TEACHING MANUAL

APPLICATION OF OPERATIONS RESEARCH IN AGRICULTURE

Compiled by
Dr. K.D. Sharma
Principal Scientist

DEPARTMENT OF AGRICULTURAL ECONOMICS
EXTENSION EDUCATION & RURAL SOCIOLOGY
COLLEGE OF AGRICULTURE
CSK HP KRISHI VISHVAVIDYALAYA, PALAMPUR- 176 062

2016

PREFACE

Operations Research offers useful scientific methods/tools employed in planning and management of agribusiness. This teaching manual has been prepared keeping in view the need of the students and may serve as a help book to the postgraduate students of the discipline of Agricultural Economics. An attempt has been made to illustrate various techniques like linear programming, variable programming, integer programming, dynamic programming, non-linear programming, quadratic programming, application of MOTAD model, game theory, inventory management models, simulation models, Markov chain analysis and transportation problem. All these techniques have been explained meticulously in simple language with illustrative examples. Even beginner can understand these concepts with ease and apply in their research and development fields.

An attempt has been made to present the solutions of the examples in and easy to understand form. Another distinguishing feature of this manual is that different techniques/models have been explained step wise so that the students may comprehends the methodology and understand the technique.

This manual is based upon my teaching experience and application of operation research tools in the field of research. It is hoped that the teaching manual will meet the need of faculty and students. In a teaching manual of this mathematical nature, a few misprints or errors are likely to have crept in and I shall be grateful for such notice and any suggestion for improvements will be gratefully acknowledged.

I am thankful to Sh Ajay Kumar (SRF) and Sh. Vivek Sharma (Computer Assistant) who helped me in typing this teaching manual.

September 5, 2016 K.D. Sharma
CSKHPKV, Palampur Principal Scientist

CONTENTS

UNIT-I

OPERATIONS RESEARCH- BASIC CONCEPTS

Objective of this Course

The science of Operations Research (OR) was initially developed after World War-II as a operational decision making tool for military installations but today there is hardly any field left where operations research tools are not applied. It is a modern science that provides an insight into the application of optimization techniques. The decision makers are interested in maximizing returns or minimizing costs, to achieve economic efficiency. There are different techniques/models and operations research provides appropriate knowledge to apply these models in agricultural planning. These techniques take due care of the constraints/restrictions confronted in the real life situation in agricultural sector. In the present day of growing global complexities, the operations research models are widely used in planning and management to achieve organizational and economic goals. This manual will be useful for the students to understand the application of the tools of operations research in scientific management and planning of agriculture and allied activities.

Characteristics of Operation Research (OR)

1. OR is inter-disciplinary team approach to find out optimum solutions.
2. OR uses techniques of scientific research to arrive at optimum solution.
3. OR emphasizes on the overall approach to the system i.e. it takes into account all the aspects of the problem under consideration.
4. OR tries to optimize total system i.e. maximize profit/minimize cost.
5. OR gives only bad answer to the problems having worse answers otherwise. It cannot give perfect answers. It only improves the quality of the solution.

Scope and Usefulness of Operations Research

1. OR is useful to directing authority in deciding optimum allocation of various limited resources such as men, money, machines, materials, time etc.
2. It is useful to production manager particularly in;
   - Designing, selecting and locating sites.
   - To determine number and size.
   - Scheduling and sequencing.
   - Calculating optimum product-mix.
3. It is useful in marketing field such as;
   - How to buy, how often to buy, when & what to buy at the minimum possible cost.
   - To find out distribution points to sell products.
   - Minimum per unit sale price.
   - Customers' choice regarding colour, packaging, size of stock etc.
   - Choice in selection of different media of advertising.
4. It is useful to personnel administration;
   - To employ skilled persons at minimum cost.

- Total number of persons/labour to be maintained.
- Sequencing of persons to a variety of jobs to enhance efficiency.

5. It is useful to proper financial management of the unit;
   - To find out proper plan for a farm/firm/industry.
   - To determine optimum replacement policies.
   - To find out long term capital requirements as well as ways and means to generate capital.

Application of OR in Agriculture

Earlier, OR was mainly applied in management of industries or firms. But the technique of OR model was applied for the first time in agriculture by Heady and Candler in livestock. Today OR tools are being applied in agricultural research and planning in the following fields.

- Optimum cropping plans for a region.
- Finding out optimum product-mix of different farm enterprises.
- In finding out capital, credit and inputs requirement.
- In designing feed rations for livestock.
- In designing transportation and marketing strategies.
- In storage and inventory management.
- To analyze the impact of new technology.

OR finds more relevance in agricultural production economics due to following reasons

- It takes into account complex system interlinkages.
- It is free from estimation problems like auto-correlation, multi-colinearity, simultaneous equation bias, etc.
- Estimation procedure is simple with the application of computer programming.
- Based upon fewer assumptions and has more realistic and practicable solutions.
- It provides scope for incorporating changes and, thus, is more flexible in methodology.

Understanding Matrix Algebra and its Application in Operations Research

Matrix Algebra

The concept of matrices is most important in OR. Therefore, before proceeding further we should understand the basics of matrices.

Definition

A matrix is the arrangement of elements in rows and columns. A matrix also presents the system of equations of a model.

Importance of Matrices to OR Research

A complete development and understanding of the theoretical and computational aspects of OR requires the blending of the basic matrix concepts and techniques. Matrix concept is most important which forms the basis of analysis of system relationships between decision variables and parameters or constraints.

A matrix is a rectangular array of 'm n' numbers arranged in 'm' rows and 'n' columns, where m denotes number of rows and n the number of columns.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & \vdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{mn} \end{bmatrix} \quad [i + j] \text{ e.g } \quad A \quad \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 4 \\ 5 & 2 & 3 \end{bmatrix} \text{ is a (3x3) matrix}$$

Basic Concepts in Matrix Algebra

Square matrix

The matrix is a square matrix iff m = n i.e. number of rows is equal to number of columns.

Unit or identity matrix

Unit or identity matrix is one in which all diagonal elements are unity (1) and all other elements are zero (0).

$$\text{i.e. } \begin{bmatrix} a_{ij} & 1, \forall & i & j \\ a_{ij} & 0, \forall & i \neq j \end{bmatrix}$$

Row matrix

A row matrix is one in which there is one row of elements i.e.

$$\begin{bmatrix} 1 & 4 & 2 & 3 & 8 \end{bmatrix}_{1\times 5} \text{ is a row matrix.}$$

Column matrix

A column matrix is one having one column of elements i.e.

$$\begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}_{4\times 1} \quad \text{is a column matrix.}$$

Transpose of a matrix

The transpose of a matrix is determined by interchanging rows and columns.

A = $[a_{ij}]$, then $A' or A^T \quad [a_{ji}]$

e.g. $A \quad \begin{bmatrix} 1 & 0 & 4 & 3 \\ 2 & 1 & 8 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 5 & 8 \end{bmatrix} \qquad A' \quad \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 4 & 1 \\ 4 & 8 & 3 & 5 \\ 3 & 1 & 2 & 8 \end{bmatrix}$

Triangular matrix

(a) Upper triangular matrix: If all $a_{ij} \quad 0, \quad \forall \quad i > j$ e.g. $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

(b) Lower triangular matrix: If all $a_{ij} \quad 0, \quad \forall \quad i < j$ e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$

Symmetric matrix

A symmetric matrix is one in which the element a_{ij} is same as element $a_{ji.}$ This implies that the element at 1st row and 3rd column will be same as that at 3rd row and Ist column. In the following example matrix A is a symmetric matrix.

$A \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 2 \end{bmatrix} \qquad A' \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 2 \end{bmatrix}$

i.e. $a_{ij} = a_{ji.}$ Obviously, a matrix is a symmetric matrix iff, $A = A^T$

Skew symmetric matrix

A skew symmetric matrix is one in which $\lfloor a_{ij} \rfloor \quad -\lfloor a_{ji} \rfloor$ in this case $A \quad -A'$

$A \quad \begin{bmatrix} 1 & -2 & -3 \\ 2 & 4 & -6 \\ 3 & 6 & 2 \end{bmatrix}, \qquad A' \quad \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 6 \\ -3 & -6 & 2 \end{bmatrix}$ i.e. $a_{ij} = - a_{ji}$

Null matrix

A null matrix has all zero elements

Main Properties of Matrices

Addition of matrices

(A+B) + C = A + (B+C)	Associative law
A + B = B + A	Commutative law
(α + β)A = αA + βA	Distributive law

$\alpha(A+B) \quad \alpha A+\alpha B$

Multiplication of matrices

Multiplication of matrices A and B is possible only iff number of columns in A are equal to number of rows in B. For example;

$$A \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$[A]_{2\times 3} * [B]_{3\times 2} \quad [C]_{2\times 2}$$

e.g. if $A \quad \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}_{2X3} \quad and \quad B \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 6 \end{bmatrix}_{3X2} \quad then \quad AB \quad \begin{bmatrix} 13 & 10 \\ 8 & 3 \end{bmatrix}_{2X2}$

Inverse of matrix

The inverse of a matrix A is A^{-1}, iff;

$A * A^{-1} \quad I(identity)$

Determinant

Determinant (det.) is associated with every square matrix and is single value derived through computation procedure as under;

$$\begin{vmatrix} 2 & 4 & 1 \\ 0 & 1 & 4 \\ 5 & 2 & 3 \end{vmatrix}$$

$$2(3-8)-4(0-20)+1(0-5)$$

$$-10+80-5 \quad 65$$

Some properties of determinant are;

1. If every element of any row or any column is zero $|\det|$ is zero.
2. Value of det. does not change if corresponding rows and columns are changed.
3. If $|B|$ det. is formed by interchanging two rows or columns in $|A|$ then $|B|$ = - $|A|$
4. If two rows or columns of a det. are identical than $|\det|$ has zero value.
5. If every element of a row or column of det. is multiplied by the number k then value of $|\det|$ is k times.
6. Value of $|\det|$ does not change if to every element of a column or a row we add k times the corresponding element of another column or row.

Rank of a Matrix

Rank of a matrix A is the order of the largest square array in A whose $|\det|$ does not vanish i.e. does not become zero (0)

A square matrix is called singular if det. $|A|$ = 0.

Minor: d_{ij} of the element a_{ij} is the det. obtained from the square matrix by striking i^{th} row & j^{th} column of the matrix.

Co-factor: D_{ij} of the element a_{ij} is a minor with appropriate sign which is determined by using the relation $(-1)^{i+j}$ d_{ij}. It is also known as signed minor of the element a_{ij}.

Adjoint: Adjoint of an $n \times n$ square matrix A is another $n \times n$ square matrix Adj A where i^{th} row and j^{th} column is the co-factor of the element in the j^{th} row and i^{th} column of A.

$$A \begin{bmatrix} 1 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A' \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$Adj \quad A \begin{bmatrix} 0 & -2 & 2 \\ 3 & 2 & 1 \\ -3 & 0 & 3 \end{bmatrix}$$

A = (a_{ij}), then transpose of a matrix is A' = (a_{ji}).

Then we can find inverse of A matrix as $A^{-1} \quad \frac{1}{|A|} Adj \quad A$

Where $|A|$ = determinant value of A.

Use of Matrix Algebra in Solving Linear Equations

Given the set of equations;

$$\begin{array}{ccccccccc} a_{11}X_1 & + & a_{12}X_2 & + & a_{13}X_3 & \cdots & \cdots & a_{1n}X_n & b_1 \\ a_{21}X_1 & + & a_{22}X_2 & + & a_{23}X_3 & \cdots & \cdots & a_{2n}X_n & b_2 \\ \vdots & & \vdots & & \vdots & & & \vdots & \vdots \\ \vdots & & \vdots & & \vdots & & & \vdots & \vdots \\ a_{m1}X_1 & + & a_{m2}X_2 & + & a_{m3}X_3 & \cdots & \cdots & a_{mn}X_n & b_m \end{array}$$

We can write in matrix notation as

$$AX \quad B$$

$$\text{where } A \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad X \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ \vdots \\ X_n \end{bmatrix}_{m\times 1}, \quad B \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}_{m\times 1}$$

The solution is $X \quad A^{-1}B$, so if we can find out the inverse of A, we can find solution to the equations by multiplying this inverse with matrix B.

Example: Let us consider the simultaneous equation model,

$$3X_1 + 4X_2 + 3X_3 \quad 4$$
$$X_1 + 2X_2 + 5X_3 \quad 10$$
$$X_1 + 3X_2 + X_3 \quad 6$$

We can write these equations in matrix form as;

AX = B

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}$$

where;

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix}; \qquad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}$$

There are two methods for solving equations with the use of matrices

(a) Adjoint Method

Let us consider the equations

$$X_1 \quad + 4X_2 + X_3 \quad 6$$
$$-2X_1 + X_2 + X_3 \quad 4$$
$$X_1 \quad + X_2 + X_3 \quad 10$$

We write in the form **AX = B.**

X = A^{-1}B.

$$A^{-1} = \frac{Adjoint\ A}{\det.|A|}$$

$$Adj \quad A \quad \begin{bmatrix} 0 & -2 & 2 \\ 3 & 2 & 1 \\ -3 & 0 & 3 \end{bmatrix}, \; |A| \quad 6 \;\; \textbf{;} \text{ so } A^{-1} = \frac{Adj\ A}{|A|} \quad \begin{bmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Hence, the solution is **X = A^{-1}B;** $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \begin{bmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$

$$X_1 \quad -\frac{4}{3} + \frac{10}{3} \quad \frac{6}{3} \quad 2$$

$$X_2 \quad 3 + \frac{4}{3} + \frac{10}{6} \quad 6$$

$$X_3 \quad -3 + 5 \quad 2$$

(The student can cross check by putting these values in any of the equations)

Assignment

Solve the following system of equations by using adjoint method

$$4X_1 - 2X_2 + X_3 \quad 16$$

$$3X_1 + X_2 - X_3 \quad 8$$

$$X_1 + X_2 + X_3 \quad 20$$

(b) Application of Cramer's Rule to solve Equations

The system of equations can be solved by using the Crammer's rule

$$\begin{bmatrix} 2x_1 + & 3x_2 + & 6x_3 \\ 1x_1 + & 4x_2 + & 8x_3 \\ 4x_1 + & 2x_2 + & 8x_3 \end{bmatrix} \quad \begin{matrix} 20 \\ 45 \\ 15 \end{matrix}$$

We can write these equations in matrix form as;

AX = B

$$\begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & 8 \\ 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} 20 \\ 45 \\ 15 \end{bmatrix}$$

Det. Δ (det. of matrix **A)** = 20

Det. Δ_1 (det. by replacing first column with **B**) =140

Det. Δ_2 (det. by replacing second column with **B**) = 110

Det. Δ_3 (det. by replacing third column with **B**) = -35

x_1= Det Δ_1/ Det Δ =7

x_2= Det Δ_2/ Det Δ =5.5

x_3=- Det Δ_3/ Det Δ =-1.75

(Cross check by putting these values in any of the equations)

Assignment

Solve these equations by using Crammer's rule:

Equation set 1

$$\begin{bmatrix} x_1 + & 3x_2 + & 6x_3 \\ 2x_1 + & 3x_2 + & 7x_3 \\ 4x_1 + & 2x_2 + & 8x_3 \end{bmatrix} \begin{matrix} 10 \\ 5 \\ 8 \end{matrix}$$

Equation set 2

$$\begin{bmatrix} 2x_1 + & 6x_2 + & 5x_3 \\ 2x_1 + & 3x_2 + & 4x_3 \\ 4x_1 + & 2x_2 + & 5x_3 \end{bmatrix} \begin{matrix} 20 \\ 10 \\ 16 \end{matrix}$$

UNIT-II

FORMULATION OF LINEAR PROGRAMMING PROBLEM

Assumptions of Linear Programming Problem (LPP)

1 Optimization: Objective function to maximize or minimize

2. Fixedness: At least one constraint with RHS different from 0

3. Finiteness: A finite number of activities and constraints to consider

4. Determinism: All parameters are assumed to be known constants

5. Continuity: All resources can be used and all activities produced in any fraction

6. Homogeneity: All units of the same resource or activity are identical

7. Additivity: When two or more activities are used, the total product is equal to the sum of the individual products (no interaction effects between activities)

8. Proportionality: Constant gross margin and resource requirement per unit of activity regardless of the level of activity used (constant returns to scale)

General Problem

The general linear programming problem (LPP), is to find out vector of variables (X_1, X_2,……..X_n) which maximizes or minimizes the linear objective function:

$$C_1X_1 + C_2X_2 + \cdots\cdots C_nX_n$$

Such that linear constraints are satisfied

$$\begin{array}{ccccccccc}
a_{11}X_1 & + & a_{12}X_2 & + & a_{13}X_3 & \cdots & \cdots & a_{1n}X_n > & < \; b_1 \\
a_{21}X_1 & + & a_{22}X_2 & + & a_{23}X_3 & \cdots & \cdots & a_{2n}X_n > & < \; b_2 \\
\vdots & & \vdots & & \vdots & & & \vdots & \vdots \\
\vdots & & \vdots & & \vdots & & & \vdots & \vdots \\
a_{m1}X_1 & + & a_{m2}X_2 & + & a_{m3}X_3 & \cdots & \cdots & a_{mn}X_n > & < \; b_m
\end{array}$$

In brief, the model can be written as;

Max/Min $Z_j \quad \sum_{j\ 1}^{n} C_jX_j$ (j = 1, 2, 3,……….n) Objective function

Subject to

$$\sum_{j\ 1}^{n} a_{ij}X_j >=< b_i,\ i \quad 1,2,3,\cdots m \qquad \text{Resource constraints}$$

$$X_j \geq 0 \qquad\qquad \text{Non-negativity constraint}$$

Some Definitions

1. Feasible solution to LPP is a vector $X(X_1,X_2,\ldots\ldots X_n)$ which satisfies the condition 1 and 2, i.e.

$$1 \quad \sum a_{ij} X_j >=< b_i (i \quad 1,2,3,\cdots m),(j \quad 1,2,3,\cdots n)$$

$$2 \quad X_j \geq 0$$

2. A basic solution to (1) is a solution obtained by setting 'n -m' variables equal to zero (0) and solving for the remaining m variables provided that determinant of the coefficients is non-zero and m variables are non-zero. The 'm' variables are called basic variables (X_B).
3. A basic feasible solution is a basic solution which satisfies non-negativity assumption beside restrictions imposed i.e. satisfies conditions (1) and (2) stated above.
4. Non-degenerate basic feasible solution is a basic feasible solution with exactly m positive X_j i.e. all basic variables are positive.
5. Degenerate solution occurs if one or more of the basic variables vanish or become zero.

Graphical Solution to LPP

Simple Linear Programming problem having two variables can be solved by the Graphical approach**.** In this method, we consider a set of two variables and find the feasible zone by plotting the constraints on the two axes. The values of a variable (say x_1) are taken on X-axis while the values of other variable (say x_2) are plotted on y-axis. Thus, we find concave (in case of maximization) or convex feasible zone (in case of minimization). We can find the optimum point by plotting budget/ price line and its tangency point with the co-ordinates of the vertices of points formed by the constraints.

Example 1: Find the maximum value of

Max $Z= 2X_1+3X_2$

Subject to

$2X_1+X_2<=8$

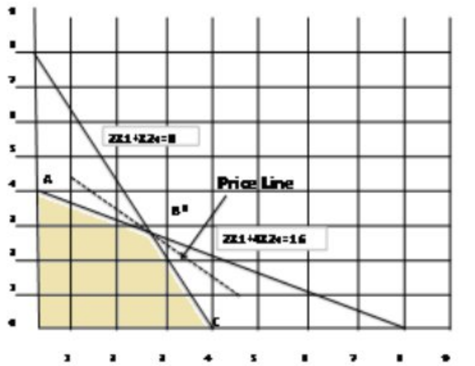
$2X_1+4X_2<=16$

$X_1, X_2>=0$

We can plot the graph of the constraint equations and find the feasible zone as shown in the shaded area.

With help of objective function (the price line), the optimum is at point B

$X_1=2.67$ $\quad X_2=2.67$ $\quad Z_{Max}=13.33$



Example 2: Find the maximum value of

$$Max\ Z = 4X_1 + 3X_2$$

Subject to

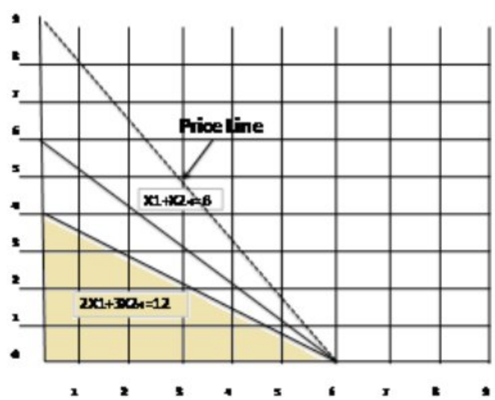
$$X_1 + X_2 <= 6$$

$$2X_1 + 3X_2 <= 12$$

$$X_1, X_2 >= 0$$

We can plot the graph of the constraint equations and find the feasible zone as shown in the shaded area of the graph. With help of objective function (the price line), the optimum point is;

$X_1=6$ $X_2=0$ $Z_{Max}=24$



Example 3: Find the maximum value of

$$Max\ Z = 4X_1 + 3X_2$$

Subject to

$$X_1 + X_2 <= 6$$

$2X_1+3X_2<=12$

$X_1, X_2>=0$

By ploting the graph of the constraint equations, the optimum point is ;

$X_1=4$ $X_2=0$ $Z_{Max}=16$

Example 4: Find the maximum value of

Max $Z= 5X_1+4X_2$

Subject to

$X_1 <=6$

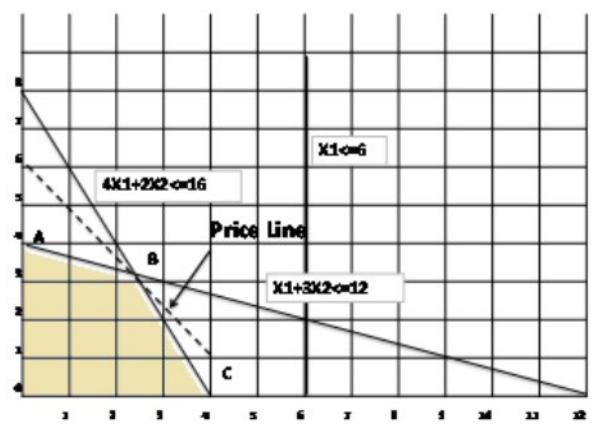
$X_1+3X_2 <=12$

$4X_1+2X_2 <=16$

$X_1, X_2 >=0$

We can plot the graph of the constraint equations and find the feasible zone as shown in the shaded area. With help of objective function (the price line), the optimum point is at B

$X_1=2.40$ $X_2=3.20$ $Z_{Max}=24.80$



Example 5: Find the maximum value of

Max $Z= 2X_1+3X_2$

Subject to

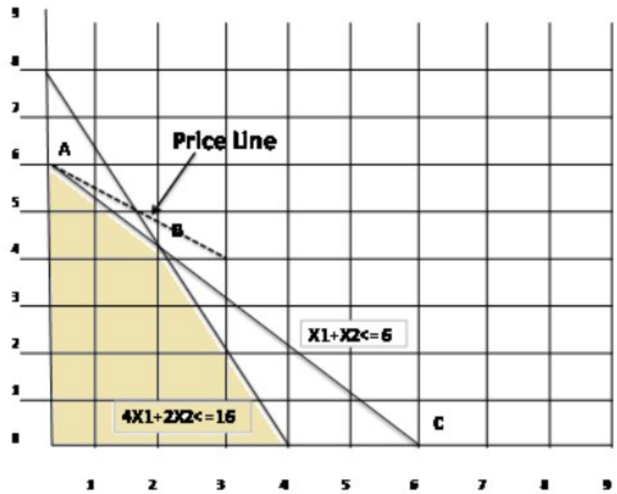
$X_1+X_2 < = 6$

$4X_1+2X_2 < =16$

$X_1, X_2 > = 0$

By plotting the graph of the constraint equations the optimum point is at A

$X_1=0$ $X_2=6$ $Z_{Max}=18$



Example 6: Find the minimum value of

$$\text{Min } Z = 2X_1 + 3X_2$$

Subject to

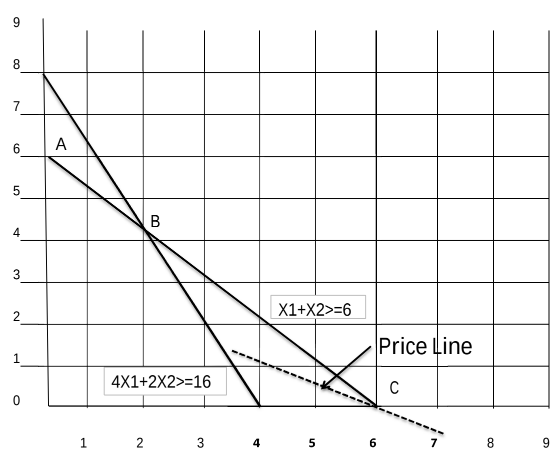
$$X_1 + X_2 >= 6$$

$$4X_1 + 2X_2 >= 16$$

$$X_1, X_2 >= 0$$

Solution: Optimum is at point C

$X_1=6$ $X_2=0$ $Z_{Min}=12$



Formulations of Linear Programming Problems

Diet Problem

The medical expert prescribes that an adult should consume at least 75 gms of proteins, 85 gms of fats and 300 gms of carbohydrate daily. The following table gives the food items and their nutrition values and cost. The total quantity should not exceed 750 gms of food stuff. Formulate the linear problem.

Food Type	Per 100 gram			Cost per kg
	Proteins	Fats	Carbohydrates	
1	8	2	35	10
2	18	15	-	30
3	16	4	7	40
4	4	20	3	20
5	5	8	40	15
6	3	-	25	30
Min	75	80	300	

Problem formulation

Objective is to minimize cost of food i.e

$$Min\,Y \quad 10X_1 + 30X_2 + 40X_3 + 20X_4 + 15X_5 + 30X_6$$

$$subject \quad to$$

$$8X_1 + 18X_2 + 16X_3 + 4X_4 + 5X_5 + 3X_6 \geq 75$$

$$2X_1 + 15X_2 + 4X_3 + 20X_4 + 8X_5 + 0X_6 \geq 80$$

$$35X_1 + 0X_2 + 7X_3 + 3X_4 + 40X_5 + 25X_6 \geq 300$$

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \leq 750$$

$$X_1, X_2, \cdots\cdots X_6 \geq 0$$

Feed Problem

A feed processing company purchases and mixes 3 types of grains each containing different amounts of 3 nutritional elements A,B and C with given cost and minimum quantity requirement.

Per unit weight of				Min. Requirement
Items	Grain 1	Grain 2	Grain 3	
A	2	4	6	125
B	0	2	5	24
C	5	1	3	80
Cost	25	15	18	

Problem Formulation

$$Min\ Z \quad 25X_1 + 15X_2 + 18X_3$$

subject to

$$2X_1 + 4X_2 + 6X_3 \geq 125$$
$$2X_2 + 5X_3 \geq 24$$
$$5X_1 + X_2 + 3X_3 \geq 80$$

X1, X2, X3 >= 0

Regional Planning Problem

CSKHPKV has farm at 3 locations of comparable productivity. The output of each farm is limited by cultivated area and by amount of water for irrigation. Data for the upcoming season are;

Farm	Cultivated area (Ha)	Water available (mm)
1	400	18000
2	200	11000
3	300	9000

There are 3 crops maize, rice and soybean, the average water required & profit is given below

	Water required (mm/ha)	Profit Rs./ha
Maize	100	8000
Rice	200	10000
Soybean	80	6000

Problem Formulation

Let i[th] farm & j[th] crop can be denoted by

$$X_{ij} \quad (i \quad 1,2,3, j \quad A,B,C)$$

Objective function

$$Max\,Z \quad 8000\sum_{i\ 1}^{3} X_{iA} + 10000\sum_{i\ 1}^{3} X_{iB} + 6000\sum_{i\ 1}^{3} X_{iC}$$

Subject to

Cultivable area

$$X_1A + X_1B + X_1C \leq 400$$
$$X_2A + X_2B + X_2C \leq 200$$
$$X_3A + X_3B + X_3C \leq 300$$

Water requirement

$$100X_1A + 200X_1B + 80X_1C \leq 18000$$
$$100X_2A + 200X_2B + 80X_2C \leq 11000$$
$$100X_3A + 200X_3B + 80X_3C \leq 9000$$
$$X_1 + X_2 + X_3 >= 0$$

Note: The typical farming linear programming problem has been formulated in appendix-I

UNIT-III

SIMPLEX METHOD

Standard Format of Linear Programming Simplex Tableau

$C_{j's}$ Cost/Returns Vector					θ ratio
Constraints ($b_{i's}$)	Activities ($X_{j's}$)			Artificial activities (in case of surplus disposal or equality)	
	Real activities (crops, livestock, etc.	Intermediate activities (purchases, borrowing, hiring, sales etc.)	Disposal activities (slack, surplus)		
	Input output (a_{ij}) coefficients				
Z_j (Profit)					
$Z_j - C_j$ (Net evaluation row					

Note: These have been described in detail in theory part

The simplex method is an iterative procedure which solves a LPP in a finite number of steps. These steps are described below:

Step1: Formulate the objective function and check whether it is to be minimized or maximized along with cost (minimization) or returns (maximization) associated with each activity.

Step 2: Check the input-output relationship and constraints.

Step 3: Convert all inequalities into equation by introducing slack variables (< inequality) and surplus variables (> inequality). The cost vector associated with slack or surplus is zero. Also introduce artificial variable in case of equality (=) constraint or for ≥ inequality in case of minimization with very high cost M (big M method).

Step 4: Obtain the initial basic feasible tableau and compute net evaluation row z_j-c_j and select the most negative z_j-c_j (in case of maximization) or most positive z_j-c_j (in case of minimization) as the incoming row.

Step 5: Compute the ratio Xbi/ aij, aij>0 for the incoming row and select the one with minimum positive ratio as the pivot element and the outgoing row.

Step 6: Convert the pivot element to unity and all other element in this column to zero and follow the same operation in other columns of the simplex table.

Step 7: Go to step 4 and repeat the computational procedure until all the z_j-c_j are positive or zero in case of maximization and negative or zero in case of minimization, in minimization see that all the artificial variables are out of the basis.

(Note: The minimization problem can also be solved by converting into a maximization problem)

Here we illustrate the formulation of simplex table to solve the LPP

Example 1: Solve the following problem by using simplex method

$$MaxZ \quad X_1 + X_2 + 3X_3$$
$$subject\ to$$
$$3X_1 + 2X_2 + X_3 \le 3$$
$$2X_1 + X_2 + 2X_3 \le 2$$
$$X_1, X_2, X_3 \ge 0$$

Now introducting slack variables *in these equations* to restore equality

$$\text{Max Z} \quad X_1 + X_2 + 3X_3 + 0S_1 + 0S_2$$
$$subject \quad to$$
$$3X_1 + 2X_2 + X_3 + S_1 \quad 3$$
$$2X_1 + X_2 + 2X_3 + S_2 \quad 2$$

Initial Simplex Tableau

	Cj		1	1	3	0	0	
C_B	X_B	b	X_1	X_2	X_3 ↓	S_1	S_2	Min R (b/Xj*)
0	S_1	3	3	2	1	1	0	3
0 ←	S_2	2	2	1	2^*	0	1	1
Z		0	0	0	0	0	0	
Z_j-C_j			-1	-1	-3	0	0	

Since, most negative Zj-Cj is -3 hence, X_3 will enter the basis. Further, min ratio R (bi/aij) is min for 2nd row, hence, S_2 will leave the basis.

First iteration

	Cj		1	1	3	0	0	
C_B	X_B	b	X_1	X_2	X_3	S_1	S_2	Min R
0	S_1	2	2	1.5	0	1	-0.5	3
3	X_3	1	1	0.5	1^*	0	0.5	1
Z		3	3	1.5	3	0	1.5	
Z_j-C_j			2	0.5	0	0	1.5	

Since all Zj-Cj are positive or zero, thus, final solution has reached.

Result X_1=0, X_2=0, X_3=1, Max Z=3

Resource constraint (2) is limiting factor while resource constraint (1) is surplus.

Example 2

$$MaxZ \quad 60X_1 + 50X_2$$
$$subject \quad to$$

$X_1 + 0.12X_2 \leq 4$

$12X_1 + 15X_2 \leq 60$

$20X_1 + 30X_2 \leq 110$

$X_1, X_2 \geq 0$

After adding slack variables, we get the equations;

$Max Z \quad 60X_1 + 50X_2 + 0S1 + 0S2 + 0S3$

subject to

$X_1 + 0.12X_2 + S1 \quad 4$

$12X_1 + 15X_2 + S2 \quad 60$

$20X_1 + 30X_2 + S3 \quad 110$

$X_1, X_2, S1, S2, S3 \geq 0$

Initial Tableau

	Cj	60	50	0	0	0	Min R
C_B	b	X_1 ↓	X_2	S_1	S_2	S_3	
0 S_1 ←	4	1*	0.12	1.5	0	1	4
0 S_2	60	12	15	0.5	1*	0	5
0 S_3	110	20	30	1.5	3	0	5.5
Z_j	0	0	0	0	0	0	
Z_j-C_j		-60	-50				

Since, most negative Zj-Cj is -60, X_1 will enter the basis. Further min ratio R (bi/aij) is min. for 2nd row, hence, S_2 will leave the basis.

I[st] Iteration

		60	50	0	0	0	R
C_B	Xb	X_1	X_2 ↓	S_1	S_2	S_3	
60 X_1	4	1	0.12	1	0	0	33.3333
0 S_2 ←	12	0	13.56*	-12	1	0	0.8849*
0 S_3	30	0	27.60	-20	0	1	0.9500
Z_j	240	60	7.20	60	0	0	
Z_j-C_j		0	-42.80	60	0	0	

Since, most negative Z_j-C_j is -42.80. Further, min. ratio R is min. for 2nd row.

2[nd] Iteration

		60	50	0	0	0	R
60 X_1	3.8938	1	0	1.1062	-0.0088	0	
50 X_2	0.8849	0	1	-0.8849	0.0737	0	
0 S_3	5.5952	0	0	4.4235	-2.0341	1	
Z_j	277.8730	60	50	22.1270	3.1570	0	
Z_j-C_j		0	0	22.1270	3.1570	0	

Since, all Z_j-C_j are positive or zero, thus, final solution has reached.

Result X_1=3.8936, X_2=0.8849, S_3 (disposal) =5.5952

Profit = Rs. 277.8730

The values of Z_j-C_j below slack variables are MVPs of resources used. The MVP of surplus/unused resource will be zero (0). We can apply Euler's theorem to verify as;

$$(22.1270)\times 4 + 3.1570\,(60) + (0)\ X\,110$$
$$88.508 + 189.4200 + 0$$
$$Rs.\ 277.928$$

Minimax Theorem

Let f(x) be a linear function of 'n' variables such that f(X*) is its minimum value for some point X*, $X^T \in R^n$. Then –f(X) attains its maximum value at the point X* or for $X^T \in R^n$

$$Min\ f(X) \quad -\max\{-f(X)\}$$

Proof: Since f(X) is minimum at point X*. Therefore

$$f(X^*) \le f(X) \qquad \forall\ X^T \in R^n$$
$$-f(X^*) \ge -f(X)$$

This shows that –f(X) attains its maximum value at point X*

Thus, Max (-f(X) = -f (X*)

$$Since,\ Min\,f(X) \quad f(X^*)$$
$$Min\,f(X) \quad -f(X^*) \quad Max(-f(X))$$
$$Hence,\ \text{minimum f(X)} \quad -Max(-f(X))$$

Artificial Variables

When there are both slack and surplus variables in the LP model and in order to get desirable number of unit column vectors, artificial variables are inserted with an obvious intention of finding basic matrix. The ancillary non-negative variables associated with the unit column vectors are called artificial variables so that they leave the basis as soon as possible.

We call this method as Charnes Method of Penalties or Big M Method. *Let* given LP

$$\text{Max Z} \quad \text{CX}$$

subject to

$$\text{AX} \quad \text{b}, \quad \text{X} \ge 0$$

then new problem will be

$$\text{MaxZ} \quad \text{CX - MA}_i$$

subject to

$$AX + I_m \quad b \quad X \ge 0, \quad A_i \ge 0$$

Example : This methodia used when we have different type of inequalities in the equations

Min Z $\quad 2X_1 + X_2$

$3X_1 + X_2 \quad 3$

$4X_1 + 3X_2 \geq 6$

$X_1 + 2X_2 \leq 3$

$X_1, X_2 \geq 0$

Since, MinZ $\quad$ -MaxZ

or $\quad$ MaxZ $\quad -2X_1 - X_2 - MA_1 - MA_2$

subject to

$3X_1 + X_2 + A_1 \quad 3$

$4X + 3X_2 - S_1 + A_2 \quad 6$

$X_1 + 2X_2 + S_2 \quad 3$

$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

		-2	-1	0	0	-M	-M	
C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	
-M	3	3*	1	0	0	0	1	I[st]
-M	6	4	3	-1	0	1	0	Iteration
0	3	1	2	0	1	0	0	
Z_j-C_j	-9M	-7M +2	-4M +1	M	0	-M	-M	
-2	1	1	1/3	0	0	0	1/3	2[nd]
-M	2	0	5/3	-1	0	1	-4/3	Iteration
0	2	0	5/3	0	1	0	-1/3	
Z_j	-2M	0	-5/3M	M	0	-M		
	-2	-2	+1/3	0	0			
-2	X_1 3/5	1	0	1/5	0		3[rd]	
-1	X_2 6/5	0	1	-3/5	0		Iteration	
0	0	0	0	1	1			
Z_j	-12/5	-2	-1	1/5	0			
		0	0	1/5	0			

Since, all Z_j-C_j are zero or positive so optimum solution has been obtained.

$$MaxZ \quad -\left(-\frac{12}{5}\right) \quad \frac{12}{5}$$

$$X_1 \quad \frac{3}{5}, \quad X_2 \quad \frac{6}{5}$$

Concept of Duality

Associated with every LP there is always a corresponding LPP called the dual problem of given LPP. The original LP is called primal.

Example:

$$Min Z \quad 3X_1 + 2.5X_2$$

subject to

$$2X_1 + 4X_2 \geq 40$$

$$3X_1 + 2X_2 \geq 50$$

$$X_1, X_2 \geq 0$$

Its dual problem can be written as;

$$Max Z \quad 40w_1 + 50w_2$$

subject to

$$2w_1 + 3w_2 \leq 3$$

$$4w_1 + 2w_2 \leq 2.5$$

$$w_1, w_2 \geq 0$$

Dual Simplex Method

We can also find the optimum solution of given LP by employing dual simplex method. By this method, minimization problem can be converted into maximization problem. The steps followed in dual simplex method are;

Step I

Convert the min. problem into max. by multiplying with negative (-) sign.

Select the most minimum bi i.e. bi with highest –ve value

Step II

Select minimum

$$\left[(\frac{Cj}{aij}), aj < 0 \right] \text{ as incoming activity}$$

Example:

Min Z $=3x_1+6x_2+x_3$

Subject to

$x_1+x_2+x_3 \geq 6$

$x_1-5x_2 -x_3 \geq 4$

$x_1+5x_2 +x_3 \geq 24$

$x_1,x_2,x_3 \geq 0$

We can solve this problem with dual simplex method;

Max z $=-3x_1-6x_2-x_3$

Subject to

$-x_1-x_2-x_3 <= -6$

$-x_1+5x_2 +x_3 <=- 4$

$-x_1-5x_2 -x_3<= -24$

We can solve this by using dual simplex method

Initial Tableau

	C_j		-3	-6	-1	0	0	0
C_B	X_b	b	X_1	X_1	X_3↓	S_1	S_2	S_3
0	S_1	-6	-1	-1	-1	1	0	0
0	S_2	-4	-1	5	1	0	1	0
0	S_3 ←	-24	-1	-5	-1*	0	0	1
Min. (c_j/ a_{ij})			-3/-1	-6/-5	-1/-1			

Select most minimum bi i.e. bi with highest –ve value, which is -24 , which will decide the outgoing row

Select the minimum

$$\left[\left(\frac{Cj}{aij}\right), aj < 0 \right]$$

In this case it is 1

$$\frac{-3}{-1} \quad \frac{-6}{-5} \quad \quad \frac{-1}{-1}$$

Last column is minimum. So it will be the incoming row. Repeat these steps and the final iteration is given below:

	C_j		-3	-6	-1	0	0	0
C_B	X_b	b	X_1	X_1	X_3 ↓	S_1	S_2	S_3
0	S_1	18	0	4	0	1	0	-1
-3	X_1	14	1	0	0	0	-0.5	-0.5
-1	X_3 ←	10	0	5	1	0	0.5	-0.5
Z_j		-52	-3	-5	-1	0	1	2
Z_j-C_j			0	1	0	0	1	2

Since, all the values are positive or zero, final solution has been obtained with the value

X_1=14 X_3 =10 S_1 (disposal)= 18 ; Min Z= -(Max Z) = 52

Assignment

Solve the following problem by using simplex method

Maximize $Z=10x_1+5x_2-2x_3$

Subject to the constraints

$x_1+ x_2- 2x_3 \leq 10$

$4x_1+ x_2+ x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

Note: solution is x1 = 0; x2 = 16.67; x3 = 3.33; Max Z = 90.0

UNIT-IV

MODIFIED SIMPLEX METHOD AND SENSITIVITY ANALYSIS

So far, we have understood how to find the maximum or minimum value of the objective function. Modified Simplex method is used to work out the exact requirement of resources. This is also known as parametric programming. The formation of the matrix is same as for LPP, the only difference is the resource supply that is allowed to vary. The optimum plan allocation is achieved through this technique which means that further increase in profit is possible until the MVP of that resource becomes zero.

Steps

The steps followed in modified simplex method are given below:

1. Prepare initial tableau with Zero availability in the resource supply for which we want to work out the exact requirement.
2. Compute $Z_j - C_j$ row
3. Compute ? = -ve $\dfrac{Z_j - C_j}{a_{ij} \quad (coefficients\ of\ 0\ resource\ vector)}$
4. Select the highest –ve ? that has the highest MVP (marginal value productivity) of resource. Thus, we decide the incoming row.
5. The min. R ratio (b_i/a_{ij}) decides the outgoing row.
6. Perform the iteration in the same way as in simplex method and workout Z_j- C_j and ? again.
7. Continue the iterations till all the entries in Z_j- C_j row become positive or zero.
8. The negative value in bi: (for zero resource supply) shows the exact requirement of that resource.

Example 1:

Given the problem:

Max Z = 60 x_1+50 x_2

Subject to

X_1+0.12 $X_2 \leq$ 4 Land

12X_1+15 $X_2 \leq$ 60 Labour

20X_1+30 $X_2 \leq$ 110 Capital

X_1, $X_2 \geq 0$

If we want to find exact amount of capital, we put 0 in the availability of capital and apply modified simplex method. We take capital at zero level so as to work out the exact requirement. The iterations shown below explains the procedure to find optimum solution to the problem. Considering the earlier problem (Example 2 chapter III), let us consider that we want to find out exact amount of capital, so we put zero (0) in availability column in simplex tableau as given below:

Initial table

	Cj	60	50	0	0	0	Min R
C_B	Xb	X_1 ↓	X_2	S_1	S_2	S_3	
0 S_1 ⟵	4	1*	0.12	1.5	0	1	4
0 S_2	60	12	15	0.5	1^*	0	5
0 S_3	0	20	30	1.5	3	0	-
Z_j	0	0	0	0	0	0	
Z_j-C_j		**-60**	-50				
$\overline{A}$ $-\frac{Z_j - C_j}{a_{ij}}$		-3	-1.67	0	0	0	

Since, most negative $\overline{A}$ is -3, thus, X_1 will be enter the basis. The min. ratio R is for I[st] row, thus, S_1 will leave the basis.

First iteration

	Cj	60	50	0	0	0	R
C_B	Xb	X_1	X_2 ↓	S_1	S_2	S_3	
60 X_1	4	1	0.12	1	0	0	33.3333
0 S_2 ⟵	12	0	13.5600	-12	1	0	0.8849^*
0 S_3	80	0	28.56	-20	0	1	0.9500
Z_j	240	60	7.20	60	0	0	
Z_j-C_j		0	-42.80	60	0	0	
$\overline{A}$ $\frac{Z_j - C_j}{a_{ij}}$	0	0	-1.4986				

Since, most negative Zj-Cj is -42.80 and min. $\overline{A}$ is for X_2, then X_2 enter the basis. Further the ratio R is min. for 2nd row, thus, S_2 will leave the basis.

Second iteration

		60	50	0	0	0	R
C_B	Xb	X_1	X_2	S_1	S_2	S_3	
60 X_1	3.8938	1	0	1.1062	-0.0088	0	
50 X_2	0.8849	0	1	-0.8849	0.0737	0	
0 S_3	-105.2756	0	0	5.2756	-2.1049	1	
Z_j	277.8780	60	50	22.1220	3.1570	0	
Z_j-C_j		0	0	22.1220	3.1570	0	

Since, all Z_j-C_j are positive or zero, thus, final solution has been obtained.

Thus, it shows that capital worth Rs. 105.2756 may be needed. The values Z_j-C_j below the slack activities show marginal value products of land and labour. So, the profit will be

$$(22.1270)\times 4 + 3.1570(60)$$
$$88.508 + 189.4200$$
$$Rs.\, 277.928$$

This technique can be used to study which resource is limited in supply and what should be the exact amount of that resource. The student may not that results are same as obtained in earlier example. It is also interesting to note that exact capital requirement is same as obtained by deducting disposal value from availability.

Sensitivity Analysis

It allows us to determine the effect of changes in the prices, resource supply and technical coefficients on the optimum solution. It is known as variable programming. When there is change in cost vector (C_j) it is known as variable price programming and when changes occur in resource constraint (b_i) it is known as variable resource programming. In this section we will study the effect of these changes on optimality of the solution obtained previously and procedure to incorporate these changes to proceed further.

1. Changes in C_j vector

When there are some discrete changes in Cjs, we can find range with in which our optimum solution is not affected. For this, we examine the following condition;

Max. (-(Zj-Cj)/Sij)< ΔCj <Min (-(Zj-Cj)/Sij)

Sij>0 Sij>0

The following steps are perfumed to examine the effect of changes in cost/price vector on optimal solution;

Step 1: When there is some change in C_j vector, put the value of new prices in final optimum simplex tableau derived earlier.

Step 2: Work out new Z_j and Z_j- C_j row and examine the sign. In case of max. problem if all the values in Z_j- C_j row are +ve or zero the solution is not affected but if some of Z_j- C_j values turn out to be –ve, the solution is no more optimum. In case of min. problem, if all the values in Z_j - C_j row are -ve or zero the solution is not affected but if some values of Z_j - C_j turn out to be +ve, the solution is no more optimum.

Step 3: In case the solution is not optimum, start from this step onward by selecting the most negative/positive value (as the case may be) and select incoming row and outgoing by following the same procedure as in simplex method.

Step 4: Perform successive iterations, examine Z_j - C_j till the optimum solution is arrived at.

Example 2:

Let us take the previous example

Max Z = 60 X_1+50 X_2

Subject to

$X_1+0.12 X_2 \leq 4$ Land

$12X_1+15 X_2 \leq 60$ Labour

$20X_1+30 X_2 \leq 110$ Capital

$X_1, X_2 \geq 0$

Now we change

$$C_1 \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

Then following steps 1 to 4 given above and apply to final solution. Table obtained previously as given below;

Starting table

		40	50	0	0	0	R
C_B	X_B	X_1	X_2	S_1	S_2	S_3	
40 X_1	3.8938	1	0	1.1062	-0.0088	0	3.5200
50 X_2	0.8849	0	1	-0.8849	0.0737	0	-
0 S_3	5.5760	0	0	4.4232	-2.0341	1	604.22
Z_j	199.9770	40	60	0.0030	3.3350	0	
Z_j-C_j	-	0	0	0.0030	3.3330	0	

Since, all Z_j-C_j are positive or zero, thus, final solution is not affected. The results are;

X_1=3.8938

X_2= 0.8849

S_3 (disposal)= 5.5752 and profit Z_j = Rs. 199.9770

Now let us change this to $C^* \begin{bmatrix} 10 \\ 50 \end{bmatrix}$

and follow the procedure given above

Starting table

			10	50	0	0	0	R
C_B	X_B		X_1	X_2	S_1 ↓	S_2	S_3	
10	X_1	3.8938	1	0	1.1062	-0.0088	0	3.5
50	X_2	0.8849	0	1	-0.8849	0.0737	0	-
0	S_3 ←	5.5752	0	0	4.4232*	-2.0341	1	1.2604
Z_j		83.1830	10	50	-33.1830	3.5970	0	
Z_j-C_j		-	0	0	-33.1830	3.5970	0	

Since, one of the values Z_j-C_j for S_1 column is negative, hence, S_1 will enter the basis and S_3 will leave the basis (having minimum ratio).

I[st] iteration

			10	50	0	0	0	R
C_B	X_B		X_1	X_2	S_1	S_2 ↓	S_3	
10	X_1 ←	2.4995	1	0	0	0.4999	-0.2501	5.00
50	X_2	2.0002	0	1	0	-0.3333	0.2001	
0	S_1	1.2604	0	0	1	-0.4589	0.2261	
Z_j		125.0050	10	50	0	-11.660	7.5040	
Z_j-C_j		-	0	0	0	-11.660	7.5040	

Since, still one of the values Z_j-C_j for S_2 column is negative, hence, S_2 will enter the basis and on the basis of min ratio R, X_1 will leave the basis.

2nd iteration

			10	50	0	0	0	R
C_B	X_B		X_1	X_2	S_1	S_2 ↓	S_3	
0	S_2	5.000	2.0004	0	0	1	-0.5003	
50	X_2	3.6667	2.0004	1	0	0	0.1167	0.03335
0	S_1	3.5598	0.9200	0	1	0	0.1111	0.0035
Z_j		183.3350	100.020	50	0	0	5.8350	
Z_j-C_j			90.020	0	0	0	5.8350	

Since all Z_j - C_j are positive or zero, thus, final solution has been obtained with the value

X_2 = 3.6667

S_1 (disposal) = 3.5599

S_2 (disposal) = 5.0 and profit Zj = 183.3350

2. Addition of a single variable

When we want to add a new activity or variable in the optimized model, instead of solving the entire model we follow the procedure given below:

$$\textit{Let}\ \text{Max Z} \quad C'X$$
$$AX \le b, \quad X \ge 0$$
$$\textit{Let}\ \text{us add an activity}\ X_{n+1}\ \text{having technical coefficient column}\ A_{n+1}\ \text{and cost}\ C_{n+1}$$

This problem will require the computation of

$$X_{n+1} \quad A^{-1}a_{n+1}\ \text{and}$$
$$Z_{n+1} - C_{n+1} \quad C_B X_{n+1} - C_{n+1}$$
$$\textit{Then}\ \text{either}\ Z_{n+1} - C_{n+1} \ge 0 \cdots \textit{optimum}$$
$$\textit{or} \quad Z_{n+1} - C_{n+1} < 0 \cdots \textit{Not optimum}$$

Example 3:

Considering the previous optimized LPP solution;

	Cj		60	50	0	0	0
C_B	X_B	bi	X_1	X_2	S_1	S_2	S_3
60	X1	3.8938	1	0	1.1062	-0.0088	0
50	X_2	0.8849	0	1	-0.8849	0.0737	0
0	S_3	5.5752	0	0	4.4232	-2.0341	1
Z_j		277.873	60	50	22.127	3.1570	
Zj-Cj			0	0	22.127	3.1570	0

Let X_3 with price 70 and input-output column a_{ij} [0.3, 11, 15] is introduced. Therefore,

$X_{n+1} = [A^{-1}]\ [a_{n+1}] =$

$$\begin{bmatrix} 1.1062 & -0.0088 & 0 \\ -0.8849 & 0.0737 & 0 \\ 4.4232 & -2.0341 & 1 \end{bmatrix} \begin{bmatrix} 0.3 \\ 11 \\ 15 \end{bmatrix} \quad \begin{bmatrix} 0.2218 \\ 0.7670 \\ -21.048 \end{bmatrix}$$

Thus, the modified starting tablue will be:

			60	50	0	0	0	70	R
C_B	X_B		X_1	X_2	S_1	S_2	S_3	X_3 ↓	
60	X1	3.8938	1	0	1.1062	-0.0088	0	0.2218	17.58
50	X_2 ←	0.8849	0	1	-0.8849	0.0737	0	0.7670	1.15
0	S_3	5.5752	0	0	4.4232	-2.0341	1	-21.048	-
Z_j		277.873	60	50	22.127	3.1570		51.658	
Zj-Cj			0	0	22.127	3.1570	0	- 18.342	

Since, one of the values Zj-Cj is negative, hence, the solution is not optimum. Therefore, we will proceed from this point forward.

(Note: This is an assignment to the students to proceed from this point forward)

3. Deletion of a variable/ activity

Sometimes any activity in the previous model may not be required and needs to be deleted. In that event following steps are followed:

1. If variable is not in the basis, it is superfluous and we can drop it as such and the solution will not be affected by deleting this variable.
2. If variable we want to delete is in the basis, then we assign very high penalty (M) in C_j so that it goes out from the basis.

4. Addition of a single constraint

A new resource constraint may emerge due to new technology while the activities remain the same. In that case, we can incorporate the new constraint in the optimized solution by following the computational procedure given below:

Consider a LPP model;

$$MaxZ \quad C^{/} X$$
$$subject\ to$$
$$AX \leq b \quad X \geq 0$$
Now let the new constraint be;

$AX \leq b_{m+1}$

Two cases will arise

1. X_B satisfies the new constraint
2. X_B does not satisfy the new constraint

In (i) it does not affect optimum solution but in (ii) solution is no more optimum. So, we follow the following procedure;

We know

$$AX \leq b$$
$$AX + X_S \quad b_{m+1}$$

we know that solution vector Xb*= $\begin{bmatrix} Xb \\ Xs \end{bmatrix}$

and matrix B*= $\begin{bmatrix} b & 0 \\ u & 1 \end{bmatrix}$

Where; u= a_{m+1}, 1; a_{m+1}, 2; a_{m+1}, 3;……a_{m+1}, n i.e. input-output row for new constraint.
Thus,

$$\{B^*\}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ -uB^{-1} & 1 \end{bmatrix}$$

The value of X_B^* will be given by

$$X_B^* = \begin{bmatrix} X_B \\ X_s \end{bmatrix} \quad [B^*]^{-1} b^*$$
$$\begin{bmatrix} B^{-1} & 0 \\ -uB^{-1} & 1 \end{bmatrix} \begin{bmatrix} b \\ b_{m+1} \end{bmatrix}$$

X* (for new constraint=$-uX_B+b_{m+1}$

Considering the previous optimized LPP

Max Ƶ = 60X + 50Y

$X + 0.12Y \leq 4$

$12X+15Y \leq 60$

$20X+30Y \leq 110$

and the optimum solution;

	C_j		60	50	0	0	0
C_B	X_B	bi	X_1	X_2	S_1	S_2	S_3
60	X1	3.8938	1	0	1.1062	-0.0088	0
50	X_2	0.8849	0	1	-0.8849	0.0737	0
0	S_3	5.5752	0	0	4.4232	-2.0341	1
Z_j		277.873	60	50	22.127	3.1570	
Zj-Cj			0	0	22.127	3.1570	0

In this case, X_b = (3.8938, 0.8849, 5.575)

If new constraint is u = $6X_1+4X_2$ <=35

Then, the X* =-[3.8938, 0.8849, 5.575] $\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$ +35 =8.2412

Thus, revised optimum solution X_B*= (3.8938, 0.8849, 5.5752, 8.2412)

Therefore, the solution is still optimum. However, if the new constraint is

u = $6X_1+4X_2$ <=25, then, the revised optimum solution will be

X* = -[3.8938, 0.8849, 5.575] $\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$ +25

= - 1.7588; which is not optimum and we can solve the problem from this point onwards with any of the methods explained earlier.

5. Change in b_i column:

When there are some discrete changes in bis, we can find range with in which our optimum solution is not affected. For this, we examine the following condition;

Max. (-Xbi/Sij) < Δbi < Min (-Xb/Sij)

Sij>0 Sij>0

Suppose b is initial resource and we add say q quantities of resources so that new supply changes to constraints is $\bar{b}i$

$i.e\ \bar{b}1 \quad b+q \qquad q \quad [q_1, q_{2,}, q_3 \cdots\cdots qm]$

The solution will be optimum if $\bar{b}i$ remain non- negative so we compute

$\overline{b*}$ = $A^{-1}\bar{b}$ = A^{-1} (b+q)

$A^{-1}b + A^{-1}q$

$b^* + A^{-1}q$

where, b* is the previous optimum solution.

Let us take the previous example;

Max Z = $60X_1 + 50X_2$

$X_1 + 0.12X_2 \le 4$

$12X_1 + 15X_2 \le 60$

$20X_1 + 30X_2 \le 110$

With optimum solution

	C_j		60	50	0	0	0
C_B	X_B	bi	X_1	X_2	S_1	S_2	S_3
60	X1	3.8938	1	0	1.1062	-0.0088	0
50	X_2	0.8849	0	1	-0.8849	0.0737	0
0	S_3	5.5752	0	0	4.4232	-2.0341	1
Z_j		277.873	60	50	22.127	3.1570	
Zj-Cj			0	0	22.127	3.1570	0

$X_1 = 3.8938$, $X_2 = 0.8849$, $S_3 = 5.5752$

Now discrete changes in bi can be calculated as;

Max. (-Xbi/Sik)< Δbi <Min (-Xbi/Sik)

Sik>0 Sik<0

In our example, b1, b2, b3 can be calculated as;

$b = \begin{bmatrix} 4 \\ 60 \\ 110 \end{bmatrix}$ Let change in resource matrix be $q = \begin{bmatrix} -2 \\ -3 \\ 10 \end{bmatrix}$

Hence, $\bar{b} = b + q = \begin{bmatrix} 2 \\ 57 \\ 120 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1.1062 & -0.0088 & 0 \\ -0.8849 & 0.0737 & 0 \\ 4.4232 & -2.0341 & 1 \end{bmatrix}$ *please see the solution table given above*

So $\overline{b^*} = b^* + A^{-1}q$

$$\begin{bmatrix} 3.8938 \\ 0.8849 \\ 5.5752 \end{bmatrix} + \begin{bmatrix} -1.1860 \\ 1.5487 \\ 7.2569 \end{bmatrix} \begin{bmatrix} 2.7078 \\ 2.4336 \\ 12.8321 \end{bmatrix} > 0$$

So, solution is still optimum and the final value b* is replaced by $\overline{b^*}$ and Zj would be Zj*.

	C_j		60	50	0	0	0
C_B	X_B	bi	X_1	X_2	S_1	S_2	S_3
60	X_1	2.7078	1	0	1.1062	-0.0088	0
50	X_2	2.4346	0	1	-0.8849	0.0737	0
0	S_3	12.1730	0	0	4.4232	-2.0341	1
Z_j		284.198	60	50	22.127	3.1570	
Z_j-C_j			0	0	22.127	3.1570	0

However, if q is decreased by

$$q=\begin{bmatrix} 0 \\ -50 \\ 0 \end{bmatrix}$$

$$\overline{b^*} = b^* + A^{-1}q^2 = \begin{bmatrix} 4.3338 \\ -2.7801 \\ 107.2802 \end{bmatrix}$$

This negative value renders the solution infeasible.

Thus, we have to multiply this by (-1) and solve the problem from this point onward.

Starting table

			60	50	0	0	0	-M*
C_B	X_B		X_1	X_2	S_1	S_2 ↓	S_3	S_4
60	X_1	4.3338	1	0	1.1062	-0.0088	0	-0.0088
50	X_2	2.7801	0	-1	-0.8849	-0.0737	0	0.0737
0	S_3 ⟵	107.2802	0	0	4.4232	-2.0341	1	-2.0341
Zj		399.0330	60	50	22.1270	-4.2310	0	3.157
Zj-Cj			0	0	22.1270	**-4.2310**	0	3.157+M

**** We use artificial variable S_4 as inequality has changed after multiplying with (-).***

Since, one Zj-Cj is negative, S_2 will enter and S_3 will leave the basis in the next iteration. (The student may solve this problem as an assignment).

Dual simplex method

As stated earlier with the help of dual simplex method, we can convert min. problem into max. problem or vice versa. We know that max. problems are easier to solve by using simplex method than the min. problems. Let us consider minimization problem

Min Z= $3X_1+2X_2+X_3+X_4$

subject to

$2X_1+4X_2+5X_3+X_4 \geq 10$

$3X_1-2X_2+7X_3-2X_4 \geq 2$

$5X_1+2X_2-X_3+6X_4 \geq 15$

$X_1, X_2, X_3 \geq 0$

We can convert into dual (max. problem by multiplying with (-1)

Max $Z=-3X_1-2X_2-X_3-X_4$

subject to

$-2X_1-4X_2-5X_3-X_4 <= -10$

$-3X_1+2X_2-7X_3+2X_4 <= -2$

$-5X_1-2X_2+X_3-6X_4 <= -15$

Initial Tableau

	Cj	-3	-2	-1	-4	0	0	0
Cb	b	X_1	X_2	X_3	X_4	S_1	S_2	S_3
0 S_1	-10	-2	-4	-5	-1	1	0	0
0 S_2	-2	-3	1	-7	2	0	1	0
0 S_3	**-15**	**-5***	-2	1	-6	0	0	1
Zj-Cj	**0**	**3**	2	1	4	0	0	0
$\frac{Zj-cj}{yij}$		**3/-5**	2/-2	1/1	4/-6			

We select the max negative bi for outgoing row and max. of the ratio $\frac{Zj-cj}{yij}, yij < 0$ for deciding incoming row

So, max negative bi = -15 and max ratio (3/-5, 2/-2, 1/1, 4/-6) is -3/5

First iteration

	C_j	-3	-2	-1	-4	0	0	0
C_b	X_b	X_1	X_2	X_3	X_4	S_1	S_2	S_3
0 S_1	-4	0	-3.2	**-4.6***	1.4	1	0	-0.4
0 S_2	7	0	2.2	-6.4	5.6	0	1	-0.6
-3 X_1	3	1	0.4	-0.2	1.2	0	0	-0.2
Z_j-c_j	-9	0	0.8	0.4	0.4	0	0	0.6
$\frac{Zj-cj}{yij}$			0.8/-3.2	0.4/-4.6				0.6/-0.4

We select the max negative bi (-4) for outgoing row and max. of the ratio $\frac{Zj-cj}{yij}, yij < 0$ (0.8/-3.2) for deciding incoming row

Second iteration

	Cj	-3	-2	-1	-4	0	0	0
Cb	X_b	X_1	X_2	X_3	X_4	S_1	S_2	S_3
0 S_1	0.87	0	0.69	1	-0.304	-0.2174	0	0.0869
0 S_2	12.57	0	6.65	0	3.6522	-1.3913	1	-0.0435
-3 X_1	2.83	1	0.26	0	1.2608	0.0435	0	-0.2174
Zj	-9.3478	0	0.5217	0	0.5217	0.0869	0	0.5652

Since, all Z_j-C_j are positive or zero and all Xb positive, so, optimum solution is obtained.

X_1 = 2.83, S1(disposal=0.87, and S2 (disposal)=12.57

Min Z (max –Z) = 9.3478

UNIT-V

APPLICATION OF INTEGER PROGRAMMING

The solution derived from LP may not always be the integer value. Although it is tempting to round off non-integer solutions in problem including indivisible resources, but such rounding can result in sub- optimal solutions. As in Fig I the best solution for non-integer will be at A while B presents the rounded off solution which is closest point to point A.

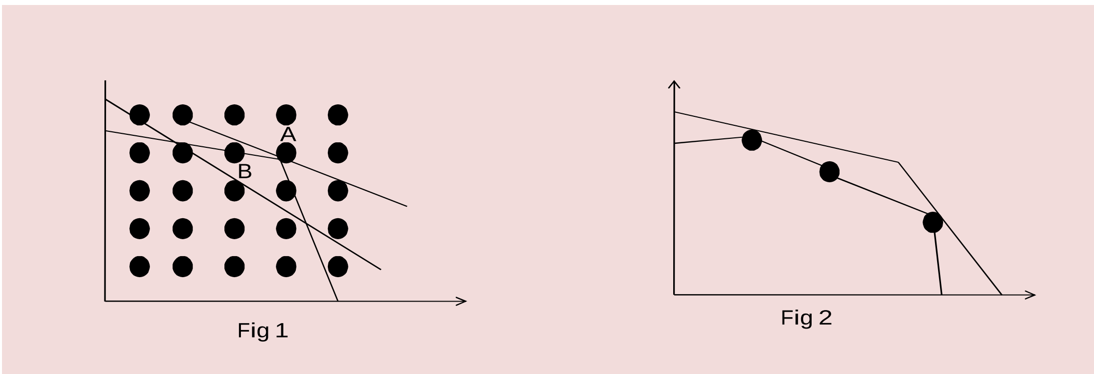


Fig 1

Fig 2

A systematic procedure was developed by Gomory R.E. (1958). He made use of dual simplex method to find optimum solution to mixed integer problems.

All Integer Programming Problem:

Let the optimum solution be;

$$b_i \quad \sum_{j\ 1}^{n} \bar{a}_{ij} Y_j \cdots\cdots (1)$$

where $\bar{b}$ is non - integer

Let $\bar{b} \quad \hat{b}_i + \beta_i \quad -(2)$ $\hat{b}$ *and* $\bar{a}_{ij}$ denote integers contained by the fraction parts from $\bar{b}$ and $\bar{a}_{ij}$

$$\bar{a}_{ij} \quad \hat{a}_{ij} + \alpha_{ij} \quad -(3)$$

Thus, β_i will be strictly the positive fraction [0< β_i <1) and α_{ij} will be a non-negative fraction $\left(0 \le \alpha_{ij} < 1\right)$

Now with (2) and (3) equation (1) can be written as

$$\left(\hat{b} + \beta_i\right) \quad \sum_{j\ 1}^{n} \left(\hat{a}_{ij} + \alpha_{ij}\right) y_j$$

$$or \quad \beta_i - \sum_{j\ 1}^{n} \alpha_{ij} y_j \quad -\hat{b}_i + \sum_{j\ 1}^{n} \hat{a}_{ij} y_j$$

Steps in integer programming:

1. Solve LPP ignoring integer condition.
2. Test the integer values of variables in final solution.
3. If all desired integer values are obtained, then an optimum solution in integer has been obtained.
4. If the optimum solution contains some non-integer values then proceed to next step
5. Examine the constraint equation

   $$\sum_{j\ 1}^{n} \overline{a_{ij}} X_j \quad \overline{b_i}$$

   and choose the largest fraction of $\overline{b_i}$ let it be f_{k0}

6. Express each of the negative fractions, if any, in the k[th] row of the optimum simplex tableau as the sum of negative integer and non-negative fraction.
7. Find the Gomory constraint i.e. the fractional cut

   $$\sum_{j\ 0}^{n} f_{kj} X_j / f_{k0}$$

   and append the equation

   $$\sigma_1 \quad -f_{k0} + \sum_{j}^{n'} f_{kj} X_j \qquad (fractional\,\text{cut})$$

   In this σ_1 is integer

8. Starting with the new set of equations, find new optimum solution so that σ_1 leaves the basis.
9. If this new solution is integer it is also feasible & optimum. If some of other are non-integer values then repeat again from step V onwards.

Example 1: Consider the LPP;

$$Max\,Z \quad X_1 + X_2$$

$$subject \quad to$$

$$3X_1 + 2X_2 \leq 5$$

$$X_2 \leq 2$$

		1	1	0	0
C_B	X_B	X_1	X_2	S_1	S_2
1	X_1 0.33	1	0	0.33	-0.67
1	X_2 2	0	1	0	1
Z_j	2.33	0	0	0.33	0.33

And this is the final solution. Now fractional cut is 0.33 of b_1 in first row

$$0.33 \quad X_1 + 0.33X_3 - 0.67X_4$$

or

$$0.33 \quad X_1 + 0.33X_3 - X_4 + 0.33X_4$$

Thus, fractional cut is given by

$$\sigma_1 \quad -0.33 + 0.33X_3 + 0.33X_4$$

			1	1	0	0	0
C_B	X_B	X_B	X_1	X_2	X_3	X_4	σ_1
1	X_1	0.33	1	0	0.33	−0.67	0
1	X_2	2	0	1	0	1	0
0	σ_1	−0.33	0	0	−0.33	−0.33	1

We use dual simplex method

drop σ_1 and introduce X_3

C_B	X_B		X_1	X_2	X_3	X_4	σ_1
1	X_1	0	1	0	0	−1	1
1	X_2	2	0	1	0	1	0
0	X_3	1	0	0	1	1	−3
		2	0	0	0	1	1

$X_1 \quad 0 \qquad X_2 \quad 2 \qquad and\ Z \max \quad 2$

Since, all Z_j-C_j are positive or zero, the final solution has reached with the values $X_2 = 2$ and $X_1 = 0$

Example 2: Let us consider the problem

$$Max Z \quad 2X_1 + 2X_2$$

subject to

$$5X_1 + 3X_2 \leq 8$$

$$X_1 + 2X_2 \leq 4$$

$$X_1, X_2 > 0$$

and the final optimum solution is:

			2	2	0	0	0
Cb	X_b		X_1	X_2	S_1	S_2	σ_1
2	X_1	$4/7$	1	0	$\frac{5}{7}$	$\frac{-3}{14}$	0
2	X_2	$12/7$	0	1	$\frac{-1}{7}$	$\frac{5}{14}$	0
0	σ_1	$-5/7$	0	0	$\frac{-6}{7}$	$\frac{-5}{14}$*	1
	Zj	$32/7$	2	2	$8/7$	$2/7$	0

Now highest fractional cut is 5/7

Thus we select S_2 & row 2

$$1+\frac{5}{7} \quad X_2 - S_1 + \frac{6}{7}S_1 + \frac{5}{14}S_2$$

Thus corresponding Gomory Constraint will be

$$\sigma_1 \quad -\frac{5}{7} + \frac{6}{7}S_1 + \frac{5}{14}S_2 \quad (\text{solve this with dual simplex method})$$

Note: Final solution is; $X_1=1, X_2=1, \ S_2=2, Z = 4$

CHAPTER-VI

GOAL PROGRAMMING

Goal Programming

Goal programming model is an algorithm and a mathematical model, consisting of linear functions and continuous or discrete variables, in which all the functions are transformed into goals. Goal Programming (GP) is a powerful and flexible technique over other programming models that can be applied to a variety of decision making problems involving multiple objectives. Thus, goal programming may be used to solve linear programs with multiple objectives, with each objective viewed as a "goal". In goal programming, Yj+ and Yj- , deviation variables, are the amounts a targeted goal j is overachieved or underachieved, respectively. The goals themselves are added to the constraint set with Yj+ and Yj- acting as the surplus and slack variables. One approach to goal programming is to satisfy goals in a priority sequence. Second-priority goals are pursued without reducing the first-priority goals, etc. For each priority level, the objective function is to minimize the (weighted) sum of the goal deviations.

Steps in Goal Programming Formulation

Step 1: Decide the priority level for each goal.
Step 2: Decide the weight on each goal. If a priority level has more than one goal, for each goal i decide the weight, Pi , to be placed on the deviation(s) Yj+ and Yj from the goal.
Step 3: Set up a linear programming problem consider new objectives (minimize deviations), subject to all functional constraints, and goal constraints
Step 4: Solve the current linear programming problem.

The basic approach of goal programming is to establish a specific numeric goal for each of the objectives, formulate an objective function for each objective, and then find a solution that minimizes the (weighted) sum of deviations of these objective functions from their respective goals. There are three possible types of goals:

1. A lower, one-sided goal sets a *lower limit* that we do not want to fall short (but exceeding the limit is fine).
2. An upper, one-sided goal sets an *upper limit* that we do not want to exceed the limit (but falling short of the limit is fine).
3. A two-sided goal sets a specific target that we do not want to underachieve or overshoot the target. on either side.

Field Application of Goal Programming

The model has been employed to develop optimum sustainable irrigated cropping system plans in the command area of Lower Baijnath Kuhl in Himachal Pradesh during 2015-16 by including different alternatives/enterprises, their inter-relations/inter-linkages, technological options and constraints. For possible optimization of cropping systems, LINGO 10.0 version was used to develop farm production plans under different set of goals and available

opportunities under different cropping systems. The programming model was designed for the minimization of penalties keeping in view the prospects of higher income, food security and boundaries of resource constraints (land, labour, capital, etc.).
The mathematical form of goal optimization model employed in this study is as under:

$$\textit{Minimize } Z \quad \sum_{j=1}^{n} P_j Y_j \qquad (j=1, 2, 3......, n)$$

Subject to;

$$\sum_{k=1}^{l} a_{ik} X_k \quad >=< \quad b_i \qquad (k = 1, 2, 3......l.; \quad i = 1, 2, 3.......m)$$

X_k >= 0; Y_j >= 0

(Non-negativity restriction)

where;

P_j = Penalty for not achieving the j[th] goal

Y_j = Planned j[th] goal

a_{ik} = Unit contribution/requirement or input-output relationship

X_k = Number of activities (cropping systems and other enterprises)

b_i = Level of i[th] constraint

The Y_j consists of two components Y_j^+ *and* Y_j^-

where;

Y_j^+ implies overshooting the j[th] assigned goal

Y_j^- implies underachieving the j[th] assigned goal

Thus, the final equations of the model are;

$$\textit{Minimise } Z \quad \sum_{j=1}^{n} P_j \quad (Y_j^+ \quad + \quad Y_j^-) \qquad (j = 1, 2, 3......, n)$$

Subject to the following resource constraints;

$$\sum_{k=1}^{l} a_{ik} X_k \quad - \quad (Y_j^+ \quad - \quad Y_j^-) \qquad b_i$$

and,

X_k, Y_j^+ *and* Y_j^- >= 0

Objective function (Z)

Objective function consists of the goals or multiple objectives considered for planning. The goals/objectives have been presented by assigning different penalties. Higher the penalty, higher will be the priority and vice-versa. The main penalties fixed for the model are:

1. Land must be fully utilized and there should not be over or under utilization of the cropped land, thus, high penalties for under or over use of land
2. The returns should be increased to the maximum feasible extent by selecting profitable cropping system/enterprise-mix
3. Food security should be achieved as far as possible. For this, the minimum production of foodgrains required for meeting family consumption was considered
4. Minimum green fodder production either for sale or use for maintaining atleast one milch animal on the farm
5. Polyhouse area per farm should not exceed 105 square metre due to management constraint
6. One milch animal per farm to augment farm income

Assigned penalties and components for goal programming are given in Table 1.

Table 1. Penalties and components of goal programming

Sr. No.	Factors/ constraints	Unit contribution/ requirement of sub-system (per ha) $S_1, S_2, S_3, \ldots\ldots, S_k$	Inequality	Goal units (per farm) (b_i)	Penalty weights assigned	
					Y_j^+	Y_j^-
1.	Cultivated land availability (ha)	a_{ik}	<=	b_1	6	10
2.	RFFR (Rs.)		>=	b_2	0	2
3.	Capital (Rs.)			b_3		
	a. No borrowing		<=		1	0
	b. Borrowing		>=		0	1
4.	Labour (days)		<=	b_4	0	0
5.	Foodgrains (q)		>=	b_5	0	1
6.	Green fodder (q)		>=	b_6	0	1
7.	Maximum polyhouse area (ha)		=	b_7	1	1
8.	One crossbred cow		=	b_8	1	1

Different optimized plans were developed to achieve these goals keeping in view the following conditions:

1. P_0 = Existing optimized cropping system plan without capital borrowing
2. P_1 = Existing optimized cropping system plan with borrowing
3. P_2 = Existing optimized cropping system plan with borrowing and a polyhouse unit of 105m^2 area

4. P_3 = Improved optimized cropping system plan with borrowing
5. P_4 = Improved optimized cropping system plan with borrowing and a polyhouse unit of $105m^2$ area
6. P_5 = Improved optimized cropping system plan with borrowing and one crossbred cow

Table 2. Cropping Systems Yield and Returns

Farm Activities	Yield (q/ha)		RFFR (Rs./ha)
	I Crop I	II Crop	
Paddy-Wheat	21.03	21.83	45936
Paddy-Berseem	25	377.27	46795
Soybean-Potato	11.43	134.05	108231
Sorghum (*chari*)-Potato	481.82	134.09	114011
Ginger-Garlic	50.5	20.07	299460
Sorghum (*chari*)-Berseem	404.76	390.48	55154
Polyhouse ($105m^2$)			8898
Crossbred cow (per cow)	7.53 litres/day		42520

Optimum Cropping System Plans

The optimum cropping system plans have been developed for the study area with multiple objectives. Table 3 depicts land allocation under optimized plans

Table 3.. Land allocation under existing and optimized cropping system plans

(Per cent area)

Sr. No.	Cropping systems	Farmers' plan	P_0	P_1	P_2	P_3	P_4	P_5
1	Paddy-Wheat	80.52	75.24	71.52	71.48	48.52	48.52	65.95
2	Paddy-Berseem	4.76	-	0.10	0.10	35.24	35.24	-
3	Soybean-Potato	4.52	-	-	-	-	-	-
4	Sorghum (*chari*)-Potato	4.33	11.43	-	-	-	-	-
5	Ginger-Garlic	1.10	-	8.86	3.90	16.24	11.24	16.43
6	Sorghum (*chari*)-Berseem	0.67	13.33	19.52	19.52	-	-	17.62
7	Polyhouse (105 m^2)	-	-	-	5.00	-	5.00	-
8	Others	4.10	-	-	-	-	-	-
	Total	100 (0.22)	100 (0.22)	100 (0.22)	100 (0.22)	100 (0.22)	100 (0.22)	100 (0.22)

Note: Figures in parentheses show cultivated area in hectare. P_5 also includes one crossbred cow in the plan

Goals Achieved under Optimized Farm Plans

Table 4 shows the goals achieved in various optimized cropping system plans. The returns to fixed farm resources can be increased from Rs. 11,678 (farmer's plan) to Rs. 86,927 (optimized plan with improved technology and a milch cow).The capital required varies from Rs. 12,043 to Rs. 48,278. Therefore, the capital borrowing increased under optimized plans from Rs. 7,659 in P_1 to Rs. 36,235 in P_5. Labour use also increased in optimized plans from existing 41.38 man-days to 141.09 man-days, maximum being in P_5. In this way, in optimized plans, the foodgrain production will reduce from 10.37q/farm to 6.30q/farm which is a goal to meet the family consumption needs. The milk availability also increased from 1409 litres/ farm (P_1 to P_4) to 2748 litres/farm in P_5 sufficient to meet nutritional requirement of family and to increase farm income also. Green fodder production will increase from existing 11q/farm to 37q/farm to feed a crossbred cow.

Table 4. Resource use and extent of goals achieved under optimized plans in LB *kuhl*

(per farm)

Sr. No.	Particulars	Farmers' plan	P_0	P_1	P_2	P_3
1	RFFR (Rs.)	11678	12453	14328	20456	37893
2	Capital (Rs.)					
	i Capital required	12043	11590	19665	15614	28123
	ii Capital borrowing	-	-	7659	3559	16297
3	Human labour used (days)	41.38	42.00	42.84	59.97	52.01
4	Cereal availability (q)	10.37	6.60	6.30	6.30	6.30
5	Milk availability (litres)	1409	1409	1409	1409	1409
6	Green fodder (q)	11.18	37.00	37.00	37.00	37.00

Note: increase in milk availability in P_5 due to inclusion of a crossbred cow

UNIT- VII

DYNAMIC PROGRAMMING

Dynamic programming problem (DPP) is concerned with multi-stage decision making process. Solution to a DPP is achieved sequentially starting from one (initial) stage to the next till the final stage is reached. DPP was developed by Richard Bellman in early 1950. DPP can be given a more significant naming i.e. a recursive optimization. There is an obvious attraction of splitting a large problem into sub problems each of which involve only few variables.

According to Bellman, "An optimal policy has the property that whatever the initial stage and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision".

Recursive Equation

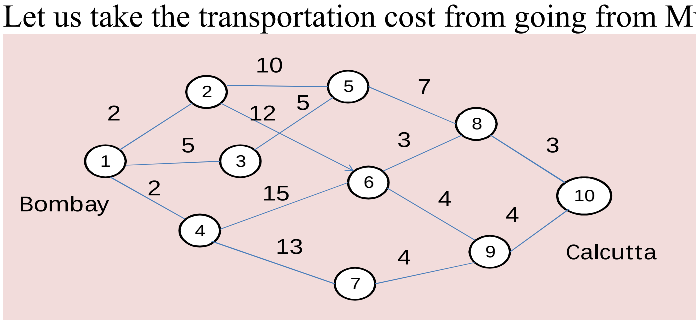
The recursive equation connects the optimal decision function for the N-stage problem with the optimal decision function for the (N-1) stage sub-problem. Thus, solution to DPP can be obtained by using the recursive equation techniques starting from the first to the last (forward solution) or from last to the first (backward solution).

Characteristic of Dynamic Programming

1. The problem can be sub divided into stages with a policy decision arrived at each stage.
2. Every stage consists of a number of states associated with it. The states are the different possible conditions in which the system may find itself at that stage of the problem.
3. Decision at each stage connects the current state into state associated with the next stage.
4. When the current state is known, an optimal policy for the remaining stages is independent of the policy of the previous ones.
5. By deriving the optimal policy for each state of the last stage, the solution procedure starts.
6. To identify the optimum policy for each state of a system a recursive equation is formulated with n stages remaining, given the optimal policy for each state with (n-1) stages remaining.
7. Using recursive system approach, each time the solution procedure moves forward or backward stage by stage for obtaining the optimum policy of each state for that particular stage, till it attains the optimum policy beginning at the initial stage.

Example 1: Minimum Path Problem

Let us take the transportation cost from going from Mumbai to Kolkata and finding cost.



Tabular Method to Solve DPP

Problem:

The above problem can be expressed in tabular form as given below;

	2	3	4
1	2	4	3

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

	8	9
5	1	4
6	6	3
7	3	3

	10
8	3
9	4

Find the shortest route of travelling so that the total travelling cost becomes minimum.

Solution:

Since, it is a four stage problem, let k_j (j= 1, 2, 3, 4) be the four decision variables.

Let f_j (s, x_j) be the total cost of the best overall policy for the remaining stages, given that the person is in stage s and selects x_j as the immediate decision. Given s and j, let x_j^* denote the value of x_j that minimizes f_j (s, x_j) and let f_j^* (s) be the corresponding minimum value of f_j (s, x_j).

Thus, f_j^* (s) = f_j (s, x_j^*). The objective is to find f_1^* (1) and the corresponding policy.

When the person has only one more stage to go, his route is entirely determinate by the final destination. Thus, for one stage problem i.e., for j=4, we have the following table.

Staring from last stage, j=4

State s	$F_4^*(s)$	X_4
8	3	10
9	4	10

j=3, s=2

F_3 (s_1 , x_3) = Cs (X_3) + F_4^* (X_3)

s	8	9	F_3^* (s)	X_3^*
5	1+3	4+4	4	8
6	6+3	3+4	7	9
7	3+3	3+4	6	8

j=2

F_2 (s_1 , x_2) = Cs (X_2) + F_3^* (X_2)

s	5	6	7	$F_2^*(s)$	X_2^*
2	7+4	4+7	6+6	11	5 or 6
3	3+4	2+7	4+6	7	5
4	4+4	1+7	5+6	8	5 or 6

j=1

F_2 (s_1 x_1) = Cs (X_1) + F_2^* (X_1)

s	2	3	4	$F_1^*(s)$	X_1^*
1	2+11=13	4+7=11	3+8=11	11	3 or 4

The optimum solution can now be written. The results for the four stage problem indicate the person should initially go from stage 1 either to state 3 or the state 4. Suppose that he chooses x_1^* =3, then the three stage problem results for s=3 is x_2^*=5. This leads to the two stage problem, which gives $x_3^* = 8$ for s = 5, and the one stage problem yields $x_4^* = 10$ for s=8. Hence, one optimal routes are, 1-3-5-8-10. Choosing $x_1^*=4$ leads to the other two optimal routes, 1-4-5-8-10 and 1-4-6-9-10. They all yield a total cost of $f_1^*(1) = 11$.

Example 2

A sales Co. is making a plan for sales promotions. There are 6 salesman and 3 market segments which are to be assigned to them to increase market penetration. The following table gives the estimated increase in the plurality of customers if it were allowed various no. of salespersons.

No. of workers	Market1	Market2	Market3
0	0	0	0
1	25	20	33
2	42	38	43
3	55	54	47
4	63	65	50
5	69	73	52
6	74	80	53

Let 3 segments be 3 stages, the decision variable is x_j, j = 1,2,3, i denote the number of workers at the i^{th} stage from the previous one.

Let $P_j(xj)$ be the expected plurality of customers of xj salesmen in segment j.

Then

Max $Z = P_1(x_1) + P_2(x_2) + P_3(x_3)$

Subject to

$x_1 + x_2 + x_3 = 6 \qquad x_1, x_2, x_3 \geq 0$

Let there be s workers available for j remaining segments and x_j be the optimal assignment. Define $f_j(x_j)$ as the value of the optimal assignment for segment 1 through 3. Thus for stage I.

$f_1(s, x_1) = [P(x_1)]$

If $f_j(s,x_j)$ be the profit associated with optimum solution $f_j^*(s)$, j=1,2,3 then

$$f_1^*(s) = \max_{0 \leq x_1 \leq s} [P(x_1)]$$

Thus, the recurrence equation will be

$f_j(s, x_j) = P_j(x_j) + f_j^*+1(s-x_j)$, j=1,2,3

$$f_j^* = \max_{0 \leq xj \leq s} [P_j(x_j) + f_j^*+1(s-xj)]$$

The solution to the problem staring in j=1 with f_3^*

s	**$f_3^*(s)$**	**x_3^***
0	0	0
1	33	1
2	43	2
3	44	3
4	50	4
5	52	5
6	53	6

Now for j=2 we have following table

S (x_j)	$f_2(s, x_2) = P_2(x_2) + f_3^* (s-x_3)$							Optimal solution	
	0	1	2	3	4	5	6	f2*(s)	x2*
0	0+0							0	0
1	0+33	20+0						33	0
2	0+43	20+33	38+0					53	1
3	0+47	20+43	38+33	54+0				71	2
4	0+50	20+47	38+43	54+33	65+0			87	3
5	0+52	20+50	38+47	54+43	65+33	73+0		98	4
6	0+53	20+52	38+50	54+47	65+43	73+33	80+0	108	4

Now j =3

s\ x_j	$f_1^*(s, x_1) = P_1(x_1) + f_2^* (s-x_2)$							Optimal solution	
	0	1	2	3	4	5	6		
1	0+108 =108	25+98 =123	42+87 =129	55+71 =126	63+59 =102	69+33 =102	74+0 =74	129	2

Thus $x_1^* = 2$, $x_2^* = 3$, $x_3^* = 1$

Example 3: Capital Investment

Let us consider the capital investment in different portfolios and the returns there from. We want allocate capital to maximize the returns. We can use DPP.

	Returns from			
Capital	I	II	III	IV
0	0	0	0	0
1000	2000	3000	2000	1000
2000	4000	5000	3000	3000
3000	6000	7000	4000	5000
4000	7000	9000	5000	6000
5000	8000	10000	5000	7000
6000	9000	11000	5000	8000
7000	9000	12000	8000	8000

Max $Z = P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4$

Subject to

$x_1 + x_2 + x_3 + x_4 = 7000$

$f_1(s,x) = [P_1(x_1)]$ + which implies

$$f_1^*(s) = \max_{0 \le x_1 \le s} [P(x_1)]$$

and $$f_j^*(s) = \max_{0 \le x_j \le s} [P_j(x_j) + f_j^*+ 1 (s-x_j)] \qquad j=1,2,3,4$$

now f4*

s	$f_3*(s)$	x_3*
0	0	0
1	1	1
2	3	2
3	5	3
4	6	4
5	7	5
6	8	6
7	8	6 or 7

	$F_3*(s_3) = [P_2x_3 + f_4* (s\text{-}x_j)]$									
	0	1	2	3	4	5	6	7	$F_3*(s)$	X_3*
0	0+0								0	0
1	0+1	2+0							2	1
2	0+3	2+1	3+0						3	0,1,2
3	0+5	2+3	3+1	4+0					5	0,1
4	0+6	2+5	3+3	4+1	5+0				7	1
5	0+7	2+6	3+5	4+3	5+1	5+0			8	1,2
6	0+8	2+7	3+6	4+5	5+3	5+1	5+0		9	1,2,3
7	0+8	2+8	3+7	4+6	5+5	5+3	5+1	8+0	10	1,2,3,4

	$F_2(s_2) = [P_2x_2 + f_3* (s\text{-}x_3)]$									
	0	1	2	3	4	5	6	7	$F_3*(s)$	X_3*
0	0+0								0	0
1	0+2	3+0							3	1
2	0+3	3+2	5+0						5	1,2
3	0+5	3+3	5+2	7+0					7	2,3
4	0+7	3+5	5+3	7+2	9+0				9	3,4
5	0+8	3+7	5+5	7+3	9+2	10+0			11	4
6	0+9	3+8	5+7	7+5	9+3	10+2	11+0		12	2,3,4,5
7	0+10	3+9	5+8	7+7	9+5	10+3	11+2	12+0	14	3,4

Lastly j for four stage problem

s	0	1	2	3	4	5	6	7	$f_1*(s)$	x_1*
7	0+4	2+12	4+11	6+9	7+7	8+5	9+3	9+0	15	2,3
	4	14	15	15	14	13	12	9		

Note: The students can do the same by employing the law of equi-marginal returns as given below and it is interesting to see that results are same from these two approaches.

Solution of LPP through DPP

Max $Z = x_1+9x_2$

Subject to

$2x_1 + x_2 \leq 25$

$x_2 \leq 11$

$x_1, x_2 \geq 0$

This problem consists of two resources and two variables. The states of Dynamic Program are, therefore, B_{1j} and B_{2j}, j=1,2

Thus,

$f_2(B_{12}, B_{22})$ = Max [9x2]

Where max is taken over $0 \leq x_2 \leq 25$ and $0 \leq x_2 \leq 11$

i.e. $f_2(B_{12}, B_{22}) = 9 \max x_2 = 9x \min [25,11]$

since maximum of x_2 satisfying relations

$x_2 \leq 25$ and $X_2 \leq 11$ is the min of 25 and 11

$\therefore\ x_2^* = 11$

Now $f_1(B_{11},B_{21}) = \max[x_1 + f_2(B_{11}-2x_1, B_{21-0})]$

Where max is taken over $0 \leq x_1 \leq \frac{25}{2}$

As it is the last stage, we substitute the values of B_{11} and B_{21} as 25 and 11.

$f_1(25,11) = \max[x_1 + 9 \min (25-2x_1,11-0)]$

Now, $\min[25-2x_1,11] = [\begin{matrix} 11 for 0 \leq x_1 \leq 7 \\ 25-2x_1 for 7 \leq x_1 \leq \frac{25}{2} \end{matrix}]$

Hence, $x_1+9 [25-2x_1,11] = [\begin{matrix} x1+99 \forall 0 \leq x_1 \leq 7 \\ 225-17x \forall 7 \leq x_1 \leq \frac{25}{2} \end{matrix}]$

Since the max of both $x_1 + 99$ and $225 - 17x$ this at $x_1 = 7$

Thus, $f_1(25,11) = 7 + 9x \min [11,11] = 106$

$x_1 = 7, x_2 = 11, Z = 106$

Example 4:

Max $Z = 50x_1 + 100 x_2$

Subject to the constraints:

$10x_1 + 5x_2 \leq 2500$

$4x_1 + 10x_2 \leq 2000$

$x_1 + 3/2x_2 \leq 450$

$x_1, x_2 \geq 0$

Solution: The problem consists of three resources and two decision variables. The states of the equivalent dynamic programming problem are B_{1j},B_{2j} and B_{3j} for j=1,2. Thus

$f_1(B_{12}, B_{21}, B_{31}) = \max \{50x_1\}$

Where max is taken over $0 \leq 10x_1 \leq 2500$, $0 \leq 4x_1 \leq 2000$ and $0 \leq x_1 \leq 450$

i.e. $f1(B_{12}, B_{21}, B_{31}) = 50 \text{ x max. } (x_1)$

$= 50 * \min. [\frac{2500-5x_2}{10}, \frac{2000-10x_2}{4}, 450 - \frac{3}{2}x_2]$

The second stage problem is to find the value of f_2

$f_2(B_{12}, b_{21}, B_{31}) = \max.[100x_2 + 50 \min. (\frac{2500-5x_2}{10}, \frac{2000-10x_2}{4}, 450 - \frac{3}{2}x_2)]$

Now, the maximum value that x_2 can assume without violating any constraint is given by

$x_2^* = \min [\frac{2500}{5}, \frac{2000}{10}, \frac{450}{3/2}] = 200$

Therefore,

$$\text{Min. }\left[\frac{2500-5x_2}{10},\ \frac{2000-10x_2}{4},\ 450-\frac{3}{2}x_2\right]$$

$$=\begin{cases}\dfrac{2500-5x_2}{10}, if\ 0 \le x_2 \le 125\\[2ex] \dfrac{2000-10x_2}{4}, if\ 125 \le x_2 \le 200\end{cases}$$

Hence f_2 (B_{12}, B_{22}, B_{32}) = max. $0 \le x_2 \le 200$

$$\left\{100x_2 + 50 \text{ X min.}\left(\frac{2500-5x_2}{10},\ \frac{2000-10x_2}{4},\ 450-\frac{3}{2}x_2\right)\right\}$$

$$= \text{min.}\begin{cases}100x_2 + 50\left(\dfrac{2500-5x_2}{10}\right), if\ 0 \le x_2 \le 125\\[2ex] 100x_2 + 50\left(\dfrac{2000-10x_2}{4}\right), if\ 125 \le x_2 \le 200\end{cases}$$

$$= \text{max.}\begin{cases}75x_2+12500, \text{ if } 0\le x_2 \le 125\\ 25000-25x_2, \text{ if } 125\le x_2 \le 200\end{cases}$$

Now max. $(75x_2+12500)=21875$ at $x_2=125$,

And max. $(25000-25x_2)=21875$ at $x_2=125$.

Therefore, f_2^0(2500,2000,450)=21875 at $x_2^0=125$

$$x_1^0 = \text{min.}\left(\frac{2500-5x_2^0}{10},\ \frac{2000-10x_2^0}{4},\ 450-\frac{3}{2}x_2^0\right)$$

= min. (187.5, 187.5, 262.5)

=187.5.

Hence the optimum solution is

$X_1^0=187.5$, $x_2^0= 125.0$

And max. z=21875.

UNIT-VIII

GAME THEORY

In many farm situations we have to take decisions when there are two or more opposite parties with conflicting interests, the action of one is dependent upon the action which opposite takes. Such a situation is known as competitive situation. Each player has a number of choices, finite or infinite called strategies. Different strategies represent a loss, gain or a draw. Each outcome of game may be represented as single pay off. A game where a gain of one player equals loss to the other is known a two persons zero- sum- game or rectangular game.

Let us consider the tossing of a coin by players A and B resulting into following pay off matrix; i.e. A gets Rs. 5 for HH and B for TT. A loses Rs. 2 for TH and B Rs. 1 for HT

		Head	Tail
Player A	Head	5	-1
	Tail	-2	5

Optimum solution of two persons zero sum game is obtained by making use of minimax or maximin principle, according to which player A (whose strategies are represented by rows) selects the strategy which maximizes his min. gain, (maximin), the minimum being taken over strategies of B. In similar way, B selects his strategy which minimizes his maximum loss (minimax). The value of the game is the maximum guaranteed gain to A and minimum possible loss to B.

When maximin = minimax value corresponding to pure strategies are called optimal strategies and the game is said to have *a* **saddle point**.

In general, maximin (lower value) $\leq$ value of the game $\leq$ minimax (upper value) i.e.

$$\underline{\vee} \leq \vee \leq \overline{\vee}$$

$$If \; \underline{\vee} \quad 0 \quad \overline{\vee} \quad fair \;\; game$$

$$\underline{\vee} \quad \vee \quad \overline{\vee} \quad strictly \;\; \det er \min able \;\; game$$

Rules for Determining Saddle Point

1. Select the minimax element of each row put (*)
2. Select the greatest element of each column put (+)
3. It there appeared an element in pay off matrix marked [*] and [+] both, the position of the element is saddle point.

Example 1: Find minimax and maximin for the following matrix

Maximum value 1

$$\begin{bmatrix} 1^* & 3^+ & 6^+ \\ 2 & 1^* & 3 \\ 6^+ & 2 & 1^* \end{bmatrix} \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}$$

Minimax value 3

Thus

$\underline{\vee} \le \vee \le \overline{\vee}$

$1 \le \vee \le 3$

the value of the game lies between 1 & 3.

$\underline{\vee} \ne \overline{\vee}$ so game is not strictly determined

Example 2: Let us consider the pay off matrix

	B_1	B_2	$Row\ \min$
A_1	2^*_+	6^+	2
A_2	-2^*	4	-2
Column max	2	6	

In this case, maximin $\underline{\vee}$ 2 minimax $\overline{\vee}$ 2

Hence, $\underline{\vee}$ $\overline{\vee}$

The best strategy for A = A_1 The best strategy for B = B_1

Value of game $(\vee)$ 2

Example 3: Find the range of value of p and q which will render (2,2) as the saddle point

		Player B		Row min.
	2	4	5	2
Player A	10	7	q	7
	4	p	6	4
Col. max	10	7	6	

$\underline{\vee}$ maximin value 7 $\overline{\vee}$ minimax value 6

Hence, $p \le 7 \le q$

Example 4: Given the (3X3) pay off matrix

	B_1	B_2	B_3	Row min.
A_1	1^{*+}	3	1^{*+}	1
A_2	0	-4	-3	-4
A_3	1	5^{+}	-1	-1
Col max.	1	5	1	

Maximin $\underline{\vee}$ 1 Minimax $\overline{\vee}$ 1

There are two saddle points (1, 1) and (1, 3). The optimal strategies A_1 for A & B_1 for B or A_1 for A and B_3 for B.

The value of the game is 1 for A and -1 for B.

Practical Exercise:

Solve the game whose pay off matrix is

Player B

		B_1	B_2	B_3	B_4
Player A	A_1	1	7	3	4
	A_2	5	6	4	5
	A_3	7	2	0	3

Game Without Saddle Point

When there is no saddle point, the mixed strategies are said to prevail. In that A & B select some strategies according to some probability such that;

$$S_A \begin{bmatrix} A_1 & A_2 & \cdots & \cdots & A_m \\ p_1 & p_2 & \cdots & \cdots & p_m \end{bmatrix}, \quad p_1 + p_2 + \cdots + p_m \quad 1$$

$$S_B \begin{bmatrix} B_1 & B_2 & \cdots & \cdots & B_n \\ q_1 & q_2 & \cdots & \cdots & q_n \end{bmatrix}, \quad q_1 + q_2 + \cdots + q_n \quad 1$$

problem then becomes ..p_i (i=1,2,…….,m) and q_j (j = 1,2,……n)

Let us consider 2X2 game played by A and B

	B_1	B_2
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

$$S_A \quad \begin{vmatrix} A_1 & A_2 \\ p_1 & p_2 \end{vmatrix}, p_1 + p_2 \quad 1$$

$$S_B \quad \begin{vmatrix} B_1 & B_2 \\ q_1 & q_2 \end{vmatrix}, q_1 + q_2 \quad 1$$

Then;

If player B choses B_1 or B_2 then net gain of A

$$E_1(p) \quad a_{11}p_1 + a_{21}p_2$$

$$E_2(p) \quad a_{12}p_1 + a_{22}p_2$$

$$E_1(p) \quad E_2(p)$$

$$a_{11}P_1 + a_{21}P_2 \quad a_{12}p_1 + a_{22}p_2$$

$$a_{11}p_1 - a_{12}p_1 \quad a_{22}p_2 - a_{21}p_2$$

$$p_1[a_{11} - a_{12}] \quad p_2[a_{22} - a_{21}]$$

$$\frac{p_1}{p_2} \quad \frac{a_{22} - a_{21}}{a_{11} - a_{12}}$$

Similary, if A choses strategies A_1 or A_2 then;

$$E_1(q) \quad a_{11}q_1 + a_{12}q_2$$

$$E_2(q) \quad a_{21}q_1 + a_{22}q_2$$

$$a_{11}q_1 - a_{21}q_1 \quad a_{22}q_2 - a_{12}q_2$$

$$q_1(a_{11} - a_{21}) \quad q_2(a_{22} - a_{12})$$

$$\frac{q_1}{q_2} \quad \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

Since $p_1 + p_2 \quad 1$ *and* $q_1 + q_2 \quad 1$

Therefore;

$$p_1 \quad (1 - p_1)\left[\frac{a_{22} - a_{21}}{a_{11} - a_{12}}\right]$$

$$p_1 \quad \frac{a_{22} - a_{21}}{a_{11} - a_{12}} - p_1\left[\frac{a_{22} - a_{21}}{a_{11} - a_{12}}\right]$$

$$p_1\left[1+\frac{a_{22}-a_{21}}{a_{11}-a_{12}}\right] \quad \frac{a_{22}-a_{21}}{a_{11}-a_{12}}$$

$$p1 \quad \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$and, p_2 \quad \frac{a_{11}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

Similary, We can also prove that

$$q_1 \quad \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$q_2 \quad \frac{a_{11}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$E_1(p)$ = V (value of game)

$$V \quad a_{11}p_1 + a_{21}p_2$$

$$a_{11}\left[\frac{a_{22}-a_{21}}{R}\right]+a_{21}\left[\frac{a_{11}-a_{12}}{R}\right]$$

$$\frac{a_{11}a_{22}-a_{11}a_{21}+a_{11}a_{21}-a_{21}a_{12}}{R}$$

$$\frac{a_{11}a_{22}-a_{21}a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

Where; R is= $(a_{11}+a_{22})-(a_{12}+a_{21})$

Solution of Game Theory by Dominance

When one of the pure strategies of either player is inferior to attract one of the remaining strategies, the superior strategies are said to dominate the inferior ones.

General rules for solution by dominance are;

(1) If all the elements of say k^{th} row are less than or equal to the corresponding elements of any other row say r^{th} then k^{th} row is dominated by r^{th} row & k^{th} row can be deleted.

(2) If all the elements of say k^{th} column are greater than r^{th} column then k^{th} column is said to dominate r^{th} and, thus, can be deleted.

Example 5: Given the pay off matrix

		B		
		B_1	B_2	B_3
	A_1	1	7	2
A	A_2	0	2	7
	A_3	5	1	6

In this game there is no saddle point. However, one observes that B_1 is better as (loss is minimum) than B_3 and B_3 can be dropped.

	B_1	B_2
A_1	1	7
A_2	0	2
A_3	5	1

In this A_1 is better (as the gain is more) than A_2 so A_2 can be deleted.

Hence, the game reduces to

	B_1	B_2
A_1	1	7
A_3	5	1

Thus, by considering mixed strategies

$$S_A \quad \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & 0 & p_3 \end{bmatrix}, \quad p_1 + p_2 \quad 1$$

$$S_B \quad \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{bmatrix}, \quad q_1 + q_2 \quad 1$$

Now using 2×2 zero sum game formula for mixed strategies

$$p_1 \quad \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$\frac{(1-5)}{(1+1)-(7+5)} \quad \frac{2}{5}$$

$$p_2 \quad 1 - \frac{2}{5} \quad \frac{3}{5}$$

$$S_A \quad \begin{bmatrix} A_1 & A_2 & A_3 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

$$q_1 \quad \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$\frac{1-7}{-10} \quad \frac{3}{5}$$

$$q_2 \quad 1 - \frac{3}{5} \quad \frac{2}{5}$$

$$S_B \quad \begin{bmatrix} B_1 & B_2 & B_3 \\ \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

$$V \quad \frac{a_{11}.a_{22} - a_{21}.a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \quad \frac{1 \times 1 - 7 \times 5}{10} \quad \frac{-34}{-10} \quad \frac{17}{5}$$

Example 6: Solve the following game by dominance

	B_1	B_2	B_3
A_1	30	40	-80
A_2	0	15	-20
A_3	90	20	50

As A_1 is better than A_2 so A_2 is dropped. Further, B_2 is better than B_1 so B_1 is deleted. We are left with 2×2 pay off matrix.

A_1	40	-80
A_3	20	50

Note: the student can solve this game by using the formula of mixed strategies.

Example 7: Solve the game by using dominance method

	B_1	B_2	B_3
A_1	-5	10	20
A_2	5	-10	-10
A_3	5	-20	-20

Since, B_1 is better than B_3 so B_3 can be deleted. Similarly A_2 is better than A_3 so A_3 can be deleted. Thus**,** 2×2 pay off matrix is

	B_1	B_2
A_1	-5	10
A_2	5	-10

Note the student may solve this problem by using the formula of mixed strategies given eached.

Practical Exercise

Solve the following game by dominance property

		Player B		
		B_1	B_2	B_3
	A_1	2	8	3
Player A	A_2	0	3	9
	A_3	6	1	7

Graphical Solution

Graphical solution enables us to reduce $2 \times n$ or $m \times 2$ games to simplex 2 x 2 games and then solve the 2 x 2 game.

Steps:

(1) Plot the A's expected pay off against B on axis 1 and 2.

(2) Find the maximin on the lower half of the envelop. Let it be H.

(3) Identify the lines (strategies) passing through H and then solve by 2x2 formula.

Example 8:

	Player B		
Player A	1	3	11
	8	5	2

$$Let\ S_A \quad \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad where\ p_2 \quad 1 - p_1\, against\ B$$

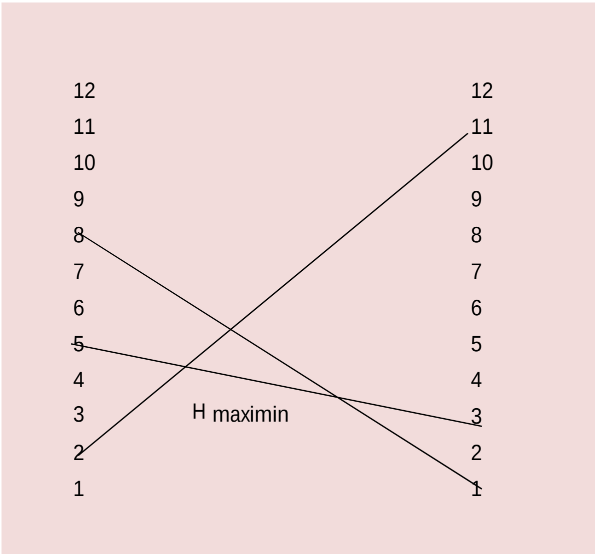
$$Hence,\ A's\ expected\ \ pay\ off\ \ against\ B's\ is$$

$$E_1(p) \quad p_1 + 8p_2 \quad p_1 + 8 - 8p_1 \quad -7p_1 + 8$$

$$E_2(p) \quad 3p_1 + 5p_2 \quad -2p_1 + 5$$

$$E_3(p) \quad 9p_1 + 2$$

$$Now\ plotting\ these\ as\ \ function\ of\ \ p_1\ on\ two\ axis(with\ 0\ \&\ 1, values)$$



Thus, two lines correspond to pay off

	B_1	B_2
A_1	3	11
A_2	5	2

We can now solve this as 2 x 2 pay off matrix with formula for mixed strategies

$$p_1 \quad \frac{3}{11}, p_2 \quad \frac{8}{11}$$

$$q_1 \quad 0, q_2 \quad \frac{2}{11}$$

$$V(value) \quad \frac{49}{11}$$

Example 9: If the game is like given below

	Player B	
Player A	B_1	B_2
	1	-3
	3	5
	-1	6
	4	1
	2	2
	-5	0

In this case plot B's pay off against A's

Therefore,

$E_1(q_1) \quad q_1 + 3q_1 - 3 \quad 4q_1 - 3$

$E_2(q_1) \quad -2q_1 + 5$

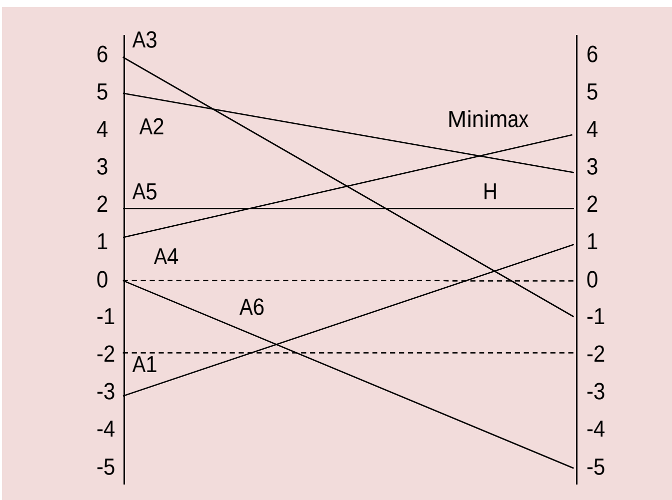
$E_3(q_1) \quad -7q_1 + 6$

$E_4(q_1) \quad 3q_1 + 1$

$E_5(q_1) \quad -2$

$E_6(q_1) \quad -5q_1$

Thus we plot and use minimax criterion on upper undeveloped of the graph



The two lines are w.r.t. pay off

	Player B	
Player A	4	1
	3	5

Now we can solve this 2 x 2 game by using formula of mixed strategies.

Solution of Game by Linear Programming

We can formulate and solve the games through LPP

We know that given the pay off matrix of the game A and B

$$\begin{bmatrix} a_{11} & a_{11} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

Then, we can have the following relation for p

$$a_{11}p_1 \quad + \quad a_{21}p_1 \quad \ldots\ldots \quad + \quad a_{m1}p_m \geq v$$
$$a_{12}p_1 \quad + \quad a_{22}p_1 \quad \ldots\ldots \quad + \quad a_{m2}p_m \geq v$$
$$a_{13}p_1 \quad + \quad a_{23}p_1 \quad \ldots\ldots \quad + \quad a_{m3}p_m \geq v$$
$$a_{1n}p_1 \quad + \quad a_{m}p_n \quad \ldots\ldots \quad + \quad a_{mn}p_n \geq v$$

for q same is

$$a_{11}q_1 \quad + \quad a_{21}q_1 \quad \ldots\ldots \quad + \quad a_{m1}q_n \leq v$$
$$a_{12}q_1 \quad + \quad a_{22}q_1 \quad \ldots\ldots \quad + \quad a_{m2}q_n \leq v$$
$$a_{13}q_1 \quad + \quad a_{23}q_1 \quad \ldots\ldots \quad + \quad a_{m3}q_n \leq v$$
$$a_{1n}q_1 \quad + \quad a_{m}q_n \quad \ldots\ldots \quad + \quad a_{mn}q_n \leq v$$

Dividing both sides by v i.e. value of game

$$a_{11}\frac{p_1}{v} + a_{21}\frac{p_2}{v}\ldots\ldots\ldots\ldots\ldots + a_{m1}\frac{p_m}{v} \geq 1$$

.

.

and for q

$$a_{11}\frac{q_1}{v} + a_{12}\frac{q_2}{v}\ldots\ldots\ldots\ldots\ldots + a_{1n}\frac{q_n}{v} > \leq 1$$

..

..

Let $\frac{p_i}{v}$=p'

Thus, we frame the problem as

For player A

Max v or min 1/v =p_1' + p_2'…………………….p_m'

Subject to

$$a_{11}p_1' + a_{21}p_2'\ldots\ldots\ldots\ldots\ldots\ldots a_{2n}p_m' \geq 1$$

. . .

. . .

. . .

$$a_{1n}p_1' + a_{2n}p_2'\ldots\ldots\ldots\ldots\ldots\ldots a_{mn}p_m' \geq 1$$

and pi =p'* v

Similarly, for player B the problem is

Min v or max 1/v =q_1' + q_2'…………………….q_n'

Subject to

$$a_{11}q_1' + a_{21}q_2' \ldots\ldots\ldots\ldots\ldots\ldots a_{2n}q_n' \leq 1$$
$$a_{1n}q_1' + a_{2n}q_2' \ldots\ldots\ldots\ldots\ldots\ldots a_{mn}q_n' \leq 1$$
$$\vdots$$
$$a_{m1}q_1' + a_{m2}q_2' \ldots\ldots\ldots\ldots\ldots\ldots a_{mn}q_n' \leq 1$$

We can give the value of 1/v =q_0 And the q_j' and then find out

q_j'= [illegible] and v= [illegible]

and we can find out the value of p1' directly from dual values of the final tableau

p_i= [illegible]

Example 10: Given the following pay off matrix, solve the game by using LPP

$$\begin{array}{cc} & B \\ A & \begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 4 & 1 \end{vmatrix} \end{array}$$

Then, we can formulate the LP problem is;

[illegible] $\frac{1}{v}$ [illegible] q_1' + q_2' + q_3'

Subject to

$$3q_1' + q_2' + q_3' \leq 1$$
$$q_1' + q_2' + 5q_3' \leq 1$$
$$q_1' + 4q_2' + q_3' \leq 1$$
$$q_1' \quad q_2' \quad q_3' \geq 0$$

Initial Tableau

S	Xb	b	1	1	1	0	0	0	R min
			q_1'	q_1'	q_1'	S_1	S_2	S_3	
0	S_1	1	3	1	1	1	0	0	1/3
0	S_2	1	1	1	**5**	0	1	0	1/5
0	S_3	1	1	4	1	0	0	1	1/4
q_0		0	0	0	0	0	0	0	

Solve this by simplex iterations

And the final solution is

Cb	Xb		1	1	1	0	0	0
			q_1	q_1	q_1	S_1	S_2	S_3
1	q_1'	6/25	1	0	0	19/25	-3/50	-2/25
1	q_2'	3/25	0	0	1	-3/50	11/50	-1/25
1	q_3'	4/25	0	1	0	-2/25	-1/25	7/25
	q_0	13/25	0	0	0	6/25	3/25	4/25

$$Value\,of\;\;game\,v\quad \frac{1}{q_0}\quad 1*\frac{25}{13}$$

The strategies are ;

$$q_1\quad \frac{q_1'}{q_0}\quad \frac{6}{25}*\frac{25}{13}\quad \frac{6}{13}$$

$$q_2\quad \frac{4}{25}*\frac{25}{13}\quad \frac{6}{13}$$

$$q_3\quad \frac{3}{25}*\frac{25}{13}\quad \frac{3}{13}$$

The strategy for A can be found from dual value as

$$p_1'\quad \frac{6}{13}\quad or\,\frac{6}{25}*\frac{25}{13}\quad \frac{6}{13}$$

$$p_2'\quad \frac{3}{25}\quad or\,\frac{3}{25}*\frac{25}{13}\quad \frac{3}{13}$$

$$p_3'\quad \frac{4}{25}\quad or\,\frac{4}{25}*\frac{25}{13}\quad \frac{4}{13}$$

So $S_A=\begin{bmatrix} A_1 & A_2 & A_3 \\ 6/13 & 3/13 & 4/13 \end{bmatrix}$

$S_B=\begin{bmatrix} B_1 & B_2 & B_3 \\ 6/13 & 4/13 & 3/13 \end{bmatrix}$

Value of game $=\frac{25}{13}$

UNIT IX

INVENTORY MANAGEMENT

Inventory, in wider sense, is defined as idle resource stock of an enterprise. It is a physical stock of goods kept for future use. It may be raw material, packaging, spares and other stock in order to meet an unexpected demand or distribution in future.

Importance of Keeping Inventory

(1) Smooth and efficient running of business

(2) Providing adequate service to customers

(3) Take advantage of price discounts of bulk.

(4) Serve as a buffer in case of delayed delivery of raw materials

(5) Provide flexibility to allow changes in production plans

(6) Make possible economies of scale in transportation, clearing & forwarding

Inventory Control

Inventory control includes the following pertinent aspects;

(a) When should an order for materials be placed?

(b) How much flow should be produced & what quantity should be ordered each time?

(c) Whether or not to avail quantity or price discounts?

Concepts/Definition

1. **Set up Cost (CS):** Cost associated with setting up of machinery before starting production. It is independent of quantity ordered or produced.
2. **Ordering Cost:** Cost associated with ordering of raw material, advertisement, postage, stationary, telephone, TA/DA etc.
3. **Production Cost/Purchase Price Cost:** When large production runs are in process, these results in reduction of production cost/quantity or price discounts.
4. **Holding or Storage Cost:** Cost of storage i.e. lose when goods are in short supply
5. **Shortage Cost:** These are the costs involved with the inadequacy of stock to the demand. Shortage may result in cancellation of orders & losses in sale & or goodwill or profit.
6. **Demand:** No. of required per period and may be known or unknown. DD may be deterministic (if fixed) and probabilistic if variable.
7. **Lead Time:** Time between placing an order and its actual arrival in inventory is known as lead time.
8. **Stock Replenishment:** The actual replacement of stock by new inventory.

Important Notations

Q = Lot size in each production run

C_S = Set up cost of production run

C_1 = Inventory holding cost/unit/time

C_2 = Shortage cost/unit/time

I = Cost of carrying one Re in inventory for a unit time (Storage cost).

P = Cost of one item of product in Rs.

r = Demand rate

k = Production/supply rate

L = Lead time

C_A = Average total cost per unit time

B = Buffer stock

t = Time interval between two consecutive replenishment

Inventory Models

Model I: Lot size problem with no shortages

Demand is known & Q denote lot size

No shortages permitted

Production or supply is instantaneous

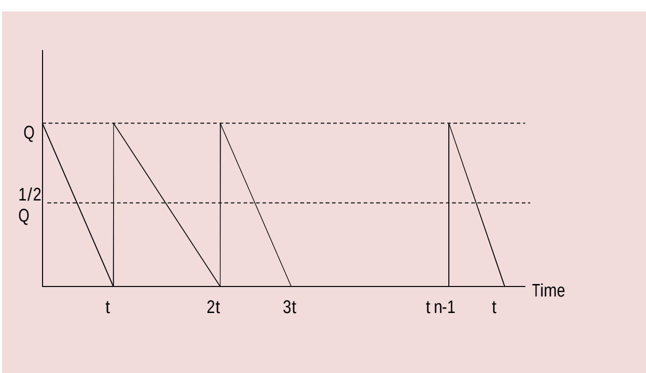
Holding cost C_1 & set up C_s

D (Demand)= n.Q

Average inventory = $\frac{1}{2}Q$

Total inventory over time t is $\frac{1}{2}Q$

Inventory Model with no shortages

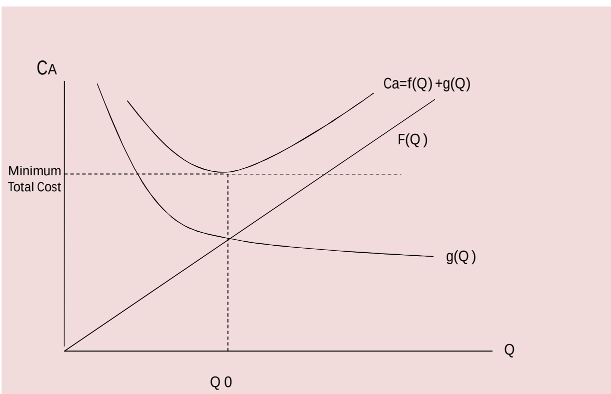


So inventory on any day = $\left(\dfrac{\frac{1}{2}Qt}{t} \right) \quad \dfrac{1}{2}Q$

Holding cost f (Q) = $\dfrac{1}{2}QC_1$ (C_1= holding cost per unit)

Set up cost g (Q) = nC_s = $\dfrac{D}{Q}C_s$

$$Total\ \cos t \quad C_A \quad \frac{1}{2}QC_1 + \frac{D}{Q}C_S$$



Minimum Total Cost

We want to minimize total cost.

$$\frac{\partial C_A}{\partial Q} \quad 0 \quad \frac{1}{2}C_1 - \frac{D}{Q^2}C_S$$

$$\frac{D}{Q^2}C_S \quad \frac{1}{2}C_1$$

$$\frac{1}{2}Q^2C_1 \quad DC_S$$

$$Q^2C_1 \quad 2DC_S$$

$$Q^2 \quad \frac{2DC_S}{C_1}$$

$$Q^0 \quad \sqrt{\frac{2DC_S}{C_1}}$$

$$t \quad \frac{Q}{D} \quad \sqrt{\frac{2DC_S}{C_1}} \cdot \frac{1}{D} \quad \sqrt{\frac{2C_S}{C_1D}}$$

$$C_A \quad \frac{1}{2}QC_1 + \frac{D}{Q}C_S$$

$$\frac{1}{2}\sqrt{\frac{2DC_S}{C_1}} \cdot C_1 + \frac{D}{\sqrt{\frac{2DC_S}{C_1}}} \cdot C_S$$

$$\frac{1}{2}\sqrt{2DC_SC_1} + \sqrt{\frac{2DC_SC_1}{4}}$$

$$\sqrt{2DC_SC_1}\left(\frac{1}{2}+\frac{1}{2}\right) \quad \sqrt{2DC_SC_1}$$

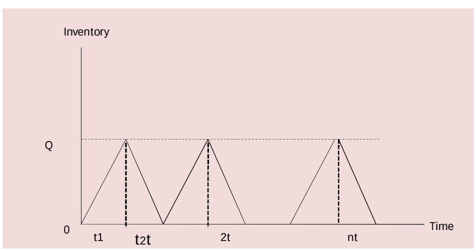
Model II: Inventory with finite rate of replenishment & no shortages.

Let rate of replenishment is k units per unit time. Let each production of length t consists of two parts t_1 & t_2 such that

i.) At t_1 inventory is building at constant rate (k-r)

ii.) At t_2 there is no replenishment and inventory decreasing at r per unit time

Inventory model with replenishment



At the end of t_1, let the level of inventory be S and at the end of t_2, let it be zero; then

$$S \quad t_1(k-r) \, and \; S \quad t_2 r$$

if Q is lot size we must have

$$S \quad Q - rt \text{ where } Q \text{ is total quantity and } rt_1 \text{ is utilized in time } t_1$$

$$thus; \quad S \quad (Q-S)(k-r)/r$$

$$rS \quad kQ - kS - rQ + S^{/}r$$

$$kQ - kS - rQ \quad 0$$

$$kS \quad kQ - rQ$$

$$S \quad \left(\frac{(k-r)Q}{k}\right)$$

$$Total \; inventory \, in \; t(t_1+t_2) \; (total \; area \; of \; 1^{st} \; triangle)$$

$$\frac{1}{2}S\,(t_1+t_2)$$

$$and \; average \, inventory \, is \; \frac{1}{2}S\,(t_1+t_2)/(t_1+t_2) \quad \frac{1}{2}S$$

Thus the annual inventory holding cost is given by

$$\frac{1}{2}SC_1 \quad \frac{1}{2}(k-r)Q\frac{C_1}{k}$$

$$\textit{set up } \cos t \textit{ for unit time } t \textit{ is } \frac{C_S}{(t_1+t_2)}$$

$$\sin ce\, t_1 + t_2 \quad \frac{S}{k-r}+\frac{S}{r} \quad \frac{Sk}{r(k-r)}$$

$$\frac{(k-r)Q/k}{r(k-r)}\cdot k \quad \frac{Q}{r}$$

$$\textit{so total } \cos t \textit{ per unit is}$$

$$C_A \quad \frac{1}{2}\left(\frac{k-r}{k}\right)QC_1 + \frac{r}{Q}C_S$$

$$\frac{\partial C_A}{\partial Q} \quad \frac{1}{2}\left(\frac{k-r}{k}\right)C_1 + \frac{r}{Q^2}C_S \quad 0$$

$$\frac{r}{Q^2}C_S \quad \frac{1}{2}\left(\frac{k-r}{k}\right)C_1$$

$$Q^2(k-r)C_1 \quad 2krC_S$$

$$Q^0 \quad \sqrt{\frac{2C_S}{C_1}.\frac{rk}{k-r}} \quad (Optimum quantity)$$

$$t^0 \quad \frac{Q^0}{r} \quad \sqrt{\frac{2C_S}{C_1}\cdot\frac{rk}{k-r}}\cdot\frac{1}{r}$$

$$\sqrt{\frac{2C_S}{rC_1}\cdot\frac{k}{k-r}}$$

$$C_A \quad \sqrt{2C_SC_1r}\sqrt{\frac{k-r}{k}}$$

Example 1: Let us consider the following inventory (lot size) problem with no shortages.

$D \quad 600\, units\ per\ year$

$C_1 \quad Rs.\,0.60 / unit / year$

$C_S \quad Rs.\,80.00\ per\ production\,run. Thus, optimum\,lot\,size\,is$

$$Q^0 \quad \sqrt{\frac{2DC_S}{C_1}} \quad \sqrt{\frac{2\times600\times80\times100}{0.60}}$$

$$\sqrt{160000} \quad 400\, units;$$

$$t \quad \frac{Q^0}{D} \quad \frac{400}{600} \quad \frac{2}{3}\, years\ or\ 8\, months$$

$$C_A \quad \sqrt{2DC_SC_1} \quad \sqrt{\frac{2\times600\times80\times0.60}{100}}$$

$$\sqrt{57600} \quad 240$$

$No. of\ orders$

$$\frac{D}{Q^0} \quad \frac{600}{400} \quad 1.5\, orders\ per\ year$$

Example 2: Let us consider the following inventory problem with finite rate of replenishment of no shortages.

$C_1 \quad Rs\,0.01\ per\ day$

$C_S \quad Rs\,100\ per\ set\ up$

$r \quad 25\, units\ per\ day$

$k \quad 50\, units\ per\ day$

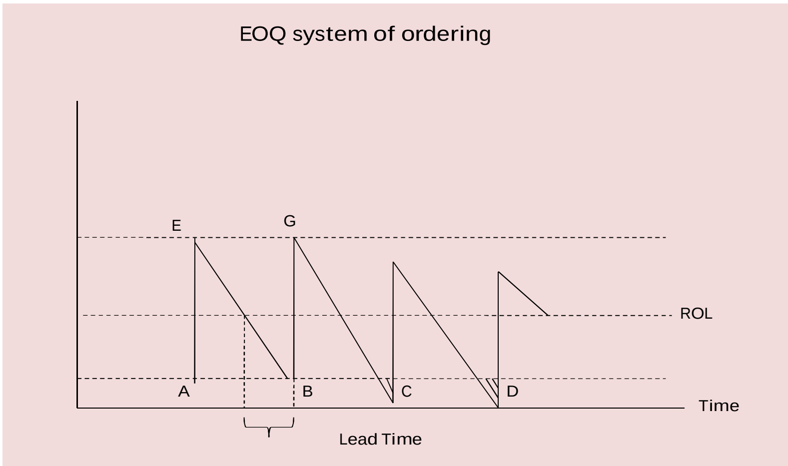
$$Q^0 \quad \sqrt{\frac{2C_S}{C_1}}\sqrt{\frac{rk}{k-r}} \quad \sqrt{\frac{2\times100}{0.01}}\cdot\sqrt{\frac{25\times50}{50-25}} \quad \sqrt{20000\times50} \quad \sqrt{1000000} \quad 1000\, units$$

$$t^0 \quad \frac{Q}{r} \quad \frac{1000}{25} \quad 40\, days$$

$$C_A \quad \sqrt{2C_SC_1r}\sqrt{\frac{k-r}{k}} \quad \sqrt{2\times100\times0.01\times25}\times\sqrt{\frac{50-25}{50}} \quad \sqrt{50\times\frac{1}{2}} \quad Rs\,5\ per\,day$$

$so\ total\ \mathrm{cost}\ per\ unit \quad 5\times40 \quad Rs.\,200$

Model III: Economic Order Quantity (EOQ) System of Ordering



1. E.O.Q $\Rightarrow Q^0 = \sqrt{\dfrac{2DC_S}{C_1}}$
2. Buffer stock (B)= [Max. lead time – Normal lead time] $\times r$, where r is the rate of consumption or use.
3. Re-order level (ROL) will be

R.O.L = Buffer stock + Normal lead time consumption

a. The maximum inventory will be B + Q^0

b. Minimum inventory = B

c. Average inventory = B + $\dfrac{1}{2}Q^0$

Example 3: From the following example we can find out the economic order quality (EOQ).

D = 10000 units cost of one unit = Rs. 100

C_1 = Rs 0.24/unit

C_s = Rs 12.00 per run

Past lead time = 15 days & 30 days

$$Q^0(EOQ) \quad \sqrt{\frac{2C_S D}{C_1}} \quad \sqrt{\frac{2\times 12\times 10000}{0.24}} \quad 1000\,units$$

$$Optimum\ Buffer\ Stock \quad (ML - NL)r$$

$$\frac{(30-15)}{30}\times\frac{10000}{12} \quad 416.66\ or\ say\ 450\ units$$

$$Normal\ lead\ time\ consumption \quad \frac{15}{30} \times \frac{10000}{12}$$

$$416.66\ units$$

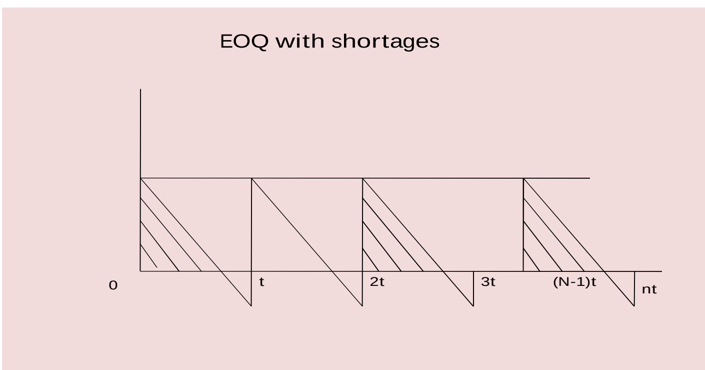
$$ROL \quad 450 + 417 \quad 867\ units$$

$$Max\ Inventory \quad 450 + 1000 \quad 1450\ , Minimum\ \ inventory \quad 450\ units$$

$$Average\ inventory \quad 450 + \frac{1000}{2}$$

$$450 + 500 \quad 950\ units \quad or \quad \frac{1450 + 450}{2} \quad 950\ units$$

Model IV: E.O.Q with shortage



$$t \quad t_1 + t_2$$

$$Q \quad Q_1 + Q_2$$

Total inventory over time = $\frac{1}{2}Q_1 t_1$

$$Average\ inventory \quad \frac{\left(\frac{1}{2}Q_1 t_1\right)}{t}$$

$$Holding\ \text{cost} \quad \frac{C_1\left(\frac{1}{2}Q_1 t_1\right)}{t}$$

Total amount of shortage in t_2 = $\frac{1}{2}Q_2 t_2$

Shortage cost = $\dfrac{C_2\left(\dfrac{1}{2}\right)Q_2 t_2}{t}$

Set up cost associated with run of size Q = $\dfrac{DC_S}{Q}$

Therefore, total cost C_A will be;

$$C_A \quad \frac{C_1\left(\frac{1}{2}Q_1 t_1\right)}{t} + C_2\left(\frac{\frac{1}{2}Q_2 t_2}{t}\right) + \frac{D}{Q}C_S \cdots\cdots(A)$$

Now triangles are symmetrical so

$$\frac{t_1}{t} \quad \frac{Q_1}{Q}; \frac{t_2}{t} \quad \frac{Q_2}{Q}$$

$$t_1 \quad \frac{Q_1}{Q}t \;,\; t_2 \quad \frac{Q_2}{Q}t$$

substituting these values in (C_A) *we get*

$$C_A \quad \frac{1}{2}C_1\left(\frac{Q_1^{\,2}}{Q}\right) + \frac{1}{2}C_2\left[\frac{(Q-Q_1)^2}{Q_1}\right] + C_S\frac{D}{Q}$$

$$\frac{\partial C_A}{\partial Q_1} \quad 0$$

$$\frac{\partial C_A}{\partial Q} \quad 0$$

on simplification the optimum quantity will be

$$Q^{0} \quad \sqrt{\frac{2C_S D}{C_1}} \cdot \sqrt{\frac{(C_1+C_2)}{C_2}}$$

$$Q_1 \quad \sqrt{\frac{2C_S D}{C_1}} \cdot \sqrt{\frac{C_2}{(C_1+C_2)}}$$

$$\quad \frac{C_2}{(C_1+C_2)} \cdot \sqrt{\frac{(C_1+C_2)}{C_2}}\sqrt{\frac{2C_S D}{C_1}}$$

$$C_A \quad \sqrt{2C_S C_1 D} \cdot \sqrt{\frac{C_2}{(C_1+C_2)}}$$

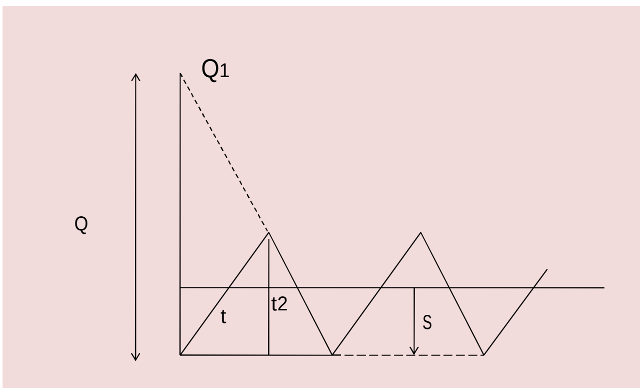
and if inventory is replenished then

$$Q^0 \quad \sqrt{\frac{2C_S(C_1+C_2)}{C_1C_2}}\sqrt{\frac{rk}{k-r}}$$

$$t^0 \quad \frac{Q}{r}$$

$$Q_1 \quad \sqrt{\frac{2C_2C_S}{C_1(C_1+C_2)}}\cdot\sqrt{\frac{r(k-1)}{k}}$$

$$S \quad \sqrt{\frac{2C_SC_1}{(C_1+C_2)C_2}}\sqrt{\frac{r(k-r)}{k}}$$



$$C_A \quad \sqrt{\frac{2C_1C_2C_S}{C_1+C_2}}\cdot\sqrt{\frac{r(k-r)}{k}}$$

Example 4: Let us consider the following example

$C_1 \quad Rs.16.00 \ per\ item\ per\ month$

$C_2 \quad Rs\,10.00\ per\ item\ per\ day$

$C_S \quad Rs\,10{,}000$

$D \quad 25\, items\ per\ day$

$$Q^0 \quad \sqrt{\frac{2C_SD}{C_1}}\sqrt{\frac{C_2}{C_1+C_2}} \quad \sqrt{\frac{2\times 10000\times 25}{16/30}}\sqrt{\frac{10}{\left(\frac{16}{30}+10\right)}} \quad 943\, items\,(approx)$$

$$t^0 \quad \frac{Q^0}{D} \quad \frac{943}{25} \quad 38\, days$$

Model V: Inventory Model with Price Breaks

Quantity Discounts:

The basic EOQ model assumes a fixed price, therefore, the total cost equation does not include the price of the item, because it remains constant and is not a relevant factor in deciding the quantity to be ordered/stored. Let us now consider a model that includes the value of an item as a factor in order to take into account the quantity discounts. The incremental cost associated with such a system is then;

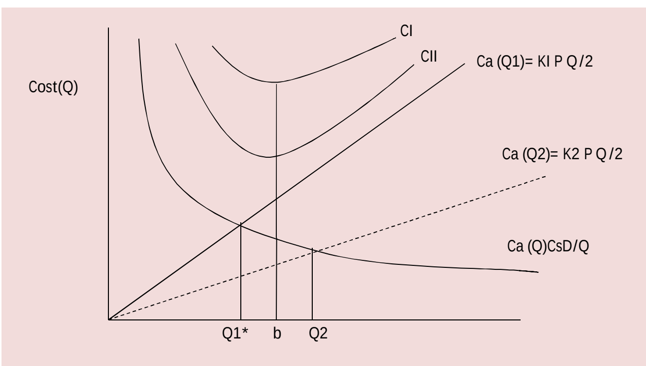
$C_A(Q)$ = Annual cost of placing order + Annual purchase cost of D items + Annual holding cost for inventory

$$C_A(Q) \quad \frac{C_s D}{Q} + DK + KP\frac{Q}{2}$$

where K is the cost/price per unit of P fraction of inventory value ($KP = C_1$)

The optimum quantity will be (Q_n^0)

Let the price breaks are available at b_n quantity.



Steps:

1. Compute Q_n^0 $if\ Q_n \geq b_{n-1}$, the optimum purchase quantity is reached.
2. If $Q_n<b_{n-1}$, then compute Q_{n-1}^0. If $Q_{n-1}^0 \geq b_{n-2}$ avail the price break and compare $C_A(Q_{n-1})$ with $C_A(b_{n-1})$.
3. If $Q_{n-1}<b_{n-2}$, compute Q_{n-2}^0, if $Q_{n-2} \geq b_{n-3}$ avail two price breaks by comparing $C_A(Q_{n-2})$ and $C_A(b_{n-2})$ and $C_A(b_{n-1})$, continue in this way.

Example 5: Consider the following inventory model.

C_s = 5.00, D = 1600 units, b = 800

P = Rs. 0.10, K_1 = Rs. 1, K_2 = Rs. 0.98 (with price breaks)

$$Q_2^0 \quad \sqrt{\frac{2C_s D}{K_2 P}} \quad \sqrt{\frac{2\times5\times1600}{(0.10)(0.98)}} \quad 404\,units$$

$$Now\,Q_2 \quad 404, b \quad 800 \quad Q_2 < b$$

So using step 2 we compute Q_1

$$Q_1^0 \quad \sqrt{\frac{2\times 5\times 1600}{(0.10)(1)}} \quad 400\, units$$

Now $Q_1^0 < b$, we compare the cost by computing least quantity which will entail in a price break

$$C_A\left(Q_1^0\right) \quad \sqrt{2C_S DKP} + DK_1$$
$$\sqrt{2\times 5\times 1000\times 1\times 0.10} + 1600\times 1$$
$$Rs.1640$$

$$and\ C_A(b) \quad C_S\frac{D}{b} + DK_2 + K_2 P\frac{b}{2}$$
$$5\times\frac{1600}{800} + 1600\times 0.98 + \frac{1}{2}(0.98)\times 0.1\times\frac{800}{2}$$
$$Rs.1617.20$$

This shows that

$$1617 < Rs.1640\ i.e \quad C_A(b) < C_A(Q_1)$$

Hence, the optimum purchase quantity should be $b \quad 800\, units$

Example 6: Let us consider inventory model with two price breaks is;

Twopricebreaks	Unit Cost
$0 \le Q_1 < 50$	10
$50 \le Q_2 < 100$	9.00
$100 \le Q_3$	8.00

$$D \quad 200, \quad C_S \quad 20, \quad P \quad 0.25,$$

$$Q_3 \quad \sqrt{\frac{2C_S D}{K_3 P}} \quad \sqrt{\frac{2\times 20\times 200}{8\times 0.25}} \quad 63$$

$$63 < 100 \quad Q_3 < b,\ so\ we\ compute$$

$$Q_2 \quad \sqrt{\frac{2C_S D}{K_2 P}} \quad \sqrt{\frac{2\times 20\times 200}{9\times 0.25}} \quad 60$$

$Q_2 > b_1(50)$, the optimum purchase quantity is determined by comparing $C_A(Q_2)$ with $C_A(b_2)$

$$C_A(Q_2) \quad 20\times\frac{200}{60} + 200\times 9 + 9\times 0.25\times\frac{60}{2}$$
$$1934.16$$

$$C_A(b_2) \quad 20\times\frac{200}{100} + 200\times 8 + 8\times 0.25\times\frac{100}{2} \quad 1740$$

$$\sin ce\ C_A(Q_2) > C_A(b_2), \quad so\ optimum\ purchase\ is\ 100\, units$$

Example 7: Consider 3 price breaks as given below;

$D \quad 400 \; units \; per \; month$

$P \quad 20\% , C_S \quad Rs. 25.00$

Range	Price Breaks
$0 \le Q_1 < 100$	$Rs\ 20.00$
$100 \le Q_2 < 200$	$Rs.18.00$
$200 \le Q_3$	$Rs.16.00$

$$Q_3^{\,0} \quad \sqrt{\frac{2C_S D}{K_3 P}} \quad \sqrt{\frac{2\times 25\times 400\times 10}{16\times 0.2}} \quad 79 \; units$$

Since Q2 < 200, so we examine Q2

$$Q_2^{\,0} \quad \sqrt{\frac{2C_S D}{K_2 P}} \quad \sqrt{\frac{2\times 25\times 400}{18\times 0.2}} \quad 75$$

since $Q_2 < b_1 \quad 100$, so we examine Q_1

$$Q_1^{\,0} \quad \sqrt{\frac{2C_S D}{K_1 P}} \quad \sqrt{\frac{2\times 25\times 400}{20\times 0.2}} \quad 70$$

$$Now\ C_A(Q_1) \quad 25\times\frac{400}{70}+400\times 20+20\times 0.2\times\frac{400}{2} \quad 8283$$

$$or\ C_A(Q) \quad \sqrt{2C_S D K_1 P}+DK_1$$

$$\sqrt{2\times 25\times 400\times 20\times 0.2}+400\times 20$$

$$283+8000 \quad Rs.8283$$

$$C_A(b_1) \quad 25\times\frac{400}{100}+400\times 18+18\times\frac{0.2}{10}\times\frac{100}{2}$$

$$100\times 7200+180 \quad 7480$$

$$C_A(b_2) \quad 25\times\frac{400}{200}+400\times 16+16\times\frac{0.2}{10}\times\frac{200}{2}$$

$$50+6400+320 \quad Rs.6770$$

$Hence,\ Q^0 \equiv b_2 \quad 200\ units$

UNIT- X

NON-LINEAR PROGRAMMING

Non Linear programming deals with the optimizing an objective function in the presence of equality and inequality constraints

let Ƶ be a real valued function of a variable

Ƶ $f(x_1, x_2, x_3, \ldots\ldots x_n)$

let $(b_1, b_2, b_3 \quad b_m)$ be a set of constraints such that

$g_1 (x_1, x_2, x_3, \ldots\ldots x_n) \leq \geq = b_1$

$g_2 (x_1, x_2, x_3, \ldots\ldots x_n) \leq \geq = b_2$

$g_m (x_1, x_2, x_3, \ldots\ldots x_n) \leq \geq = b_m$

$x_j \geq 0 \qquad j = 1,2,3, \ldots\ldots n$

in generalized form, GNLPP may be written as;

Ƶ = f(x) where f (x) is non-linear

Subject to

$gi (x) \leq \geq = b_i$

$x_j \geq = 0$

where f(x) or g(x) or both are non-linear, sometimes, it is convenient to write as

$gi (x) \leq \geq = b_i$ or $hi (x) \leq \geq = 0$

where; $h_i (x) = g_i (x) - b_i$

Kuhn Tucker Conditions

Kuhn Tucker conditions offer the necessary and sufficient conditions for solving non-linear problems. Kuhn Tucker conditions for maximisation are:

$$f_i \quad \sum_{i \quad 1}^{m} \lambda_i h_i (i \quad 1,2,3,\ldots\ldots m)$$

$$\lambda_i h' \quad 0 \quad (i \quad 1,2,3,\ldots\ldots m)$$

$$h_i = \leq 0 \quad (i \quad 1,2,3,\ldots\ldots m)$$

$$\lambda_i \quad 0 \quad (i \quad 1 > 2,3,\ldots\ldots m)$$

$$h_j' \quad \frac{\partial h^i}{\partial x_j}$$

Constrained minima or maxima

Ƶ= f(x)

Subject to

g(x) =c

hx= gx-c

We define Langrangian function $L(x_1, x_2, x_3, \ldots\ldots \lambda)$ as;

$L(x_1, x_2, x_3, \ldots\ldots \lambda) = f(x_1, x_2, x_3 \ldots\ldots) - \lambda h(x_1, x_2, x_3 \ldots\ldots)$

Find partial derivatives

$$\frac{\partial L}{\partial x_1} \quad f_1 - \lambda h_1 \quad 0$$

$$\frac{\partial L}{\partial x_2} \quad f_2 - \lambda h_2 \quad 0$$

……………………

……………………

……………………

$$\frac{\partial L}{\partial x_n} \quad f_n - \lambda h_n \quad 0$$

$$\frac{\partial L}{\partial \lambda} \quad -h \quad 0$$

We can solve these equations and find the values of $x_1, x_2, \ldots..$ and $.\lambda$

The necessary and sufficient conditions for max. (min.) of the objective functions are concave (convex) and the constraints are equalities.

The sufficient conditions are given by the evaluation of principal minors

$$\begin{matrix} -\lambda \\ x_1 \\ \cdots \\ x_n \end{matrix} \begin{bmatrix} L'_{\lambda\lambda} & L'_{\lambda x1} & .. & L'_{\lambda xn} \\ L'_{x1\lambda} & L'_{x1x1} & \cdots & L'_{x1xn} \\ \cdots & \cdots & \cdots & \cdots \\ L'_{\lambda n} & L'_{\lambda xnx1} & \cdots & L'_{xnxn} \end{bmatrix}$$

Necessary and sufficient conditions for max. or min. of a function

For Single variable

		Maxima	Minima
I	$f^{/}(y)\ \frac{\partial y}{\partial \lambda}$	=0	=0
II	$f^{//}(x)\ \frac{\partial^2 y}{\partial x^2}$	<0	>0

For Two variables

I	$f^{/}(x)$, $f^{/}(y)$,	=0	=0
II	$f^{//}(x)*f^{//}(y)$	<0	>0
III	$f^{//}(xy)^2 < f^{//}xx\ f^{//}yy$		$f^{//}(xy) < f^{//}xx\ f^{//}yy$

Example 1:

Examine whether the function Ƶ= $x^2\ y^2$-2x-4y has a minima or maxima

$\frac{\partial z}{\partial x}$ = 2x-2=0 x=1

$\frac{\partial z}{\partial y}$ =2y-4=0 y=2

Now $\frac{\partial^2 z}{\partial x^2}$ 2 > 0 $\frac{\partial^2 z}{\partial y^2}$ => 0

$\frac{\partial^2 z}{\partial x \partial y}$ 0 $\frac{\partial^2 z}{\partial y \partial x}$ 0

Now $(f^{//}xy)^2 < f^{//}(xx)\ f^{//}(yy)$

As $(o)^2 < (2)\ (2)$; so the function has a minimum value

Example 2:

Examine whether the function Ƶ = 2x +4y-y^2-x^2 has minimum or maximum value

$\frac{\partial z}{\partial x}$ = 2-2x=0

$\frac{\partial z}{\partial x}$ = 4-2y=0

x=1, y=2

$\frac{\partial^2 z}{\partial x^2}$ =-2<0 $\frac{\partial^2 z}{\partial y^2}$ =-2<0

$\frac{\partial^2 z}{\partial x \partial y}$ 0 $\frac{\partial^2 z}{\partial y \partial y}$ 0

Thus, $f^{//}xx) < 0$ $f^{//}yy) < 0$

$f^{//}(xy)^2 < f^{//}(xx).\ f^{//}(yy)^2$

$(0)^2 < (-2)\ (-2) = 4$, thus function has minimum value

Function f(x) Non-Linear & Constraint Linear

Ƶ= $f(x_1, x_2) = 3.6\ x_1 - 0.4\ x_1^2 + 1.6\ x_2 - 0.2\ x_2^2$

Subject to

$2\ x_1+x_2=10$

$x_1,\ x_2 \geq 0$

Now Max. Ƶ = $f(x_1,\ x_2)$

Subject to

$h\ (x_1,\ x_2\ )=0$

Where, $h\ (x_1,\ x_2\ ) = 2x_1, + x_2 -10=0$

We define lagrangian function L (x)

$L(x_1,\ x_2,\ \lambda)= f(x_1,x_2) - \lambda\ h(x_1,\ x_2)$, and find derivatives

$\frac{\partial L}{\partial x_1}$ = 3.6-0.8x_1-2 λ=0

$\frac{\partial L}{\partial x_2}$ = 1.6-0.8x_2- λ =0

$\frac{\partial L}{\partial \lambda}$ = -2x_1-x_2=10=0 solving these equations by crammer's rule, values are;

x_1= 3.5
x_2= 3
λ= 0.4
Ƶ =10.7

To examine whether it is maximum or minimum, we form bordered Hessian Det.

$$H^{BO} \begin{bmatrix} 0 & -2 & -1 \\ 2 & -0.8 & 0 \\ 1 & 0 & -0.4 \end{bmatrix}$$

and examine the signs of principal minors

$$\Delta_2 \quad \begin{bmatrix} 0 & 2 \\ 2 & -0.8 \end{bmatrix} =(-0.8-4)=-4.8<0$$

Δ_3= 0 [Δ]-2(-0.8-0)+1(0+0.8)

0+1.6+0.8= 2.4>0

So Δ_2, Δ_3 have alternate signs, so function has maximum value.

Note: if principal minors are of alternate signs, the function has a maxima.

Example 3:

Min Z= $2x_1^2-24x_1+2x_2^2-8x_2+2x_3^2-12x_3+200$

Subject to

$x_1+x_2+x3=11$

$x_1,x_2,x_3>=0$

We can formulate Lagrangian function as:

$L\ (x_1, x_2, x_3, \lambda) = 2x_1^2-24x_1+2x_2^2-8x_2+2x_3^2-12x_3+200-\lambda\,(x_1+x_2+x_3-11)$

The necessary conditions for the stationary point are;

$$\frac{\partial L}{\partial x_1} \quad 4x_1 - 24 - \lambda \quad 0$$

$$\frac{\partial L}{\partial x_2} \quad 4x_2 - 8 - \lambda \quad 0$$

$$\frac{\partial L}{\partial x_3} \quad 4x_3 - 12 - \lambda \quad 0$$

$$\frac{\partial L}{\partial \lambda} \quad -(x_1 + x_2 + x_3 - 11) \quad 0$$

The solution of these simultaneous equations yield the stationary point

$X_0 = (x_1, x_2, x_3) = (6,2,3);\ \lambda = 0$

The sufficient conditions for the stationary point to be a minimum is that the minors Δ_3 and Δ_4 be both negative. Now we have

$$\Delta_3 \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix} \quad -8$$

and

$$\Delta_4 \quad \begin{matrix} \lambda \\ X_1 \\ X_2 \\ X_3 \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \quad -48$$

which are both negative. Thus x_0 (6,2,3) provides the solution to the NLPP.

Solution to NLPP with Inequality Constraints

To solve inequality constraints, we use lagrangian function and in addition we examine the Kuhn Tucker conditions to find out whether the function is minimum or maximum.

The Kuhn Tucker conditions are

$$\frac{\partial f(x_j)}{\partial x_j} - \lambda_i \frac{\partial h_j^i(x_j)}{\partial x_j} \quad 0 \qquad 1$$

$$\lambda_i h_i(x_j) \quad 0 \qquad 2$$

$$hi(x_j) \le 0 \qquad 3$$

$$\lambda \ge 0 \quad \cdots\cdots \qquad 4$$

It is easy to observe that for maximisation h(x) is convex in (x) and f(x) is concave in (x), thus, Kuhn Tucker conditions are the sufficient conditions for maximum.

Example 4

Max Ƶ $=3.6x_1-0.04x_1^2 +1.6x_2-0.2x_2^2$

$2x_1+ x_2 <10$

Here f(x) = $3.6x_1-0.4x_1^2 +1.6x_2-0.2x_2^2$

g(x) $=2x_1+x_2$; c=10

h(x) = g(x)-c= $2x_1+x_2-10=0$

The Kuhn Tucker conditions are

$$\frac{\partial f(x_j)}{\partial x_j} - \lambda_i \frac{\partial h_j^i(x_j)}{\partial x_j} \quad 0$$

$3.6-0.8x_1-2\lambda =0$ 1

$1.6-0.4x_2 - \lambda =0$ 2

$\lambda\, [2x_1+2-10]=0$ 3

$2x_1+x_2-10\le 0$ 4

$\lambda >0$ 5

from equation 3

Either $\lambda =0$ or $2x_1+x_2-10=0$

if

$\lambda =0$ then (2) and (1) yield $x_1= 4.5$

$x_2=4$ with these values of x_1 and x_2

equation (4) cannot be satisfied. Thus, $\lambda \neq 0$

This implies that

$2x_1+x_2-10 =0$, this together with (1) and (2) yield

$$\left.\begin{array}{l} x_1=3.5 \\ x_2=3 \\ Ƶ=10.7 \end{array}\right\}$$ that satisfy all the equations

Note: We use Cramer's rule to solve these equations as given below;

$$\begin{bmatrix} 0.8 & 0 & 2 \\ 0 & 0.4 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \lambda \end{bmatrix} \begin{matrix} 3.6 \\ 1.6 \\ 10 \end{matrix}$$

$x_1=3.5$

$x_2=3$

$\lambda=1.6-0.4(3)$

$1.6-12=0.4$

$2\lambda=3.6-0.8(3.5)$

$3.6-2.80=0.8$

$\lambda=0.8/2=0.4$

UNIT -XI

QUADRATIC PROGRAMMING

Quadratic Programming (Q.P.) tackles the problems of random variations in enterprise outputs as caused by annual fluctuations in yields and prices. It provides the maximum profit at any given level of income variation or risk. Thus, in Quadratic Programming the farmer can take into account not only the average profit but also the minimum profit (or maximum loss) that is likely to occur in bad years. This technique uses modification of simplex method. In Quadratic Programming along with returns we also allow for variance, covariance matrix. The general form of Quadratic Programming is;

$$Max\ f(X) \quad C'X + \frac{1}{2} X' \Omega X \cdots\cdots (1)$$

$$s.t$$

$$Ax \le B \cdots\cdots (2)$$

$$X \ge 0 \cdots\cdots (3)$$

$$\text{Where } X \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad C \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \Omega\ (\text{variance} - \text{covariance matrix}) \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1} & d_{n2} & \ldots & d_{nn} \end{bmatrix}$$

$$\text{A (input-output coefficients)} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Kuhn Tucker Conditions for Quadratic Programming (Non-Linear)

Max Z= f(x)

Subject to

h(x) ≤ 0

and –x≤ 0

we add slack variables

$h^i(x) + S_i^2 = 0$

$-x_j + S^2 m+j = 0$

Lagrangian function

$$L(X,S,\lambda) = f(x) - \sum_{i\ 1}^{m} \lambda_i \left[h^i(x) + S_i^2 \right]$$

$$- \sum_{j\ 1}^{n} \lambda_{m+j} \left[-x + S_{m+j}^2 \right]$$

$$\frac{\partial L}{\partial x_j} \quad f_i - -\sum_{i\ 1}^{m} \lambda_i h^i + \sum_{j\ 1}^{n} \lambda_{m+j} \quad 0$$

$$\frac{\partial L}{\partial Si} \quad -2\lambda_i S_i = 0$$

$$\frac{\partial L}{\partial \lambda i} \quad -[h(x) + S_i^2] = 0$$

$$\frac{\partial L}{\partial \lambda m + j} \quad -[-xj + S_{m+j}^2] \quad 0$$

The Kuhn Tucker conditions on the basis of these equations after simplification are;

$$a \quad f' \quad \sum_{i\ 1}^{m} \lambda_i h' - \lambda_{m+1}$$

$$b \quad \lambda_i h^i(x) \quad 0$$

$$c \quad -\lambda_{m+j} x_j \quad 0$$

$$d \quad h^i(x) \le 0$$

$$e \quad \lambda i, \lambda m - j, x_j \ge 0$$

Thus,

1. If x'π x is +[ve] semi- definite then it is convex. If it is –[ve] semi definite then it is concave.
2. If x'πx is positive definite, it is strictly convex. If it is –[ve] definite then it is strictly concave. (where π is variance covariance matrix)

We can write

$$a \quad c_j + \sum_{k\ 1}^{n} c_{ij} x_k - \sum_{i\ 1}^{m} \lambda_i a_{ij} + \lambda_{m+j} \quad 0$$

$$b \quad \lambda_i [\sum_{j\ 1}^{n} a_{ij} x_j - b_i] \quad 0$$

$c \quad (-x_j)\lambda_{m+j} \quad 0$

$d \quad \sum_{j-1}^{n} a_{ij}\, x_j - bi \le 0$

$e \quad \lambda_i, \quad \lambda_{m+j}, \quad x_j \ge 0$

Constraints with inequality sign

$Z \quad f(x)$

$s.t\ g(x) \le \ge = bc$

$for\ this\ we\ define\ new\ number\ S\ such\ that$

$S^2 \quad -h(x)\ such\ that \quad h(x) + S^2 \quad 0$

$so \quad Z \quad f(x)$

$subject\ to$

$h(x) + S^2 \quad 0$

$x \ge 0$

$L(X, S, \lambda) \quad f(x) - \lambda[h(x) + S^2]$

$The\ \ conditions\ for\ stationary\ parts\ are$

$1 \quad \frac{\partial L}{\partial X_j} - \frac{\lambda \partial h}{\partial X_j} \quad 0$

$2 \quad \frac{\partial L}{\partial \lambda} \quad -[h(x) + S^2] \quad 0$

$3 \quad \frac{\partial L}{\partial S} \quad -2S\lambda \quad 0$

in eq. 3 $-S\lambda \quad 0$ such that either $\lambda \quad 0$ or S=0. If S=0 implies h(x)=0, then (2) and (3) together imply

$4 \quad \lambda h(x) \quad 0$

If S=0 it can be deleted (as it was introduced for removing inequality), $S^2 \ge 0$ so $h(x) \le 0$

So, whenever h(x) <0 we get from equation (4) then $\lambda \quad 0$ & when $\lambda > 0$ we get h(x) = 0, but λ is unrestricted in sign. Thus, we get set of equations known as *kuhn Tucker conditions* as given below

$$f_j - \lambda h_j \quad 0 \qquad j \quad 1,2,3 \cdots\cdots n$$

$$\lambda h \quad 0 \qquad \text{maximise } f$$

$$h \le 0 \qquad \text{subject to the constraints}$$

$$\lambda \ge 0 \quad \cdots\cdots\cdots \quad h \le 0$$

similar arguments hold for minimisation

$$f_j - \lambda h_j \quad 0 \qquad j \quad 1,2,3 \cdots\cdots n$$

$$\lambda h \quad 0 \qquad \min .\, f$$

$$h \ge 0 \qquad \text{subject to the constraints}$$

$$\lambda \ge 0 \quad \cdots\cdots\cdots \quad h \le 0$$

$$\text{now } L(X, S, \lambda) \quad f(x) - \lambda\left[h(x) + S^2\right]$$

$$(1) \quad \frac{\partial L}{\partial X_j} \quad \frac{\partial f}{\partial X_j} - \sum_{j\ 1}^{m} \lambda_i \frac{\partial h_i}{\partial X_j} \quad 0 \qquad (j \quad 1,2 \cdots\cdots n)$$

$$(2) \quad \frac{\partial L}{\partial \lambda} \quad h^i + Si^2 \quad 0 \qquad (i \quad 1,2 \cdots\cdots m)$$

$$(3) \quad \frac{\partial L}{\partial S^2} \quad -2S_i\lambda_i \quad 0$$

$$\text{where} \quad \lambda_i \quad [\lambda_1, \lambda_2 \cdots\cdots \lambda_m]$$

$$S_i \quad [S_1, S_2 \cdots\cdots S_m]$$

Example 1:

$$Max\ Z \quad 2X_1 + 3X_2 - 2X_1^2$$

subject to

$$X_1 + 4X_2 \le 4$$

$$X_1 + X_2 \le 2$$

$$X_1 \ge 0$$

$$X_2 \ge 0$$

We introduce surpuls variables, thus,

$$\left.\begin{array}{l} X_1 + 4X_2 + S_1^2 \quad 4 \\ X_1 + X_2 + S_2^2 \quad 2 \end{array}\right] S_1, S_2 \ \text{surplus}$$

$$\left.\begin{array}{l} -X_1 + S_3^2 \quad 0 \\ -X_2 + S_4^2 \quad 0 \end{array}\right] S_3, S_4 \ \text{surplus}$$

Construct the Lagrangian function

$$L\left(X_1, X_2, S_1, S_2, S_3, S_4, \frac{\lambda_1, \lambda_2}{\lambda_i}, \frac{\lambda_3, \lambda_4}{\lambda_{m+1}}\right)$$

$$\left(2X_1 + 3X_2 - 2X_1^{\ 2}\right) - \lambda_1\left(X_1 + 4X_2 + S_1^{\ 2} - 4\right) - \lambda_2\left(X_1 + X_2 + S_2^{\ 2} - 2\right) - \lambda_3\left(-X_1 + S_3^{\ 2}\right)$$

$$-\lambda_4\left(-X_2 + S_4^{\ 2}\right)$$

$$\frac{\partial L}{\partial X_1} \quad 2 - 4x_1 - \lambda_1 - \lambda_2 + \lambda_3 \quad 0$$

$$\frac{\partial L}{\partial X_2} \quad 3 - 4\lambda_1 - \lambda_2 + \lambda_4 \quad 0$$

$$\frac{\partial L}{\partial S_1} \quad -2\lambda_1 S_1 \quad 0$$

$$\frac{\partial L}{\partial S_2} \quad -2\lambda_2 S_2 \quad 0$$

$$\frac{\partial L}{\partial S_3} \quad -2\lambda_3 S_3 \quad 0$$

$$\frac{\partial L}{\partial S_4} \quad -2\lambda_4 S_4 \quad 0$$

$$\frac{\partial L}{\partial \lambda_1} \quad X_1 + 4X_2 + S_1^{\ 2} - 4 \quad 0 \qquad (multiply\ by\ (-)\ sign)$$

$$\frac{\partial L}{\partial \lambda_2} \quad x_1 + x_2 + S_2^{\ 2} - 2 \quad 0$$

$$\frac{\partial L}{\partial \lambda_3} \quad -x_1 + S_3^{\ 2} \quad 0$$

$$\frac{\partial L}{\partial \lambda_4} \quad -x_2 + S_4^{\ 2} \quad 0$$

upon simplification we get

$$4x_1 + \lambda_1 + \lambda_2 - \lambda_3 \quad 2$$

$$4\lambda + \lambda_2 - \lambda_4 \quad 3$$

$$x_1 + 4x_2 + S_1^{\ 2} \quad 4$$

$$x_1 + x_2 + S_1^{\ 2} \quad 2$$

Adding other equations together & multiplying by S_{js}

$$\lambda_1 S_1^{\ 2} + \lambda_2 S_2^{\ 2} + \lambda_3 S_3^{\ 2} + \lambda_4 S_4^{\ 2} \quad 0$$

$$\lambda_1 S_1^{\ 2} + \lambda_2 S_2^{\ 2} + \lambda_3 X_1 + \lambda_4 X_2 \quad 0$$

$$\lambda_1 S_1^{\ 2} + \lambda_2 S_2^{\ 2} + \lambda_3 X_1 + \lambda_4 X_2 \quad 0$$

$$Note: \frac{S_3}{-2}\begin{pmatrix} -2\lambda_3 S_3 \quad 0 \\ -X_1 + S_3^{\ 2} \quad 0 \end{pmatrix}$$

$$\therefore X_1 \quad S_3^{\ 2}$$

$$Similarly, X_2^{\ 2} \quad S_4^{\ 2}$$

$$\lambda_3 S_3^{\ 2} \quad 0$$

$$\lambda_1 X_1 \quad 0$$

$$Similarly,$$

$$-X_2 + S_4^{\ 2} \quad 0 \Rightarrow X_2 \quad S_4^{\ 2}$$

$$Thus\ \lambda_4 S_4^{\ 2} \quad 0$$

$$\lambda_4 X_2 \quad 0$$

so the complementory conditions are

$$\lambda_i S_i \quad 0$$

$$\lambda_{m+j} X_j \quad 0$$

i.e when S_i *is in basis* λ_i *cannot enter and also when* λ_{m+j} *is in basis* x_j *cannot enter the basis*

Therefore, solution to x_1, x_2 be necessarily an optimal solution that maximize L. To determine the solution, we introduce artificial variables w_1 and w_2 in the first two constraints & construct the dummy objective function to min variance (risk)

$$Min\ Z \quad w_1 + w_2$$

subject to

$$4X_1 + \lambda_1 + \lambda_2 - \lambda_3 + w_1 \quad 2$$

$$4\lambda_1 + \lambda_2 - \lambda_4 + w_2 \quad 3$$

$$X_1 + 4X_2 + S_1^{\ 2} \quad 4$$

$$X_1 + X_2 + S_2^{\ 2} \quad 2$$

$$X_1, X_2, S_1, S_2, w_1, w_2, \lambda_i \geq 0$$

Also satisfying the complementary condition

$$\sum_{i \ 1}^{4} \lambda_i S_i^{\ 2} \quad 0, \sum \lambda_i X_j \quad 0$$

The initial table to solve this problem is given below

			0	0	0	0	0	0	0	0	-1	-1
C_B	X_B	X_B	X_1	X_2	S_1^2	S_2^2	λ_1	λ_2	λ_3	λ_4	w_1	w_2
-1	w_1	2	4	0	0	0	1	1	-1	0	1	0
-1	w_2	3	0	0	0	0	4	1	0	-1	0	1
0	S_1^2	4	1	4	1	0	0	0	0	0	0	0
0	S_2^2	2	1	1	0	1	0	0	0	0	0	0
Zj		-5	-4	0	0	0	-5	-2	1	1	-1	-1
Z_j-C_j			-4	0	0	0	-5	-2	1	1	0	0

λ_1 is most negative but it cannot enter the basis since S_1^2 is in the basis. So we select X_1

(The students can solve this problem as an assignment)

Wolfe's Method to solve Q.P.P

We can also use Wolfe's Method to solve Q.P.P by writing the problem given in **Example 1:** as

$$Max\ Z \quad \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} X_1 & X_2 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} <= \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Example 2:

$$Max\ Z \quad 4X_1 + 2X_2 - X_1^2 - X_2^2 - 5$$

subject to

$$X_1 + X_2 \le 4$$

$$X_1, X_2 \ge 0$$

Now we can write this Q.P.P as

$$Max\ Z \quad \begin{bmatrix} 4 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} X_1 & X_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \leq 4$$

The Kuhu Tucker conditions by Wolfe's criteria are;

$$C'X - Iv + A'u - P \quad 0$$

$$AX + Y \quad b \qquad \cdots\cdots Y : Slack$$

$$v'X + u'Y \quad 0 \quad \cdots\cdots Complementary\ condition$$

$$u, v, X, Y \geq 0$$

Thus,

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad 0$$

$$X_1 + X_2 + Y_1 \quad 4$$

$$v'X + u'Y \quad 0$$

$$v_1 X_1 + v_2 X_2 + u_1 Y_1 \quad 0$$

Then Q.P.P becomes Min Z $\quad w_1 + w_2$

subject to

$$2X_1 - v_1 + u_1 + w_1 \quad 4$$

$$2X_2 - v_2 + u_2 + w_2 \quad 2$$

$$X_1 + X_2 + Y_1 \quad 4$$

$$v_1 X_1 + v_2 X_2 + u_1 Y_1 \quad 0$$

			0	0	0	0	0	0	0	1	1
C_B		X_B	X_1	X_2	Y_1	u_1	u_2	v_1	v_2	w_1	w_2
-1	w_1	4	2	0	0	1	0	-1	0	1	0
-1	w_2	2	0	2	0	0	1	0	-1	0	1
0	Y_1	4	1	1	1	0	0	0	0	0	0
Z_j		6	2	2	0	1		-1	-1	1	1
Z_j-C_j			2	2	0	1		-1	-1	0	0

Solution of this problem is

$$X_1 \quad 2, X_2 \quad 1 \qquad Z\min \quad 0$$

Example 3:

$$Max\,Z \quad 4X_1 + 6X_2 - 2X_1^{\,2} - 2X_1X_2 - 2X_2^{\,2}$$

$$subject\ to$$

$$X_1 + 2X_2 \le 2$$

$$X_1, X_2 \ge 0$$

$$We\ can\ write\ this\ equation\ as;$$

$$Max\,Z \quad \begin{bmatrix} 4 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$subject\ to$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \le 2 \qquad X_1, X_2 \ge 0$$

$$X_1 + 2X_2 + Y_1 \quad 2$$

$$v_1X_1 + v_2X_2 + u_1Y_1 \quad 0$$

$$we\ can\ use\ artical\ \mathrm{var}iables\ and\ formulate the\ tables$$

$$C'X - Iv + A'u - p \quad 0$$

$$AX + Y \quad b$$

$$v'X + u'Y \quad 0$$

$$u, v, X, Y \ge 0$$

$$which\ is\ ana\log ous\ to$$

$$Min\,Z \quad w_1 + w_2$$

$$subject\ to$$

$$4X_1 + 2X_2 - v_1 + u_1 + w_1 \quad 4$$

$$2X_1 + 4X_2 - v_2 + 2u_2 + w_2 \quad 6$$

$$X_1 + 2X_2 + S_3^{\,2} \quad 2$$

$$v_1X_1 + v_2X_2 + u_1Y_1 + u_2Y_2 \quad 0$$

$$We\ can\ formulate the\ table\ and\ solve by\ u\sin g$$

Simplex method

			0	0	0	0	0	0	1	1
C_B		X_B	X_1	X_2	Y_1	u_1	v_1	v_2	w_1	w_2
1 w_1	w_1	4	4	2	0	1	-1	0	1	0
1 w_2	w_2	6	2	4	0	2	0	-1	0	1
0 y_1	Y_1	2	1	2	1	0	0	0	0	0
Z_j		0	6	6	0	3	-1	-1	0	0

X_1 will enter the basis and w_1 will leave the basis

Example 4:

$$Max\ Z \quad 2X_1 + X_2 - X_1^2$$

subject to

$$2X_1 + 3X_2 \le 6$$

$$2X_1 + X_2 \le 4$$

$$X_1, X_2 \ge 0$$

we can write this problem as

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Subject to

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \le \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Now applying Kuhn Tucker conditions, we get

$$Min\ Z \quad w_1 + w_2$$

subject to

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Thus, equations are

$$2X_1 - v_1 + 2u_1 + 2u_2 + w_1 \quad 2$$

$$-v_2 + 3u_1 + u_2 + w_2 \quad 1$$

$$2X_1 + 3X_2 + u_1 \quad 6$$

$$2X_1 + X_2 + u_2 \quad 4$$

Formulate simple Tableau

Solution is X_1=2/3, X_2=14/9, Opt Z= 22/9

We now apply the simplex method:

Starting the Table

c_b	y_b	x_b	x_1	x_2	y_1	y_2	v_1	v_2	u_1	u_2	z_1	z_2
1	z_1	2	2*	0	0	0	-1	0	2	2	1	0
1	z_2	1	0	0	0	0	0	-1	3	1	0	1
0	y_1	6	2	3	1	0	0	0	0	0	0	0
0	y_2	4	2	1	0	1	0	0	0	0	0	0
		3	2	0	0	0	-1	-1	5	5	0	0

↑

Max, z_j-c_j is 5 but we can't introduce u_1 (and similarly u_2) in the basis because of the equality (1)

c_b	y_b	x_b	x_1	x_2	y_1	y_2	v_1	v_2	u_1	u_2	z_1	z_2
0	x_1	1	1	0	0	0	-1/2	0	1	1	½	0
1	z_2	1	0	0	0	0	0	-1	3	1	0	1
0	y_1	4	0	3	1	0	1	0	-2	-2	-1	0
0	y_2	2	0	1	0	1	1	0	-2	-2	-1	0
		1	0	0	0	0	0	-1	3	1	-1	0

↑

Again we introduce u_1 or u_2 because y_1 and y_2 are in the basis. So x_2 enters the basis.

Second iteration

c_b	y_b	x_b	x_1	x_2	y_1	y_2	v_1	v_2	u_1	u_2	z_1	z_2
0	x_1	1	1	0	0	0	-1/2	0	1	1	½	0
1	z_2	1	0	0	0	0	0	-1	3*	1	0	1
0	x_2	4/3	0	1	1/3	0	1/3	0	-2/3	-2/3	-2/3	0
0	y_2	2/3	0	0	-1/3	1	2/3	0	-4/3	-4/3	-2/3	0
		1	0	0	0	0	0	-1	3	1	0	0

Third iteration

c_b	y_b	x_b	x_1	x_2	y_1	y_2	v_1	v_2	u_1	u_2	z_1	z_2
0	x_1	2/3	1	0	0	0	-1/2	1/3	0	2/3	½	-1/3
1	u_1	1/3	0	0	0	0	0	-1/3	1	1/3	0	1/3
0	x_2	14/9	0	1	1/3	0	1/3	-2/9	0	-4/9	-1/3	2/9
0	y_2	10/9	0	0	-1/3	1	2/3	-4/3	0	-8/9	-2/3	4/9
		0	0	0	0	0	0	0	0	0	-1	-1

Hence the optimum solution is $x_1=2/3$, $x_2=14/9$; opt. z =22/9

Unit -XII

MINIMIZATION OF TOTAL ABSOLUTE DEVIATION MODEL (MOTAD)

Hazzel (1971) developed MOTAD model as a linear iterative quadratic and semi variance programming for farm planning under risk. This model uses linear decision criterion with expected returns and mean absolute deviations. He observed that the MOTAD model could be solved with conventional linear programming packages. Risk is incorporated in the model as mean absolute deviation of farm profit.

The mathematical form, of the MOTAD programming model is as follows.

$$\text{MinZ}=\sum_{h\ 1}^{S} y^{-}{}_{h} + y_{h}^{+} \quad \text{(objective function)}$$

Subject to

$$\sum_{h\ 1}^{n}(Ch_j - g_j)x_j - y_h^{-} + y_h^{+} \quad 0 \quad \text{Linear constraints}$$

$$\sum_{j\ 1}^{n} a_{ij}x_j \le b_i \quad (resource \quad constra\text{int}\, s)$$

$$\sum_{j\ 1}^{n} f_j x_j \quad \lambda_i \quad (Parametric \quad constra\text{int}\, s)$$

X_j, y^{-}_{h}, $y^{+}_{h} \geq$ =0 (non negativity constraints)

where;

Z= sum of the absolute values of the deviations of the gross returns of each activity from their means.

y^{-}_{h}, is the absolute value of the negative total gross returns devotions for each activity in the h^{th} year from their mean.

Ch_j = gross returns of the j^{th} farm activity in the h^{th} year.

G_j = mean value of the gross returns of the j^{th} farm activity

X_j = level of j^{th} farm activity.

f_j= The expected gross returns per unit of the j^{th} activity.

λ = the total returns from all farm activities

a_{ij} = Technical requirements of the j^{th} activity for the i^{th} resource

b_i = The constraint level of the i^{th} resource

s= number of time series observations

n = Number of farm activities

Steps involves in solving the MOTAD

1. Work out average gross returns for different farm activities.
2. Work out deviation of gross returns from the mean for different enterprises/ activities over the years.
3. Prepare initial table for simple linear programming and work out maximum value of λ
4. Prepare initial table for MOTAD by including deviations of gross returns in simple linear programming.
5. Put the value of λ and get mean absolute deviations and level of different activities
6. Work out standard deviation with statistic as;

$$d\left[\pi s/2(s-1)\right]^{1/2} \quad or \quad \sqrt{\frac{1}{s-1}\sum_{h\ 1}^{s}\left[\sum_{j\ 1}^{n} Ch_j x_j - \sum_{j\ 1}^{n} g_j x_j\right]^2}$$

7. Workout coefficient of variation as $CV \quad \frac{Mean\,gross\,returns}{SD} X100$

Example of MOTAD model

Four vegetable activities: carrot (x_1), celery (x_2), cucumber (x_3) and pepper (x_4) and three less than or equal to constraints on the variable acreage of land (b_1), hours of labour (b_2) and a rational and market output constraints (b_3). The technical requirement coefficient matrix A and constraint vector b are

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 25 & 36 & 27 & 87 \\ -1 & 1 & -1 & 1 \end{bmatrix} \qquad b \begin{bmatrix} 200 \\ 10000 \\ 0 \end{bmatrix}$$

Activities gross margins

Year	X1	X_2	X_3	X_4
t_1	292	-128	420	579
t_2	179	560	187	639
t_3	114	648	366	379
t_4	247	544	249	924
t_5	426	182	322	5
t_6	259	850	159	569
Average	253	443	284	516

Initial Tableau

Row & unit	x_1	x_2	x_3	x_4	Y_1^-	Y_2^-	Y_3^-	Y_4^-	Y_5^-	Y_6^-	Constraints
$A(dollers)$					1	1	1	1	1	1	*Minimise*
$b_1(acres)$	1	1	1	1							≤ 200
$b_2(hours)$	25	36	27	87							≤ 10000
$b_3(acres)$	-1	1	-1	1							≤ 0
$t_1(dollers)$	39	-571	136	63	1						≥ 0
$t_2(dollers)$	-74	117	-97	123		1					≥ 0
$t_3(dollers)$	-139	205	82	-137			1				≥ 0
$t_4(dollers)$	-6	101	-35	408				1			≥ 0
$t_5(dollers)$	173	-261	38	-511					1		≥ 0
$t_6(dollers)$	6	407	-125	53						1	≥ 0
$E(dollers)$	253	443	284	516							λ

E= Average gross returns

Results

Cropping plan	I	II	III	IV	V
E dollars	62769	73574	77329	77529	77996
A dollars	2753	9301	12533	12787	13479
SD dollars	4702	16358	21442	21792	22372
X1 Acres	72.26	3285	19.15	16.59	-
X2 Acres	26.80	28.03	28.46	26.80	27.45
X3 Acres	83.92	81.64	80.85	83.41	100.00
X4 Acres	17.02	57.48	71.54	73.20	72.55

Standard deviation is worked out from previous example as follows

$$SD= \sqrt{\frac{1}{s-1}\sum_{h\ 1}^{s}[\sum_{j\ 1}^{h} C_{hj}X_j - \sum_{j\ 1}^{h} g_j X_j]^2}$$

Year-wise gross margins per acre [*level of* j^{th} *activity*]

$$\begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\begin{bmatrix}292 & -128 & 420 & 579\\179 & 560 & 187 & 639\\114 & 648 & 366 & 379\\247 & 544 & 249 & 924\\426 & 182 & 322 & 5\\259 & 850 & 159 & 569\end{bmatrix} \qquad xj\begin{bmatrix}72.26\\26.80\\83.92\\17.02\end{bmatrix}$$

[Average gross margin]

g_i [253 443 284 516]

$$Ch_jx_j = \begin{pmatrix} 292*72.26 & + & (-128)*26.80 & + & 420*83.92 & + & 579*17.02 \\ 179*72.26 & + & 560*26.80 & + & 187*83.92 & + & 639*17.02 \\ 114*72.26 & + & 648*26.80 & + & 366*83.92 & + & 379*17.02 \\ 247*72.26 & + & 544*26.80 & + & 249*83.92 & + & 924*17.02 \\ 426*72.26 & + & 182*26.80 & + & 322*83.92 & + & 5*17.02 \\ 259*72.26 & + & 850*26.80 & + & 159*83.92 & + & 569*17.02 \end{pmatrix}$$

$$C_{hj}x_j \quad \begin{bmatrix} 21099.92 & - & 3430.40 & + & 35246.40 & + & 9854.58 \\ 12934.54 & + & 15008 & + & 15693.04 & + & 10875.78 \\ 8237.64 & + & 17366.40 & + & 30714.72 & + & 6450.58 \\ 17848.22 & + & 14579.20 & + & 20896.08 & + & 15726.48 \\ 30782.76 & + & 4877.60 & + & 27022.24 & + & 85.10 \\ 18715.34 & + & 22780 & + & 13343.28 & + & 9684.38 \end{bmatrix}$$

$$\sum_{j\ 1}^{n} C_{hj}x_j \quad \begin{bmatrix} 62770.50 \\ 54511.36 \\ 62769.34 \\ 69049.98 \\ 62767.70 \\ 64523.00 \end{bmatrix}$$

$$g_j x_j \quad [253 \times 72.26 + 443 \times 26.80 + 284 \times 83.92 + 516 \times 17.02]$$

$$g_j x_j \quad [18281.78 + 11872.40 + 23833.28 + 8782.32]$$

$$\sum_{j\ 1}^{n} g_j x_j \quad [62769.78]$$

$$\sum_{j\ 1}^{n} C_{hj}x_j - \sum_{j\ 1}^{n} g_j x_j \quad \begin{bmatrix} 62770.50 & - & 62769.78 \\ 54511.36 & - & 62769.78 \\ 62769.34 & - & 62769.78 \\ 69049.98 & - & 62769.78 \\ 62767.70 & - & 62769.78 \\ 64523.00 & - & 62769.78 \end{bmatrix} \quad \begin{bmatrix} 0.72 \\ -8258.42 \\ -0.44 \\ 6280.20 \\ -2.08 \\ 1753.22 \end{bmatrix}$$

$$\left(\sum_{j\ 1}^{n} C_{hj}x_j - \sum_{j\ 1}^{n} g_j x_j \right)^2 \quad \begin{bmatrix} 0.5184 \\ 68201500.90 \\ 0.1936 \\ 39440912.04 \\ 4.3264 \\ 3073780.37 \end{bmatrix}$$

$$\sum_{h\ 1}^{s}\left(\sum_{j\ 1}^{n} C_{hj}x_j - \sum_{j\ 1}^{n} g_j x_j\right)^2 \quad [110716198\ \ .30]$$

$$\frac{1}{s-1}\sum_{h\ 1}^{s}\left(\sum_{j\ 1}^{n} C_{hj}x_j - \sum_{j\ 1}^{n} g_j x_j\right)^2 \quad \left[\frac{110716198.30}{5}\right] \quad 22143239.67$$

$$SD \quad \sqrt{\frac{1}{s-1}\sum_{h\ 1}^{s}\left(\sum_{j\ 1}^{n} C_{hj}x_j - \sum_{j\ 1}^{n} g_j x_j\right)^2} \quad \sqrt{22143239.67} \quad 4706$$

UNIT XIII

MARKOV CHAIN MODEL, SIMULATED SAMPLING, MONTE CARLO METHOD AND PRACTICAL APPLICATION

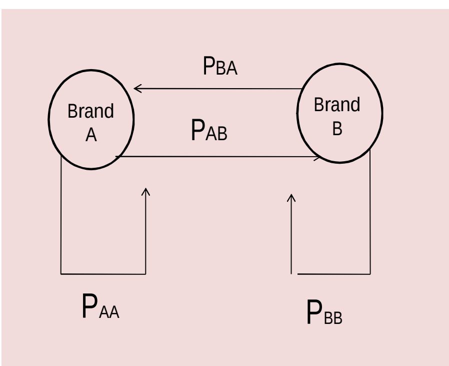
Markov Chain Analysis

In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a Markov chain. Markov chain analysis is a way of analyzing the current movements of some variables in order to forecast the future movement of some of variables. This method is being employed in the field of marketing and accounting.

It can be used in predicting the loyalty of customers to one brand and their switching pattern to other brands.

Brand Switching Analysis:

In order to explain the Markov chain model, let us consider the time behaviour of customers who make repeated purchases of a product class but may switch over from time to time from one brand to another. The brand switching model can be explained in term of following diagram



To / From	Transition Matrix A	B
A	P_{AA} (loyal to A)	P_{AB} (shift from A to B)
B	P_{BA} (shift from B to A)	P_{BB} (loyal to B)

or for n brands the transition probability P_{ij} will be;

$$
P_{ij} \quad \begin{array}{c|ccccc|}
 & A_1 & A_2 & A_3 & \cdots\cdots & A_n \\
A_1 & p_{11} & p_{12} & p_{13} & \cdots\cdots & p_{1n} \\
A_2 & p_{21} & p_{22} & p_{23} & \cdots\cdots & p_{2n} \\
A_3 & p_{31} & p_{32} & p_{33} & \cdots\cdots & p_{3n} \\
\vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots \\
A_n & p_{n1} & p_{n2} & p_{n3} & \cdots\cdots & p_{nn}
\end{array}
$$

This behaviour can be explained in probalistic terms. This probalistic description can be given a transition matrix, P_{ij} from brand i to brand j from one period to another.

The significant characteristic of transition matrix is that $\sum_{j\ 1}^{n} P_{ij} \quad 1$

Here, we illustrate the Markov chain model with the help of an example.
Suppose two brands of a product A & B with each of these brands having exactly 50% share in market in the given period. The transition matrix is given below:

To / From	Transition Matrix	
	A	B
A	0.90	0.10
B	0.50	0.50

Transition matrix indicates that in one period ahead period brand A will retain 90% of its customer and the attract 50% of B's. Similarly, B will retain 50% of its customers and extract 10% of A's. Thus, in first period the market share will be

$$\begin{bmatrix}50 & 50\end{bmatrix} \quad \begin{bmatrix}0.9 & 0.1 \\ 0.5 & 0.5\end{bmatrix} \quad \begin{bmatrix}70 & 30\end{bmatrix}$$

In second period

$$\begin{bmatrix}70 & 30\end{bmatrix} \quad \begin{bmatrix}0.9 & 0.1 \\ 0.5 & 0.5\end{bmatrix} \quad \begin{bmatrix}78 & 22\end{bmatrix}$$

Or [50 50] $[P_{ij}]^2$

and in third period

[50 50] $[P_{ij}]^3$

Steady State Equilibrium

Steady state is the long run equilibrium and reveals the ultimate position of each brand in the market. The following steps are followed to find out steady state:

1. Transpose the transition probability matrix $[P_{ij}]=[P_{ji}]$
2. Workout P_{ji} – I (unit matrix) of the same order as P_{ji} and set the relation as P_{ji} X - I =O where X = column matrix of different brands ($x_1, x_2, \ldots x_n$) and O is the null matrix, with all zero elements.
3. Put the relation $x_1+x_2+x_3\ldots\ldots x_n = 1$ where x_1, x_2, x_3 are different brands.
4. Replace the first row of the expression given in step 2 by 3 and solve the new formulation for $x_1, x_2, x_3\ldots\ldots x_n$ by using Cramer's rule.

Example1: Steady state equilibrium

Let the transition probability of three brands A, B, C

$$\alpha \begin{bmatrix} 0.60 & 0.15 & 0.25 \\ 0.20 & 0.45 & 0.35 \\ 0.10 & 0.18 & 0.72 \end{bmatrix}$$

Step I.

α=**[** α**]** = α'

$$\alpha^T \begin{bmatrix} 0.60 & 0.20 & 0.10 \\ 0.15 & 0.45 & 0.18 \\ 0.25 & 0.35 & 0.72 \end{bmatrix}$$

Step II α' –unit matrix

$$\alpha \begin{bmatrix} -0.40\,A & +0.20\,B & +0.10\,C \\ 0.15\,A & -0.55\,B & +0.18\,C \\ 0.25\,A & 0.35\,B & -0.28\,C \end{bmatrix} \begin{matrix} 0 & (i) \\ 0 & (ii) \\ 0 & (iii) \end{matrix}$$

A + B + C =1 (iv)

Step III

Replacing (i) by (iv)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.15 & 0.55 & 0.18 \\ 0.25 & 0.35\,B & 0.28 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Step IV

Solve the expression by Cramer's rule to find out A, B and C

Determinant of the above matrix D= 0.368

$$Det\ D_1 \begin{bmatrix} 1 & 0 & 0.10 \\ 0 & -0.55 & 0.18 \\ 0 & 0.35 & -0.28 \end{bmatrix} \quad 0.091$$

$$Det\ D_2 \begin{bmatrix} 1 & 1 & 1 \\ 0.15 & 0 & 0.18 \\ 0.25 & 0 & -0.28 \end{bmatrix} \quad 0.087$$

$$Det\ D_3 \begin{bmatrix} 1 & 1 & 1 \\ 0.15 & -0.55 & 0 \\ 0.25 & 0.35 & 0 \end{bmatrix} \quad 0.191$$

A= $D_1/D = 0.25$

B= $D_2/D = 0.24$

C= $D_3/D = 0.51$

$$Hence, steady\ state\ market share\ will\ be \begin{matrix} A \\ B \\ C \end{matrix} \begin{bmatrix} 25\,\% \\ 24\,\% \\ 51\,\% \end{bmatrix}$$

Practical Exercise

Let us take the data of 3 dairies A, B and C having their share A=22%, B=49% and C=29% and the observations of customers for June & July are given below

Dairy	June 1 customers	Gains from			Loses to			July1 customers
		A	**B**	**C**	**A**	**B**	**C**	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290
	1000							1000

Find out market share for August and also the steady state.

Based on data, the Transitional Matrix is

From \ to	A	B	C
A	160/200	20/200	20/200
B	35/500	450/500	15/500
C	25/300	20/300	255/300

=

From \ To	A	B	C
A	0.80	0.10	0.10
B	0.07	0.90	0.03
C	0.083	0.067	0.85

Thus, market share in August

$$\begin{bmatrix} 0.22 & 0.49 & 0.29 \end{bmatrix} \begin{bmatrix} 0.80 & 0.10 & 0.10 \\ 0.07 & 0.90 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix} \quad \begin{bmatrix} 0.234 & 0.483 & 0.283 \end{bmatrix}$$

and steady state equilibrium

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0.80 & 0.10 & 0.10 \\ 0.07 & 0.90 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix} \quad \begin{bmatrix} 0.273 & 0.454 & 0.273 \end{bmatrix}$$

Practical Assignment

On Jan 1, 2013 Adarsh Bakery has 40% of local Palampur market and other two bakeries Shiva and Cremica 40% and 20% share, respectively. Based upon the study conducted by Amit and Atul it was found that during one year, Adarsh retains 90% of old customers while gaining 5% of Shiva and 10% of Cremica. Shiva retains 85% and gained 5% of Adarsh and 7% of Cremica. Cremica retains 83% and gains 5% of Adarsh and 10% of Shiva. What will be the share in January 2014, 2015 and also find out the share of these bakeries at equilibrium level.

Simulation

Simulation is a method of solving decision-making problems by designing, constructing and manipulating a model of a real system. It is defined to be the action of performing experiments on a model of a given system. Here system is defined as a collection of components which act and interact together towards the accomplishment of some logical end or goal and a model as a representation of the system.

Many problems in agriculture, industry or administration in reality are complex in nature. To solve them scientifically three cases may arise.

(a) The problem is amenable to both description and analysis by a mathematical model

(b) The problem is amenable to description by a mathematical model, however, correct analysis is beyond the level of mathematical sophistication.

(c) Even description by a mathematical model is beyond the capability of an analyst. In that situation the simulation can be used.

Advantages of simulation;

(1) It is useful in solving problems where all the values of variables are not known or partly known, there is easy way to find these values.

(2) The model of system once constructed may be employed as often to analyse different situations.

(3) Simulation methods are quite useful for analyzing proposed system where information is sketchy.

(4) The effect of using model can be observed without actually using in the real situation.

(5) Data for further analysis can be easily generated from simulation model.

(6) Simulation methods are easier to apply. Even non-technical executives can simulate.

Disadvantages of Simulation

(1) Adequate knowledge of parts of a system does not guarantee complete knowledge of the system behaviour.

(2) Simulation model is ‘run’ rather then ‘solved’.

(3) Simulation does not produce optimal answers only satisfactory answers.

(4) Simulation is time consuming as number of experiments need to be conducted.

(5) People develop tendency of using it even if better analytical techniques are available.

Event Type Simulation

Let us assume arrival of customers for a first come first serve service at a shop. Each customer requiring 1.8 hours for a service. We can simulate the service pattern and waiting time as given in the table below:

Time	**Event**	**Customer**	**Waiting time**
0.00	Ea	1	
1.8	Ea	2	
3.6	Ea	3	
4.0	Ed	1	4-1.8 = 2.2 (out 2)
5.1	Ea	4	
7.2	Ea	5	
8.0	Ed	2	8-3.6 = 4.4 (out 3)
9.0	Ea	6	
10.8	Ea	7	
12.0	Ed	3	12-5.4 = 6.6 (out 4)
13.6	Ea	8	14-7.2 = 6.8 (out 5)
14.0	End		14-9.0 = 5.0 (out 6)
			14-10.8 = 3.2 (out 7)
			14-13.6 = 0.4 (out 8)

Ea= arrival time Ed= departure time

Average waiting time $\frac{2.2+4.4+6.6+6.8+5.0+3.2+0.4}{8} \quad \frac{28.6}{8} \quad 3.57\ \textit{hours}$

Average waiting time for those who has to wait $\frac{28.6}{7} \quad 4.08\ \textit{hours}$

% idle time of facility =Nil

Generation of Random Phenomenon

Monte Carlo Technique:

Simulation of values of a discrete random variable using Monte Carlo Approach

Suppose the demand for a newspaper is governed by the following discrete random variable:

Demand	**Probability**	**Cumulative probability**	**Random Numbers Assigned**
5,000	0.10	0.10	< 0.10
15,000	0.35	0.45	Greater than or equal to 0.10- 0.44
35,000	0.30	0.75	Greater than or equal to 0.45- 0.74
55,000	0.25	1.00	Greater than or equal to 0.75 and above

Note: The students can generate 50 random numbers and verify the frequency distribution of the variable that will be nearer to the given probability distribution.

We can generate or simulate this demand for newspaper many times. For this, we associate each possible values of the RAND function with a possible demand for newspaper. This table shows that a demand of 5000 will occur 10 percent of the time, and demand of 15,000 will occur 35 per cent of the time and so on.

This technique has become so much a part of Simulation models. The technique involves selection of random observations within the simulation model. It involves two approaches:

(1) Generate random observation from a uniform distribution (0,1)

(2) Using (1) to generate random observation from desired probability.

Now generating (0,1) random observation we may use

i) random table

ii) computer

iii) using congruential method based on modulo arithmetic

$$\gamma_n \quad a.\gamma_{n-1}(\text{mod}\, ulo\ m)$$

where a, γ, m are positive integers (a <m). γ_n represents remainder when $a.\gamma_{n-1}$ is divided by m. In other words γ_n is that number between 0 and 1 which differs from $a.\gamma_{n-1}$ by an integer multiple of m.

The first no. is γ_1 is obtained by selecting any large integer for γ_0 using the recursive algorithm, then, we obtain $\gamma_1, \gamma_2, \gamma_3, \cdots\cdots$

Example 2:

Let $a \quad 4, m \quad 8, \gamma_0 \quad 1$ then we generate random numbers as $\gamma_n \quad a.\gamma_{n-1}[\text{mod}\, ulo\ 8]$

$$4X1[\text{mod}\, ulo\, 8]$$

γ_n	Normalized No. (0,1) $\gamma_n / 7$
γ_0 1	0.143
γ_1 4	0.571
γ_2 2	0.286
γ_3 1	0.143
γ_4 4	0.571
γ_5 2	0.286
⋮	⋮
⋮	⋮

If a = 10 m = 13 γ_1 1

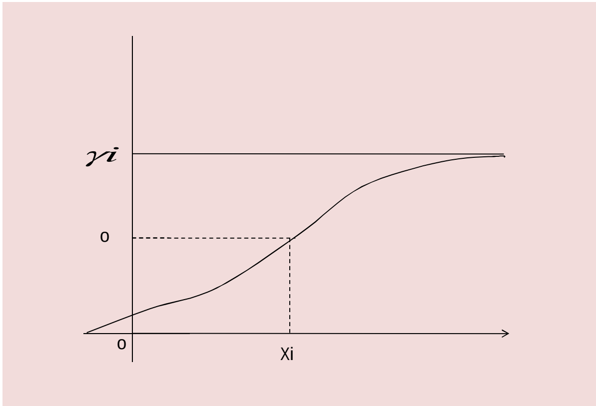
γ_n	Normalized No. (0,1) $\gamma_n / 12$
1	0.083
10	0.833
9	0.750
12	1.000
3	0.250
4	0.333
1	0.083
10	0.833
9	

We can generate large series of random numbers.

If a = 65539, m = $2^{31}$and γ_0 any odd no. less than 9999 then we can divide by 2^{31}-1 = 2,147,483,647 and get fairly largest cycle of random numbers.

Steps for Generating Random Observations

1. Construct cdf (cumulative density function) of the random variable X i.e $F_x(X)$.
2. To generate (0,1) normalized random numbers say $\gamma_1, \gamma_2, \gamma_3$ so on
3. Set $F_x(X)$ equal to the random decimal numbers and solve for X. The value of X thus obtained will be desired random observation from the probability distribution.



$F_x(X)$.is uniform over (0,1) regardless of the distribution of X. For every γ_i there is a unique value of x_i

$F_x(X) = \gamma$ so $X = F_x^{-1}(\gamma)$

Example 3:

Let us consider the cumulative density function

X	0	1	2	3
P_x	0.4	0.3	0.2	0.1
cdf	0.4	0.7	0.9	1.0

$$X_n \begin{cases} 0 < \gamma \le 0.4 \\ 0.4 < \gamma \le 0.7 \\ 0.7 < \gamma \le 0.9 \\ 0.9 < \gamma \le 1.0 \end{cases}$$

Generation of Monte Carlo Series using random table

Consider a model $Y_t = b_0 + b_1X_t + u_t$ where b_0= 0.5, b_1= 0.75 and u_t is a random variable based on random no (0,1). Thus we can generate the series as demonstrated in the table below

Time t	X_t	Random no.	U_t (from normal distribution)	$Y_t = b_0 + b_1X_t + u_t$	Time t	X_t	Random no.	U_t	$Y_t = b_0 + b_1X_t + u_t$
1	0.5	0.014	-2.192	-1.317	14	7	0.421	-0.200	5.550
2	1	0.955	1.698	2.948	15	7.5	0.311	-0.492	5.633
3	1.5	0.566	0.166	1.791	16	8	0.218	-0.779	5.721
4	2	0.925	1.439	3.439	17	8.5	0.824	0.930	7.805
5	2.5	0.869	1.120	3.495	18	9	0.889	1.219	8.469
6	3	0.975	1.952	4.702	19	9.5	0.440	-0.152	7.473
7	3.5	0.543	0.108	3.233	20	10	0.571	0.178	8.178
8	4	0.609	0.278	3.778	21	10.5	0.888	1.217	9.592
9	4.5	0.104	-1.256	2.619	22	11	0.871	1.133	9.883
10	5	0.392	-0.275	3.975	23	11.5	0.148	-1.045	8.080
11	5.5	0.381	-0.302	4.323	24	12	0.581	0.203	9.703
12	6	0.914	1.368	6.368	25	12.5	0.796	0.826	10.701
13	6.5	0.294	-0.540	4.835	26	13	0.423	-0.193	10.057

UNIT XIV

TRANSPORTATION PROBLEMS, ASSIGNMENT MODELS AND JOB SEQUENCING

Transportation Model

In the transportation problems, we have models with equalities and inequalities in the production and requirement.

Let there be m origins and n outlets/distribution points. Take X_{ij} as no. of units shipped from i^{th} origin to j^{th} destination. In this way, we have rim X_{ij} values. Since negative amount cannot be shiped so $X_{ij} \geq 0$.

Let a_i be the number of units of product available at origin i and b_j the number of units required at j. We cannot ship more goods from any origin. Hence;

$$\sum_{j \quad 1}^{n} X_{ij} \quad X_{i1} + X_{i2} \cdots\cdots X_{in} \leq a_i \quad \left(i \quad 1,2,\cdots\cdots m\right)$$

We must supply each destination with number of desired units

Thus, $$\sum_{i \quad 1}^{m} X_{ij} \quad X_{1j} + X_{2j} \cdots\cdots + X_{mj} \quad b_j \quad \left(j \quad 1,2\cdots\cdots n\right)$$

The needs of outlet can be satisfied iff,

$$\sum_{i \quad j}^{m} a_i \geq \sum_{j \quad 1}^{n} b_j$$

The cost C_{ij} from i^{th} origin to j^{th} destination

The total cost is

$$Z \quad \sum_{i \quad 1}^{m} \sum_{j \quad 1}^{n} C_{ij} X_{ij}$$

$$C_{11}X_{11} + C_{12}X_{12} \cdots\cdots + C_{1n}X_{1n}$$
$$+ C_{21}X_{21} + C_{22}X_{22} \cdots\cdots + C_{2n}X_{2n}$$
$$\cdots\cdots\cdots + C_{m1}X_{m1} + C_{m2}X_{m2} \cdots\cdots + C_{mn}X_{mn}$$

We can now summarize the problem as:

$$find\, X_{ij} \geq 0$$

$$\min Z \quad \sum_{i\ 1}^{m}\sum_{j}^{n} C_{ij}X_{ij}$$

$$s.t \quad \sum_{j\ 1}^{n} X_{ij} \leq a_i \quad i \quad 1,2\cdots\cdots m$$

$$\sum_{i\ 1}^{m} X_{ij} \quad b_j \quad j \quad 1,2\cdots\cdots n$$

Transportation problem is a special LP problem in m, n variables with m+n constraints.

They are easy to solve in comparison to general L.P. Further, as long as availability and requirements are integer values, the solution will also be integer value. We can identify the surplus and deficit zones as given below:

Zone	Production	Consumption	Surplus	Deficit
I	900	800	100	-
II	320	400	-	80
III	620	650	-	30
IV	635	725	-	90
V	1045	1020	25	-
VI	1065	990	75	-

And the cost structure of shipping

From origin (surplus zone)	To destination (deficit zone)		
	II	III	IV
I	5	10	2
V	3	7	5
VI	6	8	4

The I^{st} step is to formulate matrix with one row for each origin and one column for each destination plus two extra rows/columns. Each column /row labels with origin and destination and the additional rows/columns as demand & supply. Surplus in supply column & demand shows deficit. The figures in demand/supply rows/columns are called rim values.

After completing this matrix, the next step is to obtain the initial feasible tableau.

Methods of Obtaining Initial Feasible Solution

We apply three methods as given below;

1. North-West corner method.

2. Lowest cost entry method.

3. Vogel's approximation method.

(1) North-West Corner Rule (NWCR)

Step1: The north-west corner being upper left corner of matrix. Place in the north-west corner square, the smaller of the rim values for that row & column. The rim value for 1^{st} row is 100 & 1^{st} column is 80.

First assignment is made in the cell occupying the upper left hand (north- west) corner of the transportation table. The maximum feasible amount is allocated there. That is x_{11}= min. (a_1, b_1). This value of x_{11} is then entered in the cell (1, 1) of the transportation table.

Step 2: If $b_1>a_1$, move right horizontally to second column & make second allocation X_{21} = min ($a_{2,}$ b_1-X_{11}) in cel (2,1)

If b1=a1, there is a tie for the second allocation. One can make the second allocation of magnitude. If $b_1<a_1$, then X_{12}= min (a_1-X_{11},b_2)

x_{12} =min. (a_1-a_1, b_1) =0 in the cell (1, 2)

x_{21} =min. (a_2, b_1- b_1) =0 in the cell (2, 1)

Step 3: Repeat the steps 1 & 2 & lower right corner of LP.

Example 1: Obtain the initial basic feasible tableau for the problem given below:

		Cost to Destinations			
Cost from Origins	D	E	F	G	Total supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Total demand	200	225	275	250	950

We can use NWCR method for initial allocations by following above steps;

Origin	Destinations				Supply
	A	B	C	D	
1	11 200	13 50	17	14	250
2	16	18 175	14 125	10	300
3	21	24	13 150	10 250	400
Demand	200	225	275	250	950

(2) Lowest Cost Entry Method (LCEM)

Step 1: Determine smallest cost in cost matrix of the transportation table. Let it be c_{ij}. Allocate $X_{ij} \quad \min \left(a_i, b_j\right)$ in the cell (i,j).

Step 2: If x_{ij} = a_i cross off the ith row and decrease b_j by a_j.

If $x_{ij} = a_i = b_j$ cross off either i^{th} row or j^{th} column.

Step 3: Repeat steps 1 & 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, take an arbitrary choice among the maximum cost.

Example 2: Considering the above example 1, the initial allocation with LCEM will be;

Origin	Destinations				Supply
	A	B	C	D	
1	11 200	13 50	17	14	250
2	16	18 25	14 275	10	300
3	21	24 150	13 250	10	400
Demand	200	225	275	250	950

3. Vogel's Approximation Method

The Vogel's Approximation Method takes into account not only the least cost c_{ij} but also the cost that just exceed c_{ij}. The steps of the method are given below.

Step 1: For each row of the transportation table identify the smallest and the next to smallest costs. Determine the difference between them for each row and column.

Step 2: Identify the row or column with the largest difference among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the greatest difference corresponds to i^{th} row and let c_{ij} be the smallest cost in the i^{th} row. Allocate the maximum feasible amount x_{ij} =min (a_i,b_j) in the $(I, j)^{th}$ cell and cross off the i^{th} row or the j^{th} column in the usual manner.

Step 3: Re-compute the columns and row differences for the reduced transportation table and go to step 2. Repeat the procedure until all the rim requirements are satisfied.

Example 3: Given the previous problem, we can use VAM to prepare initial feasible table.

	A	B	C	D		
I	11 200	13	17	14	250	(2)
II	16	18	14	10	300	(4)
III	21	24	13	10	400	(3)
OD	200	225	275	250	950	
	(5)	(5)	(1)	(0)		

We allocate 200 to upper most & strike off 1st column. Now allocate 50 and strike the row as given below

	B		C		D		
I	50	13		17		14	(1)
II		18		14		10	(4)
III		24		13		10	(3)
	(5)		(1)		(0)		

The remaining allocation will be:

	B		C	D
I	175	18	14	10
II		24	13	10
	(5)		(1)	(0)

C	D	
14	125	10
13		10
(1)	(0)	

And the last allocation is:

275	13	125	10

Thus, the basic feasible solution table is:

	A		B		C		D		
I	200	11	50	13		17		14	250
II		16	175	18		14	125	10	300
III		21		24	275	13	125	10	400
Total	200		225		275		250		950

And the total transportation cost will be Rs. 12,075

Note: The initial basic feasible table will be nearer to the minimum cost but there is no guarantee that it will be the least cost. Hence, we follow iterative procedure to find the least cost.

Method to Solve the Transportation Algorithm

Various steps involved in solving any transportation problem may be summarized in the following iterative procedure:

Step 1: Construct a transportation table entering the origin capacities a_i, the destination requirements b_j and the costs c_{ij}.

Step 2: Determine an initial basic feasible solution using any of the three methods discussed above. Enter the solution in the upper left corners of the basic cells.

Step 3: For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$ for all i, k for which (i,j) is in the basis starting initially with some $u_i = 0$ and entering successively the values u_i and v_j on the transportation table.

Step 4: Compute the net evaluation $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for all the non basic cells and enter them in the upper right corner of the corresponding cells.

Step 5: Examine the sign of each $z_{ij} - c_{ij}$. If all $z_{ij} - c_{ij} \leq 0$, then the current basic feasible solution is an optimum one. If atleast one $z_i, -c_{ij} > 0$, select the variable x_{rs}, having the largest positive net evaluation to enter the basis.

Step 6: let the variable xrs enter the basis. Allocate an unknown quantity, say Θ, to the cell (r,s). Identify a loop that starts and end the cell (r,s) and connect some of the basic cells. Add and subtract interchangeably, Θ to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

Step 7: Assign a maximum value to Θ in such a way that the value of one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero, leaves the basis.

Step 8: Return to step 3 and repeat the process until an optimum basic feasible solution has been obtained.

Example 4: Solve the following transportation problem:

		To				
		A	B	C		
	I	50	30	220	1	
From	II	90	45	170	3	Availability
	III	270	200	50	4	
		4	2	2		
		Requirement				

Solution: Using the Lowest Cost Entering Method, an initial basic feasible solution to the TP is obtained which is shown in Table 1. The transportation cost is 855.

Starting Table. Determine the variables u_i (i=1,2,3) and v_j (j =1,2,3) using successively the relations $u_i + v_j = c_{ij}$ for all the basic cells, starting with $u_3 = 0$. These values are then used to compute the evaluation $z_{ij} - c_{ij}$ for all the non basic cells. Since all the net evaluations are not negative, we choose the maximum of these. Thus, the non basic cells (1,1) will enter the basis. The next step is to ascertain the optimum solution. An optimum is one which is least cost. This step has two stages:

(A). Computation of row and column values and (B) Computation of the values of unoccupied squares.

(A). For row and column values only occupied squares are used. For this the sum of row value and column value is equal to the value in the sub square of an occupied cell.

Row value + column value = sub cell value

50	1 / 30	220
2 / 90	1 / 45	170
2 / 250	200	2 / 50

Table 1

				ui
	Θ (25) / 50	1 - Θ / 30	(-ve) / 220	-175
	2 -Θ / 90	1 +Θ / 45	(-ve) / 170	-160
	2 / 250	(5) / 200	50	0
v_j	250	205	50	

Table 2

First Iteration. Introduce the cell (1,1) in the basis and drop the basic cells (1,2).

1 / 50	(-ve) / 30	(-ve) / 220	-200
1 +Θ / 90	2 - Θ / 45	(-ve) / 170	-160
2 -Θ / 250	Θ (5) / 200	2 / 50	2 0
250	205	50	

Table 3

1 / 50	(-ve) / 30	(-ve) / 220	-200
3 / 90	(-ve) / 45	(-ve) / 170	-160
0 / 250	2 / 200	2 / 50	0
250	205	50	

Table 4

Second Iteration. Introduce the cell (3,2) into the basis and drop the basic cell (2,2).

Now, since all the current net evaluations are non-positive, the current solution is optimum one.

Hence, the optimum allocation is given by

$x_{11} = 1$, $x_{21} = 3$, $x_{31} = 0$, $x_{32} = 2$ and $x_{33} = 2$.

The transportation cost according to the above root is given by

Z = 1 x 50 + 3 x 90 + 2 x 200 + 2 x 50 = 820.

Note:

(1) Negative value in an unoccupied cell square indicates that better solution can be found by moving units into unoccupied cells.

(2) Zero value in unoccupied cell indicates another solution of equal total value is available.

(3) Positive value indicates that sub-optimal solution will result if values are moved into that square.

The example of matrix show that improvement can be made by moving only with one of the cell

ASSIGNMENT PROBLEMS

It is a special case of Transportation Problem P in which the objective is to assign number of jobs to number of persons/machines. The assignment is to be made in such a way that one person gets one job to minimize cost or maximize profit.

Mathematically; Let Xij be the variables defined by

$X_{ij} = $ [illegible]
[illegible]

Since one job is to be assigned to each machine/person, we have

$$\sum_{i=1}^{n} Xij = 1 \text{ and } \sum_{j=1}^{n} Xij = 1$$

Also total assignment out

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} CijXij$$

Thus the model becomes

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} CijXij$$

Subject to

$$\sum_{i=1}^{n} Xij = 1$$

$$\sum_{j=1}^{n} Xij = 1$$

Xij = 0 or 1

Steps:

1. Subtract the min cost of each row of the cost matrix from all the element of representive row. Then again modify the resulting matrix by subtracting the min cost of each column from the entire column to obtain starting table.
2. Draw the least possible no. of horizontal and vertical lines to cover all the zeros of starting matrix.

Let no. of lines be N.
If N = n (order of matrix) optimum assignment has reached
If N< n go to next step.

3. Determine the smallest cost in the starting table, not covered by IV lines subtract their cost from the surviving elements (uncovered) and add the same to all those which are lying at the intersection of horizontal and vertical lines, thus obtaining the second starting table.
4. Now redraw the horizontal and vertical lines covering zeros and examine that N=n.
5. Examine the row until a row with exactly one unmarked zero is formed. Enclose the zero ⓪ and ⓧ other zero lying in the column as they cannot be taken for future assignment
6. Examine the column successively until a column with exactly one uncovered zero ⓪ is formed as assignment and ⓧ as other row in that row
7. Repeat step 5& 6 successively until one of the following arises
   No. unmarked zero is left. This method is known as Hungarian Assignment Method

Example 1: Solve the assignment problem

Jobs

		1	2	3	4
Persons	A	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

```
 ?  ?  ?  ?                  ?  ?  ?  ?
 ?  ?  ?  ?                  ?  ?  ?  ?
?            ? Row diff     ?            ?N=n
 ?  ?  ?  ?                  ?  ?  ?  ?
 ?  ?  ?  ?                  ?  ?  ?  ?
```

So optimum assignment has reached

	1	2	3	4
A	ⓧ	⓪		
B	ⓧ		⓪	
C			ⓧ	⓪
D	⓪			

Then A------2
B------3
C------4
D------1 cost will be 12+7+11+8=38

Example 2:

| 1 | 2 | 3 | 4 | 5 |

A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Row diff

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Column diff N<n

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

N=n

	1	2	3	4	5
A		(0)			
B	(0)			(x)	
C			(x)	(x)	(0)
D			(0)		
E				(0)	

A ----2
B---1
C---5
D---3
E---4
Cost = 5+1+8+12+8 = 34

Practical Exercise

Solve the following assignment problem

Jobs

		1	2	3	4	5
Person	A	11	19	8	16	22
	B	9	7	12	6	17
	C	13	16	15	12	16
	D	21	24	17	28	26
	E	15	10	12	11	15

SEQUENCING PROBLEMS

The selection of an appropriate order for series of jobs to be done on a finite number of services facilities is called sequencing. In this we have to determine order (sequence) of performing the jobs in such a way that the total cost (time) may be min.

Assumptions

1) No machine can process more than one job at a time
2) Processing times (Mij's) i.e ith (1,2….n) job on jth (1,2…..m) machines are independent of processing the jobs
3) The time taken in moving the jobs from one machine to another is negligibly small
4) Each job once started on a machine to be continued till it is completed
5) All machines are of different types
6) All jobs are completely known & are ready for processing before the period under consideration begins

Problems with N jobs and two machines

Let there be n jobs and M1 & M2 two machines in order M1and M2

Steps

1) Examine Mi1's and Mi2's (i=1,2….n) and find the min (Mi1, Mi2)
2) If this min be Mki for some i=k & then take kth jobs first of all. If this min is Mr2 for some i=r do rth job last of all
3) If there is a tie for min Mk1=Mr2 process kth first and rth last. If tie occur in Mi1's select the corresponding jobs min of Mi1's & process first of all. If tie for Mi2's select corresponding to the min Mi2 & process last of all
4) Cross if the jobs already assigned & repeat steps 1 to 3

Problem with n jobs & m machines

Step1: Find min Mi_1 and Mi_2, & max of Mi_2, Mi_3….Mi (m-$_1$)

Step2: Check whether

min Mi1 ≥ max Mij j=2,3….m-1

Or

min Mim ≥ max Mij j=2,3….m-1

Step3: If in equation of step2 are not satisfied, method fails, otherwise go to next step

Step4: convert m machine problem into two machine by introducing two fictitious machines G & H such that

MiG = Mi1 + Mi2…..+Mim-1

MiH = Mi2 + Mi3…….+ Mim

Determine optimal sequence of n jobs through 2machines as given above

Example 1: A = press shop, B = finishing

	Jobs (days)		
	1	2	3
Dept. A	8	6	5
Dept. B	8	3	4

Min (Ai,Bi) = 3 which is for B2 so B2 will be done last of all

		II

Now out of remaining Min (Ai, Bi) = 4 which is again for last of all

I	III	II

Thus sequence of min time is

Job	Machine A		Machine B	
	Time in	Time out	Time in	Time out
I	0	8	8	16
III	8	13	16	20
II	13	19	20	23

Min time to finish all 3
Jobs = 23 days
Idle time for A = 4 days
Idle time for B = 8 days

Example 2:

Jobs	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4
	(ii)	(i)	(v)	(iii)	(iv)

2	4	3	5	1

Job	Machine A		Machine B		Idle time
	Time in	Time out	Time in	Time out	B
2	0	1	1	7	1
4	1	4	7	15	0
3	4	13	15	22	0
5	13	23	23	27	1
1	23	28	28	30	1

Idle time for A = 2 Hours
Idle time for B = 3 Hours
Example 3: 3 machines

Jobs	1	2	3	4	5
A	5	7	6	9	5
B	2	1	4	5	3

C	3	7	5	6	7

Now Min $A_i = 5$, Min C=3 Max Bi = 5

Since Min $A_i \geq$ Max Bi given problem cab be solved as two machines G & H

As $G = A_i + B_i$ $H = B_i + C_i$

Jobs	1	2	3	4	5
G	7	8	10	14	8
H	5	8	9	11	10

So jobs sequence will be

2	5	4	3	1

Or

5	4	3	2	1

A	In	0	5	14	20	27
	Out	5	14	20	27	32
B	In	5	14	20	27	32
	Out	8	19	24	28	34
C	In	8	19	25	30	37
	Out	15	25	30	37	40

Idle time for A = 8 Hours

Idle time for B = 25 Hours

Idle time for C = 12 Hours

Example 4: Determine the optimal sequence of jobs that minimize total time on 3 machines M1M2M3.

Jobs	A	B	C	D	E	F	G
M1	3	9	6	4	9	8	7
M2	4	3	2	5	2	4	3
M3	6	7	5	10	5	6	11

Min Mi1 = 3 MinMi3 = 5

MaxMi2 = 5

Since Mi3 $\geq$ Mi2

Hence, we can solve as two we have

G =Mi1+Mi2

H=Mi2+Mi3

Example 5: Solve the following problem

		Jobs		
Machine		1	2	3
	A	8	6	5
	B	8	3	4

1	1	3	2

Jobs	A		B	
	Jobs in	Jobs out	Jobs in	Jobs out
1	0	8	8	16
3	8	13	16	20
2	13	19	20	23

Total 23, idle A= 4 days, idle B = 8days

	1	2	3	4	5
A	3	1	9	3	10
B	2	6	7	8	4

2	4	3	5	1

3	1	2	3	4	5
Machines					
A	5	7	6	9	5
B	2	1	4	5	3
C	3	7	5	6	7

Min Ai = 5 Min Ci = 3
Max Bi = 5
Hence Min Ai ≥ Max Bi so we can solve this

Gi = 7 8 10 14 8

Hi = 5 8 9 11 10

2	5	4	3	1

A	In	0	5	14	20	27
	Out	5	14	20	27	32
B	In	5	14	20	27	32
	Out	8	19	24	28	34
C	In	8	19	25	30	37
	Out	15	25	30	37	40

Idle Time for A=8
B=19
C=20

APPENDIX- I

Typical Agricultural Planning Linear Programming Model

1. Returns to Fixed Farm Resources (RFFR) Rs.;ha									
		Cj	35000	40000	-12000	-18000	35000	-15000	-12000
	Constraints	Resource supply (b)	Maize	Rice	Cherry (f)	Bajra (f)	Wheat	Barseem (f)	Oat (f)
1	Kharif land	1.5	1	1	1	1	0	0	0
2	Rabi land	1.5	0	0	0	0	1	1	1
3	Kharif Capital	120000	25000	30000	12000	18000	0	0	0
4	Rabi capital	150000	0	0	0	0	25000	15000	12000
5	Labour (days)	3 person (1095)	250	350	150	125	350	150	150
6	Bullock labour days	60	6	10	3	4	8	3	3
7	FYM (tonnes)	40	24	20	10	8	25	8	8
8	Urea(kg)	400	250	150	150	100	200	100	100
9	SSP(kg)	250	150	200	100	150	150	100	100
10	MOP(kg)	200	100	100	50	50	150	50	50
11	Green fodder	0	0	0	-280	-250	0	-300	-200
12	Dry fodder	6	-60	-35	0	0	-40	0	0
13	Min Area (maize)	0.50	1	0	0	0	0	0	0
14	Min Area (paddy)	0.50	0	1	0	0	0	0	0
15	Min Animal (dairy)	1	0	0	0	0	0	0	0
16	Max area vegetable	0.50	0	0	0	0	0	0	0
17	Max area potato	0.25	0	0	0	0	0	0	0

		95000	100000	55000	45000
	Constraints	Vegetable		Potato	
		Kharif	Rabi	Kharif	Rabi
1	Kharif land	1	0	1	0
2	Rabi land	0	1	0	1
3	Kharif Capital	45000	0	4000	0
4	Rabi capital	0	55000	0	25000
5	Labour (days)	300	350	350	350
6	Bullock labour days	12	12	15	18
7	FYM (tonnes)	36	30	30	25
8	Urea(kg)	400	300	100	100
9	SSP(kg)	200	225	200	225
10	MOP(kg)	200	200	150	150
11	Green fodder	-20	-15	0	0
12	Dry fodder	0	0	0	0
13	Min Area (maize)	0	0	0	0
14	Min Area (paddy)	0	0	0	0
15	Min Animal (dairy)	0	0	0	0
16	Max Area vegetable	1	1	0	0
17	Max Area potato	0	0	1	1

Contd…

		50000	23000	17000	27000	0	-0.05	-0.05
	Constraints	Gram	Livestock (per unit)			Capital Transfer	Capital borrowing	
			CBC	Local	Buffalo		kharif capital	Rabi capital
1	Kharif land	0	0	0	0	0	0	0
2	Rabi land	1	0	0	0	0	0	0
3	Kharif Capital	30000	5000	7000	10000	+1	-1	0
4	Rabi capital	0	4000	7000	10000	-1	0	-1
5	Labour (days)	200	100	100	100	0	0	0
6	Bullock labour days	10	0	0	0	0	0	0
7	FYM (tonnes)	35	-20	-15	-25	0	0	0
8	Urea(kg)	225	0	0	0	0	0	0
9	SSP(kg)	200	0	0	0	0	0	0
10	MOP(kg)	150	0	0	0	0	0	0
11	Green fodder	10	28	30	60	0	0	0
12	Dry fodder	0	15	18	30	0	0	0
13	Min Area (maize)	0	0	0	0	0	0	0
14	Min Area (paddy)	0	0	0	0	0	0	0
15	Min Animal (dairy)	0	1	1	1	0	0	0
16	Max Area vegetable	0	0	0	0	0	0	0
17	Max Area potato	0	0	0	0	0	0	0

		-600	-250	-450	-6.90	-8.00	-12.00
	Constraints	Purchase activities					
		Dry fodder (q)	Labour hiring (manday)	Bullock labour hiring (BPD)	urea (kg)	SSP	MOP
1	Kharif land	0	0				
2	Rabi land	0	0				
3	Kharif Capital	0	250	450	6.90	8.00	12.00
4	Rabi capital	600	0	0	0	0	0
5	Labour (days)	0	-1	0	0	0	0
6	Bullock labour days	0	0	-1	0	0	0
7	FYM (tonnes)	0	0	0	0	0	0
8	Urea(kg)	0	0	0	-1	0	0
9	SSP(kg)	0	0	0	0	-1	0
10	MOP(kg)	0	0	0	0	0	-1
11	Green fodder	0	0	0	0	0	0
12	Dry fodder	-1	0	0	0	0	0
13	Min Area (maize)	0	0	0	0	0	0
14	Min Area (paddy)	0	0	0	0	0	0
15	Min Animal (dairy)	0	0	0	0	0	0
16	Max Area vegetable	0	0	0	0	0	0
17	Max Area potato	0	0	0	0	0	0

Note: Peak periods can be identified for human and bullock labour and hiring activities can be split accordingly. Similarly purchase of inputs can be split into kharif and rabi seasons