A COMPREHENSIVE STUDY OF ELECTRICAL POWER QUANTITIES

By

CHARLES WILLIAM JILES

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THESIS AND ABSTRACT APPROVED:

Chas. F. Cameron
Thesis Adviser

A. Naeter
Faculty Representative

D. C. McIntosh
Dean of the Graduate School

PREFACE

In the past, much research and investigations have been conducted in an attempt to explain some of the discrepancies in the use of complex number theory as applied to alternating current circuit quantities. There has been a great deal of confusion and incomplete understanding of the exact nature of reactive power and how it is measured. Also, no uniform procedure has been accepted in deciding whether reactive power shall have a positive or a negative sign. The application of complex number theory to obtain power has caused confusion in the minds of many.

When harmonics have been present in the circuit being considered, the equations for power factor and power have been found untrue and useless. In this thesis, the author has attempted to show why these equations failed, and the geometric picture of the conditions causing this failure is presented.

ACKNOWLEDGEMENTS

A study of this nature requires the availability of material which has been published on the subject for the past several decades. For the use of this material and for many ideas, the author is deeply indebted to Prof. Charles F. Cameron. Also, the author wishes to express his gratitude to Prof. Cameron for his guidance and continuous inspiration in the preparation of this thesis. The time and effort spent by Dr. H. L. Jones in checking Chapter II and the tireless proofreading of Miss Opal Earl Baber were both greatly appreciated.

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CHAPTER I

INTRODUCTION

Often in the early development of a particular branch of science, rules and regulations will be established which govern the existing knowledge of the field at that time. Later, however, invariably many new, and sometimes radically different, aspects of the science are discovered. These newly discovered truths may necessitate modifications in the statement of the rules and regulations formerly established, and in some instances may even render them void and useless. For example, the development of the theory of light propagation and the flow of light energy is yet to be completely explained. At present, the use of both the Wave and the Quantum Theories are required to explain all the phenomena of light. This is the way that knowledge has progressed from the very earliest beginning; first, the most elementary ideas and concepts, and then, as the known facts increase, the more complex aspects are explained, leading to the ultimate establishment of the science on a firm and sound theoretical basis, the theory being substantiated by observation and experiment.

The development of the field of electricity and magnetism has been no exception to this general rule. The first ideas were based almost entirely upon experimental data, but since that time more powerful mathematical tools have been discovered and developed that have tended to clarify and explain the wonders of electricity. Many of the presently existing ideas in the field of alternating current electricity are even yet inconsistent.

The student of elementary electrical technology learns Ohm's law in the symbolic form $I = E/R$, which is equivalent to the verbal statement that the current in an electric circuit is exactly proportional to the voltage applied to it. By assuming the truth of this statement of the law, the student is enabled to solve a large number of simple-circuit problems, and it is not until he progresses a little further in his studies that he finds its applications to be limited and true only under important qualifications.

In simple metalic circuits with steady voltages, for instance, this expression of the law holds good only when the temperature of the material of the circuit is maintained constant. Again, with alternating voltages and with inductive circuits, it is true only if the frequency is also constant. If the magnetic flux causing the inductance flows in iron this interpretation of the law is not correct; even with constant temperature and frequency, current is not exactly proportional to applied voltage. Further, Ohm's law is known not to hold in certain circuits of an electrolytic character.

The discrepancy in cases like these is sometimes explained by using the fiction of a back EMF, but this begs the whole question. It is better to recognize that there are two distinct kinds of electrical circuits, the one in which Ohm's law as defined above is obeyed and in which current is exactly proportional to voltage, and the other in which this proportionality does not hold. Circuits of the first class are generally said to have a linear impedance; in circuits of the other class the impedance is designated non-linear.[1]

This paper is concerned in pointing out the inconsistencies in the meaning of power factor, real power, reactive power, and apparent power under different adverse conditions. An attempt is made to explain the appearance of distortion power and mesh power, and more general definitions to include the possibility of these quantities in describing power factor and power factor angle are recommended.

POWER FACTOR

In dealing with power in different types of circuits, attention is necessarily focused upon a quantity which in Electrical Engineering literature is termed "power factor." The question arises in connection with the definition of this quantity.

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[1] G. W. Stubbings, "Harmonics and P. F.," Electrical Review, (June 5, 1942).

Elementary a.c. theory is based upon the fundamental assumption of sinusoidal voltage wave forms and linear impedances. With these assumptions, current wave forms are also sinusoidal and power factor is the cosine of the angle of lead or lag. Because of this power factor is generally denoted as cos θ, a symbol that was chosen most unfortunately, because it obscures the fact that in a large group of circuits power factor is not determined solely by phase angle. It is regrettable that a characteristic symbol was not chosen for such an important ratio as power factor. Had this been done the inaccurate and unqualified identification of power factor with the cosine of an angle would not have arisen and much difficulty might have been avoided.

If power factor is defined as the cosine of the angle between the voltage wave and the current wave, difficulties may easily arise. Consider the case in a single-phase circuit where the current and voltage are in-phase, and are sinusoids. The power factor is obviously unity. If now the circuit is modified in such a manner that the third, fifth and seventh harmonics of current are allowed to flow, the effective value of the current is apparently increased. It should be recalled here that this power factor is the quantity that when multiplied by the product of the current and voltage waves (effective values) gives real or active power. Now it is a well known fact that no increase in active power is experienced in a circuit when harmonics of either current or voltage are introduced, unless like harmonics of the other are present also. It is to be noticed that in the above example when harmonics of the current wave were introduced no phase displacement in current resulted, and hence the power factor remained unity. But, this same power factor multiplied by an increased effective volt-amperage would indicate an increase in active power. Since this is known to be false, it is evident that an inconsistency exists in the definition when applied to a single-phase circuit containing harmonics in either the current or voltage wave forms.

Power factor is not the cosine of the angular measure of a phase difference but the ratio of watts to the product of effective values of voltage and current. Using accepted definitions of real and reactive powers, it can easily be shown that in the above example the square of the apparent power is not equal to the sum of the squares of the real and reactive powers. If we follow certain suggestions and define a new type of power, distortion power, so that the square of the apparent power is equal to the sum of the squares of the real, reactive, and distortion powers, it is evident from geometric intuition that the apparent power is no longer in the same plane as the real and reactive powers.[2] Notice that now, using the above definition, the power factor angle is also in a different plane from the real and reactive powers, and is, in fact, in the plane of the apparent and real powers. But this violates the definition of reactive power, for reactive power is no longer equal to the apparent power multiplied by the sine of the power factor angle, for this is what is now called ficititious power.

If one now were to become so bold as to attempt to apply the above definition (the latter one) of power factor to a polyphase, unbalanced circuit with nonsinusoidal conditions imposed upon the voltage and current wave forms, it is seen that the definition of power factor and the location of the power factor angle is even more vague. For now, in addition to distortion power, the appearance of a new type of power, termed mesh power, is noted. Mesh power being defined so that the square of the apparent power is equal to the sum of the squares of the real, reactive, distortion, and mesh powers. Thus it is evident that the exact location of the power factor angle under these conditions has become quite elusive, and there is ample room for speculation as to just what is meant when the term power factor angle is used.

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[2] V. G. Smith, "Reactive and Fictitious Power," A. I. E. E. Transactions, LII (1933), 748-751.

It must not be assumed that waves of current and voltage which are not sinusoidal and which have no apparent angular displacement with respect to each other, or that dissimilar waves in which the zero ordinates occur simultaneously will necessarily give power factor of unity. In fact, power factor of unity can be obtained only when the current and voltage waves are exactly similar and have no displacement with respect to each other. Waves of current and voltage that are not similar, but are not angularly displaced, produced a power factor less than unity. Curves of current and voltage of relative symmetry, such as a semicircle associated with a parabola, yield their maximum power factor when their zeros occur simultaneously. Their maxima, then, likewise occur simultaneously. When the current and voltage waves are not of such relative symmetry, the maxima will not be coincident in time when the zeros are, and the maximum power factor is yielded when neither zeros nor maxima are coincident.

VECTOR REPRESENTATION OF SINUSOIDALLY VARYING ELECTRICAL QUANTITIES

The exact name to be applied to sinusoidally varying currents and voltages has been under discussion for a long time. The difficulty seems to have arisen due to peculiar characteristics of these quantities and also perhaps because of a lack of sufficient knowledge of just what a vector really is. The definitions of vector and scalar quantities from the American Standard Definitions of Electrical Terms are as follows:

Scalar Quantity

> A scalar quantity is any quantity which has magnitude only. The relationship between any physical quantity and the unit used to measure it is completely described by a real number. Examples of quantities for which the relationship can be represented by either a positive or negative real number are: time, temperature, and quantity of electricity.

Vector Quantity

A vector quantity is a quantity which has both magnitude and direction. Examples of quantities that are vector quantities are: displacement, velocity, force, and magnetic intensity.

The most common methods of describing a vector are by means of the projections on a system of rectangular coordinates, or by stating the magnitude by means of spherical coordinates. In rectangular coordinates, if i, j, and k represent unit vectors along the X, Y, and Z axes, respectively, and if V_x, V_y, and V_z are the scalar values of the projections of the vector V on the axes, then $V = V_x i + V_y j + V_z k$, where the plus sign denotes addition.

Currents and voltages are not completely described by a real number, as stipulated in the definition of a scalar quantity, since they may have relative phase positions with respect to each other. The error in terminology seems to have arisen from the fact that a sinusoidally varying quantity is the scalar horizonal projection of a revolving vector that is constant in magnitude and turning at a constant speed. Moreover, the sinusoidal quantity had both magnitude and direction, that is, it has direction in the sense that it is either directed in one direction or opposite to that direction. Perhaps better terminology would be that the sinusoid has magnitude and sense. Therefore, the sinusoid is in actuality a directed magnitude or line segement. Consequently, the term directed line or sinor has been suggested by some authors as a better name instead of vector.

A directed magnitude or line segement is what today is called the modulus or magnitude of a complex number. Sinusoidally changing quantities are then in truth complex quantities. This has been known for some time, but the terminology of calling them vectors has persisted.

However, in alternating current circuit theory care should be taken in dealing with the complex quantities which arise. The impedance of an inductive circuit is often very glibly termed a complex quantity in one breath and a sinusoidally varying quantity called a complex quantity in the next breath.

Here extreme caution should be observed. For, while both these quantities are essentially complex quantities, they are basically different in that one has a characteristic frequency of oscillation while the other is purely a complex number. As long as operations are performed between two impedances represented in complex form no difficulty is observed. Furthermore, when a sinusoid in complex form is multiplied by an impedance in complex form no inconsistency is again found, since the "scalar complex quantity," the impedance, does not effect the frequency of the "vector complex quantity," i.e., current. Here the term "vector complex quantity" is used to designate those quantities having a frequency of vibration, i.e., voltage, current, and power, and the term "scalar complex quantity" is used to refer to pure complex numbers, i.e., impedance. However, when two vector complex quantities are multiplied together, i.e., voltage and current, the ordinary laws of multiplication of complex quantities no longer holds true. Now the product of current and voltage is known to be power, but to obtain the correct result with the two factors in complex form the conjugate of one of the two must be used. This fact is often treated as a law of nature by many authors and completely ignored. The truth of the matter is, as has been pointed out by A. S. Langsdorf, that the operator j is being used both as a turning operator and as a unit vector. Or, perhaps more vividly, the product of current and voltage is power, a double frequency quantity, in which the meaning of the operator j must be re-defined. If the operation of j represented a rotation of 90 degrees with respect to the current and voltage, it would now represent 180 degrees with respect to the power.

Thus it is seen that in electricity there are two kinds of complex quantities; those possessing frequency of oscillation and those not possessing this characteristic. Perhaps a better terminology would be to name them as used above, "complex vector" and "complex scalar" quantities. In this paper the "complex vector quantities" will be called merely vectors, and the "complex scalar quantities" will be termed complex quantities.

CHAPTER II

THE DEVELOPMENT OF MAXWELL'S DIFFERENTIAL EQUATIONS AND THE ULTIMATE LOCATION OF CURRENT AND VOLTAGE WAVES FROM THESE EQUATIONS

Statement of Gauss's Law: The total normal electrical induction over a closed surface is equal to 4π times the total charge within it. A similar relation holds for the magnetic case.

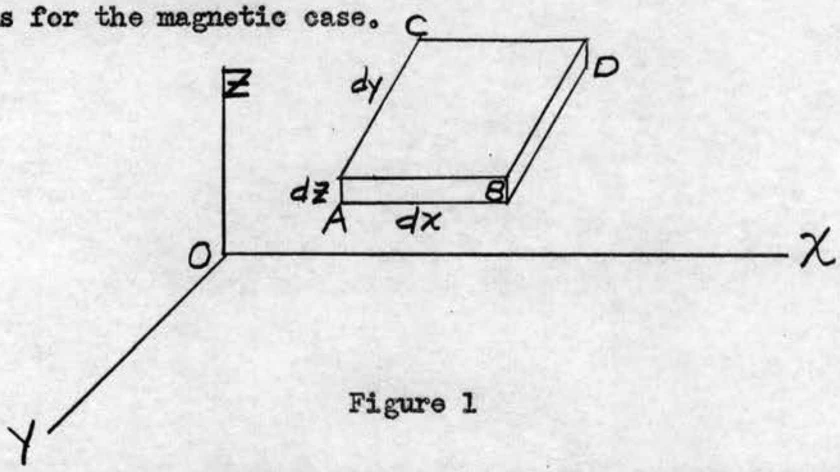

Figure 1

Consider the electric intensity, E, at any point A (shown in Figure 1) as being a vector in space. (E is defined as the force that will be exerted on an unit charge when placed in an electric field, where both the force and the electric intensity are expressed in the appropriate units). Let E be resolved into components parallel to the coordinate axes: P parallel to OX, Q parallel to OY, and R parallel to OZ. Then, if P varies as motion is experienced from point A in a direction parallel to OX, which in general it would, its rate of change would be dP/dx. If this rate of change occurs for a distance dx, then the amount of change undergone in the distance moved through (dx) would be $(dP/dx)(dx)$. The final value of the component of E parallel to OX is the sum of the original value and the change that has taken place, hence the value of P at the face BD (Figure 1) will be $P + \frac{dP}{dx}dx$. The normal induction over face AC is KPdydz, where K is the dielectric constant. Thus the normal induction over BD is $K(P + \frac{dP}{dx}dx)dydz$. The difference in these two is then the portion of the normal induction over the entire surface due to faces AC and BD. That is: $K(P + \frac{dP}{dx}dx)dydz - KPdydz = K\frac{dP}{dx}dxdydz$.

Now, if faces AB and CD are treated in exactly the same manner as above, their contribution to the total normal induction is obtained as: $K\frac{dQ}{dy}dxdydz$, and similarly for faces AD and BC the value of $K\frac{dR}{dz}dxdydz$ results. If p is the density of the electric charge, and the rectangle in Figure 1 is considered as being incremental in size, the total charge is pdxdydz. The total normal induction for all the faces is: $K(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz})dzdydz$.

From Gauss's Law:

$$K(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz})dxdydz = 4\pi pdxdydz$$

Hence

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = \frac{4\pi p}{K}$$

For the magnetic field, if L, M, and N are the components of H, the above equations in analogous form will be:

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = \frac{4\pi p}{u}$$

where p for the magnetic case is the volumetric density of the magnetic pole, and u is the magnetic permeability of the medium. This is Poisson's equation. If p be zero, the expression for the electric field is:

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = 0 \qquad (1)$$

and that for the magnetic field is:

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = 0 \qquad (2)$$

Equation (1) is frequently called the divergence of E, since it represents the rate at which E changes as motion is undergone outward from point A. Thus Div. E = $4\pi p/K$.

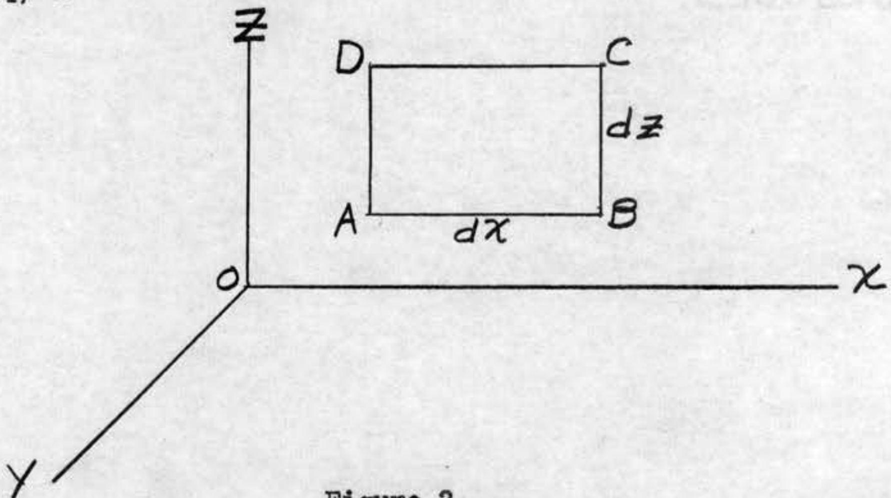

Figure 2

The work done in carrying a unit pole around a closed path through which a current is flowing is equal to 4π times the current. By definition, this is the line integral around that path. This is also the curl of H! That is, curl H = $4\pi i$, where i is the current density.

If u, v, and w be the components of i parallel to the X, Y, and Z axes respectively, then the current flowing throught the small rectangle ABCD is vdzdz. If the component of the magnetic field along AB is again L, than along DC it will be $L + \frac{dL}{dz}dz$. The work done on a unit pole as it moves along AB is Ldx, and along CD is $-(L + \frac{dL}{dz}dz)dx$. Similarly, for DA it is $-Ndz$, and for BC $(N + \frac{dN}{dx}dx)dz$. Therefore, for the entire path ABCD:

$$\text{work done} = Ldx - (L + \frac{dL}{dz}dz)dx - Ndz + (N + \frac{dN}{dx}dx)dz$$

$$\text{work done} = \left(\frac{dN}{dx} - \frac{dL}{dz}\right)dxdz$$

If the above law is applied, the following result is obtained:

$$4\pi v dxdz = \left(\frac{dN}{dx} - \frac{dL}{dz}\right)dxdz$$

$$-4\pi v = \frac{dL}{dz} - \frac{dN}{dx}$$

If the other two components of the current, u and w, are treated in an entirely analogous manner to the above procedure for v, the following group of equations are obtained for the three components.

$$-4\pi u = \frac{dN}{dy} - \frac{dM}{dz}$$

$$-4\pi v = \frac{dL}{dz} - \frac{dN}{dx} \qquad (3)$$

$$-4\pi w = \frac{dM}{dx} - \frac{dL}{dy}$$

CONSIDERATION OF THE ELECTROMOTIVE FORCE AROUND A CIRCUIT

THROUGH WHICH THE MAGNETIC FLUX IS VARYING

By Faraday's Law, the generated counter-voltage in an inductive circuit is equal to a constant times the rate of change flux interlinkages. If the units are properly chosen, the constant is unity, and using a negative sign to signify a counter-voltage, the equation for this voltage is:

$$e = -\frac{dN}{dt} \quad \text{(where N is the total normal induction)}$$

The flux N through the rectangle (ABCD) in the above figure is uMdxdz and $dxdz\frac{d(uM)}{dt}$ is the rate of change of flux.

If the component of E along AB is P, and along DC is again $P + \frac{dP}{dz}dz$, and the components along AD and BC respectively, R and $R + \frac{dR}{dx}dx$, the total electromotive force, e, around the rectangle is:

$$Pdx - (P + \frac{dP}{dz}dz)dx - Rdz + (R + \frac{dR}{dx}dx)dz \;=\; (\frac{dR}{dx} - \frac{dP}{dz})dxdz$$

Since $e = -\frac{dN}{dt}$, then $u\frac{dM}{dt} = \frac{dP}{dz} - \frac{dR}{dx}$, from the fact that

$\frac{dN}{dt} = dxdz\frac{d(uM)}{dt}$. With the two corresponding equations for the components, L and N the following equations are derived:

$$u\frac{dL}{dt} = \frac{dR}{dy} - \frac{dQ}{dz}$$

$$u\frac{dM}{dt} = \frac{dP}{dz} - \frac{dR}{dx} \qquad (4)$$

$$u\frac{dM}{dt} = \frac{dQ}{dx} - \frac{dP}{dy}$$

In equations (3) the current u, v, and w means that an electric charge is moving in a certain direction, and it is known that this motion cannot be continuous unless the medium is an electric conductor.

CONSIDERATION OF THE PROPAGATION OF PLANE WAVES

The electric intensity will be assumed to be the same over the entire plane under consideration.

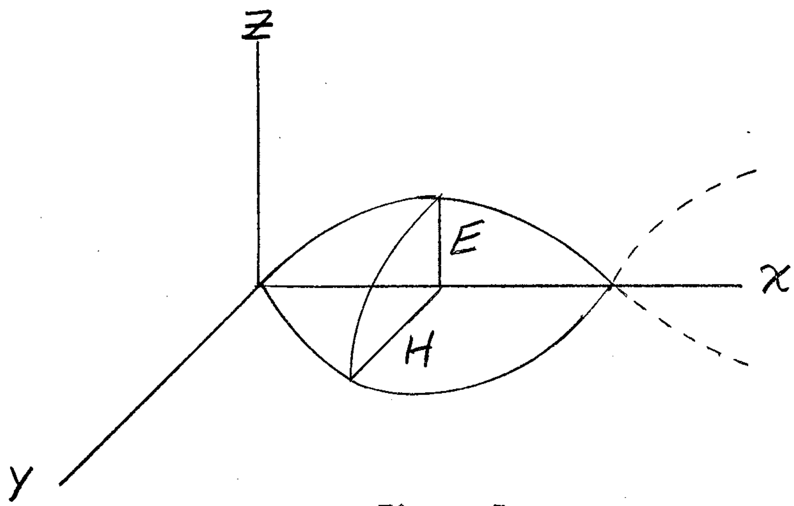

Figure 3

Let YOZ be the plane of the wave. The variation of the electric and magnetic intensities in the Y and Z directions is zero if these quantities are constant. Equations (4) then become:

$$u\frac{dL}{dt} = 0$$

$$u\frac{dM}{dt} = -\frac{dR}{dx}$$

$$u\frac{dN}{dt} = \frac{dQ}{dx}$$

and consequently L is zero or a constant. Since constant values do not enter into wave propagation, put L equal to zero.

$$D = \frac{KE}{4\pi}$$

where D is the number of Faraday tubes per square centimeter. Since there are 4π lines per tube:

$$4\pi D = N = KE$$

$$i = \frac{dD}{dt} = \frac{K}{4\pi} \cdot \frac{dE}{dt}$$

Expressing i in terms of its components, the following set of equations are obtained:

$$u = \frac{K}{4\pi} \cdot \frac{dP}{dt}$$

$$v = \frac{K}{4\pi} \cdot \frac{dQ}{dt}$$

$$w = \frac{K}{4\pi} \cdot \frac{dR}{dt}$$

Substituting these values in equation (3) gives:

$$-K\frac{dP}{dt} = \frac{dN}{dy} - \frac{dM}{dz}$$

$$-K\frac{dQ}{dt} = \frac{dL}{dz} - \frac{dN}{dx} \qquad (5)$$

$$-K\frac{dR}{dt} = \frac{dM}{dx} - \frac{dL}{dy}$$

If L from above is zero, equation (4) reduced to value on preceding page and equation (5) becomes:

$$K\frac{dP}{dt} = 0$$

$$K\frac{dQ}{dt} = \frac{dN}{dx}$$

$$K\frac{dR}{dt} = -\frac{dM}{dx}$$

Therefore, P is zero, and since it has previously been shown that L is zero, it follows that the direction of the electric and magnetic intensities are entirely in the plane of the wave. Since the direction of E in the YOZ plane is entirely arbitrary, the direction that would give the simplest result would be the logical choice for the direction of the electric intensity. Therefore, the direction of E shall be chosen as parallel to the OZ direction. Obviously then, Q is zero by virture of the choice of axes, and it is seen from either of the relations:

$$u\frac{dN}{dt} = \frac{dQ}{dx},$$

$$\text{or} \quad K\frac{dQ}{dt} = \frac{dN}{dx}$$

that in this case N must be equal to zero. But, this means, since both N and L are zero, that the magnetic intensity lies along OY. Therfore, the E and H vectors are at right angles to each other and lie along the Z and Y axes, respectively. Since R and M are now the only components of E and H, respectively, the equations reduce to:

$$u\frac{dM}{dt} = -\frac{dR}{dx}$$

$$K\frac{dR}{dt} = -\frac{dM}{dx}$$

Differentiating the first with respect to t and the second with respect to x, the following equations ensue:

$$u\frac{d^2M}{dt^2} = -\frac{d^2R}{dxdt}$$

$$K\frac{d^2R}{dxdt} = -\frac{d^2M}{dx^2}$$

Or, differentiating the first with respect to x and the second with respect to t:

$$u\frac{d^2M}{dxdt} = -\frac{d^2R}{dx^2}$$

$$K\frac{d^2R}{dt^2} = -\frac{d^2M}{dxdt}$$

$$\text{Therefore,} \quad \frac{d^2R}{dt^2} = \frac{1}{Ku} \cdot \frac{d^2R}{dx^2}$$

This is the form of the general equation of the motion of a plane wave, the direction of propagation being parallel to the axis OX, and its general solution is:

$$R = f_1(x - vt) + f_2(x + vt)$$

where $v^2 = \frac{1}{Ku}$, and f_1 and f_2 are, of course, functions.

It can be shown that $f_1(x - vt)$, and $f_2(x + vt)$, represents a wave that travels along the OX axis. The direction of propagation of $f_1(x - vt)$ is toward the right, or in the direction of the positive X axis, while the direction of propagation of $f_2(x + vt)$ is toward the left, or along the negative

X axis. Only the forward wave, namely $f_1(x - vt)$, shall be considered here, and only the most important case of such a wave, specifically that in which R and M vary harmonically.

$$\text{Let } R = Ro \sin \frac{2\pi}{A}(x - vt) \quad \text{(Where A is wavelength.)}$$

$$\text{If } t = 0: \; Ro \sin \left(\frac{2\pi}{A}x\right)$$

Although the curve is drawn with the OX axis as reference, it must be understood that the value of R at all points in any plane parallel to YOZ, is the same at each instant, and is represented by the ordinate of the curve.

If x be increased by the length A:

$$R = Ro \sin \frac{2\pi}{A}(x + A) = Ro \sin \left(\frac{2\pi}{A} x + 2\pi\right),$$

and the curve begins to repeat itself. If T is the period and t is the time;

$$R = Ro \sin 2\pi\left(\frac{x}{A} - \frac{t}{T}\right)$$

In order to make an analogous derivation for M, the following equations are used:

$$u\frac{dM}{dt} = -\frac{dR}{dx}$$

$$\frac{dR}{dx} = \frac{2\pi}{A} Ro \operatorname{Cos} 2\pi\left(\frac{x}{A} - \frac{t}{T}\right)$$

$$\text{Therefore, } \frac{dM}{dt} = -\frac{2\pi}{uA} Ro \operatorname{Cos} 2\pi\left(\frac{x}{A} - \frac{t}{T}\right.$$

Integrating which, the following is obtained:

$$M = \frac{T}{uA} Ro \sin \left(\frac{x}{A} - \frac{t}{T}\right)2\pi$$

But $\quad \frac{T}{A} = \frac{1}{v} = (Ku)^{\frac{1}{2}}$

Therefore $M = \sqrt{\frac{k}{u}}\ Ro \sin 2\pi(\frac{x}{\lambda} - \frac{t}{T})$

The maximum value of M is therefore, $\sqrt{\frac{k}{u}}\ Ro$ and calling this M_o, the following similar equation for M is obtained:

$$M = M_o \sin 2\pi(\frac{x}{\lambda} - \frac{t}{T})$$

POYNTING'S THEOREM

It is apparent that a traveling wave carries with it a transfer of power. For example, when a radio receiver picks up a wave signal a certain amount of power is required to actuate the receiving apparatus; this energy must be contained in the wave and is propagated with it from the sending station to the receiving point.

Let the symbol P represent the energy in watts per square meter that flows through an imaginary surface through which a wave is passing. Then $P \cdot a$ is the total power flowing out through an area a.

Outward flow of power $= \oint P \cdot da$

By the law of the conservation of energy, if there is an outflow of energy there must be a corresponding loss of energy within the magnetic and electric fields.

$$\text{Electric energy} = \frac{1}{2}\int D \cdot E dv$$

$$\text{Magnetic energy} = \frac{1}{2}\int B \cdot H dv$$

$$\text{Total energy} = \frac{1}{2}\int (B \cdot H + D \cdot E) dv$$

Differentiating this expression to find the rate at which this stored energy diminishes gives:

$$\text{Rate of energy decrease} = \frac{d}{dt}\frac{1}{2}\int (B \cdot H + D \cdot E)dv$$

Assuming, for simplicity, no losses of electrical energy, i.e., conversion of electrical energy into heat, the outgoing energy must equal the amount of energy given up by the electric and magnetic fields.

$$P \cdot da = \frac{d}{dt}\frac{1}{2}\int (B \cdot H + D \cdot E)dv$$

$$P \cdot da = \frac{1}{2}\int \frac{d}{dt}(uH \cdot H + eE \cdot E)dv$$ (Where D = eE, and B = uH)

Performing the indicated differentiation, the right-hand portion of the above equation becomes:

$$-\int (uH \cdot \frac{dH}{dt} + eE \cdot \frac{dE}{dt})dv = -\int (H \cdot \frac{dB}{dt} + E \cdot \frac{dD}{dt})dv$$

The derivatives in the above equation are partial derivatives.

Using Maxwell's equations to substitute for the time derivatives gives:

$$\int \left[H \cdot (\nabla \times E) - E \cdot (\nabla \times H)\right] dv$$

Which reduces to: $\int \nabla \cdot (E \times H)dv$

Since divergence is here integrated through a volume, by Gauss's Theorem a surface integral over the surface enclosing the volume may be substituted for the volume integral giving:

$$\oint P \cdot da = \oint (E \times H) \cdot da$$

Now, both sides of this equation are surface integrals over the same surface, hence the equation is obviously satisfied if:

$$P = E \times H$$

This derivation of the Poynting vector considers only the regions without conductivity, thereby, eliminating resistance loss of energy. Had conductivity been considered, the results would have been identically the same; the above expression holds for the Poynting vector regardless of whether or not the region has conductivity.

INTERPRETATION OF PRECEDING RESULTS

The general equations governing the action of magnetic and electric fields that have been derived in the preceding portion of this thesis will now be studied and analyzed to determine the relationships that exist between these fields and the quantities that produce them, namely current and voltage.

From equations (3), the expressions relating the components of the current and the components of the magnetic field are:

$$-4\pi u = \frac{dN}{dy} - \frac{dM}{dz}$$

$$-4\pi v = \frac{dL}{dz} - \frac{dN}{dx}$$

$$-4\pi w = \frac{dM}{dx} - \frac{dL}{dy}$$

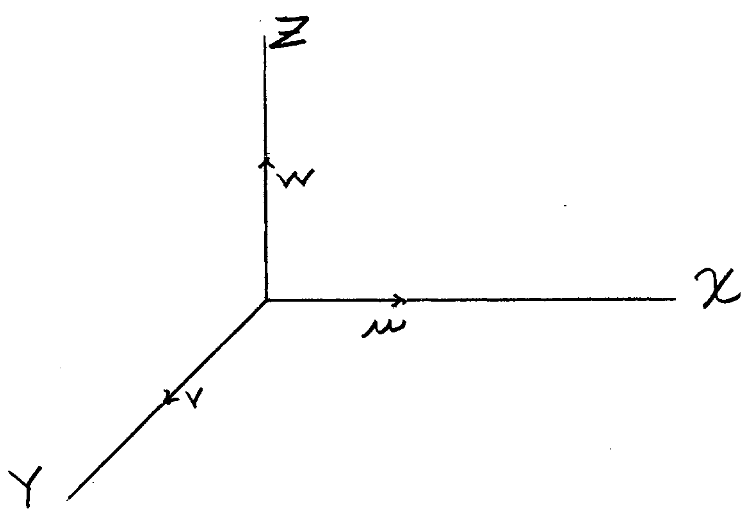

Figure 4

Where u, v, and w are the components of current density in the X, Y, and Z direction, respectively, as shown above.

Now the electromotive force around a closed path through which the magnetic flux is varying has been shown to be, from equation (4), composed of the following three components:

$$e_x = u\frac{dL}{dt} = \frac{dR}{dy} - \frac{dQ}{dz}$$

$$e_y = u\frac{dM}{dt} = \frac{dP}{dz} - \frac{dR}{dx}$$

$$e_z = u\frac{dN}{dt} = \frac{dQ}{dx} - \frac{dP}{dy}$$

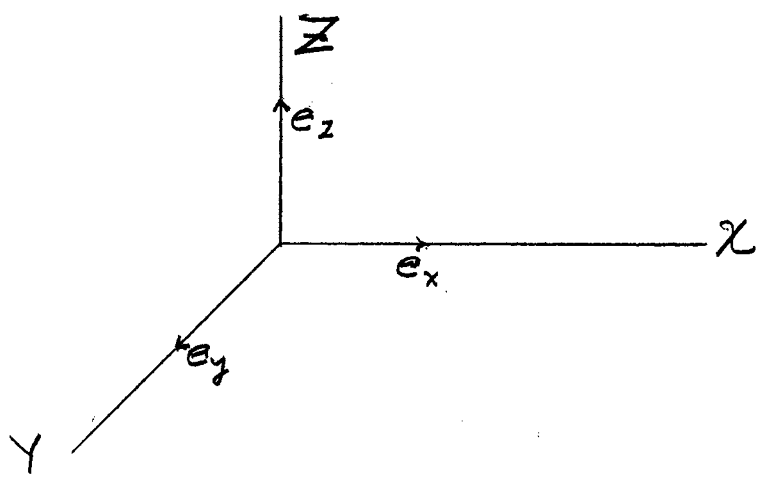

Figure 5

Now, to propagate a plane wave in the X direction, restrictions must be placed on these general equations. If it is assumed that a conductor with a circular cross-section, and with its longitudinal axis coincident with the X axis is under consideration, then within the conductor, the magnitudes of the E and H vectors would be constant in the Y and Z directions at any one particular time, i.e., the distribution of the E and H vectors in the YOZ plane would be uniform. This would be true if the conductor possessed a constant value of u and K for any small volume considered. Then the space derivatives of E and H with respect to Y and Z would be zero. Hence, the above equations for electromotive force would reduce to:

$$e_x = u\frac{dL}{dt} = \frac{dR}{dy} - \frac{dQ}{dz} = 0$$

$$e_y = u\frac{dM}{dt} = \frac{dP}{dz} - \frac{dR}{dx} = -\frac{dR}{dx} \qquad (6)$$

$$e_z = u\frac{dN}{dt} = \frac{dQ}{dx} - \frac{dP}{dy} = \frac{dQ}{dx}$$

And similarly for the current, the equations reduce to:

$$u = -\frac{1}{4\pi}\left(\frac{dN}{dy} - \frac{dM}{dz}\right) = 0$$

$$v = -\frac{1}{4\pi}\left(\frac{dL}{dz} - \frac{dN}{dx}\right) = \frac{1}{4\pi}\frac{dN}{dx} \qquad (7)$$

$$w = -\frac{1}{4\pi}\left(\frac{dM}{dx} - \frac{dL}{dy}\right) = -\frac{1}{4\pi}\frac{dM}{dx}$$

Since from equation (6), $u\frac{dL}{dt} = 0$, L must equal zero or a constant, and since constant values do not influence propagation, let L equal zero. From equation (7) $\frac{dN}{dy} - \frac{dM}{dz} = 0$, and from equation (5) $-K\frac{dP}{dt} = \frac{dN}{dy} - \frac{dM}{dt} = 0$; therefore, P = 0.

Now P, Q, and R and L, M, and N were the X, Y, and Z components of E and H, respectively. It has just been shown that the X components of both E and H are zero; therefore, the E and H vectors are contained entirely within the YOZ plane. Also, e and i (voltage and current) have been shown to have no components in the X direction, and hence, are contained in the YOZ plane.

To obtain the relationship between the E and H directions, and the e and i directions, arbitrarily place the E axis in the direction of the Z axis. Due to this orientation of axes, obviously now Q must be equal to zero, and therefore, E must equal to R, since P has previously been shown to be zero.

From equation (5)

$$-K\frac{dQ}{dt} = \frac{dL}{dz} - \frac{dN}{dx}$$

and since both Q and L have been shown to be zero, then N must be equal to zero also. Therefore, H must now be equal to M, since both L and N are zero.

From equation (6) $e_z = \frac{dQ}{dx}$, and since Q was shown above to be zero, e_z must be equal to zero. Therefore, the only component of the electromotive force remaining is e_y, which from equation (6) is:

$$e_y = -\frac{dR}{dx}$$

From equation (7)

$$v = -\frac{1}{4\pi}\frac{dN}{dx}$$

and since N was shown above to be zero, then v must be zero.

Since u and v have both been shown to be zero, then i must be equal to w, the Z component. The voltage, current, electric intensity, magnetic intensity, and the flow of energy are represented below on the coordinate axes in their proper space relationships.

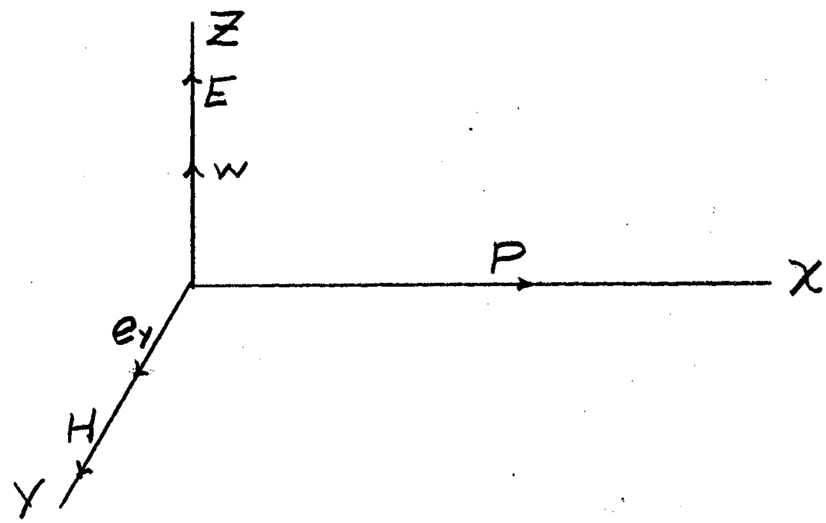

Figure 6

Notice that from the preceding section on the Poynting Theorem, the flow of energy is E x H, while from the above diagram such an energy transfer would involve the product i x e. Here power flow is obtained from the cross product of the two vectors i and e, and the resultant expression for power flow is a vector from the definition of the cross product. Moreover, it should be pointed out that this is in accordance with the usual notions of harmonically varying quantities, for here the power flow is a harmonic variation the same as the voltage or current, except that it varies at a double frequency.

The current and voltage are seen from the above derivation to be in space quadrature with each other, i.e., ninety degrees apart in space phase, exactly the same as the electric and magnetic intensities. The result of rotating the voltage vector around until it coincides with the current vector or falls within the same plane with it is to make the expression for power flow now be equal to the dot product of e and i, yield an apparent scalar quantity for the power flow. This condition is realized when, as in conventional literature, the voltage and current waves are drawn in the same plane. Actually, to be rigorous in approaching the problem, the current and voltage waves should be considered as in separate planes, just as the electric and magnetic intensities are considered. Notice that if the current and voltage are in time phase, i.e., their maxima occur at the same instant during propagation along the X axis, then the cross product will yield the same absolute value with the two vectors in space quadrature as the dot product would if the waves were considered to be in the same plane and the dot product were used. This follows from the definitions of the dot and cross products. The cross product of two vectors is a vector whose magnitude is equal to the product of the magnitudes of the two given vectors and the sine of the angle between them. The dot product of two vectors is a scalar which is equal to the product of the magnitudes of the two given vectors and cosine of the angle between them. However, the significant point not to be overlooked is, that the placing of the two waves in the same plane and using the dot product destroys, by the ordinary mathematical notations, the vector characteristics of the power, which it obviously should retain. Also, it is believed that if the voltage and current waves were considered as being in separate planes, the appearance of the mesh and distortion powers could be better explained.

In the previous section it should be noted that the derivation was valid only for conditions within the wire. If a conductor direction was assigned to the diagram showing the current, voltage, magnetic and electric field intensities, it would apparently be in the direction of the current flow, i.e., in the direction of the Z axis. The magnetic lines of flux would encircle the conductor and at any particular point a vector representing the magnetic field intensity would be drawn tangential to this circular flux line, which would appear on the diagram mentioned above as being along the Y axis. This agrees with the final position of the H vector as determined by the differential equations. From the diagram, it is seen that the flow of energy is apparently at right angles to the electric and magnetic fields, which would make the flow be in a direction perpendicular to the circular surface of the conductor, i.e., along radial lines. This is precisely as it should be, for inside the conductor the only energy being transferred is the energy flowing directly into the conductor from the outside to furnish the copper loss of the conductor. In other words, the major energy transfer takes place outside the conductor in the form of building and collapsing electric and magnetic fields parallel to the longitudinal axis of the conductor, the small amount of energy for copper losses being furnished along the conductor's length in a continuous, distributed manner.

Outside the conductor, the direction of the magnetic intensity is unchanged, but the direction of the electric intensity and the Poynting vector for energy flow are radically different. The Poynting Vector will have two components; one small component perpendicular to the conductor surface to furnish the copper loss, and a large component parallel to the conductor surface to furnish the energy being transferred or transmitted from the sending point to the receiving location. The above conditions for the inside and the outside of the wire are represented diagrammatically below.

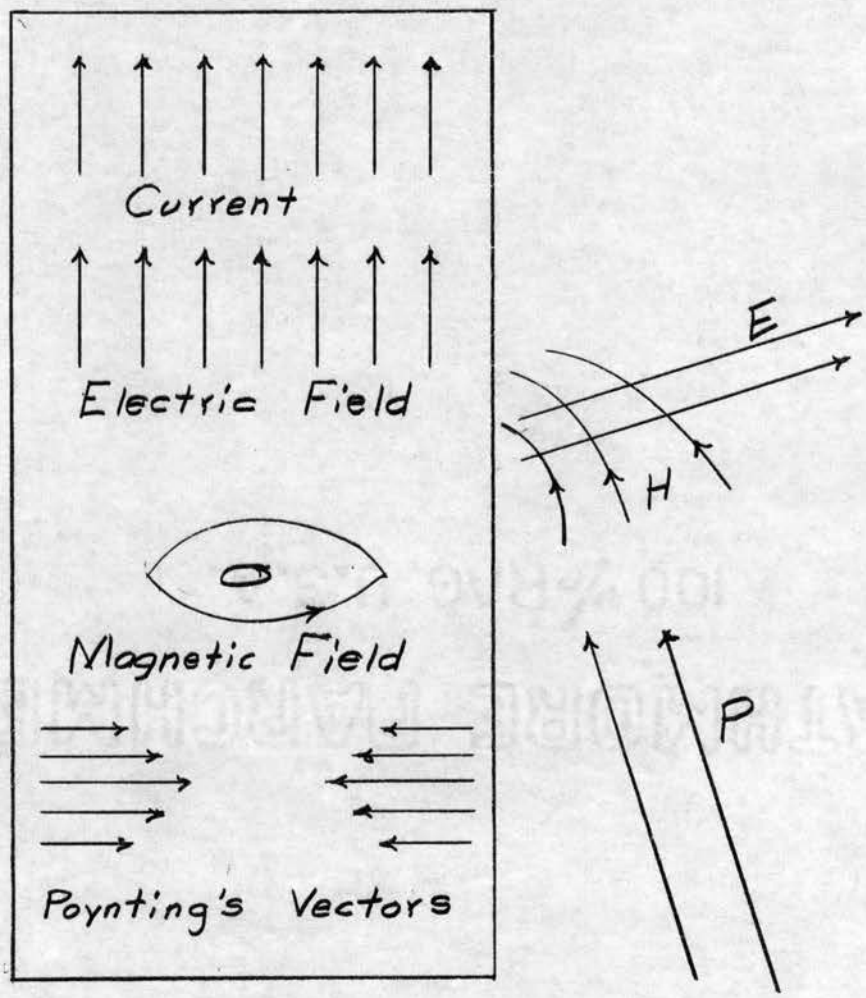

Figure 7

CHAPTER III

GEOMETRICAL INTERPRETATION OF A. C. POWER QUANTITIES

SINUSOIDAL SINGLE-PHASE CIRCUITS

It was shown in the section of this paper entitled "Trigonometric Treatment of A. C. Circuit Quantities" that the instantaneous real power in a linear circuit is composed of a constant component and a component that varies as a double-frequency cosine wave. The real or active power is the time average of the instantaneous values of the active power taken over a complete cycle of the fundamental frequency of the alternating current. The time average of a cosine wave over a complete cycle is zero, therefore, the effective real power is the constant term, $\frac{VmIm \cos \theta}{2}$.

In the same section of this paper mentioned above, it was shown that a quadrature component of instantaneous power existed in a sinusoidal circuit which varied as a double-frequency sine wave. Now the reactive power is the time average of the quadrature components of power taken over a complete cycle of the fundamental frequency of the alternating current. This time average of the sine wave is evidently zero. Reactive power in a sinusoidal circuit is arbitrarily defined as the apparent power multiplied by the sine of the angular displacement between the voltage and current wave forms. Thus the diagram shown below would correctly represent such quantities in a sinusoidal circuit.

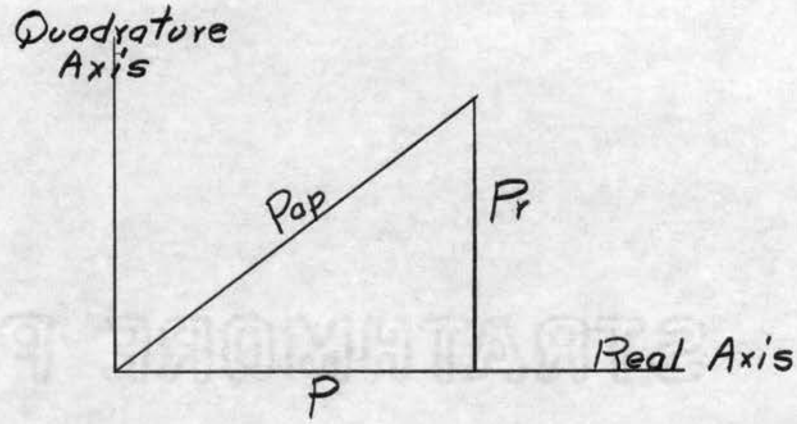

Figure 8

$$P_r = \text{Reactive power} = \frac{VmIm}{2} \sin \theta$$

$$P = \text{Active power} = \frac{VmIm}{2} \cos \theta$$

$$Pap = \text{Apparent power} = \frac{VmIm}{2}$$

SINGLE-PHASE CIRCUITS WITH NONSINUSOIDAL CURRENT AND VOLTAGE

The instantaneous power is the product of the current and potential difference as before in sinusoidal circuits. However, in this case, its variation is not a simple sine function of time, so that the instantaneous power is given by an expression which includes the product of each harmonic term of the current by the corresponding harmonic term of the voltage.

The active power under periodic conditions is equal to the algebraic sum of the active powers corresponding to the fundamental and each harmonic. The active power of the fundamental and each harmonic, as before, is the time average of the instantaneous values for that particular frequency taken over a complete cycle of the fundamental frequency of the alternating current.

Similarly, the reactive power in a circuit with periodic current and voltage is the algebraic sum of the reactive powers corresponding to the fundamental and each harmonic. Where the reactive power for each harmonic has the same meaning as in the previous section for the fundamental. The apparent power is the product of the effective current and effective voltage.

With the definitions that have been chosen, the square of the apparent power, is, in general, greater than the sum of the squares of the real and reactive powers when there are harmonics in the current and voltage wave forms. Hence, the triangle that was used to represent power under sinusoidal conditions must be modified. One method of doing this is to introduce a new quantity, "distortion power," which is so defined that the square of the apparent power is equal to the sum of the squares of the active power, the reactive power, and the distortion power. A convenient way of visualizing the three power components is to construct them parallel to the three-dimensional Cartesian Coordinate system as shown below.

Fictitious power is defined as the square root of the difference in the squares of the apparent power and the active power. From the diagram below, fictitious power is obviously equal to the square root of the sum of the squares of the reactive power and the distortion power. The vector power is defined as the square root of the sum of the squares of the real power and the reactive power. Nonreactive power is defined as the square root of the sum of the squares of the active power and the distortion power. The symbols used in the following diagram (Figure 9) are defined as follows:

Pap = Apparent power

P = Active power

P_v = Vector power

P_r = Reactive power

P_f = Fictitious power

P_n = Nonreactive power

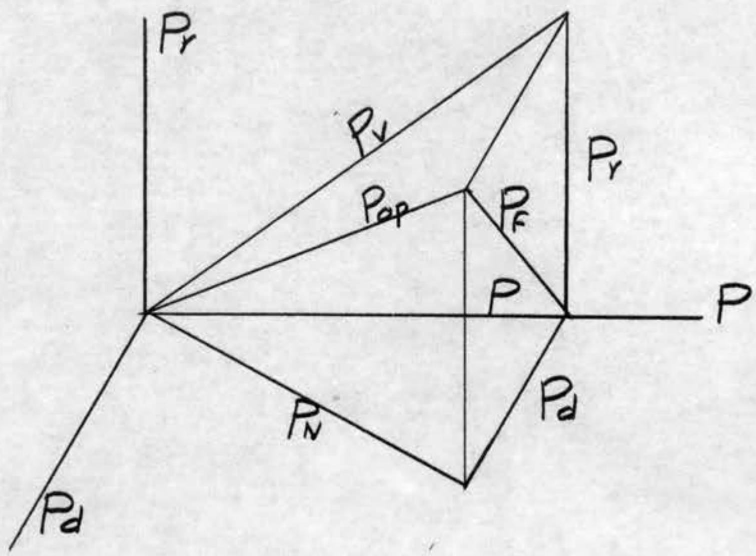

Figure 9

BALANCED POLYPHASE CIRCUITS WITH SINUSOIDAL CURRENTS AND VOLTAGES

In dealing with this type of polyphase circuit, for example the three-phase wye, the definitions for single-phase circuits will hold for each phase of the polyphase circuit. However, the instantaneous power for a balanced three-phase circuit does not vary under sinusoidal conditions as it does for a single-phase circuit.

Thus a power diagram could be constructed for each phase of the polyphase circuit exactly as was done for the single-phase circuit. The only types of power present would be the real power, reactive power and, of course, apparent power. Therefore, the square of the apparent power would necessarily be equal to the sum of the squares of the real power and the reactive power. Since the diagram for this power relationship would be identical in construction to that for the single-phase case, an additional diagram will not be drawn in this section.

BALANCED POLYPHASE CIRCUITS UNDER NONSINUSOIDAL CONDITIONS

The extension of power concepts to a balanced polyphase circuit with harmonics in the currents and voltages may be accomplished by considering the circuit as separated into a group of single-phase circuits. Each of the single-phase circuits has the same effective current and effective voltage as the others. The instantaneous power is given by a summation of the powers of each of the component single-phase circuits. In general, the instantaneous power is not constant but will have a cyclic variation with time. The power diagram for this case would be the same as for a nonsinusoidal single-phase circuit, except the quantities of one of the above discussed circuits must be multiplied by the number of circuits. Therefore, the power diagram for this section will be omitted.

UNBALANCED POLYPHASE CIRCUITS UNDER SINUSOIDAL CONDITIONS

Again, the simplest, and most common, method of attacking unbalanced sinusoidal polyphase circuits is to consider the polyphase circuit as being composed of component single-phase circuits, with a particular point being taken as reference.

It is to be noted, however, that although the insertion of a set of wattmeters in the usual manner at the terminals of an unbalanced polyphase circuit gives a theoretically correct measurement of the active power in the polyphase circuit, the corresponding method for measuring reactive power will not give a correct result, if both the current and the potential differences form unsymmetrical sets. The symmetrical components may be applied here. A theoretically possible but impractically complicated sequence network is required for a correct measurement of reactive power in this case.[1]

Algebraic apparent power is defined as the maximum active power that can exist with the given effective values of currents in the conductors and voltages between conductors. Mesh power has been defined as a quantity to relate algebraic apparent power and vector power. In the type of circuit under consideration, the square of the algebraic apparent power is equal to the sum of the squares of the active power, reactive power, and mesh power. The diagram is shown in one plane since, with sinusoidal variations, the distortion is zero. This is the vector diagram for a four-wire, three-phase circuit, where the voltages were measured from the actual neutral.

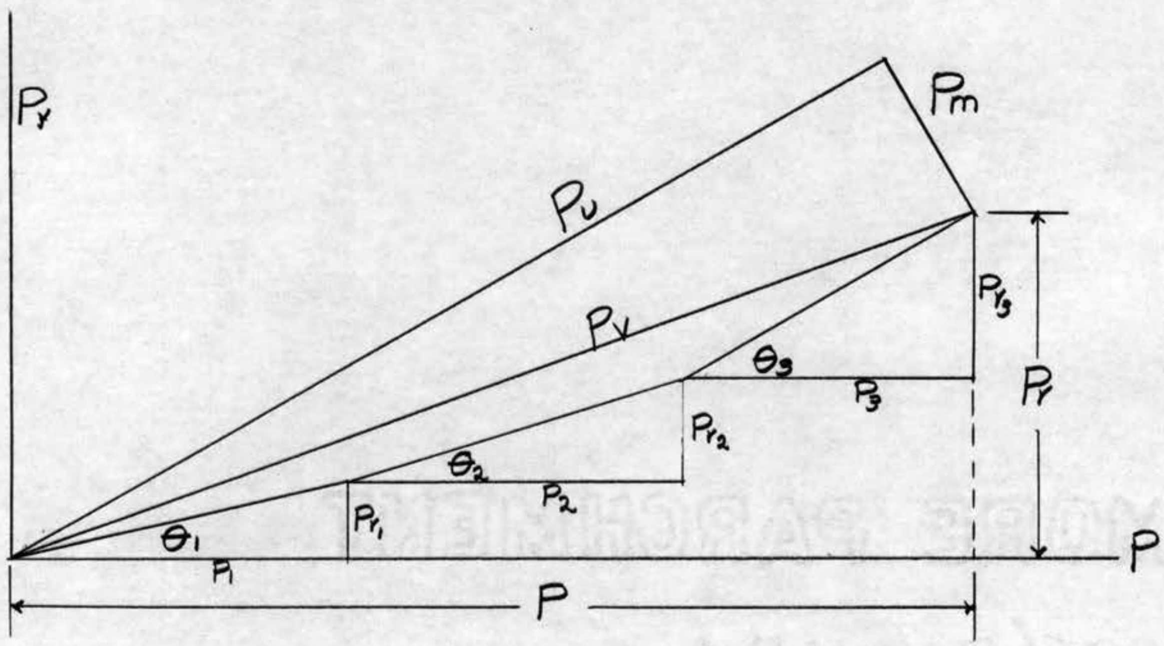

Figure 10

[1] H. L. Curtis and F. B. Silsbee, "Definitions of Power and Related Quantities," Electrical Engineering, (April, 1935), Vol. 54, pp. 394-404.

Pu = Algebraic apparent power

Pv = Vector power

Pm = Mesh power

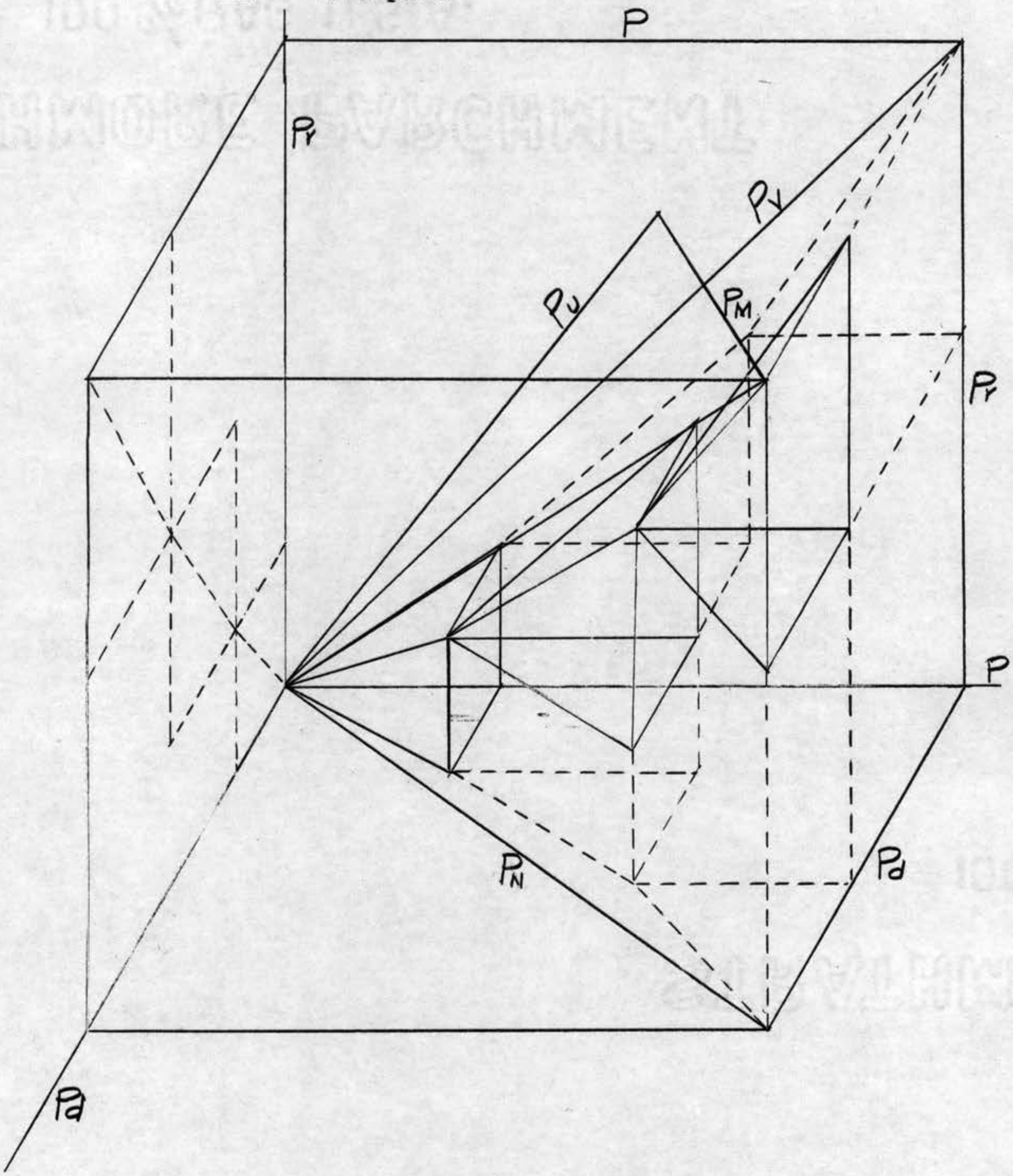

Figure 11

UNBALANCED POLYPHASE CIRCUITS UNDER NONSINUSOIDAL CONDITIONS

This type of circuit may be approached in manner similar to that utilized in the preceding sections, i.e., be broken up into component single-phase circuits. There is no unique division of an unbalanced polyphase circuit into component single-phase circuits, since no possible separation would give a group of identical circuits. The vector diagram for an unbalanced, three-wire, three-phase, nonsinusoidal circuit is shown on the preceding page.

CHAPTER IV

ALGEBRAIC VERIFICATION OF GEOMETRIC INTERPRETATION

Consider the case where a sinusoidal voltage is applied to a single-phase, non-linear circuit. For reasons of simplicity, all harmonics are neglected in current other than the third harmonic. The production of a third harmonic current with a sinusoidal voltage form might be questioned by some, but such can be produced with non-linear circuits.

> When a sinusoidal alternating voltage is applied to a non-linear impedance it is almost self-evident that, due to the nonproportionality of current to voltage, the wave form of the resulting current must be different in shape from that of the applied pressure. A non-sinusoidal wave form is said to be distorted, and the distortion is due to the superposition on a sinusoidal and fundamental component of normal frequency, of a number of other sinusoidal components having frequencies which are odd multiples of the circuit frequency. These latter components, generally called harmonics, can be conceived to flow in the non-linear circuit simultaneously with the sinusoidal fundamental component, just as in ordinary circuits active and reactive current components are conceived to be simultaneously present.[1]

Now in a circuit where the total current is composed of harmonics as well as the fundamental, it is a well known fact that the effective value of the complex current wave is equal to the square root of the sum of the squares of the effective values of each individual wave considered separately.[2] This effective value of the complex wave is the amount that an ammeter would indicate if it were placed in the circuit.

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[1] Stubbings, "Harmonic and P. F.," Electrical Review, June 5, 1942.

[2] K. V. Tang, Alternating Currents Circuits, 1st Edition, International Textbook Company, 1940, pp. 315-316.

Furthermore, by integrating over a cycle, it can easily be shown that a fundamental voltage and a third harmonic current yield no power whatsoever, either active or reactive. To better explain this point let us assume (what is conceivable, although perhaps not practically realizable) that a non-linear circuit allows the passage of a third harmonic component of current having an amplitude equal to one-half the fundamental component, when a sinusoidal voltage is applied to it. To produce the 60 cycle fundamental component will require a certain amount of power which is given by the usual power formula VI cos ϕ. Now, consider the relation of the third harmonic which has been generated by the non-linear circuit to the fundamental component. This third harmonic current is continually gaining in phase on the fundamental voltage, and in less than a half-cycle of voltage change the harmonic current will pass through a complete cycle of phase change relative to the voltage. Moreover, there will be exactly three complete cycles of this current for each cycle of the voltage, therefore, the average power required to generate this third harmonic current from the source is apparently zero. However, the instantaneous power required to deliver this third harmonic current is not zero.

In a circuit containing a third harmonic, the real power is then given by:

$$\text{(Real Power)} \quad P = \text{Rl}\ \hat{E}I_1 = EI_1 \text{ Cos } \phi$$

and the reactive power is given by:

$$\text{(Reactive power)} \quad P_r = \text{Im}\ \dot{E}I_1 = EI_1 \text{ Sin } \phi$$

Where $\hat{E}$ is the conjugate of E, and ϕ is the angle between E and I_1. The symbol I_1 here refers to the amplitude of the fundamental component of current.

In the following discussion I_1 will arbitrarily be taken as the axis of reference. The symbol I_e will be used to designate the effective value of the complex current wave.

$$I_e = \sqrt{I_1^2 + I_3^2}$$

Since the total apparent power (P_{ap}) for such a circuit does not equal to the square root of the real power (P) squared plus the reactive power (P_r) squared, a new type of power will now be defined, which shall be designated as distortion power (P_d), in such manner that the total apparent power is equal to the vector sum of P, P_r, and P_d.

$$P_{ap} = P + P_r + P_d \quad \text{(Vector sum)}$$

Before the third harmonic current flowed, i.e., when only the fundamental component flowed, the square of the total apparent power was equal to the sum of the squares of the real and reactive powers, and hence the distortion power was zero. When a circuit was devised to let this current flow the above relation no longer held true. Therefore, even though the third harmonic current does not produce any real or reactive power, it is evident that it is producing an effect that is destroying the triangle relationship among the real, reactive, and apparent power, i.e., distortion effect.

It will now be assumed that the distortion power, P_d, is due to some distortion current, I_d, then:

$$P_d = EI_d$$

The above equation for total apparent power will now be investigated to determine the relative positions of the real, reactive, and distortion powers. It should here be remembered that the total apparent power was merely defined as the vector sum of these three quantities. It is already known that the real and reactive powers are at right angles to each other, hence the remaining problem is to locate the direction of the distortion power.

$$P_{ap} = \dot{E}I_e = \dot{E}\sqrt{I_1^2 + I_{e3}^2}$$

$$P_{ap} = P + P_r + P_d$$

$$\mathring{E}\sqrt{I_1^{\,2} + I_3^{\,2}} = EI_1 \text{Cos}\,\theta - jEI_1 \sin\theta + EI_d$$

$$\sqrt{I_1^{\,2} + I_3^{\,2}} = \frac{E}{\mathring{E}}\left(I_1 \text{ Cos}\,\theta - jI_1 \sin\theta + I_d\right)$$

$$\frac{E}{\mathring{E}} = \frac{Ee^{je}}{Ee^{-je}} = e^{j2\theta} = \text{Cos } 2\,\theta + j\sin 2\,\theta$$

$$\sqrt{I_1^{\,2} + I_3^{\,2}} = I_1 (\text{Cos}\,\theta + j \sin\theta) + I_d (\text{Cos}\,\theta + j\sin\theta)^2$$

$$I_d = \frac{-I_1(\text{Cos}\,\theta + j\sin\theta) \pm \sqrt{I_1^{\,2} + I_3^{\,2}}}{(\text{Cos}\,\theta + j\sin\theta)^2}$$

If I_d is to be positive in magnitude, the positive sign on the radical must be used since I_e is greater than I_1.

$$\left| I_d \right| = \left| \sqrt{I_1^{\,2} + I_3^{\,2}} - I_1 e^{j\theta} \right|$$

$$\left| I_d \right| = \left| I_e - I_1 e^{j\theta} \right|$$

$$P_d = EI_d$$

$$I_d = I_e e^{-j2\theta} - I_1 e^{-j\theta}$$

$$P_d = EI_e e^{-j2\theta} - EI_e e^{-j\theta}$$

$$P_d = \mathring{E}I_e e^{-j\theta} - \mathring{E}I_1$$

$$P_{ap} = \mathring{E}I_e$$

$$P_v = \mathring{E}I_1 \quad \text{(Apparent power due to fundamental)}$$

$$P_d = P_{ap} e^{-j\theta} - P_v$$

This equation is interpreted on the following diagram. Notice that P_{ap} and

P_v are vectors in different directions, and in addition they are separated by the turning angle θ.

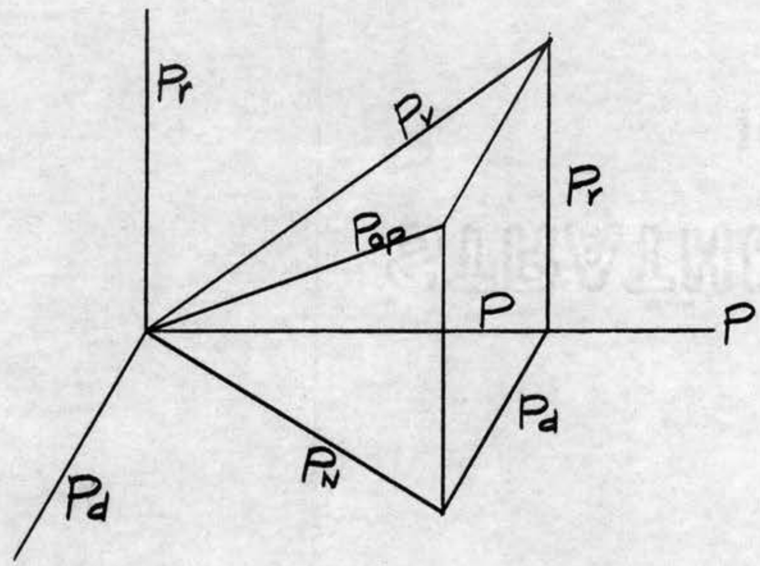

Figure 12

CHAPTER V

TRIGONOMETRIC TREATMENT OF A. C. CIRCUIT QUANTITIES

The ordinary manner in which most textbooks approach the idea of an alternating current or voltage is to represent the current as a sine wave and the voltage as a sine wave displaced from the current wave. The effective values of these waves may be represented geometrically as shown below.

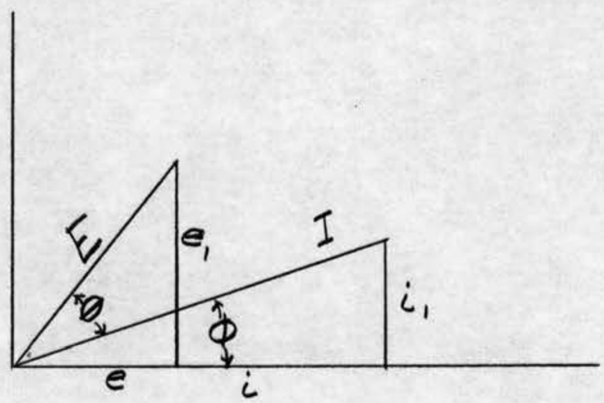

Figure 13

From which $i = I \cos \phi$

$$i_1 = I \sin \phi$$

$$e = E \cos (\theta + \phi)$$

$$e_1 = E \sin (\theta + \phi)$$

Here E and I are merely thought of as being directed lines drawn to scale such that their respective lengths represent the effective heights of the current and voltage waves. Now then, by expanding the expressions for e and e_1, the following equations are obtained.

$$e = E(\cos \phi \cos \theta - \sin \phi \sin \theta)$$

$$e_1 = E(\sin \phi \cos \theta + \cos \phi \sin \theta)$$

The two above equations follow directly from the trigonometric identities:

$$\cos (\phi + \theta) = \cos \phi \cos \theta - \sin \phi \sin \theta$$

$$\sin (\phi + \theta) = \sin \phi \cos \theta + \cos \phi \sin \theta$$

Assuming power to be given by the expression

$P = e\,i + e_1\,i_1$ the following equation is obtained.

$$P = E\,I(\cos^2\phi \cos \theta - \cos \phi \sin \phi \sin \theta) + E\,I(\sin^2\phi \cos \theta + \cos \phi \sin \phi \sin \theta)$$

From which

$$P = E\,I(\cos^2\phi + \sin^2\phi) \cos \theta = E\,I \cos \theta$$

This latter expression is recognized as being the true equation for real power only and contains no double-frequency component. Since it is well known that a double-frequency component of power is present in a general a.c. circuit, the question naturally arises as to the reason for its absence in the above expression.

At this point, the assumptions that have been made should be carefully studied. Particularly, the assumption that $P = ei + e_1 i_1$ is most interesting. As will be easily shown in the section of this paper relating to the complex representation of alternating current circuit quantities, the above expression for power actually represents only the real power developed by the voltage and current and hence should be equal to the product of their effective values and the cosine of the angle between them, as this is quite commonly known to be the expression for real or true power.

In the above computations it might at first appear that two quantities at 90° degrees with each other, ei and $e_1 i_1$, are being added arithematically to obtain the total power, and consequently the alert reader might immediately object to such a mathematical treatment. However, it must be remembered that active power is determined by the product of the in-phase component of the voltage and the current, and it should further be noticed that to take the component of the voltage in-phase with the current and multiply this by the current is identically the same as adding arithematically the product of the in-phase components of current and voltage along the x and y axes.

It might be more beneficial to study the expression for the power at any given instant.

Let $v = Vm \sin wt$, and

$$i = Im \sin (wt - \theta)$$

Then $p = vi = VmIm \sin wt \sin (wt - \theta)$

$$p = \frac{Vm\ Im}{2}\left[\cos \theta - \cos (2wt - \theta)\right]$$

from the trigonometric identities

$$\cos (a - b) = \cos a \cos b + \sin a \sin b$$

$$\cos (a + b) = \cos a \cos b - \sin a \sin b$$

$$p = \frac{Vm\ Im}{\sqrt{2} \cdot \sqrt{2}}\left[\cos \theta - \cos (2wt - \theta)\right]$$

$$p = EI \cos \theta - EI \cos (2wt - \theta)$$

$$p = EI \cos \theta - EI \cos 2wt \cos \theta - EI \sin 2wt \sin \theta$$

The last two terms on the right of the above equation are pulsating quantities of a double-frequency. Moreover, these same terms will average to zero over an integral number of cycles. The first term on the right side of the equation is seen to be the maximum value of the second term. These first two terms constitute the instantaneous real power, as their difference is never negative.

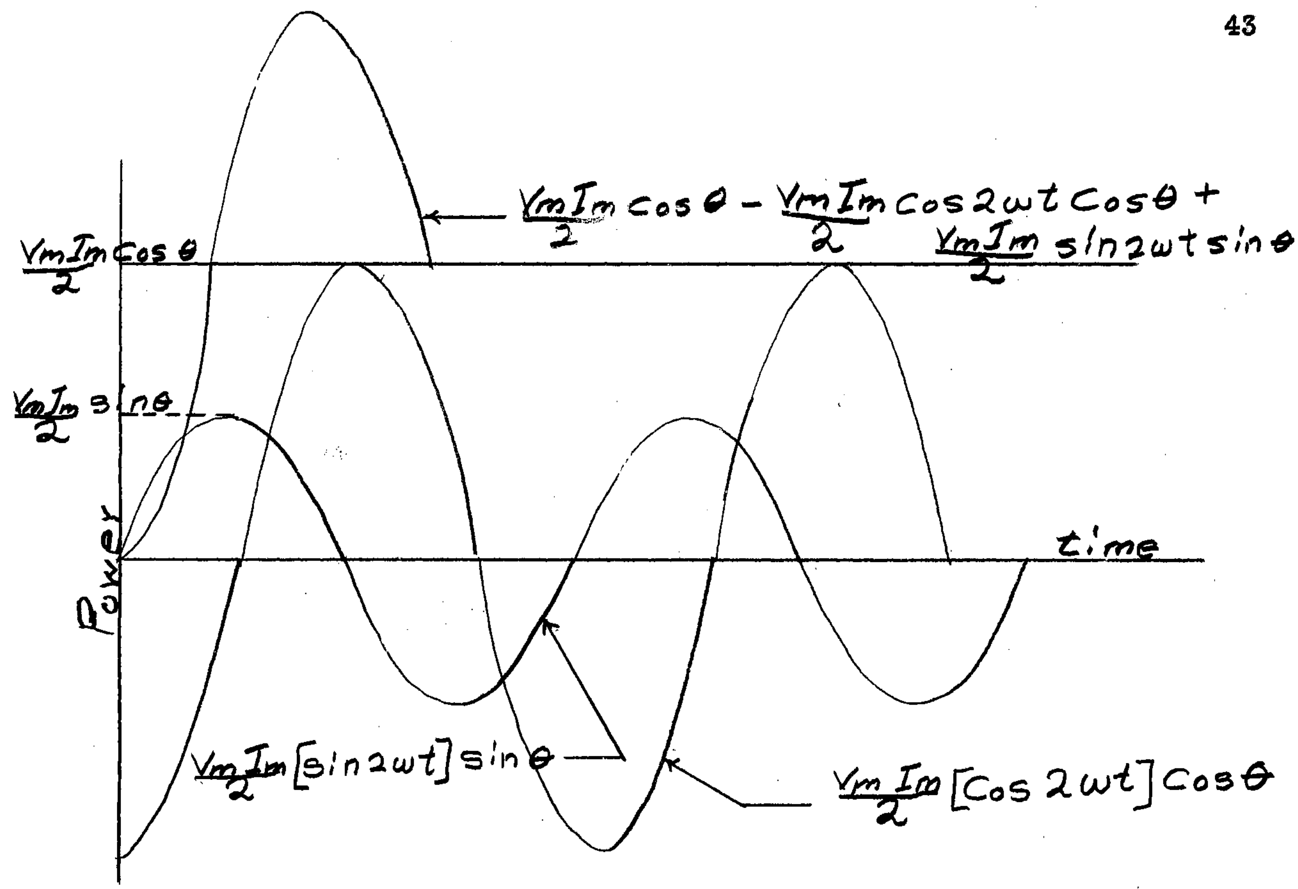

Figure 14

The term $\frac{Em\ Im}{2}$ sin 2wt sin Θ is the instantaneous reactive power. It obviously has equal positive and negative loops. It represents the oscillating energy between the generator and the inductive or capacitive reactance. This power fluctuates between $+\frac{Vm\ Im}{2}$ sin Θ and $-\frac{Vm\ Im}{2}$ sin Θ, and hence has an average value of zero.

When 2wt is an odd multiple of π, the value of the instantaneous real power is 2 VI Cos Θ . When 2 wt is an even multiple of π, the real power is zero. Thus, the real power fluctuates between 0 and 2 VI Cos Θ, and hence has the average value of VI Cos Θ.

It should be emphasized that the above discussion refers to components of the resultant power wave. These components do not exist as separate entities, but they are convenient components to consider for purposes of analysis. Actually a single power wave as determined by the equation

$$P = EI \cos \theta - EI \cos 2\,wt \cos \theta - EI \sin 2\,wt \sin \theta$$

is the only wave that has physical existance.[1]

[1] R. M. Kerchner and G. F. Corcoran, Alternating-Current Circuits, 2nd Edition, John Wiley & Sons, New York, 1943, pp. 26-32.

CHAPTER VI

COMPLEX REPRESENTATION OF A. C. QUANTITIES

Immediately after the development of complex number theory by Argand, this theory was applied to alternating current circuits by Steinmetz. However, in the rapid application of the complex number theory to a.c. circuits subsequent to the original efforts of Steinmetz, several inconsistencies have developed. For example, a sinusoidally varying current and a constant impedance are both spoken of as "vectors." Yet, immediately after calling these quantities "vectors," authors of most elementary texts promptly begin to perform operations with these quantities unknown in the algebra of true vectors. Moreover, after defining a peculiar type of product for the above mentioned basically different quantities in order to obtain the desired result, the correct one, in one instance, it has been found that to secure the desired result in other cases a new type of product had to be arbitrarily introduced, namely the substitution of a conjugate. Most authors blandly justify this violation of basic mathematical laws by the observation that such a substitution is necessary to yield the correct result. The following discussion will attempt to clarify this argument.

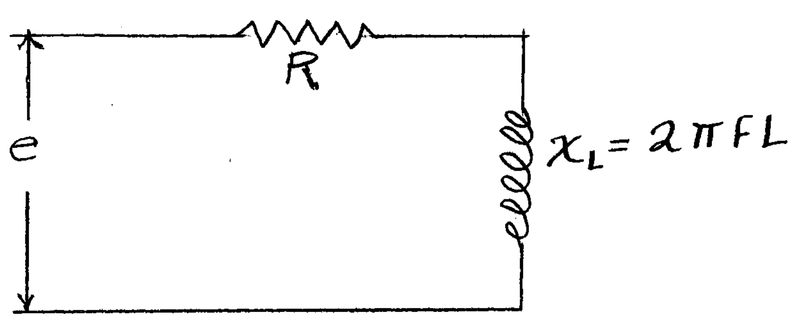

Figure 15

If a sinusoidally varying voltage is applied to the above circuit, it is a well known fact that the resulting current will also vary in a harmonic manner. Following the usual procedure, let the effective value of the current be represented in a complex manner as:

$$I = A + jB \qquad (j = \sqrt{-1})$$

and $$Z = R + jX$$

Then $$E = IZ = (A + jB)(R + jX) = AR - BX + j(BR + AX)$$

Let $$C = AR - BX, \text{ and } D = BR + AX$$

Then $$E = C + jD$$

Therefore $$P = EI = (A + jB)(C + jD) = AC - BD + j(BC + AD)$$

$$P = A(AR - BX) - B(BR + AX) + j\left[B(AR - BX) + A(BR + AX)\right]$$

$$P = A^2R - B^2R - 2ABX + j\left[2ABR + X(A^2 - B^2)\right]$$

Note that $\text{Mod. }(I) = \sqrt{(A^2 + B^2)}$

or $\text{Mod. }(I)^2 = A^2 + B^2$, where Mod. (I) refers to the absolute magnitude of the effective value of the current wave.

Now consider:

$$P = \mathring{I}\, E, \text{ where } \mathring{I} \text{ is the conjugate of } (I)$$

$$P = (A - jB)(C + jD) = AC + BD + j(AD - BC)$$

$$P = A(AR - BX) + B(BR + AX) + j\left[A(BR + AX) - B(AR - BX)\right]$$

$$P = (A^2 + B^2)R + jX(A^2 + B^2)$$

In the above complex representation a harmonic function, I, and a constant impedance, Z, were both represented by the same notation. It is seen that the ordinary product of these two quantities, I and Z, yielded the correct value for the voltage drop across the circuit. However, when this value of E so found was used further as a multiplier of the current to find the correct expression for power, P, the result obtained did not represent either real or reactive power, in fact, had no known physical significance. When the con-

A COMPREHENSIVE STUDY OF ELECTRICAL POWER QUANTITIES jugate of I, $A - jB$, was used as a multiplier of E, the resulting expression was found to be the known correct equation for power. To the student of science, the question is at once posed as to what fundamental reason should the above phenomena be ascribed.

At this point one must become rigorously critical as to the exact nature of the symbol or operator which was termed j. In mathematics, any real number along the X axis when used in conjunction with the operator j, or multiplied by the number set (0, 1), is said to be relocated along the positive imaginary axis the same distance from the origin as before. In other words, to operate on a complex number $a + jb$ with the symbol j, rotates the complex number 90° in a counter-clockwise direction, making a new complex number $-b + ja$.

Now, in applying the j operator to alternating currents and voltages, care must be observed as to the proper interpretation of j. When I was multiplied by Z to obtain E, the voltage so obtained was of the same frequency as the current. Therefore, as far as E and I are concerned, the operator j has the same meaning for each, since a 90° rotation in the Argand Plane would be interpreted as a quarter-cycle displacement of the sinusoidal waves along the time axis. But, notice that in multiplying E and I to obtain power, a quantity with a frequency twice that of the voltage or current makes its appearance as shown in the section of this paper entitled, "Trigonometric Treatment of A. C. Circuit Quantities." Now, a displacement of 90° along the time axis for the current or voltage would represent an 180° displacement for the double-frequency power.

Therefore, if an operator is to be used to indicate a rotation, one must remember that this operator will have a meaning that will depend upon the frequency of the quantity in question. In fact, it might be well to use a different operator to use in conjunction with each different frequency. These operators would then have the same meaning when applied to their fundamental frequency, as is ordinarily associated with j as used with its fundamental.

Let f_o be the frequency corresponding to which j has a meaning of a 90° counter-clockwise rotation. At a frequency of 2 f_o, j would then mean a rotation of 180°. If a new symbol, l, is used to represent a rotation of 90° at a frequency of 2 f_o, then obviously l^2 is equivalent to j. Similarly, if m represent a 90° rotation at a frequency 3 f_o, then m^3 is equivalent to j. The following table is thus derived:

Frequency	Operator	Relation to j
f_o	j	
2 f_o	l	$l^2 = j$
3 f_o	m	$m^3 = j$
4 f_o	n	$n^4 = j$
5 f_o	p	$p^5 = j$
6 f_o	r	$r^6 = j$
7 f_o	s	$s^7 = j$

The above table gives the relation that would exist between j and the other operators that represent a quadrature rotation for their particular frequency.

The idea of using other operator for higher frequencies is not altogether foreign to students of Electrical Engineering. For example, when dealing with electrical machines with more than two poles, we distinguish between "electrical" and mechanical degrees. With a slight extension, the same idea holds in the above discussion. That is, a second harmonic obviously passes through twice as many electrical degrees as its fundamental.

CHAPTER VII

SYMMETRICAL COMPONENTS

Any unbalanced three-phase system may, for purposes of analysis, be divided into three balanced systems. This was shown in the original works of Fortesque, published in the A. I. E. E. magazine in 1918.

Let Va, Vb, and Vc be the unbalanced phase voltages in a three-phase system. Then, similarly, let Ia, Ib, and Ic represent the corresponding unbalanced phase currents. Now, in dividing the given unbalanced system into three balanced systems, it is found that two of the balanced systems are of opposite phase rotation, while the third balanced system involves no phase rotation. Consequently, these three balanced systems are referred to in such a manner as to indicate the phase rotation by which they are characterized, with respect to the phase rotation of the original given unbalanced system. The balanced system having the same phase sequence as the original unbalanced system will be called the "positive sequence network," and will be designated by the symbol ABC. The other oppositely rotating system will be termed the "negative sequence network," and, similarly, will be referred to by the symbol ACB. Since the remaining balanced system has no rotating characteristics, it will be called the "zero sequence network."

The subscripts 1, 2, and 0 will be used to refer to the positive sequence, the negative sequence, and the zero sequence, respectively. In a similar manner the subscripts a, b, and c will be used to refer to phases A, B, and C, respectively.

Since the sum of positive sequence quantities or the sum of negative sequence quantities for all three phases is zero, the total vector volt-ampere imput for any three-phase load, whether balanced or unbalanced, is

$$\text{Total} \quad (\mathring{E}I) = 3(E_o I_o + E_1 I_1 + E_2 I_2)$$

It is to be noticed that the only terms that contribute to the total vector volt-amperes are those in which the sequence orders of the potential and the current are the same. Thus positive sequence and negative sequence currents produce no power, either active or reactive, in the circuit as a whole, although they do contribute to the power of each of the individual phases. That is to say, they contribute to the unbalancing of the circuit by increasing the power in one phase above the average power and decreasing the power in another phase below the average power, i.e., above and below the zero sequence component of power.

If the load is balanced, V_1 and I_1 have a definite value and the other quantities are zero. The imput per phase is $\mathring{V}_1 \cdot I_1$. The postive sequence voltages and the negative sequence currents produce no power, in the whole circuit, either active or reactive. They do contribute to the power of each phase. When voltages or currents are unbalanced, they contribute to the unbalancing of the circuit.

PHASE POWER EQUATIONS

$$Pa = \mathring{V}aIa = (V_1 + V_2 + V_0)(I_1 + I_2 + I_3)$$

Here the V's are all conjugates; the symbol not being used due to lack of appropriate capacity of typewriter.

$$Pb = \mathring{V}bIb = (aV_1 + a^2V_2 + V_0)(a^2I_1 + aI_2 + I_0)$$

Where a is merely a rotational operator indicating a rotation of 120 degrees. The operator is entirely analogous to the j operator commonly used to indicate a rotation of 90 degrees.

$$Pc = \dot{V}cIc = (a^2V_1 + aV_2 + Vo)(aI_1 + a^2I_2 + I_0)$$

$$Pa = (V_1I_1 + V_2I_2 + V_0I_0) + (V_0I_1 + V_1I_2 + V_2I_0) + (V_2I_1 + V_0I_2 + V_1I_0)$$

$$Pb = (V_1I_1 + V_2I_2 + V_0I_0) + (a^2V_0I_1 + a^2V_1I_2 + a^2V_2I_0) + (aV_2I_1 + aV_0I_2 + aV_1I_0)$$

$$Pc = (V_1I_1 + V_2I_2 + V_0I_0) + a^2(V_2I_1 + V_0I_2 + V_1I_0) + a(V_0I_1 + V_1I_2 + V_2I_0)$$

$$Pa = P_0 + P_1 + P_2$$

$$Pb = P_0 + a^2P_1 + aP_2$$

$$Pc = P_0 + a^2P_2 + aP_1$$

$$Pa + Pb + Pc = 3\,P_0 + P_1(1 + a + a^2) + P_2(1 + a + a^2)$$

But $P_1\ (1 + a + a^2) = 0$, since this represents three equal vectors 120 degrees apart. Similarly, $P_2(1 + a + a^2) = 0$.

$$P_t = P_a + P_b + P_c = 3\,P_0$$

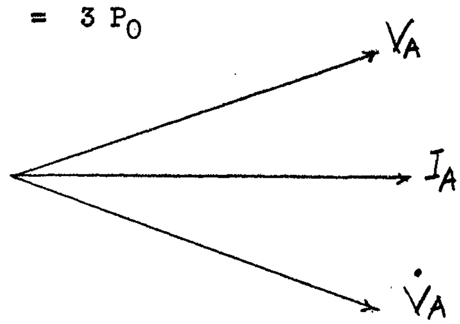

Figure 16

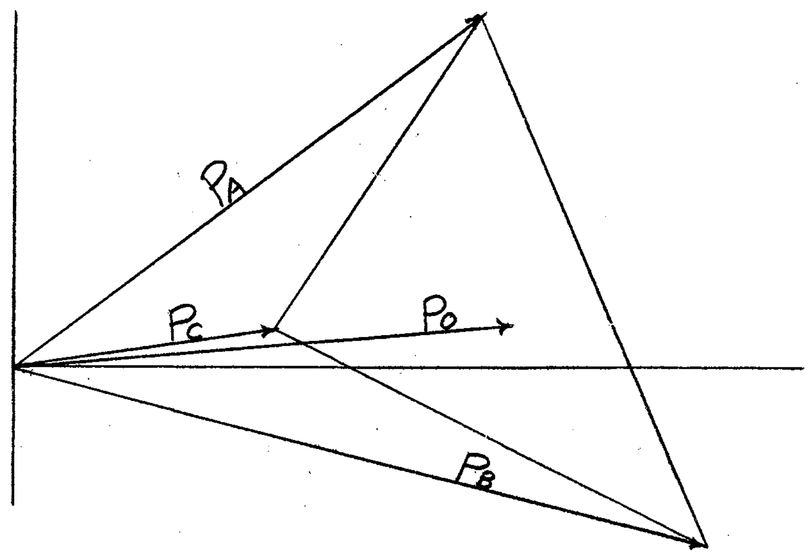

Figure 17

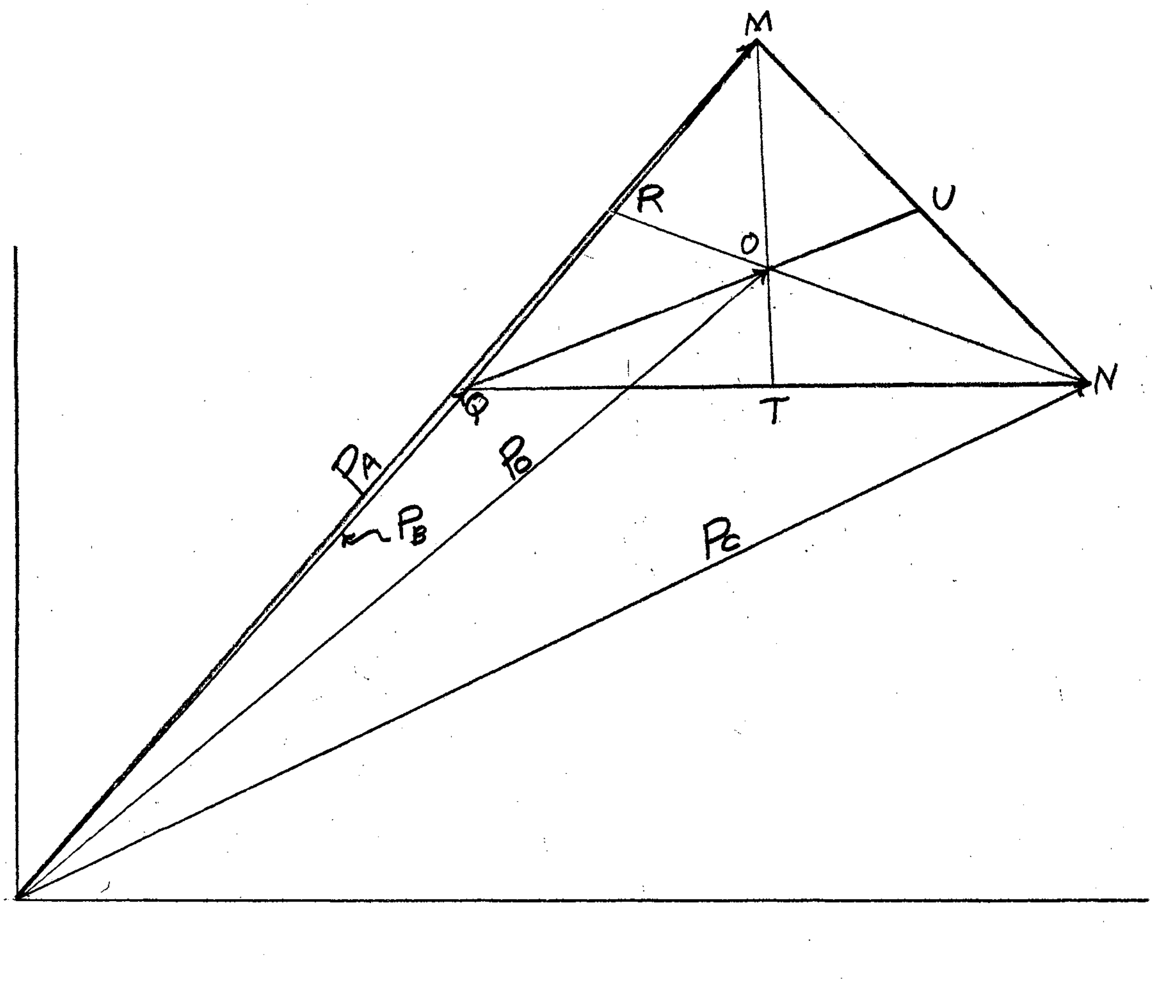

Figure 18

$$P_a = P_0 + OM$$

$$P_b = P_0 + OQ$$

$$P_c = P_0 + ON$$

But, Om + ON + OQ = 0, since the sum of three vectors which form a closed triangle is zero. Then,

$$P_t = P_a + P_b + P_c = 3\ P_0 + OM + OQ + ON = 3\ P_0$$

$$P_t = 3\ P_0 \quad \text{(Total active power)}$$

Active power is always zero sequence.

In the above derivations the power quantities were treated as vectors, which they actually are. That is to say, the different power components possess both magnitude and a definite phase relationship to each other, in other words direction, which is the definition of a vector. In using the different phase-powers in the above figure as vectors, the same result was obtained for algebraic apparent power as in the previous case using the idea of symmetrical components. That is, the above figure is merely a means of representing in a valid manner the known correct results obtained by the symmetrical components, which requires that the different powers in the different phases be considered vectors.

Since the theory developed in the foregoing discussion is primarily designed for the solution of unbalanced circuits, there should exist some procedure of using symmetrical components to solve for the magnitude of the mesh power in such a circuit. In fact, this should be the power per phase produced by the positive or negative sequence voltage and the current of opposite sequence, since it is the inter-action between these voltages and currents of unlike sequence that produces the unbalance in the circuit, in a mathematical sense.

Moreover, there might arise under different physical conditions some speculation as to the identity of the different sequence currents and voltages. In the previous development, it was merely shown that a given set of unbalanced vectors spaced 120 degrees apart could be represented by the sum of not more than three sets of vectors, two sets being balanced sets of opposite phase rotation and the third being a uni-directional set of equal magnitude. No attempt was made to formulate a circuit in which these sequence quantities might actually exist.

For example, consider an unbalanced three-phase, wye connected load placed upon an alternator with a grounded neutral. If the neutral point of the load is ungrounded, then the system can be represented by the positive and negative sequences. But, suppose that the neutral of the load is grounded, then the ground current (that in the neutral) is the zero-sequence current. Now, suppose the load is adjusted to balanced conditions; a netural current may still flow. For example, consider the load as being wye connected transformer; it is a well known fact that a third harmonic of exciting current will flow if the neutral of the primary is grounded, because of the non-linearity of the magnetization curve and the effect of distortion of the hysteresis loop.

CHAPTER VIII

APPLICATION OF QUATERNIONS TO POWER

It was shown in the section of this paper dealing with the geometric interpretation of generalized power in a.c. circuits that the power factor angle is an angle between the real or active power and the vector apparent power. This, of course, implies that the power factor angle is, in general, an angle in space, that is, it is not in the same plane as the real and reactive powers. Obviously, then, to completely describe power in the general sense would require the use of four dimensions. For example, the four dimensions could be either real, reactive, distortion and mesh powers, or they could be composed of three of these and the space-angle, i.e., power factor angle. This idea could perhaps be represented by quaternions.

A quaternion is essentially an operator; it is a magnitude and a turning factor. A quaternion expresses the relation between two vectors. The magnitude consists of the ratio of the absolute magnitudes of the two vectors. The turning factor is an operator which operates upon one of the vectors, turning it into the path or line of action of the other vector. Consider two vectors M and N. Suppose further that M is r times as great in magnitude as N. Also, assume that the two vectors are directed in different directions, the angle between their lines of action being an arbitrary angle θ. Then $M = rB^{\theta}N$, where B^{θ} represent a rotation of θ degrees from N toward M. This rotation, of course, takes place in a plane. That is, the operator B^{θ} rotates the vector N from its original position until it coincides with the direction of vector M, the planear rotation taking place in the plane M-N.

Also, the reciprocal relation to the foregoing can be written as $N = 1/rB^{-\Theta}M$. Here $B^{-\Theta}$ merely represents a rotation through the same angle Θ and in the same plane, but the rotation occurs in a negative sense to the proper direction of measuring Θ. In addition, the relation $M/N = rB^{\Theta}$ may be written. In other words, B^{Θ} rotates vector N into line with vector M and then the multiplier r makes them equal in magnitude just as the operator had made them have the same angle.

Now, Θ was any arbitrary angle, and hence could be either 0 or 90 degrees. In fact, the operator B^{Θ} may be broken into two component parts, one which rotates zero degrees and the other which rotates ninety degrees. The proper multiplier r would have to be associated with each of these rotating factors. Therefore, $B^{\Theta} = \cos\Theta \times B^{0} + \sin\Theta \times B^{90^\circ}$, or, in event the magnitude is not unity in general $rB^{\Theta} = r\cos\Theta + r\sin\Theta\, B^{90^\circ}$. The zero angle operator is omitted because it has no effect upon the quantity upon which it operates. If we use symbols to designate the $r\cos\Theta$ component and a different symbol to represent the $r\sin\Theta$ component, say p and q respectively, then $rB^{\Theta} = p + qB^{90^\circ}$. And, also, $rB^{\Theta}A = pA + qB^{90^\circ}A$. Where the relation between p and q, and r and Θ are given by $r = \sqrt{(p^2 + q^2)}$, and $\Theta = \arctan(p/q)$.

From the above discussion it is seen that a quaternion is in reality composed of two parts, a scalar part and a vector part. The scalar part in the above derivations and definitions is p, and the vector part is qB^{90°.

Example:

Let E represent a sine wave alternating electromotive force in magnitude and phase and let I denote the alternating current in magnitude and phase that would flow if the sine wave electromotive force were applied to the terminals of a series resistance-inductance circuit. Then $E = (r + 2 \times 3.14\, fLB^{90^\circ})I$. Where r is the resistance of the circuit in ohms, L is the inductance, f is the frequency of alternations per second, and I denotes the axis of the plane of representation.

$$E = rI + 6.28\ fLB^{90°}I$$

$$EI^{-1} = r + 6.28\ fLB^{90°}$$

Thus, the operator which transforms the current into the voltage is a quaternion and the inductive reactance is the vector part.

To determine the components of the reciprocal of a quaternion, consider

$$R = (p + qB^{90°})A$$

Then

$$A = \frac{1}{p + qB^{90°}} R$$

$$A = \frac{p - qB^{90°}}{(p + qB^{90°})(p - qB^{90°})} R$$

$$A = \frac{p - qB^{90°}}{p^2 + q^2} R$$

$$A = \left[\frac{p}{p^2 + q^2} - \frac{q}{p^2 + q^2} B^{90°}\right] R$$

Example:

Consider the same application as considered in the previous example. Suppose that it is desired to obtain I in terms of E.

$$E = (r + 2\pi fL \times B^{90°})I$$

Hence,

$$I = \left[\frac{r}{r^2 + (2\pi fL)^2} - \frac{2\pi fL}{r^2 + (2\pi fL)^2} B^{90°}\right] E$$

ADDITION OF COAXIAL QUATERNIONS

If the ratios of each of several vectors to one particular vector are known, the ratio of their resultant to the same reference vector is obtained by taking the sum of the ratios.

$$R_1 = (p_1 + q_1 \times B^{90°})A$$
$$R_2 = (p_2 + q_2 \times B^{90°})A$$
$$\vdots$$
$$R_n = (p_n + q_n \times B^{90°})A$$
$$\sum_{j=1}^{n} R_j = \sum_{j=1}^{n}(p_j + q_j \times B^{90°})A = \left[\sum_{j=1}^{n} p_j + \sum_{j=1}^{n}(q_j) \times B^{90°}\right]A$$
$$\text{and} \quad A = \frac{\sum p - \sum qB^{90°}}{(\sum p)^2 + (\sum q)^2} \times \sum R$$

Example:

Consider a circuit composed of a number of simple R-L circuits connected in parallel.

$$I_1 = \frac{r_1 - 2\pi f L_1 B^{90°}}{r_1^2 + (2\pi f L_1)^2} \times E_1$$
$$I_2 = \frac{r_2 - 2\pi f L_2 \times B^{90°}}{r_2^2 + (2\pi f L_2)^2} \times E_1$$
$$\sum I = \sum\left[\frac{r - 2\pi f L \times B^{90°}}{r^2 + (2\pi f L)^2}\right] \times E_1$$

PRODUCT OF QUATERNIONS

The quaternion which cnages A to R^1 is obtained by taking the product of the quaternion which changes A to R with the quaternion which changes R to R^1.

$$\text{If} \quad R = rB^{\theta}A = (p + qB^{90°})A$$
$$\text{and} \quad R^1 = r^1B^{\theta^1}R = (p^1 + q^1B^{90°})R_1$$
$$\text{then} \quad R^1 = rr^1B^{(\theta + \theta^1)}A = \left[(pp^1 - qq^1) + (pq^1 + p^1q)B^{90°}\right]A$$

Notice that the complete product is the product of the magnitudes, and also the product of the turning factors. The angles were added because they were indices of the common base B.

QUOTIENT OF TWO QUATERNIONS

The quaternion which changes R to R^1 is obtained by taking the quotient of the quaternion which changes A to R^1 by the quaternion which changes A to R.

If $\quad R = rB^{\theta}A = (p + qB^{90°})A$

and $\quad R^1 = r^1B^{\theta^1}A = (p^1 + q^1B^{90°})A$

then $\quad R^1 = \frac{r^1}{r}B^{(\theta^1 - \theta)}R$

$$R^1 = (p^1 + q^1B^{90°})\frac{1}{p + qB^{90°}}R$$

$$R^1 = (p^1 + q^1B^{90°})\frac{(p - qB^{90°})}{p^2 + q^2}R$$

$$R^1 = \frac{(pp^1 + qq^1) + (pq^1 - p^1q)B^{90°}}{p^2 + q^2}$$

PRODUCT OF TWO VECTORS

In the following paragraph are the rules commonly associated with the vector algebra. These vectors products play an important role in the field of applied mathematics. Any general vector in three dimensions is represented by its components along the coordinates of the rectangular Cartesian Coordinate system. The component of the magnitude of the vector along each of the three axes is multiplied by the appropriate unit vector, and the sum of the three vector components then obviously gives the original vector. The unit vectors along the X, Y, and Z axes shall be designated by i, j, and k respectively.

The product of two vector means to take the product of their magnitudes and direct this product in the direction normal to the plane of the two vectors in the direction that a right hand screw would advance if the screw be turned from the first vector in the product toward the second vector in the product. Then, obviously, ij is k, and ji is -k. Also, j^2 is 1, and similarly for the squares of the other two unit vectors. The following table completely describes all possible products of the three unit vectors.

i^2	is	1	j^2	is	1	k^2	is	1
ij	is	k	jk	is	i	ki	is	j
ji	is	-k	kj	is	-i	ik	is	-j

The square combination of the unit vectors give quantities which are independent of direction, and consequently are summed by simple addition.

The physical meaning of the above rules is made clearer by considering an application to the dynamo and the electric motor. In the dynamo, three principal vectors have to be considered: the peripherial velocity of the rotor conductors at an instant, the strength or intensity of the magnetic field and the vector representing electromotive force. The vectors representing the three electrical quantities above are orthogonal vectors. Therefore, by proper orientation, they can be located in the Cartesian coordinate system. Frequently all that is required is, given two of the directions, to determine the third. Suppose that the velocity of the conductor is i, and the direction of the flux is j, then the direction of the electromotive-force is k. The above formula ij is k becomes

velocity x flux is electromotive-force.

From which it easily follows that

flux x electromotive-force is velocity

and electromotive-force x velocity is flux

The corresponding equation for the electric motor is

current x flux is mechanical-force

From which by a cyclic permutation as before

flux x force is current

force x current is flux

In most elemetary courses in electricity the student is taught a thumb-and-finger rule in order to remember the above relations. The formula velocity x flux is electromotive-force is much handier and easier to remember than any thumb-and-finger rule, for it compares the three directions directly with the right hand screw.

Example:

Suppose that the electrical conductor is so orientated that it is perpendicular to the plane of the paper and its velocity is toward the bottom of the sheet. Further, suppose that the direction of the magnetic flux is toward the left of the paper. Corresponding to the rotation from the velocity to the flux, a right handed screw would advance into the paper; that then is the direction of the electromotive-force. Notice, particularly, that since velocity appears first in the product, we must turn the screw from the velocity into the direction of the flux.

On the other hand, suppose that the direction of the current along the conductor is in a direction directly out of the paper toward the reader and perpendicular to the plane of the paper. Again, assume the flux to be toward the left. Corresponding to the rotation current-flux, a right handed screw would advance toward the bottom of the page, which, therefore, must be the direction of the mechanical force which is applied to the conductor. This is the generator principle; the example above was the motor principle.

COMPLETE PRODUCT OF TWO VECTORS

$$A = a_1 i + a_2 j + a_3 k$$

and $$B = b_1 i + b_2 j + b_3 k$$

These are two general three-dimensional vectors, not necessarily of the same physical kind. Their product is then

$$\begin{aligned} AB &= (a_1 i + a_2 j + a_3 k)(b_1 i + b_2 j + b_3 k) \\ &= a_1 b_1 i^2 + a_2 b_2 j^2 + a_3 b_3 k^2 + a_2 b_3 jk + a_3 b_2 kj \\ &\quad + a_3 b_1 ki + a_1 b_3 ik + a_1 b_2 ij + a_2 b_1 ji \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j \\ &\quad + (a_1 b_2 - a_2 b_1) k \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix} \quad \text{(determinant)} \end{aligned}$$

Thus the total product breaks up into two partial products, namely, $a_1 b_1 + a_2 b_2 + a_3 b_3$, which is independent of direction, and $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$, which has a direction normal to the plane of the two original vectors, A and B. The first part is called the scalar part, and the latter, the vector part.

If one can think of the scalar part as being represented along some real axis, then the foregoing total product would very aptly represent the total power in an a.c. circuit. For example, the real part is the expression for active power, the other three could perhaps be thought of as representing the reactive power, the distortion power and the mesh power.

$$P = VI = P + P_r + P_d + P_m$$

CHAPTER IX

REPRESENTATION OF ELECTRICAL QUANTITIES IN HILBERT SPACE

While solving integral equation in connection with physical problems, Hilbert initiated what is now called a sequential Hilbert Space. Many other mathematicians developed the theory of these spaces and applied it to problems of mathematics and mathematical physics. They introduced different kinds of Hilbert spaces that are called today different realizations of the Hilbert space. It was only at the end of the third decade that J. von Neumann laid down a basis for the unified theory of the spaces developing the theory of the abstract Hilbert space. Since then the theory has undergone a great development in itself as well as in its applications to other branches of mathematics and mathematical physics.

Hilbert space appears as a generalization of the n-dimensional Euclidean or unitary spaces for n as n approaches infinity. The n-dimensional Euclidean space is a space whose points P are in a one-to-one correspondance with all sequences of real numbers (the coordinates of P). The symbol * shall be used to denote a one-to-one correspondance.

$$P * (X_1, X_2, X_3, \ldots, X_n)$$

The distance of two points $P * (X_k)$, $Q * (Y_k)$, for $k = 1, 2, 3, \ldots, n$ is:

$$\left[(Y_1 - X_1)^2 + (Y_2 - X_2)^2 + \ldots, + (Y_n - X_n)^2\right]^{\frac{1}{2}}$$

The abstract Hilbert space is based upon notions familiar in the classical vector-calculus. Recall here briefly these notions as they are used in an r-dimensional space.

Two points of the space $P \leftrightarrow (x_k)$ and $Q \leftrightarrow (y_k)$ give rise to a vector PQ = u with origin P and end-point Q. The components of the vector PQ are the differences $(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n)$. Two vectors are equivalent when their coordinates are equal. Consequently, each vector is equivalent to a vector u = OP with origin at O (0, 0, 0, ..., 0). The coordinates of the vector u = OP are the same as those of the point P.

DEFINITION OF ADDITION:

$$u \leftrightarrow (x_k), \; v \leftrightarrow (y_k)$$

$$u + v \leftrightarrow (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

MULTIPLICATION BY A NUMBER a:

$$au = (ax_1, ax_2, ax_3, \ldots, ax_n) = (ax_k)$$

SCALAR PRODUCT (u, v):

$$(u, v) = x_1y_1 + x_2y_2 + \ldots, + x_ny_n \quad \text{(real x \& y)}$$

If x and y are complex

$$(u, v) = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 + \ldots, + x_n\bar{y}_n.$$

The scalar product is never negative and is = 0, if, and only if, u = 0, i.e., u = zero vector (0, 0, 0, ..., 0). If u = OP and x = OQ, then the vector PQ is equivalent to v - u. The distance from P to Q is $|v - u|$. Any point in Euclidean space can then be represented by a vector from the origin to the point in question.

An alternating voltage source that contains harmonics of the order n will now be considered. If this voltage is applied to a circuit then currents of order corresponding to the voltage harmonics will flow, and if the circuit is non-linear this may further affect the current harmonics. Now it is a well known fact that if a voltage wave contains harmonics, the effective value of the resultant complex wave is $E = \left[E_1^2 + E_2^2 + \ldots, + E_n^2\right]^{\frac{1}{2}}$, and similarly for the current $I = \left[I_1^2 + I_2^2 + \ldots, + I_n^2\right]^{\frac{1}{2}}$. These values of E_k and I_k may be, either real or complex. Now it is evident that the two above equations define E and I as two points in the abstract Hilbert space, i.e., E represents the vector OE and I represents the vector OI.

The product of E x I represents in the ordinary sense volt-amperes. By integrating over a half-cycle it can be shown that a voltage and a current of a different integral frequency will furnish no net power. That is to say that

$$E^2 \times I^2 = E_1^2 I_1^2 + E_2^2 I_2^2 + \ldots, E_n^2 I_n^2$$

$$E \times I = \left[E_1^2 I_1^2 + E_2^2 I_2^2 + \ldots, E_n^2 I_n^2\right]^{\frac{1}{2}}$$

This is the definition of the scalar product in the abstract Hilbert space.

CHAPTER X

CONCLUSIONS

Progress is a continual struggle between the destruction or modification of the old and the development and growth of the new. In this paper the inadequacy of the old definitions of power factor, real power, reactive power, and apparent power when considering non-linear and unbalanced circuits has been discussed. Conventions and rules once established and written into textbooks are difficult and time consuming to change. However, it is the author's opinion, should the conditions merit such, that the further growth and development of the field of electricity should not be shackled and impeded by the adherence to partially inconsistent and inadequate rules and conventions.

It is not recommended here that the old definitions be completely discarded, in fact the definitions of real and reactive power could possibly be retained, but the definitions of power factor and total apparent power should be adjusted to fit the more general types of circuits rather than the simple sinusoidal circuits under balanced conditions. Further, the concept of mesh and distortion powers should evidently be incorporated into textbooks. This would at least warn the student of elementary electricity of the possibility of the right triangle relationship among the different types of power being false.

There are probably some who would argue the unpracticality of further complication of the theory of a.c. circuits, especially from the viewpoint of the practical engineer. This argument is ill-founded. For, should the concepts be extended, the practical engineer would be in no greater dilemma than he is at present even if he knew nothing of the extension, as he could continue using his old practices without undue change. Moreover, individuals not having the capacity to extend their knowledge and change their ideas do not deserve the title "engineer."

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