \frac{U_s}{\varrho}.$$

- *Mean busy period of the server*
  Since

$$U_s = 1 - P_0 = \frac{E\delta}{\frac{1}{n\lambda} + E\delta},$$

  thus

$$E\delta = \frac{1-P_0}{n\lambda P_0} = \frac{U_s}{n\lambda(1-U_s)}.$$

  In computer science and reliability theory application we often need the following measure

- *Utilization of a given source (machine, terminal)*
  The utilization of the ith source is defined by

$$U^{(i)} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \chi(\text{at time } t \text{ the } i\text{th source is active})dt$$

  Then

$$U^{(i)} = P(\text{ there is a request in the } i\text{th source}) .$$

  Hence the *overall utilization of the sources* is

$$U_n = \sum_{k=0}^{n}(n-k)P_k = \overline{m} = \frac{\mu}{\lambda}(1-P_0).$$

  Thus the utilization of any source is

$$U_t = \frac{\mu}{n\lambda}(1-P_0) = \frac{\overline{m}}{n}.$$

  This can be obtained in the following way as well,

$$U^{(i)} = \sum_{k=1}^{n}\frac{n-k}{n}P_k = \frac{\overline{m}}{n},$$

  since the source are homogeneous we have

$$U_t = U^{(i)} .$$

- *Mean waiting time*

  By using the result of Tomkó 1 we have

$$U_t = \frac{1/\lambda}{1/\lambda + \overline{W} + 1/\mu} = \frac{\overline{m}}{n}.$$

  Thus

$$\lambda\overline{m} = \frac{n}{1/\lambda + \overline{W} + 1/\mu},$$

  and

$$\lambda\overline{m}\overline{W} = n - \overline{m}\left(1 + \frac{\lambda}{\mu}\right) = n - \frac{U_s}{\varrho}(1+\varrho) = \overline{Q},$$

  which the **Litle's law** for the mean waiting time. Hence

$$\overline{W} = \frac{\overline{Q}}{\lambda\overline{m}} = \frac{1}{\mu}\left(\frac{n}{U_s} - \frac{1+\varrho}{\varrho}\right).$$

  The mean response can be obtained as

$$\overline{T} = \overline{W} + \frac{1}{\mu} = \frac{1}{\mu}\left(\frac{n}{1-P_0} - \frac{1}{\varrho}\right) = \frac{1}{\mu}\left(\frac{n}{U_s} - \frac{1}{\varrho}\right).$$

  It is easy to prove that

$$\overline{m}\lambda\overline{T} = \overline{N},$$

  which is the **Little's law** for the mean response time. Clearly we have

$$\overline{m}\lambda\left(\overline{W} + \frac{1}{\mu}\right) = \overline{Q} + \overline{m}\varrho =$$
$$= n - \frac{U_s}{\varrho}(1+\varrho) + U_s = n - \frac{U_s}{\varrho} = \overline{N}\ .$$

- *Further relations*

$$U_s = 1 - P_0 = n\varrho U_t = \overline{m}\varrho,$$

  and thus

$$\overline{m}\lambda = \mu U_s = \lambda_t\ .$$

It should be noted that the utilization of the server plays a key role in the calculation of all the main performance measures.

Distribution at the arrival instants

In the following we find the steady-state distribution of the system at arrival instants and in contrary to the infinity-source model is not he same as the distribution at a random point. To show this use the Bayes's theorem, that is

$$\Pi_k(n) = \lim_{h\to 0}\frac{(\lambda_k h + o(h))P_k}{\sum_{j=0}^{n-1}(\lambda_j h + o(h))P_j} = \frac{\lambda_k P_k}{\sum_{j=0}^{n-1}\lambda_j P_j} = \frac{\frac{n(n-1)\cdots(n-k)\lambda^k}{\mu_1\cdots\mu_k}P_0}{\sum_{j=0}^{n-1}\frac{n(n-1)\cdots(n-j)\lambda^j}{\mu_1\cdots\mu_j}P_0}$$
$$= \frac{\frac{(n-1)\cdots(n-k)\lambda^k}{\mu_1\cdots\mu_k}}{1+\sum_{j=1}^{n-1}\frac{(n-1)\cdots(n-j)\lambda^j}{\mu_1\cdots\mu_j}} = \frac{\frac{(n-1)\cdots(n-1-k+1)\lambda^k}{\mu_1\cdots\mu_k}}{1+\sum_{i=1}^{n-1}\frac{(n-1)\cdots(n-1-i+1)\lambda^i}{\mu_1\cdots\mu_i}} = P_k(n-1)$$

irrespective to the number of servers. It should be noted that this relation shows a very important result, namely that at arrivals the distribution of the system containing n sources is not he same as its distribution at random points, but equals to the random point distribution of a system with $n-1$ sources.

Distribution at the departure instants

We are interested in the distribution of the number of customers a departing customer leaves behind in the system. This calculations are independent of the number of servers. By applying the Bayes's theorem we have

$$D_k(n) = \lim_{h\to 0} \frac{(\mu_{k+1}h + o(h))P_{k+1}}{\sum_{j=1}^{n}(\mu_j h + o(h))P_j} = \frac{\mu_{k+1}P_{k+1}}{\sum_{j=1}^{n}\mu_j P_j} = \frac{\frac{\mu_{k+1}n(n-1)\cdots(n-k)\lambda^{k+1}}{\mu_1\cdots\mu_{k+1}}P_0}{\sum_{j=1}^{n}\frac{\mu_j n(n-1)\cdots(n-j+1)\lambda^j}{\mu_1\cdots\mu_j}P_0}$$
$$= \frac{\frac{(n-1)\cdots(n-k)\lambda^k}{\mu_1\cdots\mu_k}}{1+\sum_{j=2}^{n}\frac{(n-1)\cdots(n-j+1)\lambda^{j-1}}{\mu_1\cdots\mu_{j-1}}} = \frac{\frac{(n-1)\cdots(n-1-k+1)\lambda^k}{\mu_1\cdots\mu_k}}{1+\sum_{i=1}^{n-1}\frac{(n-1)\cdots(n-1-i+1)\lambda^i}{\mu_1\cdots\mu_i}} = P_k(n-1)$$

in the case when there is customer left in the system

$$D_0(n) = \frac{1}{1+\sum_{i=1}^{n-1}\frac{(n-1)\cdots(n-1-i+1)\lambda^i}{\mu_1\cdots\mu_i}} = P_0(n-1)$$

if the system becomes empty.

Recursive Relations

Similarly to the previous arguments it is easy to see that the density function of the response time can be obtained as

$$f_T(x) = \sum_{k=0}^{n-1} f_T(x|k)\Pi_k(n) = \sum_{k=0}^{n-1}\frac{\mu(\mu x)^k}{k!}e^{-\mu x}P_k(n-1).$$

Hence the mean value is

$$\overline{T}(n) = \sum_{k=0}^{n-1}\frac{k+1}{\mu}P_k(n-1) = \frac{1}{\mu}(\overline{N}(n-1)+1).$$

Similarly, for the waiting time we have

$$f_W(x) = \sum_{k=0}^{n-1} f_W(x|k)\Pi_k(n) = \sum_{k=0}^{n-1}\frac{\mu(\mu x)^{k-1}}{(k-1)!}e^{-\mu x}P_k(n-1),$$

thus its mean is

$$\overline{W}(n) = \sum_{k=0}^{n-1}\frac{k}{\mu}P_k(n-1) = \frac{1}{\mu}(\overline{N}(n-1)),$$

which is clear.
We want to verify the correctness of the formula

$$\overline{T}(n) = \frac{1}{\mu}(\overline{N}(n-1)+1).$$

As we have shown earlier the utilization can be expressed by the Erlang's loss formula, hence

$$\overline{N}(n) = n - \frac{1 - B(n, \frac{1}{\varrho})}{\varrho}.$$

Using the well-known recursive relation we have

$$B(n, \frac{1}{\varrho}) = \frac{\frac{1}{\varrho}B(n-1, \frac{1}{\varrho})}{n + \frac{1}{\varrho}B(n-1, \frac{1}{\varrho})} = \frac{B(n-1, \frac{1}{\varrho})}{n\varrho + B(n-1, \frac{1}{\varrho.})}.$$

Since

$$\overline{N}(n-1) = n - 1 - \frac{1 - B(n-1, \frac{1}{\varrho})}{\varrho}$$

thus

$$\varrho\overline{N}(n-1) = (n-1)\varrho - 1 - B\left(n-1, \frac{1}{\varrho}\right)$$

$$B\left(n-1, \frac{1}{\varrho}\right) = 1 + \varrho\overline{N}(n-1) - (n-1)\varrho.$$

After substitution we have

$$B\left(n, \frac{1}{\varrho}\right) = \frac{\frac{1}{\varrho}(1 + \varrho\overline{N}(n-1) - (n-1)\varrho)}{n + \frac{1}{\varrho}(1 + \varrho\overline{N}(n-1) - (n-1)\varrho)}$$

$$= \frac{1 + \varrho\overline{N}(n-1) - (n-1)\varrho}{n\varrho + 1 + \varrho\overline{N}(n-1) - (n-1)\varrho} = \frac{1 + \varrho\overline{N}(n-1) - (n-1)\varrho}{1 + \varrho\overline{N}(n-1) + \varrho}.$$

Therefore

$$\overline{N}(n) = \frac{n\varrho - 1 + B(n, \frac{1}{\varrho})}{\varrho} = \frac{n\varrho - 1 + \frac{1+\varrho\overline{N}(n-1)-(n-1)\varrho}{1+\varrho\overline{N}(n-1)+\varrho}}{\varrho}$$

$$= \frac{n\varrho - \frac{n\varrho}{1+\varrho\overline{N}(n-1)+\varrho}}{\varrho} = n - \frac{n}{1 + \varrho\overline{N}(n-1) + \varrho.}$$

Fnally

$$n - \overline{N}(n) = \frac{n}{1 + \varrho\overline{N}(n-1) + \varrho}$$

$$1 + \varrho(\overline{N}(n-1)+1) = \frac{n}{n - \overline{N}(n)}$$

$$\varrho(\overline{N}(n-1)+1) = \frac{\overline{N}(n)}{n - \overline{N}(n)},$$

which is a recursion for the mean number of customers in the system.

Now we able to prove our relation regarding the mean response time. Keeping in mind the recursive relation for $\overline{N}(n-1)$ we get

$$\overline{T}(n) = \frac{1}{\mu}(\overline{N}(n-1)+1)$$
$$\lambda\overline{T}(n) = \varrho(\overline{N}(n-1)+1) = \frac{\overline{N}(n)}{n-\overline{N}(n)}$$
$$\lambda(n-\overline{N}(n))\overline{T}(n) = \overline{N}(n),$$

which was proved earlier.

Now let us show how we can verify $\overline{T}(n)$ directly. It can easily be seen that

$$U_S(n) = 1 - B(n, \frac{1}{\varrho}) = 1 - \frac{\frac{1}{\varrho}B(n-1,)}{n+\frac{1}{\varrho}B(n-1,\frac{1}{\varrho})}$$

$$= \frac{n}{n+\frac{1}{\varrho}B(n-1,\frac{1}{\varrho})} = \frac{n\varrho}{n\varrho+B(n-1,\frac{1}{\varrho})} = \frac{n\varrho}{n\varrho+1-U_S(n-1)},$$

that is there is a recursion for the utilization as well. It is also very important because by using this recursion all the main performance measures can be obtained. Thus if λ, μ, n are given we can use the recursion for $U_S(n)$ and finally substitute it into the corresponding formula. Thus

$$U_S(n-1) = n\varrho + 1 - \frac{n\varrho}{U_S(n)} = 1 + n\varrho\left(1 - \frac{1}{U_S(n)}\right).$$

Since

$$\overline{N}(n-1) = n - 1 - \frac{U_S(n-1)}{\varrho},$$

we proceed

$$\overline{T}(n) = \frac{1}{\mu}(\overline{N}(n-1)+1) = \frac{1}{\mu}\left(n-1-\frac{U_S(n-1)}{\varrho}+1\right)$$

$$= \frac{1}{\mu}\left(n - \frac{U_S(n-1)}{\varrho}\right) = \frac{1}{\mu}\left(n - \frac{1+n\varrho(1-\frac{1}{U_S(n)})}{\varrho}\right) = \frac{1}{\mu}\left(\frac{n}{U_S(n)} - \frac{1}{\varrho}\right),$$

which shows the correctness of the formula.

In the following let us show to compute $\overline{T}(n), \overline{W}(n), \overline{N}(n)$ recursively. As we have seen

$$\overline{T}(n) = \frac{1}{\mu}(\overline{N}(n-1)+1)$$
$$\overline{W}(n) = \overline{T}(n) - \frac{1}{\mu} = \frac{1}{\mu}\overline{N}(n-1),$$

we have to know how $\overline{N}(n)$ can be expressed in term of $\overline{T}(n)$.
It can be shown very easily, namely

$$\begin{aligned}\overline{N}(n) &= \lambda(n-\overline{N}(n))\overline{T}(n) = \lambda n\overline{T}(n) - \lambda\overline{N}(n)\overline{T}(n)\\ \overline{N}(n)(1+\lambda\overline{T}(n)) &= \lambda n\overline{T}(n)\\ \overline{N}(n) &= \frac{\lambda n\overline{T}(n)}{1+\lambda\overline{T}(n)}.\end{aligned}$$

The initial values are

$$\begin{aligned}\overline{T}(1) &= \frac{1}{\mu}\\ \overline{N}(1) &= \frac{\varrho}{1+\varrho}.\end{aligned}$$

Now the iteration proceeds as

$$\begin{aligned}\overline{W}(n) &= \frac{1}{\mu}\overline{N}(n-1)\\ \overline{T}(n) &= \frac{1}{\mu} + \overline{W}(n)\\ \overline{N}(n) &= \frac{\lambda n\overline{T}(n)}{(1+\lambda\overline{T})(n)}\end{aligned}$$

that is we use a double iteration. The main advantage is that only the mean values are needed. This method is referred to as **mean value analysis.**

In the previous section we have derived a recursion for $U_S(n)$ and thus we may expect that there is direct recursive relation for the other mean values as well since they depends on the utilization. As a next step we find a recursion for the *mean number of customers in the source* $\overline{m}(n)$. It si quite easy since

$$\overline{m}(n) = \frac{U_s(n)}{\rho} = \frac{n}{n\rho + 1 - U_s(n-1)} =$$

$$= \frac{\frac{n}{\rho}}{n + \frac{1}{\rho} - \overline{m}(n-1)} = \frac{n}{n\rho + 1 - \rho\overline{m}(n-1).}$$

By using this relation for the *utilization of the source* can be expressed as

$$U_t(n) = \frac{\overline{m}(n)}{n} = \frac{n}{n\rho + 1 - \rho\overline{m}(n-1)}\frac{1}{n} =$$

$$= \frac{\frac{1}{n-1}}{\frac{n\rho+1}{n-1} - \rho U_t(n-1)} = \frac{1}{n\rho + 1 - (n-1)\rho U_t(n-1)}.$$

For the *mean number of customers in the system* we have

$$\overline{N}(n) = n - \frac{U_s(n)}{\rho} = \frac{n\rho - U_s(n)}{\rho} = \frac{n\rho - \dfrac{n\rho}{n\rho + 1 - U_s(n-1)}}{\rho} =$$

$$= \frac{n^2\rho + n - nU_s(n-1) - 1}{n\rho + 1 - U_s(n-1)} = \frac{n(n\rho - U_s(n-1))}{n\rho + 1 - U_s(n-1)}.$$

Since

$$\overline{N}(n-1) = n - 1 - \frac{U_s(n-1}{\rho} = \frac{n\rho - U_s(n-1)}{\rho} - 1$$

$$\rho(\overline{N}(n-1) + 1) = n\rho - U_s(n-1)$$

$$U_s(n-1) = n\rho - \rho(\overline{N}(n-1) + 1),$$

thus after substitution we get

$$\overline{N}(n) = \frac{n\rho(\overline{N}(n-1) + 1)}{1 + \rho(\overline{N}(n-1) + 1)}.$$

Finally find the recursion for the *mean response time* . Starting with

$$\overline{T}(n) = \frac{1}{\mu}\left(\frac{n}{U_s(n)} - \frac{1}{\rho}\right)$$

using that

$$\overline{T}(n-1) = \frac{1}{\mu}\left(\frac{n-1}{U_s(n-1)} - \frac{1}{\rho}\right)$$

$$\mu\overline{T}(n-1) = \frac{n-1}{U_s(n-1)} - \frac{1}{\rho}$$

$$\mu\overline{T}(n-1) + \frac{1}{\rho} = \frac{n-1}{U_s(n-1)}$$

$$U_s(n-1) = \frac{(n-1)\rho}{\lambda\overline{T}(n-1) + 1},$$

substituting into the recursion for $U_s(n)$ we obtain

$$\overline{T}(n) = \frac{1}{\mu}\frac{n\rho - U_s(n-1)}{\rho} = \frac{1}{\mu}\frac{n\lambda\overline{T}(n-1) + 1}{\lambda\overline{T}(n-1) + 1}.$$

Obviously the missing initial values are

$$\overline{m}(1) = U_t(1) = \frac{1}{1 + \varrho}.$$

Distribution Function of the Response Time and Waiting Time

This subsection is devoted to one of the major problems in finite-source queueing systems. To find the distribution function of the response and waiting time is not easy. As it is expected the theorem of total probability should be used.

Let us determine the density function and then the distribution function. As we did many times in earlier chapters the law of total probability should be applied for the conditional density functions and the distribution at the arrival instants. So we can write

$$f_T(n,x) = \sum_{k=0}^{n-1} \mu \frac{(\mu x)^k}{k!} e^{-\mu x} \frac{\frac{(\frac{\mu}{\lambda})^{n-1-k}}{(n-1-k)!} e^{-\frac{\mu}{\lambda}}}{\underbrace{\sum_{i=0}^{n-1} \frac{(\frac{\mu}{\lambda})^i}{i!} e^{-\frac{\mu}{\lambda}}}_{Q(n-1,\frac{\mu}{\lambda})}} = \mu \frac{(\mu x + \frac{\mu}{\lambda})^{n-1}}{(n-1)!} \frac{e^{-(\mu x + \frac{\mu}{\lambda})}}{Q(n-1,\frac{\mu}{\lambda})}$$

$$= \frac{\mu P(n-1, \mu x + \frac{\mu}{\lambda})}{Q(n-1,\frac{\mu}{\lambda})}.$$

Similarly for the waiting time

$$f_W(n,x) = \sum_{k=1}^{n-1} \mu \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} P_k(n-1) = \frac{\sum_{i=0}^{n-2} \mu \frac{(\mu x)^i}{i!} e^{-\mu x} \frac{(\frac{\mu}{\lambda})^{n-2-i}}{(n-2-i)!} e^{-\frac{\mu}{\lambda}}}{Q(n-1,\frac{\mu}{\lambda})}$$

$$= \frac{\mu P(n-2, \mu x + \frac{\mu}{\lambda})}{Q(n-1,\frac{\mu}{\lambda})}.$$

To get the distribution function we have to calculate the integral

$$F_T(n,x) = \int_0^x f_T(n,t)\, dt.$$

Using the substitution $y = \mu t + \frac{\mu}{\lambda}, \quad t = (y - \frac{\mu}{\lambda})\frac{1}{\mu}, \quad \frac{dt}{dy} = \frac{1}{\mu}$.
Hence

$$F_T(n,x) = \frac{\int_{\frac{\mu}{\lambda}}^{\mu x + \frac{\mu}{\lambda}} \frac{y^{n-1}}{(n-1)!} e^{-y}\, dy}{Q(n-1,\frac{\mu}{\lambda})} = \frac{\left[1 - \sum_{i=0}^{n-1} \frac{y^i}{y!} e^{-y}\right]_{\frac{\mu}{\lambda}}^{\mu x + \frac{\mu}{\lambda}}}{Q(n-1,\frac{\mu}{\lambda})} = 1 - \frac{Q(n-1, \mu x + \frac{\mu}{\lambda})}{Q(n-1,\frac{\mu}{\lambda})}.$$

Similarly for the waiting time we have

$$F_W(n,x) = 1 - \frac{Q(n-2, \mu x + \frac{\mu}{\lambda})}{Q(n-1,\frac{\mu}{\lambda})}.$$

Now let us determine the distribution function by the help of the conditional distribution functions. Clearly we have to know the distribution function of the Erlang distributions, thus we can proceed as

$$F_T(x) = \sum_{k=0}^{n-1} \left(1 - \sum_{j=0}^{k} \frac{(\mu x)^j}{j!} e^{-\mu x}\right) P_k(n-1)$$

$$= 1 - \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \frac{(\mu x)^j}{j!} e^{-\mu x}\right) P_k(n-1)$$

$$= 1 - \sum_{k=0}^{n-1} Q(k, \mu x) P_k(n-1) = 1 - \sum_{k=0}^{n-1} Q(k, \mu x) \frac{\frac{(\frac{\mu}{\lambda})^{n-1-k}}{(n-1-k)!} e^{-\frac{\mu}{\lambda}}}{Q(n-1, \frac{\mu}{\lambda})}$$

$$= 1 - \frac{Q(n-1, \mu x + \frac{\mu}{\lambda})}{Q(n-1, \frac{\mu}{\lambda})}$$

Meantime we have used that

$$\int_0^{\lambda} \frac{t^j}{j!} e^{-t}\, dt = 1 - \sum_{i=0}^{j} \frac{\lambda^i}{i!} e^{-\lambda}$$

and thus

$$\sum_{j=0}^{l} \frac{\mu^{l-j}}{(l-j)!} e^{-\mu} \sum_{i=0}^{j} \frac{\lambda^i}{i!} e^{-\lambda}$$

can be written as

$$\sum_{j=0}^{l} \left(1 - \int_0^{\lambda} \frac{t^j}{j!} e^{-t}\, dt\right) \frac{\mu^{l-j}}{(l-j)!} e^{-\mu}$$

$$= \underbrace{\sum_{j=0}^{l} \frac{\mu^{l-j}}{(l-j)!} e^{-\mu}}_{Q(l,\mu)} - \int_0^{\lambda} \frac{(t+\mu)^l}{l!} e^{-(t+\mu)} = Q(l, \mu) - \int_{\mu}^{\lambda+\mu} \frac{y^l}{l!} e^{-y}\, dy$$

$$= Q(l, \mu) - \left[1 - \sum_{i=0}^{l} \frac{y^i}{i!} e^{-y}\right]_{\mu}^{\lambda+\mu} = Q(l, \lambda + \mu).$$

During the calculations we could see that the derivative of $Q(k,t)$ is $-P(k,t)$, which can be used to find the density function, that is

$$f_T(x) = \frac{\mu P(n-1, \mu x + \frac{\mu}{\lambda})}{Q(n-1, \frac{\mu}{\lambda})}.$$

Generating Function of the Customers in the System

Using the definition the generating function $G_N(s)$ can be calculated as

$$\begin{aligned} G_N(s) &= \sum_{k=0}^{n} s^k \frac{\left(\frac{\mu}{\lambda}\right)^{n-k}}{(n-k)!} P_0 \\ &= s^n \sum_{k=0}^{n} \frac{\left(\frac{1}{s\rho}\right)^{n-k}}{(n-k)!} P_0 \\ &= s^n e^{-\frac{1}{\rho}\left(1-\frac{1}{s}\right)} \frac{Q\left(n, \frac{1}{\rho s}\right)}{Q\left(n, \frac{1}{\rho}\right)}. \end{aligned}$$

This could be derived in the following way. Let denote by F the number of customers in the source. As we have proved earlier its distribution can be obtained as the distribution of an Erlang loss system with traffic intensity $\frac{1}{\rho}$. Since the generating function of this system has been obtained we can use this fact. Thus

$$\begin{aligned} G_N(s) = E(s^N) = E(s^{n-F}) = s^n E(s^{-F}) = s^n G_F\left(\frac{1}{s}\right) \\ = s^n e^{-\frac{1}{\rho}\left(1-\frac{1}{s}\right)} \frac{Q\left(n, \frac{1}{\rho s}\right)}{Q\left(n, \frac{1}{\rho}\right)}. \end{aligned}$$

To verify the formula let us compute the mean number of customers in the system. By the property of the generating function we have

$$\begin{aligned} \overline{N}(n) = G'_{N(n)}(1) = \left(s^n G_{F(n)}\left(\frac{1}{s}\right)\right)'_{s=1} \\ G'_{N(n)}(s) = n \cdot s^{n-1} G_{F(n)}\left(\frac{1}{s}\right) + s^n G'_{F(n)}\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right), \end{aligned}$$

thus

$$\overline{N}(n) = nG_{F(n)}(1) - G'_{F(n)}(1) = n - \frac{1}{\rho}\left(1 - B\left(n, \frac{1}{\rho}\right)\right) = n - \frac{U_S(n)}{\rho}.$$

Laplace-transform of the Response Time and Waiting Time

Solution 1

By the law of the total Laplace-transforms we have

$$L_T(s) = \sum_{k=0}^{n-1} \left(\frac{\mu}{\mu+s}\right)^{k+1} P_k(n-1)$$

since the conditional response time is Erlang distributed with parameters $(k+1, \mu)$. Substituting $P_k(n-1)$ we get

$$\begin{aligned}
L_T(s) &= \sum_{k=0}^{n-1} \left(\frac{\mu}{\mu+s}\right)^{k+1} \frac{\left(\frac{\mu}{\lambda}\right)^{n-1-k}}{(n-1-k)!} \frac{e^{-\frac{\mu}{\lambda}}}{Q\left(n-1, \frac{\mu}{\lambda}\right)} \\
&= \frac{\sum_{k=0}^{n-1} \left(\frac{\mu+s}{\mu}\right)^{-k-1} \cdot \left(\frac{\mu+s}{\mu}\right)^{n} \cdot \frac{\left(\frac{\mu}{\lambda}\right)^{n-1-k}}{(n-1-k)!} e^{-\frac{\mu}{\lambda}}}{\left(\frac{\mu+s}{\mu}\right)^{n} Q\left(n-1, \frac{\mu}{\lambda}\right)} \\
&= \left(\frac{\mu}{\mu+s}\right)^{n} e^{-\frac{\mu}{\lambda}} \frac{\sum_{k=0}^{n-1} \left(\frac{\mu+s}{\mu} \cdot \frac{\mu}{\lambda}\right)^{n-1-k} \cdot \frac{1}{(n-1-k)!}}{Q\left(n-1, \frac{\mu}{\lambda}\right)} \\
&= \left(\frac{\mu}{\mu+s}\right)^{n} e^{-\frac{\mu}{\lambda}} \frac{Q\left(n-1, \frac{\mu+s}{\lambda}\right) \cdot e^{\frac{\mu+s}{\lambda}}}{Q\left(n-1, \frac{\mu}{\lambda}\right)} \\
&= \left(\frac{\mu}{\mu+s}\right)^{n} e^{\frac{s}{\lambda}} \frac{Q\left(n-1, \frac{\mu+s}{\lambda}\right)}{Q\left(n-1, \frac{\mu}{\lambda}\right)}.
\end{aligned}$$

Solution 2

Let us calculate $L_T(s)$ by the help of the density function. Since the denominator is a constant we have to determine the Laplace-transform of the numerator, that is

$$\begin{aligned}
L_{Num}(s) &= \int_0^{\infty} \mu \frac{\left(\mu x + \frac{\mu}{\lambda}\right)^{n-1}}{(n-1)!} e^{-\left(\mu x + \frac{\mu}{\lambda}\right)} \cdot e^{-sx} \mathrm{d}x \\
&= e^{-\frac{\mu}{\lambda}} \int_0^{\infty} \mu \frac{\left(\mu x + \frac{\mu}{\lambda}\right)^{n-1}}{(n-1)!} e^{-(\mu+s)x} \mathrm{d}x.
\end{aligned}$$

By using the binomial theorem we get

$$
\begin{aligned}
L_{sz}(s) &= \frac{e^{-\frac{\mu}{\lambda}}}{(n-1)!}\int_0^\infty \mu \sum_{k=0}^{n-1}\binom{n-1}{k}(\mu x)^k\left(\frac{\mu}{\lambda}\right)^{n-1-k}\cdot e^{-(\mu+s)x}\mathrm{d}x \\
&= e^{-\frac{\mu}{\lambda}}\sum_{k=0}^{n-1}\frac{\left(\frac{\mu}{\lambda}\right)^{n-1-k}}{(n-1-k)!}\int_0^\infty \mu\frac{(\mu x)^k}{k!}e^{-(\mu+s)x}\mathrm{d}x \\
&= e^{-\frac{\mu}{\lambda}}\sum_{k=0}^{n-1}\frac{\left(\frac{\mu}{\lambda}\right)^{n-1-k}}{(n-1-k)!}\left(\frac{\mu}{\mu+s}\right)^{k+1} \\
&= e^{-\frac{\mu}{\lambda}}\left(\frac{\mu}{\mu+s}\right)^n\sum_{k=0}^{n-1}\frac{\left(\frac{\mu}{\lambda}\cdot\frac{\mu+s}{\mu}\right)^{n-1-k}}{(n-1-k)!} \\
&= e^{-\frac{\mu}{\lambda}}\left(\frac{\mu}{\mu+s}\right)^n Q\left(n-1,\frac{\mu+s}{\lambda}\right)\cdot e^{\frac{\mu+s}{\lambda}} \\
&= \left(\frac{\mu}{\mu+s}\right)^n Q\left(n-1,\frac{\mu+s}{\lambda}\right)\cdot e^{\frac{s}{\lambda}}.
\end{aligned}
$$

Since

$$
L_T(s) = \frac{L_{Num}(s)}{Q\left(n-1,\frac{\mu}{\lambda}\right)},
$$

thus

$$
L_T(s) = \left(\frac{\mu}{\mu+s}\right)^n e^{\frac{s}{\lambda}}\frac{Q\left(n-1,\frac{\mu+s}{\lambda}\right)}{Q\left(n-1,\frac{\mu}{\lambda}\right)}.
$$

Solution 3

The Laplace-transform of the numerator be can obtained as

$$
L_{Num}(s) = \int_0^\infty \mu\frac{\left(\mu x+\frac{\mu}{\lambda}\right)^{n-1}}{(n-1)!}e^{-\left(\mu x+\frac{\mu}{\lambda}\right)}\cdot e^{-sx}\mathrm{d}x
$$

Substituting $t = \mu x + \frac{\mu}{\lambda}$ we get

$$
x = \frac{1}{\mu}\left(t-\frac{\mu}{\lambda}\right), \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\mu},
$$

and thus

$$L_{Num}(s) = \int_{\frac{\mu}{\lambda}}^{\infty} \mu \frac{t^{n-1}}{(n-1)!} e^{-t} e^{-\frac{s}{\mu}\left(t-\frac{\mu}{\lambda}\right)} \frac{1}{\mu} \mathrm{d}t$$

$$= e^{\frac{s}{\mu}} \int_{\frac{\mu}{\lambda}}^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-\left(1+\frac{s}{\mu}\right)t} \mathrm{d}t$$

$$= e^{\frac{s}{\mu}} \left(\frac{\mu}{\mu+s}\right)^{n-1} \int_{\frac{\mu}{\lambda}}^{\infty} \frac{\left(\left(1+\frac{s}{\mu}\right)t\right)^{n-1}}{(n-1)!} e^{-\left(1+\frac{s}{\mu}\right)t} \mathrm{d}t.$$

Substituting again $y = \frac{\mu+s}{\mu}t$ $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{\mu}{\mu+s}$, thus

$$L_{Num}(s) = e^{\frac{s}{\lambda}} \left(\frac{\mu}{\mu+s}\right)^{n} \int_{\frac{\mu+s}{\lambda}}^{\infty} \frac{y^{n-1}}{(n-1)!} e^{-y} \mathrm{d}y$$

$$= e^{\frac{s}{\lambda}} \left(\frac{\mu}{\mu+s}\right)^{n} \cdot Q\left(n-1, \frac{\mu+s}{\lambda}\right),$$

therefore

$$L_T(s) = \frac{L_{Num}(s)}{Q\left(n-1, \frac{\mu}{\lambda}\right)} = e^{\frac{s}{\lambda}} \left(\frac{\mu}{\mu+s}\right)^{n} \cdot \frac{Q\left(n-1, \frac{\mu+s}{\lambda}\right)}{Q\left(n-1, \frac{\mu}{\lambda}\right)}.$$

That is all 3 solutions gives the same result. Thus, in principle the higher moments of the response time can be evaluated.

Since $L_T(s) = L_W(s) \cdot \frac{\mu}{\mu+s}$, thus

$$L_W(s) = \left(\frac{\mu}{\mu+s}\right)^{n-1} e^{\frac{s}{\lambda}} \frac{Q\left(n-1, \frac{\mu+s}{\lambda}\right)}{Q\left(n-1, \frac{\mu}{\lambda}\right)}.$$

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1KK/MM1KK.html`
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MHypo1KK/MHypo1KK.html`

Example 12 *Consider* 6 *machines with mean lifetime of* 40 *hours. Let their mean repair time be* 4 *hours. Find the performance measures.*

Solution: $\lambda = \frac{1}{40}$ per hour, $\mu = \frac{1}{4}$per hour, $\rho = \frac{\lambda}{\mu} = \frac{4}{40} = 0.1$, $n = 6$, $P_0 = 0.484$

Failed machines	0	1	2	3	4	5	6
Waiting machines	0	0	1	2	3	4	5
P_n	$0,484$	0.290	0.145	0.058	0.017	0.003	0.000

$$\overline{Q} = 0.324, \quad U_{sz} = 0.516, \quad \overline{W} = 2.51 \text{ hour}, \quad \overline{T} = 2.51 + 0.25 = 6.51 \text{ hour}$$

$$\overline{e} = 40 \text{ hour}, \quad U_g = 0.86$$

$$\overline{m} = n \times U_g = 5.16, \quad \overline{N} = 6 - 5.16 = 0.84$$

$$E\delta = \frac{0.516}{6 \times \frac{1}{40} \times 0.484} = \frac{4 \times 5.16}{6 \times 0.484} \approx \frac{7}{10} \text{ hour}$$

■

Example 13 *Change the mean lifetime to* 2 *hours in the previous Example. Find the performance measures.*

Solution: $\frac{1}{\lambda} = 2$, $\frac{1}{\mu} = 4$, $\frac{\lambda}{\mu} = 2$, $n = 6$, $P_0 = \frac{1}{75973}$, which shows that a single repairman is not enough. We should increase the number of repairmen.

Failed machines	0	1	2	3	4	5	6
Waiting machines	0	0	1	2	3	4	5
P_k	$\frac{1}{75973}$	$\frac{1}{75973}$	0.001	0.012	0.075	0.303	0.606

$$U_s \approx 0.999, \quad \overline{Q} \approx 4.5, \quad \overline{W} \approx 22.5\text{hours}, \quad \overline{T} = 26.5 \text{ hours}$$

$$\overline{e} = 2 \text{ hours}, \quad U_g \approx 0.08, \quad \overline{m} \approx 0.5, \quad \overline{N} \approx 5.5, \quad E\delta \approx \infty.$$

All these measures demonstrate what we have expected because 1 is greater than 1. To decide how many repairmen is needed there are different criterias as we shall see in Section 3.4. To avoid this congestion we must ensure the condition $\frac{\lambda}{r\mu} < 1$ where r is the number of repairmen.

■

3.3 Heterogeneous Queues

The results of this section have been published in the paper of Csige and Tomkó [16]. The reason of its introduction is to show the importance of the service discipline.

Let us consider n heterogeneous machines with exponentially distributed operating and repair time with parameter $\lambda_k > 0$ and $\mu_k > 0$, respectively for the kth machine, $k =$

$1, \cdots, n$. The failures are repaired by a single repairman according to Processor Sharing, FIFO, and Preemptive Priory disciplines. All involved random variables are supposed to be independent of each other.
Let $N(t)$, denote the number of failed machines at time t. Due to the heterogeneity of the machines this information is not enough to describe the behavior of the system because we have to know which machine is under service. Thus let us introduce an $N(t)$-dimensional vector with components $\left(x_1(t), \ldots, x_{v(t)}(t)\right)$ indicating the indexes of the failed machines. Hence for $N(t) > 0$ using FIFO discipline machine with index $x_1(t)$ is under service. Under Processor Sharing discipline when all machines are serviced by a proportional service rate, that is if $N(t) = k$ then the proportion is $1/k$ the order of indexes $(x_1(t), \ldots, x_n(t))$ is not important, but a logical treatment we order them as $\left(x_1(t) < x_2(t) < \ldots < x_{v(t)}(t)\right)$. In the case of Preemptive Priority assuming that the smaller index means higher priority we use the same ordering as before mentioning that in this case the machine with the first index is under service since he has the highest priority among the failed machines.
Due to the exponential distributions the process

$$X(t)(t) = \left(v(t); x_1(t), \ldots, x_{v(t)}(t)\right), \qquad (t \geq 0),$$

is a continuous-time Markov where the ordering of $x_1(t), \ldots, x_{v(t)}(t)$ depends ot the service discipline.
Since $X(t)(t)$ is a finite state Markov chain thus if the parameters λ_k, μ_k, $(1 \leq k \leq n)$ are all positive then it is ergodic and hence the steady-state distribution exists. Of course this heavily depends on the service discipline.

3.3.1 The $\vec{M}/\vec{M}/1/n/n/PS$ Queue

Let the distribution of the Markov chain be denoted by

$$P_0(t), \quad , P'_{i_1,\ldots,i_k}(t).$$

It is not difficult to see that for this distribition we have

$$P'_0(t) = -\left[\sum_{i=1}^{n} \lambda_i\right] P_0(t) + \sum_{i=1}^{n} \mu_i P_i(t),$$

$$P'_{i_1,\ldots,i_k}(t) = \sum_{r=1}^{k} \lambda_{i_r} P_{i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_k}(t) -$$

$$-\left[\nu_{i_1\ldots i_k} + \frac{1}{k}\sum_{r=1}^{k} \mu_{i_r}\right] P_{i_1,\ldots,i_k}(t) + \sum_{r \neq i_1\ldots i_k} \frac{\mu_r}{k+1} P_{i'_1 i'_2 \ldots i'_{k+1}}(t)$$

where $i'_1, \ldots, i'_{k+1}$ is the ordering of the indexes $i_1, \ldots, i_k, r$ and

$$\nu_{i_1\ldots i_k} = \sum_{r \neq i_1\ldots i_k} \lambda_r, \quad k = 1, \ldots, n-1.$$

$$P'_{1,\ldots,n}(t) = \sum_{r=1}^{n} \lambda_r P_{1,\ldots,r-1,r+1,]\ldots,n}(t) - \left[\frac{1}{n}\sum_{r=1}^{n}\mu_r\right] P_{1,\ldots,n}(t).$$

The steady-state distribution which is denoted by

$$P_0 = \lim_{t\to\infty} P_o\,(t)\,,$$

$$P_{i_1\ldots i_k} = \lim_{t\to\infty} P_{i_1\ldots i_k}\,(t)$$

$$(1 \le i_1 < i_2 < \ldots < i_k \le n, \;\; 1 \le k \le n).$$

is the solution of the following set of equations

$$\left[\sum_{i=1}^{n}\lambda_i\right] P_0 = \sum_{i=1}^{n}\mu_i P_i,$$

$$\left[\nu_{i_1\ldots i_k} + \frac{1}{k}\sum_{r=1}^{k}\mu_{i_r}\right] P_{i_1\ldots i_k} = \sum_{r=1}^{k}\lambda_{i_r} P_{i_1\ldots i_{r-1}i_{r+1}\ldots i_k} +$$

$$+ \sum_{r\neq i_1\ldots i_k} \frac{\mu_r}{k+1} P_{i'_1 i'_2\ldots i'_{k+1}},$$

$$\left[\frac{1}{n}\sum_{r=1}^{n}\mu_r\right] P_{1,\ldots,n} = \sum_{r=1}^{n}\lambda_r P_{1\ldots,r-1,r+1,\ldots,n}$$

with normalizing condition

$$P_0 + \sum P_{i_1\ldots i_k} = 1$$

where the summation is mean by all possible combinations of the indexes.

The surprising fact is it can be obtained as

$$P_{i_1\ldots i_k} = Ck! \prod_{r=1}^{k} \frac{\lambda_{i_r}}{\mu_{i_r}},$$

where C can be calculated from the normalizing condition.

For the FIFO and Preemptive Priority disciplines the balance equations and the solution is rather complicated and they are omitted. The interested reader is referred to the cited paper. However for all cases the performance measures can be computed the same way.

Performance Measures

- *Utilization of the server*

$$U_s = \frac{\mathbb{E}(\delta)}{\mathbb{E}(\delta) + \left[\sum_{i=1}^{n}\lambda_i\right]^{-1}} = 1 - P_0.$$

- *Utilization of the machines*

  Let $U^{(i)}$ denote the utilization of machine i. Then

$$U^{(i)} = \frac{\frac{1}{\lambda_i}}{\frac{1}{\lambda_i} + \overline{T}_i} = 1 - P^{(i)},$$

  where $\overline{T}_i$ denotes the mean response time for machine i, that is the mean time while it is broken, and

$$P^{(i)} = \sum_{k=1}^{n} \sum_{i \in (i_1,\ldots,i_k)} P_{i_1,\ldots,i_k},$$

  is the probability that the ith machine is failed. Thus

$$\overline{T}_i = \frac{P^{(i)}}{\lambda_i \left(1 - P^{(i)}\right)},$$

  and in FIFO case for the main waiting time we have

$$\overline{W}_i = \overline{T}_i - \frac{1}{\mu_i}.$$

  Furthermore it is easy to see that the mean number of failed machines can be obtained as

$$\overline{N} = \sum_{i=1}^{n} P^{(i)}.$$

  In addition

$$\sum_{i=1}^{n} \lambda_i \left(1 - P^{(i)}\right) \overline{T}_i = \sum_{i=1}^{n} P^{(i)}$$

  which is the **Little's formula** for heterogeneous customers. In particular, for homogeneous case we

$$(n - \bar{N})\lambda \bar{T} = \bar{N}$$

  which was proved earlier.

Various generalized versions of the machine interference problem with heterogeneous machines can be found in Pósafalvi and Sztrik [62, 63].

Let us see some sample numerical results for the illustration of the influence of the service disciplines on the main performance measures

Inpur parameters	Machine utilizations		
	FIFO	PROC-SHARING	PRIORITY
$n = 3$			
$\lambda_1 = 0.3 \quad \mu_1 = 0.7$	0.57	0.57	0.70
$\lambda_2 = 0.3 \quad \mu_2 = 0.7$	0.75 0.57	0.74 0.57	0.74 0.58
$\lambda_3 = 0.3 \quad \mu_3 = 0.7$	0.57	0.57	0.44
Overall machine utilization	1.72	1.72	1.72
$n = 3$			
$\lambda_1 = 0.5 \quad \mu_1 = 0.9$	0.48	0.51	0.64
$\lambda_2 = 0.3 \quad \mu_2 = 0.7$	0.75 0.56	0.76 0.56	0.77 0.56
$\lambda_3 = 0.2 \quad \mu_3 = 0.5$	0.62	0.58	0.44
Overall machine utilization	1.669	1.666	1.656
$n = 4$			
$\lambda_1 = 0.5 \quad \mu_1 = 0.9$	0.38	0.429	0.64
$\lambda_2 = 0.4 \quad \mu_2 = 0.7$	0.41	0.423	0.49
	0.903	0.906	0.922
$\lambda_3 = 0.3 \quad \mu_3 = 0.6$	0.46	0.451	0.36
$\lambda_4 = 0.2 \quad \mu_4 = 0.5$	0.54	0.500	0.24
Overall machine utilization	1.814	1.804	1.751

Table 3.1: Numerical results

3.4 The $M/M/r/n/n$ Queue

Consider the homogeneous finite-source model with $r, \quad r \leq n$ independent servers. Denoting by $N(t)$ the number of customers in the system at time t similarly to the previous sections it can easily be seen that it is a birth-death process rates

$$\lambda_k = (n-k)\lambda, \qquad 0 \leq k \leq n-1,$$

$$\mu_k = \begin{cases} k\mu & , 1 \leq k \leq r, \\ r\mu & , r < k \leq n, \end{cases}$$

intenzitásokkal.
The steady-state distribution can be obtained as

$$P_k = \binom{n}{k} \rho^k P_0, \qquad 0 \leq k \leq r,$$

$$P_k = \frac{k!}{r! r^{k-r}} \binom{n}{k} \rho^k P_0, \qquad r \leq k \leq n$$

with normalizing condition

$$\sum_{k=0}^{n} P_k = 1$$

To determine P_0 we can use the following simpler recursion.

Let $\frac{a_k=P_k}{P_0}$ and using the relation for the consecutive elements of the birth-death process our procedure operates as follows

$$a_0 = 1,$$
$$a_k = \frac{n-k+1}{k}\varrho a_{k-1}, \qquad 0 \le k \le r-1,$$
$$a_k = \frac{n-k+1}{r}\varrho a_{k-1}, \qquad r \le k \le n.$$

Since

$$\sum_{k=0}^{n} P_k = 1$$

must be satisfied thus we get

$$P_0 = 1 - \sum_{k=1}^{n} P_k.$$

Dividing both sides by P_0 we have

$$1 = \frac{1}{P_0} - \sum_{k=1}^{n} \frac{P_k}{P_0} = \frac{1}{P_0} - \sum_{k=1}^{n} a_k,$$

hence

$$P_0 = \frac{1}{1 + \sum\limits_{k=1}^{n} a_k}.$$

Finally

$$P_k = a_k P_0.$$

Let us determine *the main performance measures*

- *Mean number of customers in the systems* can be computed as

$$\overline{N} = \sum_{k=0}^{n} k P_k.$$

- *Mean queue length* can be obtained by

$$\overline{Q} = \sum_{k=r+1}^{n} (k-r) P_k = \frac{r^r P_0}{r!} \sum_{k=r+1}^{n} \frac{(k-r)k!}{r^k} \binom{n}{k} \varrho^k.$$

- *Mean number of customers in the source* can be calculated by

$$\overline{m} = n - \overline{N}.$$

- *Utilization of the system* is computed by

$$U_r = 1 - P_0.$$

- *Mean busy period of the systems* can be obtained by

$$E\delta^{(n)} = \frac{1-P_0}{n\lambda P_0} = \frac{U_r}{n\lambda P_0}.$$

- *Mean number of busy servers* can be calculated by

$$\overline{r} = \sum_{k=1}^{r} kP_k + \sum_{k=r+1}^{n} rP_k = \sum_{k=1}^{r-1} kP_k + r\sum_{k=r}^{n} P_k = \sum_{k=1}^{r-1} kP_k + rP(W>0).$$

  Furthermore

$$U_s = \frac{\sum_{k=1}^{r} kP_k + r\sum_{k=r+1}^{n} P_k}{r} = \frac{\overline{r}}{r}.$$

- *Mean number of idle servers*

$$\overline{S} = r - \overline{r}.$$

  Additional relation is

$$\overline{N} = \sum_{k=1}^{r} kP_k + \sum_{k=r+1}^{n} (k-r)P_k + r\sum_{k=r+1}^{n} P_k = \overline{Q} + \overline{r} = \overline{Q} + r - \overline{S} = n - \overline{m}.$$

- *Utilization of the sources* can be calculated by

$$U_t = \sum_{k=1}^{n} \frac{n-k}{n} P_k = \frac{\overline{m}}{n}.$$

- *The mean waiting and response times* can be derived by

$$U_t = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \overline{W} + \frac{1}{\mu}} = \frac{\overline{m}}{n},$$

  thus for the mean waiting time we have

$$\overline{W} = \frac{\overline{N}}{\overline{m}}\frac{1}{\lambda} - \frac{1}{\mu} = \frac{1}{\mu}\left(\frac{\overline{N}}{\overline{m}\varrho} - 1\right).$$

  Hence the mean response time is

$$\overline{T} = \overline{W} + \frac{1}{\mu} = \frac{\overline{N}}{\overline{m}\lambda},$$

  consequently we get

$$\overline{m}\lambda\overline{T} = \overline{N},$$

  which is the well-known **Little's formula**. Thus we get

$$\overline{m}\lambda\left(\overline{W} + \frac{1}{\mu}\right) = \overline{Q} + \overline{r},$$

that is

$$\overline{m}\lambda\overline{W} + \overline{m}\varrho = \overline{Q} + \overline{r}.$$

Show that

$$\overline{r} = \overline{m}\varrho,$$

because from this follows

$$\overline{m}\lambda\overline{W} = \overline{Q}$$

which is the **Little's formula** for the waiting time.

Since

$$P_{k+1} = \frac{(n-k)\lambda}{\mu_{k+1}} P_k,$$

where

$$\mu_j = \begin{cases} j\mu & , j \leq r, \\ r\mu & , j > r. \end{cases}$$

Furthermore, it is well-known that

$$\overline{r} = \sum_{k=1}^{r-1} kP_k + r\sum_{k=r}^{n} P_k.$$

We can proceed as

$$\varrho\overline{m} = \sum_{k=0}^{n} \varrho(n-k)P_k = \sum_{k=0}^{r-1} \varrho(n-k)P_k + \sum_{k=r}^{n-1} \varrho(n-k)P_k =$$

$$= \sum_{k=0}^{r-1} \frac{\lambda(n-k)(k+1)}{(k+1)\mu} P_k + r\sum_{k=r}^{n-1} \frac{\lambda(n-k)}{r\mu} P_k =$$

$$= \sum_{k=0}^{r-1} (k+1)P_{k+1} + r\sum_{k=r}^{n-1} P_{k+1} = \sum_{j=1}^{r} jP_j + r\sum_{j=r+1}^{n} P_j = \sum_{j=1}^{r-1} jP_j + r\sum_{j=r}^{n} P_j = \overline{r}.$$

Finally we get

$$\varrho\overline{m} = \overline{r},$$

or in another form

$$\lambda\overline{m} = \mu\overline{r},$$

that is

$$\text{mean arrival rate} = \text{mean service rate},$$

which was expected because the system is in steady state. Consequently

$$\overline{W} = \frac{\overline{Q}}{\overline{m}\lambda} = \frac{\overline{Q}}{\overline{r}\lambda\varrho} = \frac{\overline{Q}}{\mu\overline{r}}.$$

- *Mean idle period of a server* can be computed as follows.
  If the idle servers start their busy period in the order as they finished the previous busy period, then their activity can be written as follows. If a server becomes idle and finds other $j-1$ servers idle, then his busy period start at the instant of the arrival of the j th customer.

  Let $\overline{e}$ denote the mean idle period of the server and let $\overline{e}_j$ denote the mean conditional idle period mention above. Clearly

$$\overline{e}_j = \frac{j}{\lambda},$$

  and $\overline{e}$ can be computed by the help of the theorem of total expectation, namely

$$\overline{e} = \sum_{j=1}^{r} \frac{P_{r-j}}{P(e)} \frac{j}{\lambda} = \frac{\overline{S}}{P(e)\lambda},$$

  where

$$P(e) = \sum_{j=0}^{r-1} P_j = 1 - P(W > 0),$$

  is the probability that there is an idle server.

- *Mean busy period of the servers* can be calculated as follows.
  Since

$$U_s = \frac{E\delta}{\overline{e} + E\delta},$$

  thus

$$E\delta = \frac{U_s}{1-U_s}\overline{e} = \frac{\frac{\overline{r}}{r}}{1-\frac{\overline{r}}{r}}\overline{e} = \frac{\frac{\overline{r}}{r}}{\frac{r-\overline{r}}{r}} \frac{\overline{S}}{P(e)\lambda} = \frac{\overline{r}}{\overline{S}} \frac{\overline{S}}{P(e)\lambda} = \frac{\overline{r}}{P(e)\lambda} = \frac{\overline{m}}{\mu P(e)}.$$

  That is

$$E\delta = \frac{\overline{m}}{\mu P(e)}.$$

Distribution Function of the Waiting and Response Time

This subsection is devoted to the most complicated problem of this system, namely to the determination of the distribution function of the waiting and response times. First the density function is calculated and then we obtain the distribution function. You may remember that the distribution has been given in the form

$$P_k = \begin{cases} \binom{n}{k} \rho^k P_0 \\ \\ \binom{n}{k} k! \rho^k \\ \hline r! r^{k-r} P_0. \end{cases}$$

Introducing $z = \frac{1}{\rho}$, this can be written as

$$P_k = \begin{cases} \binom{n}{k} z^{-k} P_0 \\ \\ \binom{n}{k} k! z^{-k} \\ \hline r! r^{k-r} P_0 \end{cases}$$

thus

$$\begin{aligned} P_k &= \frac{\binom{n}{k} k! r^r (rz)^{-k}}{r!} P_0 \\ &= \frac{n! r^r (rz)^{n-k} \cdot e^{-rz}}{(n-k)! r! (rz)^n \cdot e^{-rz}} P_0 \\ &= \frac{r^r}{r!} \frac{P(n-k, rz)}{P(n, rz)} P_0, \qquad k \geq r. \end{aligned}$$

Since

$$\begin{aligned} &\Pi_k(n) = P_k(n-1), \quad \text{thus} \\ &\Pi_k(n) = \frac{r^r}{r!} \frac{P(n-1-k, rz)}{P(n-1, rz)} P_0(n-1), \qquad \text{ha } k = r, \ldots, n-1. \end{aligned}$$

It is easy to see that the probability of waiting is

$$\sum_{k=r}^{n-1} \Pi_k(n) = \sum_{k=r}^{n-1} P_k(n-1) = P_W$$

Inserting z this can be rewritten as

$$\begin{aligned} P_W &= \sum_{k=r}^{n-1} \frac{r^r}{r!} \frac{P(n-1-k, rz)}{P(n-1, rz)} P_0(n-1) \\ &= \frac{r^r}{r!} P_0(n-1) \frac{\sum_{i=0}^{n-1-r} P(i, rz)}{P(n-1, rz)} \\ &= \frac{r^r}{r!} \frac{Q(n-1-r, rz)}{P(n-1, rz)} P_0(n-1). \end{aligned}$$

We show that the distribution function of the waiting time can be calculated as

$$F_W(x) = 1 - \frac{r^r Q(n-1-r, r(z+\mu x))}{r! P(n-1, rz)} P_0(n-1),$$

and thus

$$F_W(0) = 1 - \frac{r^r Q(n-1-r, rz)}{r! P(n-1, rz)} P_0(n-1)$$

which is probability that an arriving customer finds idle server. For the density function we have

$$f_W(0) = 1 - P_W,$$
$$f_W(x) = \mu r \frac{r^r P(n-1-r, r(z+\mu x))}{r! P(n-1, rz)} P_0(n-1), \qquad x > 0.$$

If we calculate the integral $\int_{0+}^{\infty} f_W(x)\mathrm{d}x$-t that is 0 is not considered then

$$\int_{0+}^{\infty} f_W(x)\mathrm{d}x = \frac{r^r P_0(n-1)}{r! P(n-1, rz)} \cdot \int_{0+}^{\infty} \mu r \frac{(r(z+\mu t))^{n-1-r}}{(n-1-r)!} e^{-r(z+\mu t)} \mathrm{d}t.$$

By the substitution $y = r(z+\mu t)$ we have $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{\mu}$ for the integral part we get

$$\int_{rz}^{\infty} \frac{y^{n-1-r}}{(n-1-r)!} e^{-y} \mathrm{d}y = Q(n-1-r, rz)$$

that is

$$\int_{0+}^{\infty} f_W(x)\mathrm{d}x = \frac{r^r Q(n-1-r, rz)}{r! P(n-1, rz)} P_0(n-1) = P_W,$$

as it was expected. Thus

$$\int_{0}^{\infty} f_W(x)\mathrm{d}x = f_W(0) + \int_{0+}^{\infty} f_W(x)\mathrm{d}x = 1.$$

Let us determine the density function for $x > 0$. That is

$$\begin{aligned}
f_W(x) &= \sum_{k=r}^{n-1} r\mu \frac{(r\mu x)^{k-r}}{(k-r)!} e^{-r\mu x} P_k(n-1) \\
&= \sum_{k=r}^{n-1} r\mu \frac{(r\mu x)^{k-r}}{(k-r)!} e^{-r\mu x} \frac{r^r}{r!} \frac{P(n-1-k, rz)}{P(n-1, rz)} P_0(n-1) \\
&= \frac{r\mu r^r P_0(n-1)}{r! P(n-1, rz)} \sum_{k=r}^{n-1} \frac{(r\mu x)^{k-r}}{(k-r)!} \frac{(rz)^{n-1-k}}{(n-1-k)!} e^{-r(z+\mu x)} \\
&= \frac{r\mu r^r P_0(n-1) e^{-r(z+\mu x)}}{r! P(n-1, rz)} \sum_{i=0}^{n-1-r} \frac{(r\mu x)^i}{i!} \frac{(rz)^{n-1-r-i}}{(n-1-r-i)!} \\
&= \frac{r\mu r^r P_0(n-1)}{r! P(n-1, rz)} \frac{(r(z+\mu x))^{n-1-r}}{(n-1-r)!} \cdot e^{-r(z+\mu x)} \\
&= \frac{r\mu r^r P_0(n-1) P(n-1-r, r(z+\mu x))}{r! P(n-1, rz)},
\end{aligned}$$

as we got earlier, but we have to remember that

$$f_W(0) = 1 - P_W.$$

Therefore

$$\begin{aligned} P(W > x) &= \int_x^\infty f_W(t)\mathrm{d}t \\ &= \frac{r^r P_0(n-1)}{r!P(n-1,rz)} \int_x^\infty r\mu \frac{(r(z+\mu t))^{n-1-r}}{(n-1-r)!} e^{-r(z+\mu t)}\mathrm{d}t \\ &= \frac{r^r P_0(n-1)}{r!P(n-1,rz)} \int_{r(z+\mu x)}^\infty \frac{y^{n-1-r}}{(n-1-r)!} e^{-y}\mathrm{d}y \\ &= \frac{r^r P_0(n-1)Q(n-1-r, r(z+\mu x))}{r!P(n-1,rz)}. \end{aligned}$$

Thus for the distribution function we have

$$F_W(x) = 1 - P(W > x)$$

which was obtained earlier.

To verify the correctness of the formula let $r = 1$. After substitution we get

$$P(W > x) = \frac{P_0(n-1)Q(n-2, z+\mu x)}{P(n-1,z)},$$

but

$$P_0(n-1) = \frac{P(n-1,z)}{Q(n-1,z)},$$

thus

$$P(W > x) = \frac{Q(n-z, z+\mu x)}{Q(n-1,z)}.$$

The derivation of the distribution function of the response time is analogous. Because the calculation is rather lengthly it is omitted, but can be found in the Solution Manual for Kobayashi [50].
As it can be seen in Allen [2], Kobayashi [50], the following formulas are valid for $r \geq 2$

$$F_T(x) = 1 - C_1 e^{-\mu x} + C_2 Q(n-r-1, r(z+\mu x)),$$

where

$$\begin{aligned} C_1 &= 1 + C_2 Q(n-r-1, rz), \\ C_2 &= \frac{r^r P_0(n-1)}{r!(r-1)(n-r-1)!P(n-1,rz)}. \end{aligned}$$

Hence the density function can be obtained as

$$f_T(x) = \mu C_1 e^{-\mu x} - C_2 r\mu P(n-r-1, r(z+\mu x)).$$

It should be noted that for the normalizing constant we have the following recursion

$$P_0^{-1}(n) = 1 + \frac{n}{rz} P_0^{-1}(n-1) + \frac{n}{z} \sum_{i=0}^{r-1} \frac{\binom{n-1}{i}}{z^i} \left(\frac{1}{i+1} - \frac{1}{r} \right), \qquad n > r,$$

with initial value

$$P_0^{-1}(r) = \left(1 + \frac{1}{z}\right)^r. \qquad r \geq 1.$$

Laplace-transform of the Waiting and Response Times

First determine the Laplace-transform of the waiting time.
It is easy to see that by using the theorem of total Laplace-transform we have

$$L_W(s) = 1 - P_W + \sum_{k=r}^{n-1} \left(\frac{r\mu}{r\mu + s} \right)^{k-r+1} P_k(n-1).$$

We calculate this formula step-by-step. Namely we can proceed as

$$\sum_{k=r}^{n-1} \left(\frac{r\mu}{r\mu+s} \right)^{k-r+1} \frac{r^r P_0(n-1) P(n-1-k, rz)}{r! P(n-1, rz)}$$
$$= \frac{r^r P_0(n-1) e^{-rz}}{r! P(n-1, rz)} \sum_{k=r}^{n-1} \left(\frac{r\mu}{r\mu+s} \right)^{k-r+1} \frac{(rz)^{n-1-k}}{(n-1-k)!}.$$

Then

$$\sum_{k=r}^{n-1} \left(\frac{r\mu}{r\mu+s} \right)^{k-r+1} \frac{(rz)^{n-1-k}}{(n-1-k)!}$$
$$= \sum_{i=0}^{n-1-r} \left(\frac{r\mu}{r\mu+s} \right)^{i+1} \frac{(rz)^{n-1-r-i}}{(n-1-r-i)!},$$

where $i = k - r$. Thus the last equation can be written as

$$\left(\frac{r\mu}{r\mu+s} \right)^{n-r} \cdot \sum_{i=0}^{n-1-r} \frac{\left(\frac{r\mu+z}{\lambda}\right)^{n-1-r-i}}{(n-1-r-i)!} = \left(\frac{r\mu}{r\mu+s} \right)^{n-r} e^{\frac{r\mu+z}{\lambda}} Q\left(n-1-r, \frac{r\mu+z}{\lambda} \right).$$

Finally collecting all terms we get

$$L_W(s) = 1 - P_W + \left(\frac{r\mu}{r\mu+s} \right)^{n-r} \frac{r^r P_0(n-1) e^{-rz}}{r! P(n-1, rz)} e^{\frac{r\mu+s}{\lambda}} Q\left(n-1-r, \frac{r\mu+s}{\lambda} \right)$$
$$= 1 - P_W + \frac{r^r e^{\frac{s}{\lambda}} P_0(n-1) Q\left(n-1-r, \frac{r\mu+s}{\lambda}\right)}{r! P(n-1, rz)} \left(\frac{r\mu}{r\mu+s} \right)^{n-r}.$$

To verify the correctness of the formula let $r = 1$.
Thus after inserting we have

$$\begin{aligned}
L_W(s) &= P_0(n-1) + \left(\frac{\mu}{\mu+s}\right)^{n-1} \frac{e^{\frac{s}{\lambda}} P_0(n-1) Q\left(n-2, \frac{\mu+s}{\lambda}\right)}{P(n-1,z)} \\
&= \frac{P(n-1,z)}{Q(n-1,z)} + \left(\frac{\mu}{\mu+s}\right)^{n-1} \frac{e^{\frac{s}{\lambda}} Q\left(n-2, \frac{\mu+s}{\lambda}\right)}{Q(n-1,z)} \\
&= \frac{\left(\frac{\mu}{\mu+s}\right)^{n-1}}{Q(n-1,z)} \left[\frac{\left(z\left(\frac{\mu+s}{\mu}\right)\right)^{n-1}}{(n-1)!} e^{-z} + e^{\frac{s}{\lambda}} Q\left(n-2, \frac{\mu+s}{\lambda}\right) \right] \\
&= \frac{\left(\frac{\mu}{\mu+s}\right)^{n-1}}{Q(n-1,z)} \left[\frac{\left(\frac{\mu+s}{\lambda}\right)^{n-1} e^{-\frac{\mu+s}{\lambda}} e^{\frac{s}{\lambda}}}{(n-1)!} + e^{\frac{s}{\lambda}} Q\left(n-2, \frac{\mu+s}{\lambda}\right) \right] \\
&= \frac{\left(\frac{\mu}{\mu+s}\right)^{n-1} e^{\frac{s}{\lambda}} Q\left(n-1, \frac{\mu+s}{\lambda}\right)}{Q\left(n-1,z\right)},
\end{aligned}$$

as we got earlier.

Keeping in mind the relation between the waiting time and the response time and the properties of the Laplace-transform we have

$$L_T(s) = \left(\frac{\mu}{\mu+s}\right) L_W(s),$$

which is in the case of $r = 1$ reduces to

$$L_T(s) = \left(\frac{\mu}{\mu+s}\right)^{n} \frac{e^{\frac{s}{\lambda}} Q\left(n-1, \frac{\mu+s}{\lambda}\right)}{Q(n-1,z)}.$$

Java applets for direct calculations can be found at
`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcKK/MMcKK.html`

Example 14 *A factory possesses* 20 *machines having mean lifetime of* 50 *hours. The mean repair time is* 5 *hours and the repairs are carried out by* 3 *repairmen. Find the performance measures of the system.*

Solution:

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{\mu}}{\frac{1}{\lambda}} = \frac{5}{50} = \frac{1}{10} = 0.1$$

By using the recursive approach we get

$$a_0 = 1$$

$$a_1 = \frac{20-0}{0+1} \times 0.1 \times 1 = 2$$

$$a_2 = \frac{20-1}{1+1} \times 0.1 \times 2 = 1.9$$

$$a_3 = \frac{20-2}{2+1} \times 0.1 \times 1.9 = 1.14$$

$$a_4 = \frac{20-3}{3} \times 0.1 \times 1.14 = 0.646$$

$$\vdots$$

and so on.
Hence

$$P_0 = \frac{1}{1 + \sum_{k=1}^{n} a_k} = \frac{1}{1 + 6.3394} = 0.13625.$$

Innen

$$P_1 = a_1 \times P_0 = 2 \times 0.13625 = 0.2775$$

$$P_2 = a_2 \times P_0 = 1.9 \times 0.13625 = 0.2588 \text{ etc}$$

The distribution can be seen in the next Table for
$n = 20$, $r = 3$, $\rho = 0.1$

K	Number of busy under repair repairmen	Number of waiting machines (Q)	Number of idle repairmen (S)	Steady-state. distribution (P_k)
0	0	0	3	0.13625
1	1	0	2	0.27250
2	2	0	1	0.25888
3	3	0	0	0.15533
4	3	1	0	0.08802
5	3	2	0	0.04694
6	3	3	0	0.02347
7	3	4	0	0.01095
8	3	5	0	0.00475
9	3	6	0	0.00190
10	3	7	0	0.00070
11	3	8	0	0.00023
12	3	9	0	0.00007

Hence the performance measures are

$$\overline{Q} = 0.339, \qquad \overline{S} = 1.213, \qquad \overline{N} = \overline{Q} + r - \overline{S} = 2.126$$

$$P(W>0)=0.3323, \qquad P(e)=0.6677, \qquad \overline{W}=\frac{\overline{Q}}{\lambda(n-\overline{N})}=0.918 \text{ hours, } 58 \text{ minutes}$$

$$\overline{m}=20-2.126=17.874, \qquad U^{(n)}=0.844$$

$$E\delta^{(n)}=\frac{U^{(n)}}{n\lambda P_0}=\frac{5}{2}\times\frac{0.844}{0.136}\approx 15.5 \text{ hours}, \qquad \overline{r}=1.787, \qquad \overline{s}=1.213$$

$$U_S=\frac{\overline{r}}{r}=\frac{1.787}{3}=0.595, \qquad \overline{e}=\frac{\overline{s}}{P(e)\lambda}=\frac{1.213}{0.667\times\frac{1}{50}}=\frac{50\times 1.213}{0.667}\approx 90.8 \text{ hours}$$

$$E\delta=\frac{\overline{r}}{P(e)\lambda}=\frac{1.787}{0.667\times\frac{1}{50}}=\frac{50\times 1.787}{0.667}\approx 132.1 \text{ hours}$$

$$U_g=\frac{\overline{m}}{n}=\frac{17.874}{20}\approx 0.893$$

$$\overline{T}=\overline{W}+\frac{1}{\mu}=0.981+5=5.981 \text{ hours}$$

$$K_1=\frac{\text{mean number of waiting machines}}{\text{total number of machines}}=\frac{\overline{Q}}{n}=\frac{0.339}{20}=0.0169$$

$$K_2=\frac{\text{mean number of idle repairmen}}{\text{total number of repairmen}}=\frac{\overline{S}}{r}=\frac{1.213}{3}=0.404$$

Let us compare these measures to the system where we have 6 machines and a single repairman. The lifetime and repair time characteristics remain the same. The result can be seen in the next Table

Number of machines	6	20
Number of repairman	1	3
Number of machines per repairman	6	$6\frac{2}{3}$
Waiting coefficient for the servers K_2	0.4845	0.4042
Waiting coefficient for the machines K_1	0.0549	0.01694

■

Example 15 *Let us continue the previous Example with cost structure. Assume that the waiting cost is* 18 000 *Euro/hour and the cost for an idle repairman is* 600 *Euro/hour. Find the optimal number of repairmen. It should be noted that different cost functions can be constructed.*

Solution:

The mean cost per hour as a function of r can be seen in the next Table which are calculated by the help of the distribution listed below for $r=3,4,5,6,7$.

r	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
3	0.136	0.272	0.258	0.155	0.088	0.047	0.023	0.011	0.005
4	0.146	0.292	0.278	0.166	0.071	0.028	0.010	0.003	0.001
5	0.148	0.296	0.281	0.168	0.071	0.022	0.006	0.001	0.000
6	0.148	0.297	0.282	0.169	0.072	0.023	0.006	0.001	$\cdots$
7	0.148	0.297	0.282	0.169	0.072	0.023	0.006	$\cdots$	$\cdots$

The mean cost per hour is

r	$\overline{Q}$	$\overline{S}$	$E(Cost)$ Euro
3	0.32	1.20	6480
4	0.06	2.18	2388
5	0.01	3.17	2082
6	0	4.17	2502
7	0	5.16	3096

Hence the optimal number is $r = 5$.
This simple Example shows us that there are different criteria for the optimal operation.
■

3.5 The $M/M/r/K/n$ Queue

This system is an combination of the finite-source systems considered in the previous sections. It is the most general system since for $K = r$ we have the Engset system treated in Section 3.1, for $r = 1, \quad K = n$ get the system analyzed in Section 3.2, for $K = n$ we obtain the system of Section 3.4. For the value $r < K < n$ we have delay-loss system, that is customers can arrive into the system until the number of customers in the system is $K - 1$ but then the must return to the source because the system is full.
As before it is easy to see that the number of customers in the systems is a birth-death process with rates

$$\lambda_k = (n-k) \quad , \qquad 0 \le k < K,$$

$$\mu_k = \begin{cases} k\mu & , \qquad 1 \le k \le r, \\ r\mu & , \qquad r < k \le K \end{cases}$$

where $1 \le r \le n$, $r \le K \le n$. It is qiute complicated system and have not been investigated, yet. The main problem is that there are no closed form formulas as before, but using computers all the performance measures can be obtained. The normalizing constant $P_0(n, r, K)$ should satisfies the normalizing condition

$$\sum_{i=0}^{K} P_k(n, r, K) = 1.$$

As before it can easily be seen that

$$P_k(n, r, K) = \begin{cases} \binom{n}{k} \rho^k P_0(n, r, K) & , \qquad 0 \le k < r, \\ \dfrac{\binom{n}{k} k! \rho^k}{r! r^{k-r}} P_0(n, r, K) & , \qquad r \le k \le K \end{cases}$$

The main *performance measures* can be computed as

$$\overline{N} = \sum_{k=0}^{K} kP_k, \qquad \overline{Q} = \sum_{k=r}^{K} (k-r)P_k, \qquad \overline{r} = \sum_{k=1}^{r-1} kP_k + r\sum_{k=r}^{K} P_k, \qquad \overline{m} = n - \overline{N},$$

$$U_S = \frac{\overline{r}}{r}, \qquad U_t = \frac{n-\overline{N}}{n}, \qquad \overline{\lambda} = \overline{\mu} = \mu\overline{r},$$

$$\overline{T} = \frac{\overline{N}}{\overline{\lambda}}, \qquad \overline{W} = \frac{\overline{Q}}{\overline{\lambda}}, \qquad \overline{W} = \overline{T} - \frac{1}{\mu},$$

$$\frac{n-\overline{N}}{n} = \frac{\mathbb{E}(\tau)}{\mathbb{E}(\tau)+\overline{T}}, \qquad \mathbb{E}(\tau) = \frac{(n-\overline{N})\overline{T}}{\overline{N}}, \qquad \overline{N_R} = \mathbb{E}(\tau)\lambda.$$

By using the Bayes's rule it is easy to see that for the *probability of blocking* we have

$$P_B(n,r,K) = \frac{(n-K)P_K(n,r,K)}{\sum_{i=0}^{K}(n-i)P_i(n,r,K)} = P_K(n-1,r,K).$$

In particular, if $K = n$, then

$$\overline{\lambda} = \lambda(n-\overline{N}) = \mu\overline{r},$$

thus

$$\overline{T} = \frac{\overline{N}}{\lambda(n-\overline{N})}, \qquad \mathbb{E}(\tau) = \frac{1}{\lambda}, \qquad P_B = 0,$$

as it was expected.
Furthermore, by elementary calculations it can be seen that the normalizing constant $P_0(n,r,K)$ can be expressed recursively with respect to K under fixed r, n. Namely we have

$$(P_0(n,r,K))^{-1} = (P_0(n,r,K-1))^{-1} + \frac{\binom{n}{K}K!\rho^K}{r!r^{K-r}},$$

with initial value

$$(P_0(n,r,r))^{-1} = \sum_{i=0}^{r}\binom{n}{i}\rho^i.$$

By using the Bayes's rule it is easy to see that the probability that an arriving customer finds k customers in the system is

$$\Pi_k^*(n,r,K) = P_k(n-1,r,K), \qquad k = 0,\cdots,K$$

but the probability that a customer arriving into the systems finds k customers there is

$$\Pi_k(n,r,K) = \frac{(n-k)P_k(n,r,K)}{\sum_{i=0}^{K-1}(n-i)P_i(n,r,K)}, \qquad k = 0,\ldots,K-1.$$

Hence the probability of waiting and the density function of the waiting time can be expressed as

$$\begin{aligned} P_W(n,r,K) &= \sum_{k=r}^{K-1} \Pi_k(n,r,K) \\ f_W(0) &= 1 - P_W(n,r,K) \\ f_W(x) &= \sum_{k=r}^{K-1} \frac{(r\mu)(r\mu x)^{k-r+1}}{(k-r+1)!} e^{-r\mu x} \cdot \Pi_k(n,r,K) \end{aligned}$$

By using the Bayes's rule it can easily be verified that

$$\Pi_k(n,r,K) = \frac{P_k(n-1,r,K)}{1-P_K(n-1,r,K)},$$

and analogously to the earlier arguments for the density function we obtain

$$f_W(x) = \frac{\mu r r^r P(K-1-r,rz)}{r!P(K-1,rz)} \frac{P_0(n-1)}{1-P_K(n-1,r,K)}.$$

In paricular, if $K = n$, that is all customer may enter into the system, then $P_K(n-1,r,K) = 0$ and thus we got the formulas derived before.

By reasonable modifications for the distribution function we have

$$F_W(x) = 1 - \frac{r^r Q(K-1-r, r(z+\mu x))}{r!P(K-1,rz)} \frac{P_0(n-1,r,K)}{1-P_K(n-1,r,K)}.$$

The corresponding Laplace-transform can be computed as

$$L_W(s) = 1 - P_W(n,r,K) + \left(\frac{r\mu}{r\mu+s}\right)^K \frac{r^r e^{\frac{s}{\lambda}}}{r!} \frac{Q\left(K-1-r, \frac{r\mu+s}{\lambda}\right) P_0(n-1,r,K)}{P(K-1,rz)(1-P_K(n-1,r,K))}.$$

3.6 The $M/G/1/n/n/PS$ Queue

This system is a generalization of system $M/M/1/n/n/FIFO$ treated in Section 3.2. The essential differences are the distribution of the service time and the service disciple. Since the service times are not exponentially distributed the number of customers as a stochastic process is not a Markov chain. In this Section we introduce the model which has been published in Yashkov [101].

The requests arrive from a finite-source where they spend an exponentially distributed time with parameter λ. The required service time S is generally distributed random variable with $\mathbb{E}S < \infty$. Let us denote by $G(x)$ and $g(x)$ its distribution function, density function, respectively, assuming that $(G(0^+) = 0)$. The service discipline is Processor Sharing, that is all customers in the service facility are being served but the rate is proportional to the number of customers in service.

The method of *supplementary variables* is used for the description of the behavior of the system.
Let us introduce the following random variables.
Let $\nu(t)$ denote the number of customers in the system at time t, and for $\nu(t) > 0$ let $\xi_1(t), \ldots, \xi_{\nu(t)}(t)$ denote the elapsed service time of the requests.

The stochastic process

$$X(t) = \big(\nu(t); \xi_1(t), \ldots, \xi_{\nu(t)}(t)\big)$$

is a continuous-time Markov process with discrete and continuous components which are called *piecewise linear Markov process.*

It should be noted the many practical problems can be modeled by the help of these processes and the interested reader is referred to the book of Gnedenko–Kovalenko [31].

Let

$$P_k(t, x_1, \ldots, x_k)\, dx_1 \ldots dx_k = P\left(\nu(t) = k; x_i \le \xi_i < x_i + dx_i, i = 1, \ldots, k\right),$$

that is $P_k(t, x_1, \ldots, x_k)$, $k = 1, \ldots, n$ denotes the density function that at time t there are k customers in the system and their elapsed service times are $x_1, \ldots, x_k$.
Let δ be a small positive real number. Then for the density functions $P_k(t, x_1, \ldots, x_k)$ we have the following set of equations

$$P_k(t; x_1, \ldots, x_k) =$$

$$= P_k\left(t - \delta; x_1 - \frac{\delta}{k}, \ldots, x_k - \frac{\delta}{k}\right) \prod_{i=1}^{k} \frac{1 - G(x_i)}{1 - G\left(x_i - \frac{\delta}{k}\right)} \left[1 - \lambda(n-k)\,\delta\right] +$$

$$+ (k+1) \int_0^\infty P_{k+1}\left(t - \delta; x_1 - \frac{\delta}{k}, \ldots, x_{k+1} - \frac{\delta}{k+1}\right) \times$$

$$\times \prod_{i=1}^{k} \frac{1 - G(x_i)}{1 - G\left(x_i - \frac{\delta}{k+1}\right)} \cdot \frac{G(x_{k+1}) - G\left[x_{k+1} - \frac{\delta}{k+1}\right]}{1 - G\left[x_{k+1} - \frac{\delta}{k+1}\right]} dx_{k+1}.$$

Dividing both sides by $\prod_{i=1}^{k} [1 - G(x_i)]$ and taking the limits as $\delta \to 0$, $t \to \infty$ we have the stationary equations, namely

$$\left[\frac{1}{k} \sum_{i=1}^{k} \frac{\partial}{\partial x_i} + \lambda(n-k)\right] q_k(x_1, \ldots, x_k) =$$

$$\int_0^\infty q_{k+1}(x_1,\ldots,x_{k+1})\,g(x_{k+1})\,dx_{k+1},\ k=1,\ldots,n-1,$$

where

$$q_k(x_1,\ldots,x_k)=\lim_{t\to\infty}P_k(t;x_1,\ldots,x_k)\,/\prod_{i=1}^{k}[1-G(x_i)]$$

are called *normalized density functions.*

Similarly, for P_0 and $q_n(x_1,\ldots,x_n)$ we obtain

$$\lambda n P_0=\int_0^\infty q_1(x_1)\,g(x_1)\,dx_1,$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}q_n(x_1,\ldots,x_n)=0.$$

Beside these equation we need the boundary conditions which are

$$q_1(0)=\lambda n P_0,$$

$$q_k(0,x_1,\ldots,x_{k-1})=\lambda(n-k+1)\,q_{k-1}(x_1,\ldots,x_{k-1}),$$

$$k=1,\ldots,n.$$

The solution to these set of integro-differential equations is surprisingly simple, namely

$$q_k(x_1,\ldots,x_k)=P_0\lambda^k n!/\left[(n-k)!\right],$$

which can be proved by direct substitution.
Consequently

$$P_k(x_1,\ldots,x_k)=P_0\lambda^k\frac{n!}{(n-k)!}\prod_{i=1}^{k}[1-G(x_i)],$$

$$i=1,\ldots,n.$$

Let us denote by P_k the steady-state probability of the number of customers in the system. Clearly we have

$$P_k=\int_0^\infty\ldots\int_0^\infty P_k(x_1,\ldots,x_k)\,dx_1\ldots dx_k=P_0\frac{n!}{(n-k)!}(\lambda\mathbb{E}S)^k.$$

Probability P_0 can be obtained by using the normalizing condition $\sum_{i=0}^{n}P_i=1$.

Recall that it is the same as the distribution in the $M/M/1/n/n$ system if $\varrho=\lambda\mathbb{E}S$.

It is not difficult to see that for this $M/G/1/n/n/PS$ system the *performance measures* can be calculated as

$$(i) \qquad \overline{N} = \sum_{k=1}^{n} kP_k$$

$$(ii) \quad U^{(i)} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \overline{T}} = \frac{n - \overline{N}}{n},$$

thus

$$\overline{T} = \frac{1}{\lambda} \frac{\overline{N}}{n - \overline{N}},$$

hence

$$\lambda(n - \overline{N})\overline{T} = \overline{N}$$

which is the **Little's formula**.
Clearly, due to the Processor Sharing discipline the response time is longer then the required service time, and there is no waiting time since each customer are being served. The difference is $\overline{T} - \mathbb{E}(S)$.

It can be proved, see Cohen [14], that for an $\vec{G}/\vec{G}/1/n/n/PS$ system the steady-state probability that customers with indexes $i_1, \ldots, i_k$ are in the system can be written as

$$P(i_1, \ldots, i_k) = C \cdot k! \prod_{j=1}^{k} \rho i_j, \qquad \rho_i = \frac{\mathbb{E}(S_i)}{\mathbb{E}(\tau_i)}, \quad i = 1, \ldots, n.$$

For homogeneous case we get

$$P_k = C \cdot k! \binom{n}{k} \rho^k.$$

3.7 The $\vec{G}/M/r/n/n/FIFO$ Queue

This section is devoted to a generalized version the finite-source model with multiple servers where the customers are supposed to have heterogeneous generally distributed source times and homogeneous exponentially distributed service times. They are served according to the order of their arrivals. The detailed description of this model can be found in Sztrik [77].

Customers arrive from a finite source of size n and are served by one of r ($r \leq n$) servers at a service facility according to a first-come first-served (FFIFO) discipline. If there is no idle server, then a waiting line is formed and the units are delayed. The service times of the units are supposed to be identically and exponentially distributed random variables with parameter μ. After completing service, customer with index i returns to the source and stays there for a random time τ_i having general distribution function $F_i(x)$ with density $f_i(x)$. All random variables are assumed to be independent of each other.

Determination of the steady-state distribution

As in the previous section the modeling is more difficult since the involved random times are not all exponentially distributed and thus we have to use the *method of supplementary variables.*

Let the random variable $\nu(t)$ denote the number of customers staying in the source at time t and let $\left(\alpha_1(t), \ldots, \alpha_{\nu(t)}\right)$ indicate their indexes ordered lexicographically, that is in increasing order of their indexes.

Let us denote by $\left(\beta_1(t), \ldots, \beta_{n-\nu(t)}\right)$ the indexes of the requests waiting or served at the service facility in the order of their arrival. It is not difficult to see that the process

$$Y(t) = \left(\nu(t); \alpha_1(t), \ldots, \alpha_{\nu(t)}; \beta_1(t), \ldots, \beta_{n-\nu(t)}\right), \qquad (t \geq 0)$$

is not Markovian unless the distribution functions $F_i(x), \quad i = 1, \ldots, n$ are exponential.

To use the supplementary variable technique let us introduce the supplementary variable $\xi\alpha_i(t)$ to denote the elapsed source time of request with index $\alpha_i, \quad i = 1, \cdots, n$. Define

$$X(t) = \left(\nu(t); \alpha_1(t), \ldots, \alpha_{\nu(t)}; \xi\alpha_1(t), \ldots, \xi\alpha_{\nu(t)}; \beta_1(t), \ldots . \beta_{n-\nu(t)}\right)$$

This is a multicomponent piecewise linear Markov process.

Let V_k^n and C_k^n denote the set of all variations and combinations of order k of the integers $1, 2, \ldots, n$, respectively, ordered lexicographically. Then the state space of process $(X(t), t \geq 0)$ consists of the set of points

$$(i_1, \ldots, i_k; x_1, \ldots, x_k; j_1, \ldots, j_{n-k})$$

where

$$(i_1, \ldots, i_k) \in C_k^n, (j_1, \ldots, j_{n-k}) \in V_k^n, x_i \in R_+, i = 0, 1, \ldots, k, k = 0, 1, \ldots, n.$$

Process $X(t)$ is in state $(i_1, \ldots, i_k; x_1, \ldots, x_k; j_1, \ldots, j_{n-k})$ if k customers with indexes $(i_1, \ldots, i_k)$ have been staying in the source for times $(x_1, \ldots, x_k)$, respectively while the rest need service and their indexes in the order of arrival are $(j_1, \ldots, j_{n-k})$.

To derive the Kolmogorov-equations we should consider the transitions that can occur in an arbitrary time interval $(t, t+h)$. For $0 \leq n-k < r$ the transition probabilities are then the following

$$P\left[X(t+h) = (i_1, \ldots, i_k; x_1 + h, \ldots, x_k + h; j_1, \ldots, j_{n-k}) \mid \right.$$

$$\left. X(t) = (i_1, \ldots, i_k; x_1, \ldots, x_k; j_1, \ldots, j_{n-k})\right]$$

$$= (1 - (n-k)\mu h) \prod_{l=1}^{k} \frac{1 - F_{i_l}(x_l + h)}{1 - F_{i_l}(x_l)} + o(h),$$

$$P[X(t+h) = (i_1, \ldots, i_k; x_1 + h, \ldots, x_k + h; j_1, \ldots, j_{n-k}) \mid$$

$$X(t) = (i_1^{'}, \ldots, j_{n-k}^{'}, \ldots, i_k^{'}; x_1^{'}, \ldots, y^{'}, \ldots, x_k^{'}; j_1, \ldots, j_{n-k-1})]$$
$$= \frac{f_{j_{n-k}}(y)\, h}{1 - F_{j_{n-k}}(y)} \prod_{l=1}^{k} \frac{1 - F_{i_l}(x_l + h)}{1 - F_{i_l}(x_l)} + o(h),$$

where $(i_1^{'}, \ldots, j_{n-k}^{'}, \ldots, i_k^{'})$ denotes the lexicographical order of indexes $(i_1, \ldots, i_k, j_{n-k})$ while $(x_1{}^{'}, \ldots, y^{'}, \ldots, x_k^{'})$ indicates the corresponding times.
For $r \leq n-k \leq n$ the transition probabilities can be obtained as

$$P[X(t+h) = (i_1, \ldots, i_k; x_1 + h, \ldots, x_k + h; j_1, \ldots, j_{n-k}) \mid$$
$$X(t) = (i_1, \ldots, i_k; x_1, \ldots, x_k; j_1, \ldots, j_{n-k})]$$
$$= (1 - r\mu h) \prod_{l=1}^{k} \frac{1 - F_{i_l}(x_l + h)}{1 - F_{i_l}(x_l)} + o(h),$$

$$P\Big[X(t+h) = (i_1, \ldots, i_k; x_1 + h, \ldots, x_k + h; j_1, \ldots, j_{n-k}) \mid$$
$$X(t) = \Big(i_1^{'}, \ldots, j_{n-k}^{'}, \ldots, i_k; x_1, \ldots, y^{'}, \ldots, x_k^{'}; j_1, \ldots, j_{n-k-1}\Big)\Big] =$$
$$= \frac{f_{j_{n-k}}(y)\, h}{1 - F_{j_{n-k}}(y)} \prod_{l=1}^{k} \frac{1 - F_{i_l}(x_l + h)}{1 - F_{i_l}(x_l)} + o(h).$$

For the distribution of $X(t)$ introduce the following functions

$$Q_{0;j_1,\ldots,j_n}(t) = P(\nu(t) = 0; \beta_1(t) = j_1, \ldots, \beta_n(t) = j_n),$$
$$Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k; t) =$$
$$P(\nu(t) = k; \alpha_1(t) = i_1, \ldots, \alpha_k(t) = i_k; \xi_{i_1} \leq x_1, \ldots, \xi_{i_k} \leq x_k;$$
$$\beta_1(t) = j_1, \ldots, \beta_{n-k}(t) = j_{n-k}).$$

Let λ_i is defined by $1/\lambda_i = \mathbb{E}(\tau_i)$. Then we have

Theorem 2 *If $1/\lambda_i < \infty$, $i = 1, \ldots, n$, then the process $(X(t), t \geq 0)$ possesses a unique limiting (stationary, steady-state) distribution independent of the initial conditions, namely*

$$Q_{0;j_1,\ldots,j_n} = \lim_{t\to\infty} Q_{0;j_1,\ldots,j_n}(t),$$
$$Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k) = \lim_{t\to\infty} Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k; t).$$

Notice that $X(t)$ belongs to the class of piecewise-linear Markov processes, subject to discontinuous changes treated by Gnedenko and Kovalenko [31]. Our statement follows from a theorem on page 211 of this monograph.

Since by assumption $F_i(x)$ has density function, for fixed k Theorem 2 provides the existence and uniqueness of the following limits

$$q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k =$$

$$= P\left(\nu\left(t\right) = k; \alpha_1\left(t\right) = i_1, \ldots, \alpha_k\left(t\right) = i_l; x_l \leq \xi_{i_l} < x_l + dx_l, l = 1, \ldots, k;\right.$$
$$\left.\beta_1\left(t\right) = j_1, \ldots, \beta_{n-k}\left(t\right) = j_{n-k}\right), \qquad k = 1, \ldots, n$$

where $q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right)$ denotes the density function of state $(i_1, \ldots, i_k; x_1, \ldots, x_k; j_1, \ldots, j_{n-k})$ when $t \to \infty$.

Let us introduce the so-called *normed density function* defined by

$$\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right) = \frac{q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right)}{\left(1 - F_{i_1}\left(x_1\right)\right)\ldots\left(1 - F_{i_k}\left(x_k\right)\right)}.$$

Then we have

Theorem 3 *The normed density functions satisfy the following system of integro-differential equations (3.1), (3.3) with boundary conditions (3.2), (3.4)*

(3.1)
$$\left[\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_k}\right]^* \tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right)$$
$$= -\left(n-k\right)\mu\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right) +$$
$$+\int_0^\infty \tilde{q}_{i_1',\ldots,j_{n-k}',\ldots,i_k';j_1,\ldots,j_{n-k-1}}\left(x_1', \ldots, y', \ldots, x_k'\right) f_{j_n}\left(y\right) dy,$$

(3.2)
$$\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_k\right) =$$
$$= \mu \sum_{V^{i_l}_{j_1,\ldots,j_{n-k}}} \tilde{q}_{i_1,\ldots,i_{l-1};i_{l+1},\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_k\right)$$
$$for\ l = 1, \ldots, k, \quad 0 \leq n-k < r$$

(3.3)
$$\left[\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_k}\right]^* \tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right) =$$
$$= -r\mu\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_k\right) +$$
$$+\int_0^\infty \tilde{q}_{i_1',\ldots,j_{n-k}',\ldots,i_k';j_1,\ldots,j_{n-k-1}}(x_1', \ldots, y', \ldots, x_k') f_{j_n}(y) dy$$

(3.4)
$$\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_k\right) =$$
$$= \mu \sum_{V^{i_l}_{j_1,\ldots,j_{r-1}}} \tilde{q}_{i_1,\ldots,i_{l-1};i_{l+1},\ldots,i_k;j_1,\ldots,j_{n-k}}\left(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_k\right),$$
$$for\ l = 1, \ldots, k, \quad r \leq n-k < n-1$$

furthermore

$$r\mu Q_{0;j_1,\ldots,j_n} = \int_0^\infty \tilde{q}_{j_n;j_1,\ldots,j_{n-1}}(y) f_{j_n}(y)\, dy.$$

The symbol $[\ \ ]^*$ *will be explained later while*

$$V^{i_l}_{j_1,\ldots,j_s} = \left[(i_l, j_1, \ldots, j_s), (j_1, i_l, j_2, \ldots, j_s), \ldots, (j_1, \ldots, j_s, i_l)\right] \in V^n_{s+1}.$$

Proof: Since the process $(X(t), t \geq 0)$ is Markovian its densities must satisfy the Kolmogorov-equations. A derivation is based on the examination of the sample paths of the process during an infinitesimal interval of width h. The following relations hold

$$q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1+h, \ldots, x_k+h) =$$

$$= q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k)(1-(n-k)\mu h)\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)} +$$

$$+\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)} + \int_0^\infty \tilde{q}_{i'_1,\ldots,j'_{n-k},\ldots,i'_k;j_1,\ldots,j_{n-k-1}}\left(x'_1, \ldots, y' \ldots, x'_k\right) \times$$

$$\times \frac{f'_{j_{n-k}}(y)\, h}{1-F_{j_{n-k}}(x_l)} dy + o(h),$$

$$q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1+h, \ldots, x_{l-1}+h, 0, x_{l+1}+h, \ldots, x_k+h)\, h =$$

$$= o(h) + \prod_{\substack{s=1\\ s\neq l}}^{k}\frac{1-F_{i_s}(x_s+h)}{1-F_{i_s}(x_s)} \times$$

$$\times \mu h \sum_{V^{i_l}_{j_1,\ldots,j_{n-k}}} \tilde{q}_{i_1,\ldots,i_{l-1};i_{l+1},\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_k)$$

$$\text{for } 0 \leq n-k < r, \quad l = 1, \ldots, k.$$

Similarly

$$q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1+h, \ldots, x_k+h) =$$

$$= q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1, \ldots, x_k)(1-r\mu h)\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)} +$$

$$+\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)} \int_0^\infty \tilde{q}_{i'_1,\ldots,j'_{n-k},\ldots,ie'_k;j_1,\ldots,j_{n-k-1}}\left(x'_1, \ldots, y', \ldots, x'_k\right) \times$$

$$\times \frac{f'_{j_{n-k}}(y)\, h}{1-F_{j_{n-k}}(x_l)} dy + o(h),$$

$$q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1+h, \ldots, x_{l-1}+h, 0, x_{l+1}+h, \ldots, x_k+h)\, h =$$

$$= o(h) + \prod_{\substack{s=1 \\ s \neq l}}^{k} \frac{1 - F_{i_s}(x_s + h)}{1 - F_{i_s}(x_s)} \times$$

$$\times \mu h \sum_{V^{i_l}_{j_1,\ldots,j_{n-k}}} \tilde{q}_{i_1,\ldots,i_{l-1};i_{l+1},\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_k)$$

$$\text{for } 0 \leq n-k < r, \quad l = 1,\ldots,k.$$

Finally

$$Q_{0;j_1,\ldots,j_n} = Q_{0;j_1,\ldots,j_n}(1 - r\mu h) +$$

$$+ \int_0^\infty \tilde{q}_{j_n;j_1,\ldots,j_{n-1}}(y) \frac{f_{j_n}(y)\, h}{1 - F_{j_n}(y)} dy + o(h).$$

Thus the statement of this theorem can easily be obtained. Dividing the left-hand side of equations by $\prod_{l=1}^{k}(1 - F_{i_l}(x_l + h))$ and taking into account the definition of the normed densities taking the limit as $h \to 0$ we get the desired result.

In the left-hand side of (3.1)(3.3) used for the notation of the limit in the right-hand side, the usual notation for partial differential quotients has been applied. Strictly considering this is not allowed, since the existence of the individual partial differential quotient is not assured. This is why the operator is notated by $[\ ]^*$. Actually this is a $(1,1,\ldots,1) \in R^k$ directional derivative, see Cohen [14].

To determine the steady-state probabilities

$$\left[Q_{0;j_1,\ldots,j_n}, Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}\right],$$

$$(i_1,\ldots,i_k) \in C^n_k, \quad (j_1,\ldots,j_{n-k}) \in V^N_{n-k}, \quad k = 1,\ldots,n.$$

we have to solve equations (3.1)(3.3) subject to the boundary conditions (3.2)(3.4). If we set

$$Q_{0;j_1,\ldots,j_n} = c_0,$$

$$\tilde{q}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k) = c_k, \qquad k = 1,\ldots,n,$$

then by direct substitution it can easily be verified that they satisfy these equations with boundary conditions. Moreover these c_k can be obtained by the help of c_n, namely

$$c_k = \left(r!\, r^{n-r-k} \mu^{n-k}\right)^{-1} c_n, \qquad 0 \leq k \leq n-r,$$

$$c_k = \left((n-k)!\, \mu^{n-k}\right)^{-1} c_n, \qquad n-r \leq k \leq n.$$

Since these equations completely describe the system, this is the required solution.

Let $Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}$ denote the steady-state probability that customers with indexes $(i_1,\ldots,i_k)$ are in the source and the order of arrivals of the rest to the service facility is $(j_1,\ldots,j_{n-k})$. Furthermore, denote by $Q_{i_1,\ldots,i_k}$ the stationary probability that requests with indexes $(i_1,\ldots,i_k)$ i are staying in the source.

It can easily be seen

$$Q_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}} = (\lambda_{i_1},\ldots,\lambda_{i_k})^{-1} c_k, \qquad k = 1,\ldots,n.$$

By using the relation we obtained for c_k we have

$$Q_{i_1,\ldots,i_k} = (n-k)!\left(r! r^{n-r-k}\mu^{n-k}\lambda_{i_1},\ldots,\lambda_{i_k}\right)^{-1} c_n,$$

$$(i_1,\ldots,i_k) \in C_k^n, \qquad k = 0,1,\ldots,n-r.$$

Similarly

$$Q_{i_1,\ldots,i_k} = \left(\mu^{n-k}\lambda_{i_1},\ldots,\lambda_{i_k}\right)^{-1} c_n,$$

$$(i_1,\ldots,i_k) \in C_k^n, \qquad k = n-r,\ldots,n.$$

Let us denote by $\hat{Q}_k$ and $\hat{P}_l$ the steady-state probability of the number of customers in the source, in the service facility, respectively. Hence it is easy to see that

$$Q_{i_1,\ldots,i_n} = Q_{1,\ldots,n} = \hat{Q}_n,$$

$$\hat{Q}_k = \hat{P}_{n-k}, \qquad k = 0,\ldots,n.$$

Furthermore

$$c_n = \hat{Q}_n\left(\lambda_1,\ldots,\lambda_n\right),$$

$$\hat{Q}_k = \sum_{(i_1,\ldots,i_k)\in C_k^n} Q_{i_1,\ldots,i_k},$$

where $\hat{Q}_n$ can be obtained by the help of the normalizing condition $\sum_{k=0}^{n} \hat{Q}_k = 1$.

In the homogeneous case these formulas reduce to

$$\hat{Q}_k = \frac{n!}{r!k!r^{n-k-r}}\left(\frac{\lambda}{\mu}\right)^{n-k}\hat{Q}_n, \qquad for \;\; 0 \le k \le n-r,$$

$$\hat{Q}_k = \binom{n}{k}\left(\frac{\lambda}{\mu}\right)^{n-k}\hat{Q}_n, \qquad for\; n-r \le k \le n,$$

which is the result of the paper Bunday and Scraton [12], and for $r = 1$ is the formulas obtained by Schatte [72]. Thus the distribution of the number of customers in the service facility is

$$\hat{P}_k = \binom{n}{k}\left(\frac{\lambda}{\mu}\right)^{k}\hat{P}_0, \qquad for \;\; 0 \le k \le r,$$

$$\hat{P}_k = \frac{n!}{r!(n-k)!r^{k-r}}\left(\frac{\lambda}{\mu}\right)^{k}\hat{P}_0, \qquad for \;\; r \le k \le n.$$

This is exactly the same result that we got for the $< M/M/r/n/n >$ model.

It should be underlined that the distribution of the number of customers in the system does not depend on the form of $F_i(x)$ but the mean $1/\lambda_i$, that is it is robust.

■

Performance Measures

- *Utilization of the sources*

  Let $Q^{(i)}$ denote the steady-state distribution that source i is busy with generating a new customer, that is

$$Q^{(i)} = \sum_{k=1}^{n} \sum_{i \in (i_1,\ldots,i_k) \in C_k^n} Q_{i_1,\ldots,i_k}.$$

  Hence the utilization of source i can be obtained as

$$U^{(i)} = Q^{(i)}.$$

- *Utilization of the servers*

  As we have calculated earlier the utilization of a server can be derived as

$$U_{CPU} = \frac{1}{r}\left(\sum_{k=1}^{r} k\hat{P}_k + r\sum_{k=r+1}^{n} \hat{P}_k\right) = \frac{\bar{r}}{r},$$

  where $\bar{r}$ denotes the mean number of busy servers. Thus the overall utilization of the servers is $\bar{r}$.

- *Mean waiting and response times*

  By the results of Tomkó [92] we have

$$Q^{(i)} = (1/\lambda_i)\left(1/\lambda_i + \bar{W}_i + 1/\mu\right)^{-1}.$$

  Thus for the mean waiting time of the customer with index i we obtain

$$\bar{W}_i = \frac{1}{\lambda_i} \cdot \frac{1 - Q^{(i)}}{Q^{(i)}} - \frac{1}{\mu}, \qquad i = 1, \ldots, n.$$

  Consequently the mean response time $\bar{T}_i$ of the ith request can be calculated as

$$\bar{T}_i = \bar{W}_i + 1/\mu = \left(1 - Q^{(i)}\right)\left(\lambda_i Q^{(i)}\right)^{-1}, \qquad i = 1, \ldots, n.$$

  Since

$$\sum_{i=1}^{n} \left(1 - Q^{(i)}\right) = \bar{N},$$

  where $\bar{N}$ denotes the mean number customers at the service facility. This can be rewritten as

$$\sum_{i=1}^{n} \lambda_i \bar{T}_i Q^{(i)} = \bar{N},$$

  which the **Little's formula** for the $\vec{G}/M/r/n/n/FIFO >$ queueing system.

It should be noted that using the terminology of the machine interference problem $U^{(i)}$, $\bar{W}_i, \bar{T}_i$ denote the utilization, mean waiting time and the mean time spent in a broken state of the ith machine.

This model can be generalized such a way that the service intensities depend on the number of customers in the source, see Sztrik [79, 80].

Part II

Exercises

Chapter 4

Infinite-Source Systems

Exercise 1 *Solve the following system of equations by the help of difference equations*

$$\lambda P_0 = \mu P_1$$
$$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1}, n \geq 1.$$

Solution:
It is easy to see that it can be rewritten as

$$\lambda P_{n-1} - (\lambda + \mu)P_n + \mu P_{n+1} = 0, n = 1, 2, \ldots$$

which is a 2-nd order difference equation with constant coefficient. Its general solution can be obtained in the form

$$P_n = c_1 x_1^n + c_2 x_2^n, n = 1, 2, \ldots$$

where x_1, x_2 are the solutions to

$$\mu x^2 - (\lambda + \mu)x + \lambda = 0.$$

It can easily be verified that $x_1 = 1, \qquad x_2 = \varrho$ and thus

$$P_n = c_1 + c_2 \varrho^n, n = 1, 2, \ldots.$$

However $P_1 = \varrho P_0$, and because $\sum_{n=0}^{\infty} P_n = 1$, thus $c_1 = 0$ and $c_2 = P_0 = 1 - \varrho$.
■

Exercise 2 *Find the generating function of the number of customers in the system for an $M/M/1$ queueing system by using the steady-state balance equations. Then derive the corresponding distribution.*

Solution:
Staring with the set of equations

$$\lambda P_0 = \mu P_1$$
$$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1}, n \geq 1$$

by multiplying both sides by s^i and then adding the terms we obtain

$$\lambda G_N(s) + \mu G_N(s) - \mu P_0 = \lambda s G_N(s) + \frac{\mu}{s} G_N(s) - \frac{\mu}{s} P_0.$$

Thus we can calculate as

$$\begin{aligned} G_N(s)\left(\lambda(1-s) + \mu\left(1 - \frac{1}{s}\right)\right) &= \mu\left(1 - \frac{1}{s}\right) P_0, \\ G_N(s)\left(\mu\left(1 - \frac{1}{s}\right) - \lambda s\left(1 - \frac{1}{s}\right)\right) &= \mu\left(1 - \frac{1}{s}\right) P_0, \\ G_N(s) &= \frac{\mu}{\mu - \lambda s} P_0. \end{aligned}$$

Since $G_N(1) = 1$, therefore

$$P_0 = \frac{\mu - \lambda}{\mu} = 1 - \varrho.$$

That is

$$G_N(s) = \frac{1 - \varrho}{1 - s\varrho},$$

which is exactly the generating function of a modified geometric distribution with parameter $(1 - \varrho)$. It can be proved as follows, if

$$P(N = k) = (1 - \varrho)\varrho^k, k = 0, \ldots$$

then its generating function is

$$G_N(s) = \sum_{k=0}^{\infty} s^k (1 - \varrho)\varrho^k = \frac{1 - \varrho}{1 - s\varrho}.$$

■

Exercise 3 *Find the generating function of the number of customers waiting in a queue for an $M/M/1$ queueing system.*

Solution:
Clearly

$$\begin{aligned} G_Q(s) &= (P_0 + P_1)s^0 + \sum_{k=2}^{\infty} s^{k-1} P_k = P_0 + \sum_{k=1}^{\infty} s^{k-1} P_k = 1 - \varrho + \sum_{k=1}^{\infty} s^{k-1} \varrho^k (1 - \varrho) \\ &= 1 - \varrho + \varrho \sum_{i=0}^{\infty} s^i (1 - \varrho)\varrho^i = 1 - \varrho + \varrho G_N(s) = 1 - \varrho(1 - G_N(s)). \end{aligned}$$

For verification let us calculate the mean queue length, thus

$$G'_Q(1) = \varrho G'_N(1) = \frac{\varrho^2}{1 - \varrho}.$$

■

Exercise 4 *Find the Laplace-transform of T and W for an $M/M/1$ queueing system.*

Solution:
It is easy to see that

$$L_T(s) = \sum_{k=0}^{\infty} \left(\frac{\mu}{\mu+s}\right)^{k+1} \varrho^k(1-\varrho) = (1-\varrho)\frac{\mu}{\mu+s}\sum_{k=0}^{\infty}\left(\frac{\mu\varrho}{\mu+s}\right)^k$$
$$= (1-\varrho)\frac{\mu}{\mu+s}\frac{1}{1-\frac{\mu\varrho}{\mu+s}} = (1-\varrho)\frac{\mu}{\mu+s}\frac{\mu+s}{\mu(1-\varrho)+s} = \frac{\mu(1-\varrho)}{\mu(1-\varrho)+s},$$

which was expected, since T follows an exponential distribution with parameter $\mu(1-\varrho)$.
To get the Laplace-transform of W we have

$$L_W(s) = \sum_{k=0}^{\infty}\left(\frac{\mu}{\mu+s}\right)^k \varrho^k(1-\varrho) = 1-\varrho+\sum_{k=1}^{\infty}\left(\frac{\mu}{\mu+s}\right)^k \varrho^k(1-\varrho)$$

$$= 1-\varrho+\frac{\varrho\mu(1-\varrho)}{\mu(1-\varrho)+s},$$

which should be

$$\frac{L_T(s)}{\frac{\mu}{\mu+s}}$$

since

$$L_T(s) = L_W(s)\frac{\mu}{\mu+s}.$$

To show this it can be calculated that

$$L_W(s) = L_T(s)\frac{\mu+s}{\mu} = \frac{\mu(1-\varrho)}{\mu(1-\varrho)+s}\frac{\mu+s}{\mu} = 1-\varrho+\varrho\frac{\mu(1-\varrho)}{\mu(1-\varrho)+s}.$$

Let us verify the result by deriving the mean values $\overline{T}$ and $\overline{W}$.

$$L_T'(0) = -\frac{1}{\mu(1-\varrho)},$$
$$L_W'(0) = \varrho L_T'(0) = -\frac{\varrho}{\mu(1-\varrho)},$$

thus

$$\overline{T} = \frac{1}{\mu(1-\varrho)}, \overline{W} = \frac{\varrho}{\mu(1-\varrho)},$$

which was obtained earlier.
■

Exercise 5 *Show that for an $M/M/1/K$ queueing system*

$$\lim_{K\to\infty}\overline{N}(K) = \frac{\rho}{1-\rho}, \qquad \rho < 1$$

.

Solution:
It is well-known if $\rho < 1$ then

$$\lim_{K\to\infty} \rho^K = 0.$$

Since $\overline{N} = \frac{\rho\left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)(1-\rho^{K+1})}$, it is enough to show that

$$\lim_{K\to\infty} K\rho^K = 0.$$

This can be proved by the L'Hospital's rule, namely

$$\lim_{K\to\infty} \frac{K}{\rho^{-K}} = \frac{\infty}{\infty}$$
$$\lim_{K\to\infty} \frac{K}{\rho^{-K}} = \lim_{K\to\infty} \frac{1}{-ln\rho\rho^{-K}} = \frac{\rho^K}{-ln\rho} = 0.$$

■

Exercise 6 *Show that for an $M/M/1/K$ queueing system the Laplace-transform*

$$L_T(s) = \frac{\mu P_0}{1-P_K} \frac{1-\left(\frac{\lambda}{\mu+s}\right)^K}{\mu-\lambda+s}$$

satisfies $L_T(0) = 1$.

Solution:

$$\begin{aligned} L_T(0) &= \frac{\mu P_0}{1-P_K} \frac{1-\rho^K}{\mu-\lambda} = \frac{P_0}{1-P_K} \frac{1-\rho^K}{1-\rho} \\ &= \frac{\frac{1-\rho}{1-\rho^{K+1}}}{1-\frac{\rho^K(1-\rho)}{1-\rho^{K+1}}} \cdot \frac{1-\rho^K}{1-\rho} \\ &= \frac{1-\rho}{1-\rho^{K+1}} \cdot \frac{1-\rho^{K+1}}{1-\rho^{K+1}-\rho^K+\rho^{K+1}} \cdot \frac{1-\rho^K}{1-\rho} \\ &= 1. \end{aligned}$$

■

Exercise 7 *Find $\overline{T}$ by the help of the Laplace-transform for an $M/M/1/K$ queueing system.*

Solution:

Since

$$L_T(s) = \frac{\mu P_0}{1-P_K} \frac{1-\left(\frac{\lambda}{\mu+s}\right)^K}{\mu-\lambda+s}$$

then

$$L'_T(s) = \frac{\mu P_0}{1-P_K}\left(\frac{K\left(\frac{\mu+s}{\lambda}\right)^{-(K+1)}\cdot\frac{1}{\lambda}(\mu-\lambda+s)-\left(1-\left(\frac{\lambda}{\mu+s}\right)^K\right)}{(\mu-\lambda+s)^2}\right)$$

$$\begin{aligned}
L'_T(0) &= \frac{\mu P_0}{1-P_K}\left(\frac{K\rho^{K+1}\left(\frac{1}{\rho}-1\right)+\rho^K-1}{(\mu-\lambda)^2}\right)\\
&= \frac{P_0\rho}{\lambda(1-P_K)(1-\rho)^2}\left(K\rho^{K+1}\left(\frac{1}{\rho}-1\right)+\rho^K-1\right)\\
&= \frac{1}{\lambda(1-P_K)}\cdot\frac{\rho P_0(K\rho^K-K\rho^{K+1}+\rho^K-1)}{(1-\rho)^2}\\
&= \frac{1}{\lambda(1-P_K)}\frac{\rho P_0((K+1)\rho^K-K\rho^{K+1}-1)}{(1-\rho)^2}\\
&= -\frac{\overline{N}}{\lambda(1-P_K)},
\end{aligned}$$

that is

$$\overline{T} = \frac{\overline{N}}{\lambda(1-P_K)},$$

which was obtained earlier.
The higher moments can be calculated, too.
■

Exercise 8 *Consider a closed queueing network with 2 nodes containing K customers. Assume that at each node the service times are exponentially distributed with parameter μ_1 and μ_2, respectively. Find the mean performance measures at each node.*

Solution:
It is easy to see that the nodes operate the same way and they can be considered as an $M/M/1/K$ queueing system. Hence the performance measures can be computed by using the formulas with $\rho_2 = \frac{\mu_1}{\mu_2}$ and $\rho_1 = \frac{\mu_2}{\mu_1}$, respectively.
Furthermore, one can easily verify that

$$U_S(1)\mu_1 = U_S(2)\mu_2,$$

where $U_S(i)$, $i = 1, 2$ is the utilization of the server.
■

Exercise 9 *Find the generating function for an $M/M/n/n$ queueing system.*

Solution:

$$G_N(s) = \sum_{k=0}^{n} s^k \frac{\varrho^k}{k!}P_0 = \sum_{k=0}^{n}\frac{(s\varrho)^k}{k!}e^{-s\varrho}P_0 e^{s\varrho} = e^{-\varrho(1-s)}\frac{Q(n,s\varrho)}{Q(n,\varrho)}.$$

To verify the formula let us calculate $\overline{T}$.
Since $\overline{N} = G'_N(1)$, therefore take the derivative, that is we get

$$G'_N(s) = \varrho e^{-\varrho(1-s)} \frac{Q(n,\varrho s)}{Q(n,\varrho)} - e^{-\varrho(1-s)} \frac{\varrho P(n,\varrho s)}{Q(n,\varrho)}.$$

hence

$$G'_N(1) = \varrho - \varrho B(n,\varrho) = \varrho(1 - B(n,\varrho)),$$

which was obtained earlier.
■

Exercise 10 *Find $Var(N)$ for an $M/M/n/n$ queueing system.*

Solution:
Since $Var(N) = \mathbb{E}(N^2) - (\mathbb{E}(N))^2$, let us calculate first $\mathbb{E}(N^2)$. That is

$$\begin{aligned}
\mathbb{E}(N^2) &= \sum_{k=1}^{n} k^2 P_k = \sum_{k=1}^{n} (k(k-1)+k) P_k = \sum_{k=1}^{n} k(k-1) P_k + \sum_{k=1}^{n} k P_k \\
&= \sum_{k=2}^{n} k(k-1) \frac{\varrho^k}{k!} P_0 + \mathbb{E}(N) = \sum_{i=0}^{n-2} \varrho^2 \frac{\varrho^i}{i!} P_0 + \mathbb{E}(N) \\
&= \varrho^2(1 - P_n - P_{n-1}) + \mathbb{E}(N) = \varrho^2 \left(1 - P_n \left(1 + \frac{n}{\varrho}\right)\right) + \mathbb{E}(N).
\end{aligned}$$

Since $\mathbb{E}(N) = \varrho(1 - B(n,\varrho))$, therefore

$$\begin{aligned}
Var(N) &= \varrho^2(1 - B(n,\varrho)(1 + \frac{n}{\varrho})) - (\varrho(1 - B(n,\varrho)))^2 + \mathbb{E}(N) \\
&= \varrho^2(1 - B(n,\varrho)(\frac{\varrho + n}{\varrho})) - (\varrho(1 - B(n,\varrho)))^2 + \mathbb{E}(N) \\
&= \varrho^2 - \varrho^2 B(n,\varrho) - n\varrho B(n,\varrho) - \varrho^2 + \varrho^2 B^2(n,\varrho) + 2\varrho^2 B(n,\varrho) + \mathbb{E}(N) \\
&= \mathbb{E}(N) + \varrho^2 B(n,\varrho) - n\varrho B(n,\varrho) - \varrho^2 B^2(n,\varrho) \\
&= \mathbb{E}(N) - \varrho B(n,\varrho)(n - \varrho(1 - B(n,\varrho))) \\
&= \mathbb{E}(N) - \varrho B(n,\varrho)(n - \mathbb{E}(N)).
\end{aligned}$$

■

Exercise 11 *Show that $B(m,a)$ is a monotone decreasing sequence and its limit is 0.*

Solution:

$$B(m,a) = \frac{aB(m-1,a)}{m + aB(m-1,a)} < \frac{a}{m},$$

and thus it tends to 0 as m increasing. The sequence is monotone decreasing iff

$$B(m,a) - B(m-1,a) < 0, \forall m$$

that is

$$\frac{aB(m-1,a)}{m+aB(m-1,a)} - B(m-1,a) < 0$$
$$\frac{B(m-1,a)(a-m-aB(m-1,a))}{m+aB(m-1,a)} < 0$$
$$a-m-aB(m-1,a) < 0$$
$$B(m-1,a) > \frac{a-m}{a},$$

which is satisfied if $a \leq m$. Since $1 \geq B(m-1,a) \geq 0$ therefore $1 \geq \frac{a-m}{a} \geq 0$, and thus $a \geq m$,$m \geq 0$, that is $0 \leq m \leq a$. It means that $B(m,a)$ is monotone decreasing for m which was expected since as the number of servers increases the probability of loss should decrease.

■

Exercise 12 *Find a recursion for $C(m,a)$.*

Solution:
Let $a = \frac{\lambda}{\mu}$, then by the help of

$$C(m,a) = \frac{B(m,a)}{1-\frac{a}{m}(1-B(m,a))}$$

we should write a recursion for $C(m,a)$ since $B(m,a)$ can be obtained recursively. First we show how $B(m-1,a)$ can be expressed by the help of $C(m-1,a)$ and then substituting into the recursion

$$B(m,a) = \frac{aB(m-1,a)}{m+aB(m-1,a)}$$

we get the desired formula. So let us express $B(m,a)$ via $C(m,a)$ that is

$$C(m,a) = \frac{mB(m,a)}{m-a(1-B(m,a))}$$

$$C(m,a)(m-a) + C(m,a)aB(m,a) = mB(m,a)$$

thus

$$B(m,a) = \frac{(m-a)C(m,a)}{m-aC(m,a)},$$

which is positive since $m > a$ is the stability condition for an $M/M/m$ queueing system. This shows that

$$B(m,a) < C(m,a),$$

which was expected because of the nature of the problem.
Consequently

$$B(m-1,a) = \frac{(m-1-a)C(m-1,a)}{m-1-aC(m-1,a)},$$

and $m-1>a$ is also valid due to the stability condition. Let us first express $C(m,a)$ by the help of $B(m-1,a)$ then substitute. To do so

$$C(m,a)=\frac{\frac{aB(m-1,a)}{m+aB(m-1,a)}m}{m-a\big(1-\frac{aB(m-1,a)}{m+aB(m-1,a)}\big)}=\frac{\frac{amB(m-1,a)}{m+aB(m-1,a)}}{m-a\big(\frac{m+aB(m-1,a)-aB(m-1,a)}{m+aB(m-1,a)}\big)}$$
$$=\frac{aB(m-1,a)}{m+aB(m-1,a)-a}=\frac{aB(m-1,a)}{m-a(1-B(m-1,a))}.$$

Now let us substitute $C(m-1,a)$ into here. Let us express the numerator and denominator in a simpler form, namely

$$NUM=a\frac{(m-1-a)C(m-1,a)}{m-1-aC(m-1,a)}$$

$$\begin{aligned}DENOM&=m-a\left(1-\frac{(m-1-a)C(m-1,a)}{m-1-aC(m-1,a)}\right)\\&=m-a\frac{m-1-aC(m-1,a)-(m-1)C(m-1,a)+aC(m-1,a)}{m-1-aC(m-1,a)}\\&=m-a\frac{(m-1)(1-C(m-1,a))}{m-1-aC(m-1,a)}\\&=\frac{m(m-1)-maC(m-1,a)-a(m-1)(1-C(m-1,a))}{m-1-aC(m-1,a)}\\&=\frac{m(m-1)-a(m-1)-aC(m-1,a)}{m-1-aC(m-1,a)}\\&=\frac{(m-1)(m-a)-aC(m-1,a)}{m-1-aC(m-1,a)}.\end{aligned}$$

Thus

$$C(m,a)=\frac{a(m-1-a)C(m-1,a)}{(m-1)(m-a)-aC(m-1,a)},$$

and the initial value is $C(1)=a$. Thus the probability of waiting can be computed recursively. It is important because the main performance measures depends on this value.

Now, let us show that $C(m,a)$ is a monotone decreasing sequence and tends to 0 as m, increases which is expected. It is not difficult to see that

$$C(m,a)<\frac{a(m-1-a)C(m-1,a)}{(m-1)(m-a)-a}$$

and if we show that

$$\frac{a(m-1-a)}{(m-1)(m-a)a}<1,$$

then we have

$$C(m,a)<C(m-1,a).$$

To do so it is easy to see that

$$\begin{aligned} a(m-1-a) &< (m-1)(m-a)-a \\ m^2-m-ma+a-a-am+a+a^2 &> 0 \\ m^2-(1+2a)m+a+a^2 &> 0 \\ m_{1,2} &= \frac{1+2a\pm\sqrt{(1+2a)^2-4(a^2+a)}}{2} \\ m_{1,2} &= \frac{1+2a\pm\sqrt{1+4a^2+4a-4a^2-4a}}{2} \\ m_{1,2} &= \frac{1+2a\pm 1}{2} \end{aligned}$$

that is if $m > a+1$ then the values of the parabola are positive. However, this condition is satisfied since the stability condition is $m-1 > a$.
Furthermore, since

$$C(m,a) = \frac{B(m,a)}{1-\frac{a}{m}(1-B(m,a))}$$

therefore $\lim_{m\to\infty} C(m,a) = 0$, which was expected.
This can be proved by direct calculations, since

$$C(m,\varrho) = \frac{\varrho^m}{m!}\frac{m}{m-\varrho}P_0(m)$$

and from

$$\lim_{m\to\infty} P_0(m) = e^{-\varrho}, \qquad \lim_{m\to\infty}\frac{\varrho^m}{m!}\frac{m}{m-\varrho} = 0,$$

the limit is 0. It is clear because there is no waiting in an infinite-server system.
■

Exercise 13 *Verify that the distribution function of the response time for a $M/M/r$ queueing system in the case of $r = 1$ reduces to the formula obtained for an $M/M/1$ system.*

Solution:

$$\begin{aligned} P(T>x) &= e^{-\mu x}\left(1+C(n,\varrho)\frac{1-e^{-\mu(r-1-\varrho)x}}{r-1-\varrho}\right) = e^{-\mu x}\left(1+\varrho\frac{1-e^{\mu\varrho x}}{-\varrho}\right) \\ &= e^{-\mu x}(1-1+e^{\mu\varrho x}) = e^{-\mu(1-\varrho)x}. \end{aligned}$$

Thus

$$F_T(x) = 1-e^{-\mu(1-\varrho)x}.$$

■

Exercise 14 *Show that $\lim_{z\to 1} G_N(z) = 1$ for an $M/G/1$ queueing system.*

Solution:

$$\lim_{z\to 1} G_N(z) = \lim_{z\to 1} (1-\rho)L_S(\lambda-\lambda z)\cdot \frac{z-1}{z-L_S(\lambda-\lambda z)} = \frac{0}{0},$$

therefor the L'Hospital's rule is applied. It is easy to see that

$$\lim_{z\to 1} \frac{z-1}{z-L_S(\lambda-\lambda z)} = \lim_{z\to 1} \frac{1}{1+\lambda L_S'(\lambda-\lambda z)} = \frac{1}{1-\rho},$$

and thus

$$\lim_{z\to 1} G_N(z) = \lim_{z\to 1} (1-\rho)L_S(\lambda-\lambda z)\cdot \lim_{z\to\infty} \frac{z-1}{z-L_S(\lambda-\lambda z)} = \frac{1-\rho}{1-\rho} = 1.$$

■

Exercise 15 *Show that if the residual service time in an $M/G/1$ queueing system is denoted by R then its Laplace-transform can be obtained as $L_R(t) = \frac{1-L_S(t)}{t\cdot\mathbb{E}(S)}$.*

Solution:

$$L_R(t) = \int_0^\infty e^{-tx}\frac{1-F_S(x)}{\mathbb{E}(S)}\mathrm{d}x.$$

Using integration by parts we have

$$L_R(t) = \left[-\frac{e^{-tx}}{t}\frac{1-F_S(x)}{\mathbb{E}(S)}\right]_0^\infty + \int_0^\infty \frac{e^{-tx}}{t}\frac{(-f(x))}{\mathbb{E}(S)}\mathrm{d}x = \frac{1-L_S(t)}{t\mathbb{E}(S)}.$$

Verify the limit $\lim_{t\to 0} L_R(t) = 1$.
It is easy to see that

$$\lim_{t\to 0} L_R(t) = \lim_{t\to\infty} \frac{1-L_S(t)}{t\mathbb{E}(S)} = \frac{0}{0},$$

therefore apply the L'Hospital's rule. Thus

$$\lim_{t\to 0} L_R(t) = \lim_{t\to 0} \frac{-L_S'(t)}{\mathbb{E}(S)} = \frac{\mathbb{E}(S)}{\mathbb{E}(S)} = 1.$$

■

Exercise 16 *By the help of $L_R(t)$ prove that if $S \in Exp(\mu)$, then $R \in Exp(\mu)$.*

Solution:

$$L_R(t) = \frac{1 - L_S(t)}{t\mathbb{E}(S)} = \frac{1 - \frac{\mu}{\mu+t}}{\frac{t}{\mu}} = \frac{\mu}{\mu + t},$$

thus $R \in Exp(\mu)$.

■

Exercise 17 *By the help of the formulas for an $M/G/1$ system derive the corresponding formulas for an $M/M/1$ system.*

Solution:

In this case

$$L_S(t) = \frac{\mu}{\mu + t},$$

therefore the Laplace-transform of the response time is

$$\begin{aligned} L_T(t) &= L_S(t)\frac{t(1-\rho)}{t - \lambda + \lambda L_S(t)} \\ &= \frac{\mu}{\mu + t}\frac{t(1-\rho)}{t - \lambda + \frac{\lambda\mu}{\mu+t}} \\ &= \frac{\mu}{\mu + t}\frac{t(1-\rho)(\mu+t)}{\mu t + t^2 - \lambda\mu - \lambda t + \lambda\mu} \\ &= \frac{t(\mu-\lambda)}{t(t+\mu-\lambda)} = \frac{\mu-\lambda}{t+\mu-\lambda} = \frac{\mu(1-\rho)}{\mu(1-\rho)+t}, \end{aligned}$$

that is $T \in Exp(\mu(1-\rho))$, as we have seen earlier.

For the generating function of the number of customers in the system we have

$$\begin{aligned} G_N(z) &= L_S(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{L_S(\lambda - \lambda z) - z} \\ &= \frac{\mu}{\lambda - \lambda z + \mu} \cdot \frac{(1-\rho)(1-z)}{\frac{\mu}{\lambda - \lambda z + \mu} - z} \\ &= \frac{\mu}{\lambda - \lambda z + \mu} \cdot \frac{(1-\rho)(1-z)(\lambda - \lambda z + \mu)}{\mu - \lambda z + \lambda z^2 - \mu z} \\ &= \frac{\mu(1-\rho)(1-z)}{\mu(1-z) - \lambda z(1-z)} = \frac{\mu(1-\rho)}{\mu - \lambda z} = \frac{1-\rho}{1-\rho z}, \end{aligned}$$

as we proved in the case of an $M/M/1$ system.

For the mean waiting and response times we get

$$\overline{W} = \frac{\rho\mathbb{E}(S)}{1-\rho} \cdot \frac{1+C_S^2}{2} = \frac{\rho}{\mu(1-\rho)} \cdot \frac{1+1}{2} = \frac{\rho}{\mu(1-\rho)},$$
$$\overline{T} = \overline{W} + \frac{1}{\mu} = \frac{1}{\mu}\left(\frac{\rho}{1-\rho}+1\right) = \frac{1}{\mu(1-\rho)}.$$

To calculate the variance we need

$$\begin{aligned}\mathbb{E}(W^2) &= 2(\overline{W})^2 + \frac{\lambda\mathbb{E}(S^3)}{3(1-\rho)} = 2\left(\frac{\rho}{\mu(1-\rho)}\right)^2 + \frac{\lambda\frac{3!}{\mu^3}}{3(1-\rho)} \\ &= 2\frac{\rho^2}{\mu^2(1-\rho)^2} + \frac{2\lambda}{\mu^3(1-\rho)} = \frac{2\mu\rho^2 + 2\lambda(1-\rho)}{\mu^3(1-\rho)^2} \\ &= \frac{2\lambda\rho + 2\lambda - 2\lambda\rho}{\mu^3(1-\rho)^2} = \frac{2\lambda}{\mu^3(1-\rho)^2},\end{aligned}$$

thus

$$\begin{aligned}Var(W) &= \frac{2\lambda}{\mu^3(1-\rho)^2} - \left(\frac{\rho}{\mu(1-\rho)}\right)^2 \\ &= \frac{2\lambda - \mu\rho^2}{\mu^3(1-\rho)^2} = \frac{2\lambda - \lambda\rho}{\mu^3(1-\rho)^2} = \frac{(2-\rho)\rho}{\mu^2(1-\rho)^2},\end{aligned}$$

as we have seen earlier.

Furthermore

$$\begin{aligned}Var(T) &= Var(W) + Var(S) = \frac{(2-\rho)\rho}{\mu^2(1-\rho)^2} + \left(\frac{1}{\mu}\right)^2 \\ &= \frac{2\rho - \rho^2 + 1 - 2\rho + \rho^2}{\mu^2(1-\rho)^2} = \frac{1}{\mu^2(1-\rho)^2} = \left(\frac{1}{\mu(1-\rho)}\right)^2.\end{aligned}$$

The variance of the number of customers in the system is

$$\begin{aligned}
Var(N) &= \frac{\lambda^3\mathbb{E}(S^3)}{3(1-\rho)} + \left(\frac{\lambda^2\mathbb{E}(S^2)}{2(1-\rho)}\right)^2 + \frac{\lambda^2(3-2\rho)\mathbb{E}(S^2)}{2(1-\rho)} + \rho(1-\rho) \\
&= \frac{\lambda^3\frac{3!}{\mu^3}}{3(1-\rho)} + \left(\frac{\lambda^2\frac{2}{\mu^2}}{2(1-\rho)}\right)^2 + \frac{\lambda^2(3-2\rho)\frac{2}{\mu^2}}{2(1-\rho)} + \rho(1-\rho) \\
&= \frac{2\lambda^3}{\mu^3(1-\rho)} + \left(\frac{\rho^2}{1-\rho}\right)^2 + \frac{\rho^2(3-2\rho)}{1-\rho} + \rho(1-\rho) \\
&= \frac{2\rho^3}{1-\rho} + \frac{\rho^4}{(1-\rho)^2} + \frac{\rho^2(3-2\rho)}{1-\rho} + \rho(1-\rho) \\
&= \frac{2\rho^3(1-\rho) + \rho^4 + \rho^2(1-\rho)(3-2\rho)}{(1-\rho)^2} + \frac{\rho(1-\rho)^3}{(1-\rho)^2} \\
&= \frac{2\rho^3 - 2\rho^4 + \rho^4 + 3\rho^2 - 2\rho^3 - 3\rho^3 + 2\rho^4 + \rho + 3\rho^3 - 3\rho^2 - \rho^4}{(1-\rho)^2} \\
&= \frac{\rho}{(1-\rho)^2},
\end{aligned}$$

as we have seen earlier.

Finally

$$\begin{aligned}
Var(Q) &= \frac{\lambda^3\mathbb{E}(S^3)}{3(1-\rho)} + \left(\frac{\lambda^2\mathbb{E}(S^2)}{2(1-\rho)}\right)^2 + \frac{\lambda^2\mathbb{E}(S^2)}{2(1-\rho)} \\
&= \frac{\lambda^3\frac{3!}{\mu^3}}{3(1-\rho)} + \left(\frac{\lambda^2\frac{2}{\mu^2}}{2(1-\rho)}\right)^2 + \frac{\lambda^2\frac{2}{\mu^2}}{2(1-\rho)} \\
&= \frac{2\rho^3}{1-\rho} + \frac{\rho^4}{(1-\rho)^2} + \frac{\rho^2}{1-\rho} = \frac{2(1-\rho)\rho^3 + \rho^4 + \rho^2(1-\rho)}{(1-\rho)^2} \\
&= \frac{2\rho^3 - 2\rho^4 + \rho^4 + \rho^2 - \rho^3}{(1-\rho)^2} = \frac{\rho^3 - \rho^4 + \rho^2}{(1-\rho)^2} = \frac{\rho^2(1+\rho-\rho^2)}{(1-\rho)^2}.
\end{aligned}$$

These verifications help us to see if these complicated formulas reduces to the simple ones.

■

Exercise 18 *Based on the transform equation*

$$G_N(z) = L_S(\lambda - \lambda z)(1-\rho)\frac{1-z}{L_S(\lambda-\lambda z) - z}$$

find $\overline{N}$*-t.*

Solution:
It is well-known that $\overline{N} = G'_N(1)$, that is why we have to calculate the derivative at the

right hand side. However, the term $\frac{1-z}{L_S(\lambda-\lambda z)-z}$ takes an indetermined value at $z=1$ hence the L'Hospital's rule is used. Let us first define a function

$$f(z) = \frac{L_S(\lambda - \lambda z) - z}{1 - z}$$

.

Hence one can see that

$$L_N(z) = (1-\rho)\frac{L_S(\lambda - \lambda z)}{f(z)}.$$

Applying the expansion procedure

$$L_S(\lambda - \lambda z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \mathbb{E}(S^k)}{k!}(\lambda - \lambda z)^k$$

we have

$$\begin{aligned} f(z) &= \frac{1 + \sum_{k=1}^{\infty} \frac{(-1)^k \mathbb{E}(S^k)}{k!}(\lambda - \lambda z)^k - z}{1 - z} \\ &= 1 - \lambda\mathbb{E}(S) + \lambda^2 \frac{\mathbb{E}(S^2)(1-z)}{2} + \dots. \end{aligned}$$

Thus $f(1) = 1 - \rho$ and $f'(1) = -\frac{\lambda^2 \mathbb{E}(S^2)}{2}$.

After these calculations we get

$$L'_N(z) = \frac{(1-\rho)f(z)L'_S(\lambda - \lambda z)(-\lambda) - L_S(\lambda - \lambda z) \cdot f'(z)}{(f(z))^2}$$

and hence

$$\begin{aligned} \overline{N} = G'_N(1) &= \frac{(1-\rho)f(1)\lambda\mathbb{E}(S) + \frac{\lambda^2\mathbb{E}(S^2)}{2}}{(1-\rho)^2} \\ &= \frac{(1-\rho)\left((1-\rho)\rho + \frac{\lambda^2\mathbb{E}(S^2)}{2}\right)}{(1-\rho)^2} \\ &= \rho + \frac{\lambda^2\mathbb{E}(S^2)}{2(1-\rho)} = \rho + \frac{\rho^2}{1-\rho} \cdot \frac{1 + C_S^2}{2}, \end{aligned}$$

which was obtained in a different way.

■

Exercise 19 *Find* $Var(W)$ *by the help of* $L_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda L_S(s)}$.

Solution:
Let us define a function

$$f(s) = \frac{s - \lambda + \lambda L_S(s)}{s},$$

which is after expansion can be written as

$$\begin{aligned} 1 - \frac{\lambda}{s} + \frac{\lambda}{s} L_S(s) &= 1 - \frac{\lambda}{s} + \frac{\lambda}{s} \cdot \sum_{i=0}^{\infty} (-1)^i \frac{\mathbb{E}(S^i)}{i!} s^i \\ &= 1 - \lambda \mathbb{E}(S) + \frac{\lambda \mathbb{E}(S^2)}{2} s - \frac{\lambda \mathbb{E}(S^3)}{3!} s^2 + \dots . \end{aligned}$$

Therefore

$$\begin{aligned} f'(s) &= \frac{\lambda \mathbb{E}(S^2)}{2} - \frac{2\lambda \mathbb{E}(S^3)}{3!} s + \frac{3\lambda \mathbb{E}(S^4)}{4!} s^2 + \dots, \\ f^{(2)}(s) &= -\frac{2\lambda \mathbb{E}(S^3)}{3!} + \frac{3 \cdot 2\lambda \mathbb{E}(S^4)}{4!} s + \dots . \end{aligned}$$

Hence

$$\begin{aligned} f(0) &= 1 - \rho, \\ f'(0) &= \frac{\lambda \mathbb{E}(S^2)}{2}, \\ f''(0) &= -\frac{\lambda \mathbb{E}(S^3)}{3}. \end{aligned}$$

Consequently, because

$$L_W(s) = \frac{1 - \rho}{f(s)}$$

we have

$$\begin{aligned} L'_W(s) &= -(1 - \rho) \frac{f'(s)}{(f(s))^2}, \\ L''_W(s) &= -(1 - \rho) \frac{f''(s)(f(s))^2 - 2f(s)(f'(s))^2}{(f(s))^4}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}(W) = -L'_W(0) &= (1 - \rho) \frac{f'(0)}{(f(0))^2} = \frac{(1 - \rho) \frac{\lambda \mathbb{E}(S^2)}{2}}{(1 - \rho)^2} \\ &= \frac{\lambda \mathbb{E}(S^2)}{2(1 - \rho)} = \frac{\rho \mathbb{E}(S)}{1 - \rho} \cdot \frac{1 + C_S^2}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \mathbb{E}(W^2) = L''_W(0) &= -(1 - \rho) \frac{f''(0)(f(0))^2 - 2f(0)(f'(0))^2}{(f(0))^4} \\ &= -\frac{(1 - \rho)(1 - \rho)^2 \left(-\frac{\lambda \mathbb{E}(S^3)}{3} \right) - 2(1 - \rho) \left(\frac{\lambda \mathbb{E}(S^2)}{2} \right)^2}{(1 - \rho)^4} \\ &= 2(\mathbb{E}(W))^2 + \frac{\lambda \mathbb{E}(S^3)}{3(1 - \rho)}. \end{aligned}$$

Thus

$$\begin{aligned} Var(W) &= \mathbb{E}(W^2) - (\mathbb{E}(W))^2 \\ &= 2(\mathbb{E}(W))^2 + \frac{\lambda \mathbb{E}(S^3)}{3(1-\rho)} - (\mathbb{E}(W))^2 \\ &= (\mathbb{E}(W))^2 + \frac{\lambda \mathbb{E}(S^3)}{3(1-\rho)}. \end{aligned}$$

Finally

$$Var(T) = Var(W+S) = Var(W) + Var(S).$$

■

Exercise 20 *By using the Laplace-transform show that*

$$\mathbb{E}(R^k) = \frac{\mathbb{E}(S^{k+1})}{(k+1)\mathbb{E}(S)}.$$

Solution:
As we have seen earlier

$$L_R(s) = \frac{1 - L_S(s)}{s\mathbb{E}(S)},$$

and it is well-known that

$$\begin{aligned} L_S(s) &= \sum_{i=0}^{\infty} \frac{L_S^{(i)}(0)}{i!} s^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \mathbb{E}(S^i)}{i!} s^i. \end{aligned}$$

Thus for $L_R(s)$ we get

$$L_R(s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}(R^k) s^k.$$

Therefore

$$\begin{aligned} L_R(s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}(R^k) s^k &= \frac{1 - L_S(s)}{s\mathbb{E}(S)} \\ &= \frac{1 - \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}(S^k) s^k\right)}{s\mathbb{E}(S)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \mathbb{E}(S^{k+1})}{(k+1)! \mathbb{E}(S)} \cdot s^k = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\mathbb{E}(Sk+1)}{(k+1)\mathbb{E}(S)} s^k. \end{aligned}$$

Consequently

$$\mathbb{E}(R^k) = \frac{\mathbb{E}(S^{k+1})}{(k+1)\mathbb{E}(S)}, \qquad k = 1, 2, \ldots$$

■

Exercise 21 *Find the generating function of the number of customers arrived during a service time for an $M/G/1$ system.*

Solution:
By applying the theorem of total probability we have

$$P(\nu_A(S) = k) = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} f_S(x)\, dx.$$

Hence its generating function can be written as

$$\begin{aligned} G_{\nu_A(S)}(z) &= z^k \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} f_S(x)\, dx = \int_0^\infty \sum_{k=0}^{\infty} \frac{(z\lambda x)^k}{k!} e^{-\lambda x} f_S(x)\, dx \\ &= \int_0^\infty e^{z\lambda x} e^{-\lambda x} f_S(x)\, dx = \int_0^\infty e^{-\lambda x(1-z)} f_S(x)\, dx = L_S(\lambda(1-z)). \end{aligned}$$

■

Chapter 5

Finite-Source Systems

Exercise 22 *If* $P(k,\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$ *and* $Q(k,\lambda) = \sum_{i=0}^{k}\frac{\lambda^i}{i!}e^{-\lambda}$, *then show the following important formula*

$$\sum_{j=0}^{k} P(k-j,a_1)Q(j,a_2) = Q(k,a_1+a_2).$$

Solution:
It is well-known that

$$Q(n,a) = \int_a^\infty P(n,y)\,dy,$$

therefore

$$\sum_{j=0}^{k} P(k-j,a_1)Q(j,a_2) = \sum_{j=0}^{k}\frac{a_1^{k-j}}{(k-j)!}e^{-a_1}\sum_{i=0}^{j}\frac{a_2^i}{i!}e^{-a_2}$$

$$\sum_{j=0}^{k}\frac{a_1^{k-j}}{(k-j)!}e^{-a_1}\int_{a_2}^\infty\frac{y^j}{j!}e^{-y}\,dy = \int_{a_2}^\infty\frac{(y+a_1)^k}{k!}e^{-(a_1+y)}\,dy = \int_{a_1+a_2}^\infty\frac{t^k}{k!}e^{-t}\,dt = Q(k,a_1+a_2),$$

where we introduced the substitution $t = y + a_1$.
■

Exercise 23 *Find the mean response time for an* $M/M/1/n/n$ *queueing system by using the Laplace-transform.*

Solution:

It is well-known that $\overline{T} = -L_T'(0)$, that is why let us calculate $L_T'(0)$.

$$\begin{aligned} L_T'(s) &= \left[\left(\frac{\mu}{\mu+s}\right)^n e^{\frac{s}{\lambda}}\frac{Q\left(n-1,\frac{\mu+s}{\lambda}\right)}{Q\left(n-1,\frac{\mu}{\lambda}\right)}\right]' \\ &= \left(\left(\frac{\mu}{\mu+s}\right)^n e^{\frac{s}{\lambda}}\right)'\cdot\frac{Q\left(n-1,\frac{\mu+s}{\lambda}\right)}{Q\left(n-1,\frac{\mu}{\lambda}\right)} + \left(\frac{\mu}{\mu+s}\right)^n e^{\frac{s}{\lambda}}\cdot\frac{\left(Q\left(n-1,\frac{\mu+s}{\lambda}\right)\right)'}{Q\left(n-1,\frac{\mu}{\lambda}\right)}. \end{aligned}$$

Thus

$$L_T'(o) = -\frac{n}{\mu} + \frac{1}{\lambda} - \frac{1}{\lambda}B\left(n-1, \frac{\mu}{\lambda}\right) = -\frac{n}{\mu} + \frac{1}{\lambda}U_S(n-1),$$

and hence

$$\overline{T}(n) = \frac{n}{\mu} - \frac{U_S(n-1)}{\lambda}.$$

Since

$$\begin{aligned}
\overline{T}(n) &= \frac{1}{\mu}(\overline{N}(n-1)+1) \\
&= \frac{1}{\mu}\left(n-1-\frac{U_S(n-1)}{\rho}+1\right) \\
&= \frac{n}{\mu} - \frac{U_S(n-1)}{\lambda},
\end{aligned}$$

which was obtained earlier. The higher moments of $T(n)$ can be obtained and hence further measured can be calculated.
Similarly, the moments of $W(n)$ can be derived.

■

Exercise 24 *Find the mean response time, denoted by $\overline{T}A$, for an $M/M/1/n/n$ system by using the density function approach.*

Solution:

$$\begin{aligned}
z &\doteq \frac{\mu}{\lambda} \\
\overline{T} &= \frac{1}{Q(n-1,z)} \int_0^\infty x\mu \frac{(\mu x+z)^{n-1}}{(n-1)!} e^{-(\mu x+z)} \mathrm{d}x \\
&= \frac{e^{-z}}{Q(n-1,z)} \int_0^\infty x\mu \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!} \frac{z^{n-1-k}}{(n-1-k)!} e^{-\mu x} \mathrm{d}x \\
&= \frac{e^{-z}}{Q(n-1,z)} \sum_{k=0}^{n-1} \frac{z^{n-1-k}}{(n-1-k)!} \int_0^\infty x\mu \frac{(\mu x)^k}{k!} e^{-\mu x} \mathrm{d}x \\
&= \frac{e^{-z}}{Q(n-1,z)} \sum_{k=0}^{n-1} \frac{z^{n-1-k}}{(n-1-k)!} \cdot \frac{k+1}{\mu} \\
&= \frac{1}{\mu} \sum_{k=0}^{n-1} (k+1) \frac{\frac{z^{n-1-k}}{(n-1-k)!} \cdot e^{-z}}{Q(n-1,z)} \\
&= \frac{1}{\mu} \sum_{k=0}^{n-1} \frac{k+1}{\mu} P_k(n-1) = \frac{1}{\mu}(\overline{N}(n-1)+1).
\end{aligned}$$

The mean waiting time can be obtained similarly, starting the summation from 1 and taking a Erlang distribution with one phase less.

■

Exercise 25 *Find $Var(N)$ for an $M/M/1/n/n$ system.*

Solution:

Let us denote by F the number of customers in the source. Hence $F + N = n$, and thus $Var(N) = Var(F)$.
As we have proved the distribution of F equals to the distribution of an $M/M/n/n$ system with traffic intensity $z = \frac{1}{\rho}$ we have

$$\begin{aligned} Var(N) &= \frac{1}{\rho}\left(1 - B\left(n, \frac{1}{\rho}\right)\right) - \frac{1}{\rho}B\left(n, \frac{1}{\rho}\right)\left(n - \frac{1}{\rho}\left(1 - B\left(n, \frac{1}{\rho}\right)\right)\right) \\ &= \frac{U_S}{\rho} - \frac{1 - U_S}{\rho}\left(n - \frac{U_S}{\rho}\right). \end{aligned}$$

If the number of sources is denoted then this formula can be written as

$$Var(N(n)) = \frac{U_s(n)}{\rho} - \frac{1 - U_s(n)}{\rho}\left(n - \frac{U_s(n)}{\rho}\right).$$

This result helps us to determine $Var(T(n))$-t and $Var(W(n))$-t. Since $W(n)$ can be consider as a random sum, where the summands are the exponentially distributed service times with parameter μ, and the counting process is the number of customers in the system at the arrival instant of a customer, denoted by $N_A^{(n)}$, we can use the formula obtained for the variance of a random sum, namely

$$\begin{aligned} Var(W(n)) &= \mathbb{E}(N_A^{(n)}) \cdot \frac{1}{\mu^2} + \frac{1}{\mu^2} Var(N_A^{(n)}) \\ &= \overline{N}(n-1) \cdot \frac{1}{\mu^2} + \frac{1}{\mu^2} Var(N(n-1)) \\ &= \frac{1}{\mu^2}(\overline{N}(n-1) + Var(N(n-1))), \end{aligned}$$

where

$$\begin{aligned} \overline{N}(n-1) &= n - 1 - \frac{U_s(n-1)}{\rho} \\ Var(N(n-1)) &= \frac{U_s(n-1)}{\rho} - \frac{1 - U_s(n-1)}{\rho}\left(n - 1 - \frac{U_s(n-1)}{\rho}\right). \end{aligned}$$

Similarly, since $T(n) = W(n) + S$, then

$$Var(T(n)) = Var(W(n)) + \frac{1}{\mu^2}.$$

■

Part III

Queueing Theory Formulas

Chapter 6

Relationships

6.1 Notations and Definitions

Table 1. Basic Queueing Theory Notations and Definitions

a	Server utilization.
a_i	Utilization of component i in a queueing network.
$A[t]$	Distribution function of interarrival time. $A[t] = P[\tau \leq t]$
b	Random variable describing the busy period for a server
$B[c, \rho]$	Erlang's B formula. Probability all servers busy in M/M/c/c system. Also called Erlang's loss formula.
c	Number of servers in in a service facility.
$C[c, \rho]$	Erlang's C formula. Probability all servers busy in M/M/c/c system. Also called Erlang's delay formula
C_X^2	Squared coefficient of a variation of a positive random variable, $C_X^2 = \frac{Var[X]}{\mathbb{E}[X]^2}$.
D	Symbol for constant (deterministic) interarrival or service time.
E_k	Symbol for Erlang-k distribution of interarrival or service time.
$\mathbb{E}[Q\|Q > 0]$	Expected (mean or average) queue length of nonempty queues.
$\mathbb{E}[W\|W > 0]$	Expected queueing time.
FCFS	First Come First Served queue discipline.
FIFO	First In First Out queue discipline. Identical with FCFS.
$F_T(t)$	The distribution function of T, $F_T(t) = P[T < t]$.
$F_W(t)$	The distribution function of W, $F_W(t) = P[W < t]$.
G	Symbol for general probability distribution of service time. Independence usually assumed.
GI	Symbol for general independent interarrival time distribution.
H_2	Symbol for two-stage hyperexponential distribution. Can be generalized to k stages.
K	Maximum number of customers allowed in queueing system. Also size of population in finite population models.

Table 1. Basic Queueing Theory Notations and Definitions (continued)

λ	Mean arrival rate of customers into the system.
$\overline{\lambda}$	Actual mean arrival rate into the system, for which some arrivals arc turned away, e.q., the M/M/c/c system.
λ_T	Mean throughput of a computer system measured in transactions or interactions per unit time.
$\ln \cdot)$	Natural logarithm function (log to base e).
$\overline{N}_s$	Expected steady state number of customers receiving service, $\mathbb{E}[N_s]$.
LCFS	Last Come First Served queue discipline.
LIFO	Last In First Out queue discipline. Identical with LCFS.
M	Symbol for exponential interarrival or service time.
μ	Mean service rate per server, that is, the mean rate of service completions while the server is busy.
μ_a, μ_b	Parameters of the two-stage hyperexponential distribution of it? for the $M/H_2/1$ system.
$\overline{N}$	Expected steady state number of customers in the queueing system, $\mathbb{E}[N]$.
$N[t]$	Random variable describing the number of customers in the system at time t.
N	Random variable describing the steady state number of customers in the system.
N_b	Random variable describing the number of customers served by a server in one busy period.
$N_s[t]$	Random variable describing the number of customers receiving service at time t.
N_s	Random variable describing the steady state number of customers in the service facility.
O	Operating time of a machine in a machine repair queueing model. The time a machine remains in operation afler repair before repair is again necessary.
$P_n[t]$	Probability there arc n customers in the system at time t.
P_n	Steady state probability that there are n customer in the system.
PRI	Symbol for priority queueing discipline.
PS	Symbol for processor sharing queueing discipline.
p_i	A parameter of a hypoexponential random variable.
π_a, π_b	Parameters of the distribution function of w for the $M/H_2/1$ queueing system.
$\pi_X[r]$	The rth percentile for random variable X.
Q	Random variable describing the steady state number of customers in the queue.

Table 1. Basic Queueing Theory Notations and Definitions (continued)

$Q[t]$	Random variable describing the number of customers in the queue at time t.
ρ	$\rho = \lambda\overline{S}$ The traffic intensity or offered load. The international unit of this is erlang, named for A.K. Erlang, a queueing theory pioneer
RSS	Symbol for queueing discipline "random selection for service".
S	Random variable describing the service time. $\mathbb{E}[S] = \frac{1}{\mu}$.
$\overline{S}$	Expected customer service time, $\mathbb{E}[S] = \frac{1}{\mu}$.
SIRO	Symbol for service in random order, which is identical to RSS. It means each customer in queue has the same probability of being served next.
T	Random variable describing the total time a customer spends in the queueing system, $T = W + S$.
$\overline{T}$	Expected steady state time a customer spends in the system, $\overline{T} = \mathbb{E}[T] = \overline{W} + \overline{S}$.
τ	Random variable describing interarrival time. $\mathbb{E}[\tau] = \frac{1}{\lambda}$.
W	Random variable describing the time a customer spends in the queue before service begins.
W'	Random variable describing time a customer who must queue spends in the queue before receiving service. Also called conditional queueing time.
$\overline{W}$	Expected steady state time a customer spends in the queue, $\overline{W} = \mathbb{E}[W] = \overline{T} - \overline{S}$.

6.2 Relationships between random variables

Table 2. Relationships between Random Variables

a	Server utilization. The probability any particular server is busy.
$N = Q + N_s$	Number of customers in steady state system.
$\overline{N} = \overline{\lambda} \cdot \overline{T}$	Mean number of customers in steady state system. This formula is often called Little's law.
$\overline{N_s} = \overline{\lambda} \cdot \overline{S}$	Mean number of customers receiving service in steady state system. This formula sometimes called Little's law.
$\overline{Q} = \overline{\lambda} \cdot \overline{W}$	Mean number in steady state queue. Also called Little's law.
$\rho = \frac{\mathbb{E}[S]}{\mathbb{E}[\tau]} = \lambda\overline{S}$	Traffic intensity in erlangs.
$T = W + S$	Total waiting time in the system.
$\overline{T} = \overline{W} + \overline{S}$	Mean total waiting time in steady state system.

Chapter 7

Basic Queueing Theory Formulas

7.1 M/M/1 Formulas

Table 3. M/M/1 Queueing System

$\rho = \lambda\overline{S}, \quad P_n = P[N = n] = (1-\rho)\rho^n, \quad n = 0, 1, \ldots.$

$P[N \geq n] = \rho^n, \quad n = 0, 1, \ldots.$

$\overline{N} = \mathbb{E}[N] = \overline{\lambda} \cdot \overline{T} = \dfrac{\rho}{1-\rho}, \quad Var(N) = \dfrac{\rho}{(1-\rho)^2}.$

$\overline{Q} = \lambda\overline{W} = \dfrac{\rho^2}{1-\rho}, \quad Var(Q) = \dfrac{\rho^2(1+\rho-\rho^2)}{(1-\rho)^2}.$

$\mathbb{E}[Q|Q>0] = \dfrac{1}{1-\rho}, \quad Var[Q|Q>0] = \dfrac{\rho}{(1-\rho)^2}.$

$F_T(t) = P[T \leq t] = 1 - \exp\left(\dfrac{-t}{\overline{T}}\right), \quad P[T > t] = \exp\left(\dfrac{-t}{\overline{T}}\right).$

$\overline{T} = \mathbb{E}[T] = \dfrac{\overline{S}}{1-\rho} = \dfrac{1}{\mu(1-\rho)}, \quad Var(T) = \overline{T}^2.$

$\pi_T[r] = \overline{T} \ln\left(\dfrac{100}{100-r}\right), \quad \pi_T[90] = \overline{T}\ln 10, \quad \pi_T[95] = \overline{T}\ln 20$

$F_T(t) = P[W \leq t] = 1 - \rho\exp\left(\dfrac{-t}{\overline{T}}\right), \quad P[W > t] = \rho\exp\left(\dfrac{-t}{\overline{T}}\right).$

$\overline{W} = \dfrac{\rho\overline{S}}{1-\rho}, \quad Var(W) = \dfrac{(2-\rho)\rho\overline{S}^2}{(1-\rho)^2}.$

$$\pi_W[r] = \max\left\{\overline{T}\ln\left(\frac{100\rho}{100-r}\right), 0\right\}.$$

$$\pi_W[90] = \max\{\overline{T}\ln(10\rho), 0\}, \quad \pi_W[95] = \max\{\overline{T}\ln(20\rho), 0\}.$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1/MM1.html`

7.2 M/M/1/K Formulas

Table 4. M/M/1/K Queueing System

$$P_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{(1-\rho^{K+1})} & \text{if } \lambda \neq \mu \\ \dfrac{1}{K+1} & \text{if } \lambda = \mu \end{cases}$$

$n = 0, 1, \ldots, K$, where $\rho = \lambda \overline{S}$.

$\overline{\lambda} = (1 - P_K)\lambda$, Mean arrival rate into system.

$$\overline{N} = \begin{cases} \dfrac{\rho[1-(K+1)\rho^K + K\rho^{K+1}]}{(1-\rho)(1-\rho^{K+1})} & \text{ha } \lambda \neq \mu \\ \dfrac{K}{2} & \text{ha } \lambda = \mu. \end{cases}$$

$\overline{Q} = \overline{N} - (1 - P_0)$, $\Pi_n = \dfrac{P_n}{1-P_K}, n = 0, 1, \ldots, K-1.$

$$F_T(t) = 1 - \sum_{n=0}^{K-1} \Pi_n Q[n; \mu t],$$

where

$$Q[n; \mu t] = e^{-\mu t} \sum_{k=0}^{n} \frac{(\mu t)^k}{k!}.$$

$$\overline{T} = \frac{\overline{N}}{\overline{\lambda}}, \quad \overline{W} = \frac{\overline{Q}}{\overline{\lambda}}.$$

$$F_T(t) = 1 - \sum_{n=0}^{K-2} \Pi_{n+1} Q[n; \mu t].$$

$$\mathbb{E}[W|W > 0] = \frac{\overline{W}}{1 - P_0}, \quad a = (1 - P_K)\rho.$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1K/MM1K.html`

7.3 M/M/c Formulas

Table 5. M/M/c Queueing System

$$\rho = \lambda \overline{S}, \quad a = \frac{\rho}{c}$$

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!(1-a)} \right]^{-1} = \frac{c!(1-a)P[N \geq c]}{\rho^c}.$$

$$P_n = \begin{cases} \dfrac{\rho^n}{n!} P_0, & \text{if } n \leq c \\ \dfrac{\rho^n}{c! c^{n-c}} P_0, & \text{if } n \geq c. \end{cases}$$

$$P[N \geq n] = \begin{cases} P_0 \left[\displaystyle\sum_{k=n}^{c-1} \frac{\rho^k}{k!} + \frac{\rho^c}{c!(1-a)} \right] & \text{if } n < c, \\ P_0 \left[\dfrac{a^c a^{n-c}}{c!(1-a)} \right] = P[N \geq c] a^{n-c} & \text{if } n \geq c \end{cases}$$

$$\overline{Q} = \overline{\lambda} \cdot \overline{W} = \frac{\rho P[N \geq c]}{c(1-a)},$$

where

$$P[N \geq c] = C[c, \rho] = \frac{\dfrac{\rho^c}{c!}}{(1 - \dfrac{\rho}{c}) \displaystyle\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}}.$$

$$Var(Q) = \frac{aC[c,\rho][1 + a - aC[c,\rho]]}{(1-a)^2}.$$

$$\overline{N} = \overline{\lambda} \cdot \overline{T} = \overline{Q} + \rho.$$

$$Var(N) = Var(Q) + \rho(1 + P[N \geq c]).$$

$$\overline{W}[0] = 1 - P[N \geq c], \quad F_T(t) = 1 - P[N \geq c] \exp[-c\mu t(1-a)],$$

$$\overline{W} = \frac{P[N \geq c]\overline{S}}{c(1-a)}.$$

Table 5. M/M/c Queueing System (continued)

$$Var(W) = \frac{[2 - C[c,\rho]]C[c,\rho]\overline{S}^2}{c^2(1-a)^2}.$$

$$\pi_W[r] = \max\{0, \frac{\overline{S}}{c(1-a)} \ln\left(\frac{100C[c,\rho]}{100-r}\right)\}.$$

$$\pi_W[90] = \max\{0, \frac{\overline{S}}{c(1-a)} \ln(10C[c,\rho])\}.$$

$$\pi_W[95] = \max\{0, \frac{\overline{S}}{c(1-a)} \ln(20C[c,\rho])\}.$$

$$\overline{W}_{W'} = P[W \le t | W > 0] = 1 - \exp\left(\frac{-ct(1-a)}{\overline{S}}\right), \quad t > 0.$$

$$\mathbb{E}[W|W>0] = \mathbb{E}[W'] = \frac{\overline{S}}{c(1-a)}.$$

$$Var([W|W>0] = \left(\frac{\overline{S}}{c(1-a)}\right)^2.$$

$$F_T(t) = \begin{cases} 1 + C_1 e^{-\mu t} + C_2 e^{-c\mu t(1-a)} & \text{if } \rho \neq c-1 \\ 1 - \{1 + C[c,\rho]\mu t\}e^{-\mu t} & \text{if } \rho = c-1 \end{cases}$$

where

$$C_1 = \frac{P[N \ge c]}{1 - c(1-a)} - 1,$$

and

$$C_2 = \frac{P[N \ge c]}{c(1-a) - 1}.$$

$$\overline{T} = \overline{W} + \overline{S}.$$

$$\mathbb{E}[T^2] = \begin{cases} \frac{2P[N \ge c][1 - c^2(1-a)^2]\overline{S}^2}{(\rho + 1 - c)c^2(1-a)^2} + 2\overline{S}^2 & \text{if } \rho \neq c-1 \\ 2\{2P[N \ge c] + 1\}\overline{S}^2 & \text{if } \rho = c-1 \end{cases}$$

$$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$$

$\pi_T[90] \approx \overline{T} + 1.3\mathbb{D}(T)$, $\pi_T[95] \approx \overline{T} + 2\mathbb{D}(T)$ (estimates due to James Martin).

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMc/MMc.html`

7.4 M/M/2 Formulas

Table 6. M/M/2 Queueing System

$\rho = \lambda \overline{S}, \quad a = \dfrac{\rho}{2}$

$P_0 = \dfrac{1-a}{1+a}.$

$P_n = 2P_0 a^n, \quad n = 1, 2, 3, \ldots$

$P[N \geq n] = \dfrac{2a^n}{1+a}, \quad n = 1, 2, \ldots$

$\overline{Q} = \lambda \overline{W} = \dfrac{2a^3}{1-a^2},$

$P[N \geq 2] = C[2, \rho]$ is the probability that ail arriving customer must queue for service. $P[N \geq 2]$ is given by

$P[N \geq 2] = C[2, \rho] = \dfrac{2a^2}{1+a}.$

$Var(Q) = \dfrac{2a^3[(1+a)^2 - 2a^3]}{(1-a^2)^2}.$

$\overline{N} = \overline{\lambda} \cdot \overline{T} = \overline{Q} + \rho = \dfrac{2a}{1-a^2}.$

$Var(N) = Var(Q) + \dfrac{2a(1+a+2a^2)}{1+a}.$

$\overline{W}[0] = \dfrac{1+a-2a^2}{1+a}.$

$F_T(t) = 1 - \dfrac{2a^2}{1+a} \exp[-2\mu t(1-a)]$

$\overline{W} = \dfrac{a^2 \overline{S}}{1-a^2}.$

$Var(W) = \dfrac{a^2(1+a-a^2)\overline{S}^2}{(1-a^2)^2}.$

$\pi_W[r] = \max\{0, \dfrac{\overline{S}}{2(1-a)} \ln\left(\dfrac{200a^2}{(100-r)(1+a)}\right)\}.$

Table 6. M/M/2 Queueing System (continued)

$$\pi_W[90] = \max\{0, \frac{\overline{S}}{2(1-a)} \ln\left(\frac{20a^2}{1+a}\right)\}.$$
$$\pi_W[95] = \max\{0, \frac{\overline{S}}{2(1-a)} \ln\left(\frac{40a^2}{1+a}\right)\}.$$

$$\overline{W}_{W'} = P[W \le t | W > 0] = 1 - \exp\left(\frac{-2t(1-a)}{\overline{S}}\right), \quad t > 0.$$
$$\mathbb{E}[W|W>0] = \mathbb{E}[W'] = \frac{\overline{S}}{2(1-a)}.$$
$$Var[W|W>0] = \left(\frac{\overline{S}}{2(1-a)}\right)^2.$$

$$F_T(t) = \begin{cases} 1 + \dfrac{1-a}{1-a^2-2a^2} e^{-\mu t} + \dfrac{2a^2}{1-a-2a^2} e^{-2\mu t(1-a)} & \text{where } \rho \neq 1 \\ 1 - \{1 + \dfrac{\mu t}{3}\} e^{-\mu t} & \text{where } \rho = 1 \end{cases}$$
$$\overline{T} = \overline{W} + \overline{S} = \frac{\overline{S}}{1-a^2}.$$

$$\mathbb{E}[T^2] = \begin{cases} \dfrac{a^2[1-4(1-a)^2]\overline{S}^2}{(2a-1)(1-a)(1-a^2)} + 2\overline{S}^2 & \text{if } \rho \neq 1 \\ \dfrac{10}{3}\overline{S}^2 & \text{if } \rho = 1 \end{cases}$$

$$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$$

$$\pi_T[90] \approx \overline{T} + 1.3\mathbb{D}(T), \quad \pi_T[95] \approx \overline{T} + 2\mathbb{D}(T)$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM2/MM2.html`

7.5 M/M/c/c Formulas

Table 7. M/M/c/c Queueing System (M/M/c loss)

$\rho = \lambda \overline{S}$

$$P_n = \frac{\frac{\rho^n}{n!}}{1 + \rho + \frac{\rho^2}{2!} + \ldots + \frac{\rho^c}{c!}} \quad n = 0, 1, \ldots, c.$$

The probability that all servers are busy, P_c is called Erlang's B formula, $B[c, \rho]$, and thus

$$B[c, \rho] = \frac{\frac{\rho^c}{c!}}{1 + \rho + \frac{\rho^2}{2!} + \ldots + \frac{\rho^c}{c!}}.$$

$\overline{\lambda} = \lambda(1 - B[c, \rho])$ Is the average arrival rate of customers who actually enter the system. Thus, the true server utilization, a, is given by

$$a = \frac{\overline{\lambda}\ \overline{S}}{c}.$$

$$\overline{N} = \overline{\lambda}\ \overline{S}.$$

$$\overline{T} = \frac{\overline{N}}{\overline{\lambda}} = \overline{S}.$$

$$F_T(t) = 1 - \exp\left(\frac{-t}{\overline{S}}\right).$$

All of the formulas except the last one arc true for the M/G/c/c queueing system. For this system we have

$F_T(t) = F_S(t)$.

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcc/MMcc.html

7.6 M/M/c/K Formulas

Table 8. M/M/c/K Queueing System

$\rho = \lambda \overline{S}$.

$$P_0 = \left[\sum_{n=0}^{c} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \sum_{n=1}^{K-c} \left(\frac{\rho}{c} \right)^n \right]^{-1}.$$

$$P_n = \begin{cases} \dfrac{\rho^n}{n!} P_0 & \text{if } n = 1, 2, \ldots, c, \\ \dfrac{\rho^n}{c!} \left(\dfrac{\rho}{c} \right)^{n-c} P_0 & \text{if } n = c+1, \ldots, K. \end{cases}$$

The average arrival rate of customers who actually enter the system is $\overline{\lambda} = \lambda(1 - P_K)$.

The actual mean server utilization, a, is given by:

$$a = \frac{\overline{\lambda}\, \overline{S}}{c}.$$

$$\overline{Q} = \frac{\rho^c r P_0}{c!(1-r)^2} \left[1 + (K-c) r^{K-c+1} - (K-c+1) r^{K-c} \right],$$

where

$$r = \frac{\rho}{c}.$$

$$\overline{N} = \overline{Q} + \mathbb{E}[N_s] = \overline{Q} + \sum_{n=0}^{c-1} n P_n + c \left(1 - \sum_{n=0}^{c-1} P_n \right).$$

By Little's Law

$$\overline{W} = \frac{\overline{Q}}{\overline{\lambda}}, \qquad \overline{T} = \frac{\overline{N}}{\overline{\lambda}}.$$

$$\Pi_n = \frac{P_n}{1 - P_K}, \quad n = 0, 1, 2, \ldots, K-1,$$

Table 8. M/M/c/K Queueing System (continued)

where Π_n is the probability that an arriving customer who enters the system finds n customers already there.

$$\mathbb{E}[W|W>0] = \frac{\overline{W}}{1-\sum_{n=0}^{c-1}\Pi_n}.$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcK/MMcK.html

7.7 M/M/∞ Formulas

Table 9. M/M/∞ Queueing System

$\rho = \lambda \overline{S}$.

$$P_n = \frac{\rho^n}{n!} e^{-\rho}, \quad n = 0, 1, \ldots .$$

Since N has a Poisson distribution,

$$\overline{N} = \rho \quad \text{and} \quad Var(N) = \rho.$$

By Little's Law

$$\overline{T} = \frac{\overline{N}}{\lambda} = \overline{S}.$$

Since there is no queueing for service,

$$\overline{W} = \overline{Q} = 0,$$

and
$F_T(t) = P[T \leq t] = F_S(t) = P[S \leq t]$

That is, T has the same distribution as S.
All the above formulas arc true for the $M/G/\infty$ system, also.

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMinf/MMinf.html`

7.8 M/M/1/K/K Formulas

Table 10. M/M/1/K/K Queueing System

The mean operating time per machine (sometimes called the the mean time to failure, MTTF) is
$\mathbb{E}[O] = \dfrac{1}{\alpha}.$

The mean repair time per machine by one repairman is
$\overline{S} = \dfrac{1}{\mu}.$

The probability, P_0, that no machines arc out of service is given by

$$P_0 = \left[\sum_{k=0}^{K} \frac{K!}{(K-k)!}\left(\frac{\overline{S}}{\mathbb{E}[O]}\right)^k\right]^{-1} = B[K, z],$$

where $B[\cdot, \cdot]$ is Erlang's B formula and

$$z = \frac{\mathbb{E}[O]}{\overline{S}}.$$

Then, P_n, the probability that n machines arc out of service, is given by

$$P_n = \frac{K!}{(K-n)!} z^{-n} P_0, \quad n = 0, 1, \ldots, K,$$

The formula for P_n can also be written in the form

$$P_n = \frac{\dfrac{z^{K-n}}{(K-n)!}}{\displaystyle\sum_{k=0}^{K} \frac{z^k}{k!}}, \quad n = 0, 1, \ldots, K.$$

$a = 1 - P_0.$

$\overline{\lambda} = \dfrac{a}{\overline{S}}.$

$\overline{T} = \dfrac{K}{\overline{\lambda}} - \mathbb{E}[O].$
$\overline{N} = \overline{\lambda} \cdot \overline{T}.$

$\overline{W} = \overline{T} - \overline{S}.$

Table 10. M/M/1/K/K Queueing System (continued)

$$\Pi_n = \frac{(K-n)P_n}{K-\overline{N}} = \frac{\dfrac{z^{K-n-1}}{(K-n-1)!}}{\displaystyle\sum_{k=0}^{K-1}\frac{z^k}{k!}}, \quad n = 0, 1, 2, \ldots, K-1,$$

where Π_n is the probability that a machine that breaks down finds n machines in the repair facility.

$$F_T(t) = P[T \le t] = 1 - \frac{Q(K-1; z+t\mu)}{Q(K-1; z)}, \quad t \ge 0,$$

where

$$Q(n; x) = e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}.$$

$$F_T(t) = P[W \le t] = 1 - \frac{Q(K-2; z+t\mu)}{Q(K-1; z)}, \quad t \ge 0,$$

$$\mathbb{E}[W|W > 0] = \frac{\overline{W}}{1-\Pi_0}.$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MM1KK/MM1KK.html`

7.9 M/G/1/K/K Formulas

Table 11. M/G/1/K/K Queueing System

The mean operating time per machine (sometimes called the the mean time to failure, MTTF) is

$$\mathbb{E}[O] = \frac{1}{\alpha}.$$

The mean repair time per machine by one repairman is

$$\overline{S} = \frac{1}{\mu}.$$

The probability, P_0, that no machines arc out of service is given by

$$P_0 = \left[1 + \frac{K\overline{S}}{\mathbb{E}[O]} \sum_{n=0}^{K-1} \binom{K-1}{n} B_n\right]^{-1},$$

where

$$B_n = \begin{cases} 1 & n = 0, \\ \prod_{i=1}^{n} \left(\frac{1-\overline{S}^*[i\alpha]}{\overline{S}^*[i\alpha]}\right) & n = 1, 2, \ldots, K-1. \end{cases}$$

and $\overline{S}^*[\theta]$ is the Laplace-Stieltjes transform of s.

$a = 1 - P_0.$

$\overline{\lambda} = \frac{a}{\overline{S}}.$

$\overline{T} = \frac{K}{\overline{\lambda}} - \mathbb{E}[O].$

$\overline{N} = \overline{\lambda} \cdot \overline{T}.$

$\overline{W} = \overline{T} - \overline{S}.$

$\overline{Q} = \overline{\lambda} \cdot \overline{W}.$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMHyper1KK/MHyper1KK.html

7.10 M/M/c/K/K Formulas

Table 12. M/M/c/K/K Queueing System

The mean operating time per machine (sometimes called the the mean time to failure, MTTF) is

$$\mathbb{E}[O] = \frac{1}{\alpha}.$$

The mean repair time per machine by one repairman is

$$\overline{S} = \frac{1}{\mu}.$$

The probability, P_0, that no machines are out of service is given by

$$P_0 = \left[\sum_{k=0}^{c} \binom{K}{k} z^{-k} + \sum_{k=c+1}^{K} \frac{k!}{c! c^{k-c}} \binom{K}{k} z^{-k} \right]^{-1},$$

where

$$z = \frac{\mathbb{E}[O]}{\overline{S}}.$$

Then,P_n, the probability that n machines are out of service is given by

$$P_n = \begin{cases} \binom{K}{n} z^{-n} P_0 & n = 0, 1, \ldots, c, \\ \frac{n!}{c! c^{n-c}} \binom{K}{n} z^{-n} P_0 & n = c+1, \ldots, K. \end{cases}$$

$$\overline{Q} = \sum_{n=c+1}^{K} (n-c) P_n.$$

$$\overline{W} = \frac{\overline{Q}(\mathbb{E}[O] + \overline{S})}{K - \overline{Q}}.$$

$$\overline{\overline{\lambda}} = \frac{K}{\mathbb{E}[O] + \overline{W} + \overline{S}}.$$

$$\overline{T} = \frac{K}{\overline{\overline{\lambda}}} - \mathbb{E}[O].$$

$$\overline{N} = \overline{\overline{\lambda}}\, \overline{T}.$$

$$\Pi_n = \frac{(K-n)P_n}{K-\overline{N}},$$

where Π_n is the probability that a machine which breaks down finds n inoperable machines already in the repair facility. We denote Π_n by $\Pi_n[K]$ to emphasize the fact that there are K machines. It can be shown that

$$\Pi_n[K] = P_n[K-1], \quad n = 0, 1, \ldots, K-1.$$

$$P_n[K-1] = \frac{c^c}{c!}\frac{p(K-n-1;cz)}{p(K-1;cz)}P_0[K-1],$$

where, of course,

$$p(k;\alpha) = \frac{\alpha^k}{k!}e^{-\alpha}.$$

$$F_T(t) = P[W \le t] = 1 - \frac{c^c Q(K-c-1;cz)P_0[K-1]}{c!p(K-1;cz)}, t \ge 0,$$

where

$$Q(k;\alpha) = e^{-\alpha}\sum_{n=0}^{k}\frac{\alpha^n}{n!}.$$

$$F_T(t) = P[T \le t] = 1 - C_1\exp(-t/\overline{S}) + C_2\frac{Q(K-c-1;c(z+t\mu))}{Q(K-c-1;cz)},$$

$t \ge 0$,

where $C_1 = 1 + C_2$ ďż˝s

$$C_2 = \frac{c^c Q(K-c-1;cz)}{c!(c-1)(K-c-1)!p(K-1;cz)}P_0[K-1].$$

The probability that a machine that breaks down must wait for repair is given by

$$D = \sum_{n=c}^{K-1}\Pi_n = 1 - \sum_{n=0}^{c-1}\Pi_n.$$

$$\mathbb{E}[W|W>0] = \frac{\overline{W}}{D}.$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcKK/MMcKK.html`

7.11 D/D/c/K/K Formulas

Table 13. D/D/c/K/K Queueing System

The mean operating time per machine (sometimes called the the mean time to failure, MTTF) is

$$\mathbb{E}[O] = \frac{1}{\alpha}.$$

The mean repair time per machine by one repairman is

$$\overline{S} = \frac{1}{\mu}.$$

$$a = \min\{1, \frac{K}{c(1+z)}\},$$

where

$$z = \frac{\mathbb{E}[O]}{\overline{S}}.$$

$$\overline{\lambda} = ca\mu = \frac{ca}{\overline{S}}.$$

$$\overline{T} = \frac{K}{\overline{\lambda}} - \mathbb{E}[O].$$

$$\overline{N} = \overline{\lambda}\,\overline{T}.$$

$$\overline{W} = \overline{T} - \overline{S}.$$

$$\overline{Q} = \overline{\lambda}\,\overline{W}.$$

The equations for this model are derived in "A straightforward model of computer performance prediction" by John W. Boyse es David R. Warn in ACM Comput. Surveys, 7(2), (June 1972).

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/DDcKK/DDcKK.html`

7.12 M/G/1 Formulas

Table 14. M/G/1 Queueing System

The z-transform of N, the steady-state number of customers in the system is given by:

$$G_N(z) = \sum_{n=0}^{\infty} P_n z^n = \frac{(1-\rho)(1-z)\overline{S}^*[\lambda(1-z)]}{\overline{S}^*[\lambda(1-z)] - z},$$

where $\overline{S}^*$ is the Laplace-Stieltjes transform of the servide time S. The Laplace-Stieltjes transforms of T and W are given by

$$W^*[\theta] = \frac{(1-\rho)\theta\overline{S}^*[\theta]}{\theta - \lambda + \lambda\overline{S}^*[\theta]},$$

dż˝s

$$\overline{W}^*[\theta] = \frac{(1-\rho)\theta}{\theta - \lambda + \lambda\overline{S}^*[\theta]}.$$

Each of the three transforms above is called the Pollaczek-Khintchine transform equation by various authors. The probability, P_0, of no customers in the system has the simple and intuitive equation $P_0 = 1 - \rho$, where the server utilization $\rho = \lambda\overline{S}$. The probability that the server is busy is $P[N \geq 1] = \rho$.

$\overline{W} = \dfrac{\lambda\mathbb{E}[S^2]}{2(1-\rho)} = \dfrac{\rho\overline{S}}{1-\rho}\left(\dfrac{1+C_S^2}{2}\right)$ (Pollaczek formula).

$\overline{Q} = \lambda\overline{W}$.

$$Var(Q) = \frac{\lambda^3\mathbb{E}[S^3]}{3(1-\rho)} + \left(\frac{\lambda^2\mathbb{E}[S^2]}{2(1-\rho)}\right)^2 + \frac{\lambda^2\mathbb{E}[S^2]}{2(1-\rho)}.$$

$$\mathbb{E}[W|W>0] = \frac{\overline{S}}{1-\rho}\left(\frac{1+C_S^2}{2}\right).$$

$$\mathbb{E}[W^2] = 2\overline{W}^2 + \frac{\lambda\mathbb{E}[S^3]}{3(1-\rho)}.$$

$Var(W) = \mathbb{E}[W^2] - \overline{W}^2$.

$\overline{T} = \overline{W} + \overline{S}$.

$\overline{N} = \overline{\lambda}\cdot\overline{T} = \overline{Q} + \rho$.

$$Var(N) = \frac{\lambda^3\mathbb{E}[S^3]}{3(1-\rho)} + \left(\frac{\lambda^2\mathbb{E}[S^2]}{2(1-\rho)}\right)^2 + \frac{\lambda^2(3-2\rho)\mathbb{E}[S^2]}{2(1-\rho)} + \rho(1-\rho).$$

$$\mathbb{E}[T^2] = \mathbb{E}[W^2] + \frac{\mathbb{E}[S^2]}{1-\rho}.$$

Table 14. M/G/1 Queueing System (continued)

$Var(T) = \mathbb{E}[T^2] - \overline{T}^2$.

$\pi_T[90] \approx \overline{T} + 1.3\mathbb{D}(T), \quad \pi_T[95] \approx \overline{T} + 2\mathbb{D}(T)$.

The z-transform of the steady-state number in the system, N, is given by

$$G_N(z) = \sum_{n=0}^{\infty} P_n z^n = C_1 \frac{z_1}{z_1 - z} + C_2 \frac{z_2}{z_2 - z},$$

where z_1 and z_2 are the roots of the next equation

$$a_1 a_2 z^2 - (a_1 + a_2 + a_1 a_2)z + 1 + a_1 + a_2 - a = 0,$$

where

$$a = \lambda \overline{S},$$

$$a_i = \frac{\lambda}{\mu_i}, \quad i = 1, 2,$$

$$C_1 = \frac{(z_1 - 1)(1 - a z_2)}{z_1 - z_2},$$

and

$$C_2 = \frac{(z_2 - 1)(1 - a z_1)}{z_2 - z_1}.$$

From $G_N(z)$ we get

$$P_n = C_1 z_1^{-n} + C_2 z_2^{-n}, \quad n = 0, 1, \ldots.$$

Specifically, $P_0 = 1 - a$.

$$P[N \geq n] = C_1 \frac{z_1^{-n+1}}{z_1 - 1} - C_2 \frac{z_2^{-n+1}}{z_2 - 1}.$$

Additionally,

$$P[N \geq 1] = a.$$

$$F_T(t) = P[W \leq t] = 1 - C_5 e^{-\rho t} - C_6 e^{-bt}, \ t \geq 0,$$

where $\rho = -\zeta_1, \ b = -\zeta_2, \ \zeta_1, \zeta_2$ are the solutions of the

$$\theta^2 + (\mu_1 + \mu_2 - \lambda)\theta + \mu_1 \mu_2 (1 - a) = 0,$$

equation,

$$C_5 = \frac{\lambda(1-a)\zeta_1 + a(1-a)\mu_1\mu_2}{\rho(\zeta_1 - \zeta_2)}$$

and

$$C_6 = \frac{\lambda(1-a)\zeta_2 + a(1-a)\mu_1\mu_2}{\rho(\zeta_2 - \zeta_1)}.$$

$$\overline{W} = \frac{\lambda\mathbb{E}[S^2]}{2(1-a)} = \frac{a\overline{S}}{1-a}\left(\frac{1+C_S^2}{2}\right).$$ (Pollaczek-formula)

$$\mathbb{E}[W|W>0] = \frac{\overline{S}}{1-a}\left(\frac{1+C_S^2}{2}\right).$$

$$\mathbb{E}[W^2] = 2\overline{W}^2 + \frac{\lambda\mathbb{E}[S^3]}{3(1-a)}.$$

In this formula we substitute

$$\mathbb{E}[S^3] = \frac{6p_1}{\mu_1^3} + \frac{6p_2}{\mu_2^3},$$

then

$$Var(W) = \mathbb{E}[W^2] - \overline{W}^2.$$

$$F_T(t) = P[T \le t] = 1 - \pi_a e^{-\mu_a t} - \pi_b e^{-\mu_b t}, \quad t \ge 0,$$

where

$$\pi_a = C_1\frac{z_1}{z_1 - 1},$$

$$\pi_b = C_2\frac{z_2}{z_2 - 1},$$

$\mu_a = \lambda(z_1 - 1),$

and

$\mu_b = \lambda(z_2 - 1).$

$\overline{T} = \overline{W} + \overline{S}.$

$\mathbb{E}[T^2] = \mathbb{E}[W^2] + \frac{\mathbb{E}[S^2]}{1-a},$

where of course

$\mathbb{E}[S^2] = \frac{2p_1}{\mu_1^2} + \frac{2p_2}{\mu_2^2}.$

$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$

$C_T^2 = \frac{\mathbb{E}[T^2]}{\overline{T}^2} - 1.$

$\overline{Q} = \overline{\lambda} \cdot \overline{W} = \frac{a^2}{1-a}\left(\frac{1+C_S^2}{2}\right).$

$Var(Q) = \frac{\lambda^3 \mathbb{E}[S^3]}{3(1-a)} + \left(\frac{\lambda^2 \mathbb{E}[S^2]}{2(1-a)}\right)^2 + \frac{\lambda^2 \mathbb{E}[S^2]}{2(1-a)}.$

$\overline{N} = \lambda\overline{T} = \overline{Q} + a.$

$Var(N) = \frac{\lambda^3 \mathbb{E}[S^3]}{3(1-a)} + \left(\frac{\lambda^2 \mathbb{E}[S^2]}{2(1-a)}\right)^2 + \frac{\lambda^2(3-2a)\mathbb{E}[S^2]}{2(1-a)} + a(1-a).$

Table 16. M/Gamma/1 Queueing System

Since S has Gamma-distribution

$$\mathbb{E}[S^n] = \frac{\beta(\beta+1)\dots(\beta+n-1)}{\alpha^n}, \quad n = 1, 2, \dots.$$

Since

$$C_S^2 = \frac{1}{\beta},$$

so

$$\mathbb{E}[S^2] = \overline{S}^2(1 + C_S^2),$$

$$\mathbb{E}[S^3] = \overline{S}^3(1 + C_S^2)(1 + 2C_S^2),$$

and

$$\mathbb{E}[s^n] = \overline{S}^n \prod_{k=1}^{n-1}(1 + kC_S^2), \quad n = 1, 2, \dots.$$

This time

$$\overline{W} = \frac{\lambda\mathbb{E}[S^2]}{2(1-a)} = \frac{a\overline{S}}{1-a}\left(\frac{1+C_S^2}{2}\right),$$

$$\overline{Q} = \overline{\lambda} \cdot \overline{W},$$

$$Var(Q) = \frac{a^2(1+C_S^2)}{2(1-a)}\left[1 + \frac{a^2(1+C_S^2)}{2(1-a)} + \frac{2a(1+2C_S^2)}{3}\right],$$

$$\mathbb{E}[W|W>0] = \frac{\overline{S}}{1-a}\left(\frac{1+C_S^2}{2}\right),$$

$$\mathbb{E}[W^2] = 2\overline{W}^2 + \frac{a\overline{S}^2(1+C_S^2)(1+2C_S^2)}{3(1-a)},$$

$$Var(W) = \mathbb{E}[W^2] - \overline{W}^2,$$

$$\overline{T} = \overline{W} + \overline{S}, \qquad \overline{N} = \overline{\lambda} \cdot \overline{T} = \overline{Q} + a,$$

$$Var(N) = \frac{a^3(1+C_S^2)(1+2C_S^2)}{3(1-a)} + \left(\frac{a^2(1+C_S^2)}{2(1-a)}\right)^2 + \frac{a^2(3-2a)(1+C_S^2)}{2(1-a)} + a(1-a).$$

$$\mathbb{E}[T^2] = \mathbb{E}[W^2] + \frac{\overline{S}^2(1+C_S^2)}{1-a}.$$

$$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$$

$$\pi_T[90] \approx \overline{T} + 1.3\mathbb{D}(T), \quad \pi_T[95] \approx \overline{T} + 2\mathbb{D}(T).$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MGamma1/MGamma1.html

Mivel S Since S has Erlang -k distribution, hence

$$\mathbb{E}[S^n] = \left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\ldots\left(1+\frac{n-1}{k}\right)\overline{S}^n, \quad n=1,2,\ldots.$$

so

$$\mathbb{E}[S^2] = \overline{S}^2\left(1+\frac{1}{k}\right),$$

and

$$\mathbb{E}[s^3] = \overline{S}^3\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right).$$

This time

$$\overline{W} = \frac{\lambda\mathbb{E}[s^2]}{2(1-a)} = \frac{a\overline{S}}{1-a}\left(\frac{1+\frac{1}{k}}{2}\right). \text{ (Pollaczek's formula)}$$

$$\overline{Q} = \overline{\lambda}\cdot\overline{W}.$$

$$Var(Q) = \frac{a^2(1+k)}{2k(1-a)}\left[1+\frac{a^2(1+k)}{2k(1-a)}+\frac{2a(k+2)}{3k}\right].$$

$$\mathbb{E}[W|W>0] = \frac{\overline{S}}{1-a}\left(\frac{1+\frac{1}{k}}{2}\right).$$

$$\mathbb{E}[W^2] = 2\overline{W}^2 + \frac{a\overline{S}^2(k+1)(k+2)}{3k^2(1-a)}.$$

$$Var(W) = \mathbb{E}[W^2] - \overline{W}^2.$$

$$\overline{T} = \overline{W}+\overline{S}, \qquad \overline{N} = \overline{\lambda}\cdot\overline{T} = \overline{Q}+a.$$

$$Var(N) = \frac{a^3(k+1)(k+2)}{3k^2(1-a)} + \left(\frac{a^2(1+\frac{1}{k})}{2(1-a)}\right)^2 + \frac{a^2(3-2a)(1+\frac{1}{k})}{2(1-a)} + a(1-a).$$

$$\mathbb{E}[T^2] = \mathbb{E}[W^2] + \frac{\overline{S}^2(1+\frac{1}{k})}{1-a}.$$

$$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$$

$$\pi_T[90] \approx \overline{T} + 1.3\mathbb{D}(T), \quad \pi_T[95] \approx \overline{T} + 2\mathbb{D}(T).$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MEk1/MEk1.html

Since S has a constant distribution

$$\mathbb{E}[S^n] = \overline{S}^n, \quad n = 1, 2, \ldots,$$

so

$$G_N(z) = \frac{(1-a)(1-z)}{1 - ze^{a(1-z)}}.$$

We suppose that

$$|ze^{a(1-z)}| < 1,$$

we can expand $G_N(z)$ in the geometric series

$$g_N(z) = (1-a)(1-z)\sum_{j=0}^{\infty} \left[ze^{a(1-z)}\right]^j.$$

This thime we can show that,

$$P_1 = (1-a)(e^a - 1),$$

and

$$P_n = (1-a)\sum_{j=1}^{n} \frac{(-1)^{n-j}(ja)^{n-j-1}(ja+n-j)e^{ja}}{(n-j)!} \quad n = 2, 3, \ldots.$$

Additionally

$$F_T(t) = \sum_{n=0}^{k-1} P_n + P_k \left(\frac{t - (k-1)\overline{S}}{\overline{S}} \right),$$

where $(k-1)\overline{S} \leq t \leq k\overline{S}, \quad k = 1, 2, \ldots.$

So,

$\overline{W}[0] = P_0.$

$\overline{W} = \dfrac{a\overline{S}}{2(1-a)}.$

$F_W[W|W>0] = \dfrac{\overline{S}}{2(1-a)}.$

$\mathbb{E}[W^2] = 2\overline{W}^2 + \dfrac{a\overline{S}^2}{3(1-a)}.$

$Var(W) = \mathbb{E}[W^2] - \overline{W}^2.$

$\overline{Q} = \lambda\overline{W} = \dfrac{a^2}{2(1-a)}.$

$Var(Q) = \dfrac{a^3}{3(1-a)} + \left[\dfrac{a^2}{2(1-a)}\right]^2 + \dfrac{a^2}{2(1-a)}.$

$$F_T(t) = \begin{cases} 0 & if \quad t < \overline{S}, \\ \sum\limits_{n=0}^{k-1} P_n + P_k\left(\frac{t-k\overline{S}}{\overline{S}}\right) & if \quad t \geq \overline{S}. \end{cases}$$

where

$k\overline{S} \leq t < (k+1)\overline{S}, \quad k = 1, 2, \ldots.$

$\overline{T} = \overline{W} + \overline{S}.$

$\mathbb{E}[T^2] = \mathbb{E}[W^2] + \dfrac{\overline{S}^2}{1-a}.$

$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$

$\overline{N} = \overline{\lambda} \cdot \overline{T} = \overline{Q} + a.$

$Var(N) = \dfrac{a^3}{3(1-a)} + \left(\dfrac{a^2}{2(1-a)}\right)^2 + \dfrac{a^2(3-2a)}{2(1-a)} + a(1-a).$

7.13 GI/M/1 Formulas

Table 19. GI/M/1 Queueing System

The steady-state probability that an arriving customer will find the system empty, is the unique solution of the equation $1-\Pi_0 = A^*[\mu\Pi_0]$ such that $0 < \Pi_0 < 1$, where $A^*[\Theta]$ is the Laplace-Stieltjes transform of r. The steady-state number of customers in the system, N has the distribution $\{P_n\}$, where $P_0 = P[N = 0] = 1 - a,\ P_n = a\Pi_0(1 - \Pi_0)^{n-1},\ n = 1, 2, \ldots,$ tovďż˝bbďż˝

$\overline{N} = \dfrac{a}{\Pi_0}$, and $Var(N) = \frac{a(2-\Pi_0-a)}{\Pi_0^2}$.

$\overline{Q} = \dfrac{(1-\Pi_0)a}{\Pi_0}$.

$Var(Q) = \dfrac{a(1-\Pi_0)(2-\Pi_0-a(1-\Pi_0))}{\Pi_0^2}$.

$\mathbb{E}[Q|Q>0] = \dfrac{1}{\Pi_0}$.

$\overline{T} = \dfrac{\overline{S}}{\Pi_0}$.

$F_T(t) = P[T \le t] = 1 - \exp(-t/\overline{T})$.

$\Pi_T[r] = \overline{T} \ln\left[\dfrac{100}{100-r}\right]$.

$\Pi_T[90] = \overline{T}\ln 10, \quad \Pi_T[95] = \overline{T}\ln 20$.

$\overline{W} = (1-\Pi_0)\dfrac{\overline{S}}{\Pi_0}$.

$Var(W) = (1-\Pi_0^2)\left(\dfrac{\overline{S}}{\Pi_0}\right)^2$.

$F_T(t) = P[W \le t] = 1 - (1-\Pi_0)\exp(-t/\overline{T})$.

$\Pi_W[r] = \max\left\{0, \overline{T}\ln\left(\dfrac{100(1-\Pi_0)}{100-r}\right)\right\}$.

W', the queueing time for those who must, has the same distribution as T.

Table 20. Π_0 versus a for GI/M/1 Queueing System [1]

a	E_2	E_3	U	D	H_2	H_2
0.100	0.970820	0.987344	0.947214	0.999955	0.815535	0.810575
0.200	0.906226	0.940970	0.887316	0.993023	0.662348	0.624404
0.300	0.821954	0.868115	0.817247	0.959118	0.536805	0.444949
0.400	0.724695	0.776051	0.734687	0.892645	0.432456	0.281265
0.500	0.618034	0.669467	0.639232	0.796812	0.343070	0.154303
0.600	0.504159	0.551451	0.531597	0.675757	0.263941	0.081265
0.700	0.384523	0.626137	0.412839	0.533004	0.191856	0.044949
0.800	0.260147	0.289066	0.284028	0.371370	0.124695	0.024404
0.900	0.131782	0.147390	0.146133	0.193100	0.061057	0.010495
0.950	0.066288	0.074362	0.074048	0.098305	0.030252	0.004999
0.980	0.026607	0.029899	0.029849	0.039732	0.012039	0.001941
0.999	0.001333	0.001500	0.001500	0.001999	0.000600	0.000095

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/GIM1/GIM1.html

[1] At the first H_2 distribution $p_1 = 0.4,\ \mu_1 = 0.5\lambda,\ \mu_2 = 3\lambda$. At the second H_2 distribution $p_1 = 0.024405,\ \mu_1 = 2p_1\lambda$, ďż˝s $\mu_2 = 2p_2\lambda$.

7.14 GI/M/c Formulas

Table 21. GI/M/c Queueing System

Let $\Pi_n, n = 0, 1, 2, \ldots$ be the steady state number of customers that an arriving customer finds in the system. Then

$$\Pi_n = \begin{cases} \sum_{i=n}^{c-1} (-1)^{i-n} \binom{i}{n} U_i, & n = 0, 1, \ldots, c-2, \\ \\ D\omega^{n-c}, & n = c-1, c, \ldots, \end{cases}$$

where ω is the unique solution of the equation $\omega = A^*[c\mu(1-\omega)]$ such that $0 < \omega < 1$, where $A^*[\theta]$ is the Laplas-Stieltjes transform of r,

$g_j = A^*[j\mu], \quad j = 1, 2, \ldots, c,$

$$C_j = \begin{cases} 1, & j = 0, \\ \prod_{i=1}^{j} \left(\frac{g_i}{1-g_i} \right), & j = 1, 2, \ldots, c, \end{cases}$$

$$D = \left[\frac{1}{1-\omega} + \sum_{j=1}^{c} \frac{\binom{c}{j}}{C_j(1-g_j)} \left(\frac{c(1-g_j)-j)}{c(1-\omega)-j} \right) \right]^{-1}$$

and

$$U_n = DC_n \sum_{j=n+1}^{c} \frac{\binom{c}{j}}{C_j(1-g_j)} \left(\frac{c(1-g_j)-j)}{c(1-\omega)-j} \right), \quad n = 0, 1, \ldots, c-1.$$

Table 21. GI/M/c Queueing System (continued)

$$F_T(t) = P[W \le t] = 1 - P[W > 0]e^{-c\mu(1-\omega)t},\ t \ge 0,$$

where

$$P[W > 0] = \frac{D}{1-\omega}. \quad \overline{W} = \frac{D\overline{S}}{c(1-\omega)^2}. \quad \mathbb{E}[W|W > 0] = \frac{\overline{S}}{c(1-\omega)}.$$

If $c(1-\omega) \neq 1$, then

$$F_T(t) = P[\omega \le t] = 1 + (G-1)e^{-\mu t} - Ge^{-c\mu(1-\omega)t}, \quad t \ge 0,$$

where

$$G = \frac{D}{(1-\omega)[1-c(1-\omega)]}.$$

When $c(1-\omega) = 1$, then

$$F_T(t) = P[\omega \le t] = 1 - \left[1 + \frac{D\mu t}{1-\omega}\right] e^{-\mu t}, \quad t \ge 0.$$

We also have

$$\overline{T} = \overline{W} + \overline{S}.$$

$$P_0 = 1 - \frac{\lambda\overline{S}}{c} - \lambda\overline{S}\sum_{j=1}^{c-1} \Pi_{j-1}\left(\frac{1}{j} - \frac{1}{c}\right).$$

$$P_n = \begin{cases} \frac{\lambda\overline{S}\Pi_{n-1}}{n,} & n = 1, 2, \ldots, c-1, \\ \frac{\lambda\overline{S}\Pi_{n-1}}{c,} & n = c, c+1, \ldots. \end{cases}$$

7.15 M/G/1 Priority queueing system

Table 22. M/G/1 Queueing System (classes, no priority)

There are n customer classes. Customers from class i arrive in a Poisson pattern with mean arrival rate $\lambda_i, i = 1, 2, \ldots, n$. Each class has its own general service time with $\mathbb{E}[S_i] = 1/\mu_i, \quad \mathbb{E}[S_i^2], \quad \mathbb{E}[S_i^3]$. All customers served on a FCFS basis with no consideration for class. The total arrival stream to the system has a Poisson arrival pattern with

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

The first three moments of service time arc given by

$$\overline{S} = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n],$$

$$\mathbb{E}[S^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^2],$$

and

$$\mathbb{E}[S^3] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^3] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^3] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^3],$$

By Pollaczek's formula,

$$\overline{W} = \frac{\lambda\mathbb{E}[S^2]}{2(1-a)}.$$

The mean time in the system for each class is given by

$$\overline{T}_i = \overline{W} + \mathbb{E}[S_i], \quad i = 1, 2, \ldots, n.$$

The overall mean customer time in the system,

$$\overline{T} = \frac{\lambda_1}{\lambda}\overline{T}_1 + \frac{\lambda_2}{\lambda}\overline{T}_2 + \ldots + \frac{\lambda_n}{\lambda}\overline{T}_n.$$

The variance of the waiting time

$$Var(W) = \frac{\lambda\mathbb{E}[S^3]}{3(1-a)} + \frac{\lambda^2(\mathbb{E}[S^2])^2}{4(1-a)^2}.$$

The variance of T is given by

$$Var(T_i) = Var(W) + Var(S_i), \quad i = 1, 2, \ldots, n.$$

The second moment of T by class is

$$\mathbb{E}[T_i^2] = Var(T_i) + \overline{T}_i^2, \quad i = 1, 2, \ldots, n.$$

Table 22. M/G/1 Queueing System (classes, no priority)
(continued)

Thus, the overall second moment of T is given by

$$\mathbb{E}[T^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[T_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[T_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[T_n^2],$$

and

$$Var(T) = \mathbb{E}[T^2] - \overline{T}^2.$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MG1NoPrio/MG1NoPrio.html

Table 23. M/G/1 Nonpreemptive (HOL) Queueing System

There are n priority classes with each class having a Poisson arrival pattern with mean arrival rate λ_i. Each customer has the same exponential service time requirement. Then the overall arrival pattern is Poiisson with mean:

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

The server utilization

$$\overline{S} = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n],$$

$$\mathbb{E}[S^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^2],$$

and

$$\mathbb{E}[S^3] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^3] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^3] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^3],$$

Let

$$\rho_j = \lambda_1\mathbb{E}[S_1] + \lambda_2\mathbb{E}[S_2] + \ldots + \lambda_j\mathbb{E}[S_j], \quad j = 1, 2, \ldots, n,$$

and notice that

$$\rho_n = \rho = \lambda\overline{S}.$$

The mean times in the queues:

$$\overline{W}_j = \mathbb{E}[W_j] = \frac{\lambda\mathbb{E}[S^2]}{2(1-\rho_{j-1})(1-\rho_j)},$$

$$j = 1, 2, \ldots, n, \quad \rho_0 = 0.$$

Table 23. M/G/1 Nonpreemptive (HOL) Queueing System
(continued)

The mean queue lengths are

$$\overline{Q}_j = \overline{\lambda}_j \cdot \overline{W}_j, \quad j = 1, 2, \ldots, n.$$

The unified time in the queue

$$\overline{W} = \frac{\lambda_1}{\lambda}\mathbb{E}[W_1] + \frac{\lambda_2}{\lambda}\mathbb{E}[W_2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[W_n].$$

The mean times of staying in the system

$$\overline{T}_j = \mathbb{E}[T_j] = \mathbb{E}[W_j] + \mathbb{E}[S_j], \quad j = 1, 2, \ldots, n,$$

and the average of the customers staying at the system is

$$\overline{N}_j = \overline{\lambda}_j \cdot \overline{T}_j, \quad j = 1, 2, \ldots, n.$$

The total time in the system

$$\overline{T} = \overline{W} + \overline{S}.$$

The total queue length

$$\overline{Q} = \overline{\lambda} \cdot \overline{W},$$

and the average of the customers staying at the system

$$\overline{N} = \overline{\lambda} \cdot \overline{T}.$$

The variance of the total time stayed in the system by class

$$Var(T_j) = Var(S_j) + \frac{\lambda\mathbb{E}[S^3]}{3(1-\rho_{j-1})^2(1-\rho_j)}$$
$$+\frac{\lambda\mathbb{E}[S^2]\left(2\sum_{i=1}^{j}\lambda_i\mathbb{E}[S_i^2] - \lambda\mathbb{E}[S^2]\right)}{4(1-\rho_{j-1})^2(1-\rho_j)^2}$$
$$+\frac{\lambda\mathbb{E}[S^2]\sum_{i=1}^{j-1}\lambda_i\mathbb{E}[S_i^2]}{2(1-\rho_{j-1})^3(1-\rho_j)}, \quad j = 1, 2, \ldots, n.$$

Table 23. M/G/1 Nonpreemptive (HOL) Queueing System
(continued)

The variance of the total time stayed in the system

$$Var(T) = \frac{\lambda_1}{\lambda}[Var(T_1) + \overline{T}_1^2] + \frac{\lambda_2}{\lambda}[Var(T_2) + \overline{T}_2^2]$$
$$+ \ldots + \frac{\lambda_n}{\lambda}[Var(T_n) + \overline{T}_n^2] - \overline{T}^2.$$

The variance of the waiting time by class

$$Var(W_j) = Var(T_j) - Var(S_j), \quad j = 1, 2, \ldots, n.$$

We know that $\mathbb{E}[W_j^2] = Var(W_j) + \overline{W}_j^2, \quad j = 1, 2, \ldots, n,$

so

$$\mathbb{E}[W^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[W_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[W_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[W_n^2].$$

Finally

$$Var(W) = \mathbb{E}[W^2] - \overline{W}^2.$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MG1Relativ/MG1Relativ.html

Table 24. M/G/1 absolute priority Queueing System

There are n customer classes. Class 1 customers receive the most favorable treatment; class n customers receive the least favorable treatment. Customers from class i arrive in a Poisson pattern with mean arrival rate λ_i,$t = 1, 2, \ldots, n$. Each class has its own general service time with $\mathbb{E}[S_i] = 1/\mu_i$, and finite second and third moments $\mathbb{E}[S_i^2]$, $\mathbb{E}[S_i^3]$. The priority system is preemptive resume, which means that if a customer of class j is receiving service when a customer of class $i < j$ arrives, the arriving customer preempts the server and the customer who was preempted returns to the head of the line for class j customers. The preempted customer resumes service at the point of interruption upon reentering the service facility. The total arrival stream to the system has a Poisson arrival pattern with

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

The first three moment of service time are given by:

$$\overline{S} = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n],$$

$$\mathbb{E}[S^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^2],$$

$$\mathbb{E}[S^3] = \frac{\lambda_1}{\lambda}\mathbb{E}[S_1^3] + \frac{\lambda_2}{\lambda}\mathbb{E}[S_2^3] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[S_n^3].$$

Let

$$\rho_j = \lambda_1\mathbb{E}[S_1] + \lambda_2\mathbb{E}[S_2] + \ldots + \lambda_j\mathbb{E}[S_j], \quad j = 1, 2, \ldots, n,$$

and notice that

$$\rho_n = \rho = \lambda\overline{S}.$$

The mean time in the system for each class is

$$\overline{T}_j = \mathbb{E}[T_j] = \frac{1}{1-\rho_{j-1}}\left[\mathbb{E}[S_j] + \frac{\sum_{i=1}^{j}\lambda_i\mathbb{E}[S_i^2]}{2(1-\rho_j)}\right],$$

$$\rho_0 = 0, \quad j = 1, 2, \ldots, n.$$

Table 24. M/G/1 absolute priority Queueing System

(continued)

Waiting times

$$\overline{W}_j = \mathbb{E}[T_j] - \mathbb{E}[S_j], \quad j = 1, 2, \ldots, n.$$

The mean length of the queue number j:

$$\overline{Q}_j = \lambda_j \overline{W}_j, \quad j = 1, 2, \ldots, n.$$

The total waiting time, $\overline{W}$, is given by:

$$\overline{W} = \frac{\lambda_1}{\lambda}\mathbb{E}[W_1] + \frac{\lambda_2}{\lambda}\mathbb{E}[W_2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[W_n].$$

The mean number of customers staying in the system for each class is

$$\overline{N}_j = \lambda_j \overline{W}_j, \quad j = 1, 2, \ldots, n.$$

The mean total time is

$$\overline{T} = \frac{\lambda_1}{\lambda}\overline{T}_1 + \frac{\lambda_2}{\lambda}\overline{T}_2 + \ldots + \frac{\lambda_n}{\lambda}\overline{T}_n = \overline{W} + \overline{S}.$$

The mean number of customers waiting in the queue is

$$\overline{Q} = \overline{\lambda} \cdot \overline{W},$$

and the average number of customers staying in the system

$$\overline{N} = \overline{\lambda} \cdot \overline{T}.$$

Table 24. M/G/1 absolute priority Queueing System
(continued)

The variance of the total time of staying in the system for each class is

$$Var(T_j) = \frac{Var(S_j)}{(1-\rho_{j-1})^2} + \frac{\mathbb{E}[S_j]\sum_{i=1}^{j-1}\lambda_i\mathbb{E}[S_i^2]}{(1-\rho_{j-1})^3}$$

$$+\frac{\sum_{i=1}^{j}\lambda_i\mathbb{E}[S_i^3]}{3(1-\rho_{j-1})^2(1-\rho_j)} + \frac{\left(\sum_{i=1}^{j}\lambda_i\mathbb{E}[S_i^2]\right)^2}{4(1-\rho_{j-1})^2(1-\rho_j)^2}$$

$$+\frac{\left(\sum_{i=1}^{j}\lambda_i\mathbb{E}[S_i^2]\right)\left(\sum_{i=1}^{j-1}\lambda_i\mathbb{E}[S_i^2]\right)}{2(1-\rho_{j-1})^3(1-\rho_j)}, \quad \rho_0 = 0, \quad j = 1,2,\ldots,n.$$

The overall variance

$$Var(T) = \frac{\lambda_1}{\lambda}[Var(T_1) + \overline{T}_1^2] + \frac{\lambda_2}{\lambda}[Var(T_2) + \overline{T}_2^2]$$

$$+\ldots+\frac{\lambda_n}{\lambda}[Var(T_n) + \overline{T}_n^2] - \overline{T}^2.$$

The variance of waiting times for each class is

$$Var(W_j) = Var(T_j) - Var(S_j), \quad j = 1,2,\ldots,n.$$

Because,

$$\mathbb{E}[W_j^2] = Var(W_j) + \overline{W}_j^2, \quad j = 1,2,\ldots,n,$$

so

$$\mathbb{E}[W^2] = \frac{\lambda_1}{\lambda}\mathbb{E}[W_1^2] + \frac{\lambda_2}{\lambda}\mathbb{E}[W_2^2] + \ldots + \frac{\lambda_n}{\lambda}\mathbb{E}[W_n^2].$$

Finally

$$Var(W) = \mathbb{E}[W^2] - \overline{W}^2.$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MG1Absolute/MG1Absolute.html

7.16 M/G/c Processor Sharing system

Table 25. M/G/1 processor sharing Queueing System

The Poisson arrival stream has an average arrival rate of λ and the average service rate is μ. The service time distribution is general with the restriction that its Laplace transform is rational, with the denominator having degree at least one higher than the numerator. Equivalently. the service time, s, is Coxian. The priority system is processor-sharing, which means that if a customer arrives when there are already $n-1$ customers in the system, the arriving customer (and all the others) receive service at the average rate μ/n. Then $P_n = \rho^n(1-\rho),\ n = 0, 1, \ldots,$ where $\rho = \lambda/\mu$. We also have

$$\overline{N} = \frac{\rho}{1-\rho}, \quad \mathbb{E}[T|S=t] = \frac{t}{1-\rho}, \text{ and } \overline{T} = \frac{\overline{S}}{1-\rho}.$$

Finally

$$\mathbb{E}[W|S=t] = \frac{\rho t}{1-\rho}, \text{ and } \overline{W} = \frac{\rho\overline{S}}{1-\rho}.$$

`http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MG1Process/MG1Process.html`

Table 26. M/G/c processor sharing Queueing System

The Poisson arrival stream has an average arrival rate of λ. The service time distribution is general with the restriction that its Laplace transform is rational, with the denominator having degree at least one higher than the numerator. Equivalently, the service time, s, is Coxian. The priority system is processor-sharing, which works as follows. When the number of customers in the service center, is less than c, then each customers is served simultaneously by one server; that is, each receives service at the rate μ. When $N > c$. each customer simultaneously receives service at the rate $c\mu/N$. We find that just as for the M/G/l processor-sharing system.

7.17 M/M/c Priority system

Table 27. M/M/c relative priority (HOL) Queueing System

There are n priority classes with each class having a Poisson arrival pattern with mean arrival rate λ_i. Each customer has the same exponential service time requirement. Then the overall arrival pattern is Poisson with mean $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. The server utilization

$$a = \frac{\lambda \overline{S}}{c} = \frac{\lambda}{c\mu},$$

$$\overline{W}_{W_1} = \frac{C[c,\rho]\overline{S}}{c(1-\lambda_1 \overline{S}/c)},$$

and thes equations are also true:

$$\overline{W}_j = \frac{C[c,\rho]\overline{S}}{c\left[1-\left(\overline{S}\sum_{i=1}^{j-1}\lambda_i\right)/c\right]\left[1-\left(\overline{S}\sum_{i=1}^{j}\lambda_i\right)/c\right]}, \quad j = 2,\ldots,n.$$

$$\overline{W}_j = \overline{W}_j + \overline{S}, \quad j = 1,2,\ldots,n.$$

$$\overline{Q}_j = \overline{\lambda}_j \cdot \overline{W}_j, \quad j = 1,2,\ldots,n.$$

$$\overline{N}_j = \overline{\lambda}_j \cdot \overline{T}_j, \quad j = 1,2,\ldots,n.$$

$$\overline{W} = \frac{\lambda_1}{\lambda} + \frac{\lambda_2}{\lambda} + \ldots + \frac{\lambda_n}{\lambda}.$$

$$\overline{Q} = \overline{\lambda} \cdot \overline{W}.$$

$$\overline{T} = \overline{W} + \overline{S}.$$

$$\overline{N} = \overline{\lambda} \cdot \overline{T}.$$

http://irh.inf.unideb.hu/user/jsztrik/education/03/EN/MMcPrio/MMcPrio.html

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