CURRENT DISTRIBUTION AND IMPEDANCE OF A

CYLINDRICAL ANTENNA IN AN

ISOTROPIC COMPRESSIBLE

PLASMA

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PREFACE

Since 1961, the subject of compressible plasmas has grown into considerable importance, as evidenced by the large number of papers being presented and published this year. This phenomenon shows that a great many researchers are very concerned that their theoretical models closely approximate the actual medium, in this case the ionosphere. Also one finds that certain results are occurring in the mathematics which severely limits extrapolation from free space or incompressible plasmas. The proper choice of a current distribution on an antenna is one of these cases, and it is this subject which was assigned to the author for research purposes as a integral part of contract Nonr-2595(04).

I wish to acknowledge Dr. K. R. Cook of Oklahoma State University for his guidance and help when the theoretical obstacles seemed insurmountable, Gary L. Johnson of Oklahoma State University for many valuable and illuminating conversations, and the School of Electrical Engineering for providing me with financial assistance in the form of a research assistanship on contract Nonr-2595(04).

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CHAPTER I

INTRODUCTION

Within the past decade considerable interest has been generated in the use of cylindrical antenna structures for diagnostic techniques which provide experimental data describing certain parameters of the upper ionosphere, as well as for communication systems for satellites orbiting in the ionosphere. However, in order to make quantitative interpretations of such data or to make it efficient as a communication tool, accurate information must be available as to the input impedance of the antenna structure, which in turn depends on the current distribution. Because of recent experimental evidence (Whale, 1963, 1964) demonstrating that the ionosphere is a compressible plasma, an investigation into the impedance and current distribution of a cylindrical antenna would not be complete if the compression or electroacoustic modes were not considered. For the purposes of this paper, we will assume that the medium external to the antenna is an isotropic compressible electron plasma.

Discussion of the Literature

The subject of radiation of electromagnetic and electroacoustic waves from sources in an isotropic compressible plasma has been considered by many investigators (Cohen, 1962; Chen, 1964; Fejer, 1964; Hessel and Shmoys, 1962; Wait, 1964a, 1964b, 1964c, 1965a, 1965b). Historically speaking, Hessel and Shmoys first considered the radiation from an infinitesimal electric dipole in a compressible plasma, with

Cohen (1961) ingeniously providing a modal decomposition procedure which greatly simplified the formulation and analysis of problems dealing with compressible plasmas. Wait (1964a) investigated the radiation problem associated with insulated point sources. Other antenna structures in compressible plasmas such as the slotted sphere (Wait, 1964b, 1964c) and the two sphere approximation of a dipole (Fejer, 1964) have also been investigated. Although Fejer requires the normal electron velocity to vanish, the results of his two sphere approximation and quasistatic approach seems to be more closely related to the slotted sphere problem than to an electric dipole.

Of these papers, Cohen, Chen, and Wait (1965a) have treated the problem of a cylindrical antenna with an assumed sinusoidal current distribution. However, the propagation constant of the current distribution for Cohen's case is the free space plane wave number; while Chen's propagation constant is the same as a plane wave in an isotropic incompressible plasma. Wait has indicated that the actual propagation constant may be quite different from that assumed in the two previous cases. All the investigators of the cylindrical antenna exclude any coupling effects which would arise from the fact that the antenna presents a solid metal boundary to the plasma and would couple the electroacoustic and electromagnetic modes at its surface. Because of the assumed nature of the propagation constant and the exclusion of any coupling effects, the results of these papers depend heavily on the validity of such assumptions, and how close these approximations are to a metal dipole is not known.

Wait (1965b) has examined the infinitely long cylindrical antenna excited by an infinitesimal gap and has found that the electroacoustic mode plays no part in the radiation process when the coupling effects are included. Whether the infinite antenna characteristics bear any relation to those of finite antennas is in question at this time.

Basis for the Current Distribution Assumptions

Theoretically, if the source of power is localized, there are no discontinuities in the wire, and any effects of size can be ignored, one finds that the current distributions existing on wires are nearly sinusoidal for a first approximation. Thus one could say that the current distribution on a transmission line approaches that of a thin linear antenna because of the physical simularity between a cylindrical antenna and a transmission line or an external waveguide. These statements are the foundation of the "waveguide theory of antennas" or the "mode theory of antennas" as advanced by Schelkunoff (1952). This theory states that if a sinusoidal current distribution is assumed, the propagation number R wave number k of the current may be obtained by solving Maxwell's equations and the appropriate force and continuity equations for waves propagating down an infinite cylinder subject to the proper boundary conditions. These conditions make the current distribution highly dependent on the external medium. Although the transition from the infinite structure to the finite antenna has not been made mathematically, a simple explanation is that a finite cylinder creates standing waves when the traveling waves excited by a source are reflected by the discontinuity at the end of the cylinder.

From these arguments, as demonstrated by Stratton (1941), the wavenumber associated with the infinite structure in free space is the plane wave number. King (1964) has used the method to determine the current distribution on an insulated dipole in a conducting medium.

In this paper we will follow this procedure by first considering an infinite cylinder of finite radius surrounded by a homogeneous isotropic compressible plasma. We then will proceed to determine the wave numbers associated with guided waves along the cylinder, under the boundary conditions that both the tangential electric field and the normal velocity of the electrons must vanish at the cylinder. Neglecting end and gap effects, these wave numbers associated with the infinite cylinder are the same, for a first order approximation, as the wave numbers of a sinusoidal current distribution existing on a cylindrical antenna. Having obtained these eigenvalues of the current distribution, the radiation resistance of a dipole can be easily found.

CHAPTER II

BASIC EQUATIONS

Linearized Model

If we assume that the plasma is an isotropic homogeneous linear electron gas and that our analysis is only valid for small amplitudes of waves with an $e^{i\omega t}$ time dependence, then the system of linearized equations, as derived in Appendix A, describing this model is:

$$\nabla \times \overline{H} - i\omega \epsilon_o \overline{E} = N_o e \overline{V} + \overline{J}^s , \tag{1.1}$$

$$\nabla \times \overline{E} + i\omega \mu_o \overline{H} = 0 , \tag{1.2}$$

$$i\omega m N_o \overline{V} = N_o e \overline{E} - m v_o^2 \nabla n , \tag{1.3}$$

$$\nabla \cdot \overline{J} + i\omega \rho^s = 0 , \tag{1.4}$$

$$N_o \nabla \cdot \overline{V} = -i\omega n , \tag{1.5}$$

where,

$\overline{E}, \overline{H}$ = electric and magnetic fields ,

$\overline{V}$ = electron velocity ,

$\overline{J}^s, \rho^s$ = current density and charge density due to the electric source ,

M, e = mass and charge of an electron ,

ϵ_o, μ_o = permittivity and permeability of free space ,

ω = radian frequency ,

n = variation in electron density ,

v_o = acoustical velocity in an electron gas = $(3KT/m)^{\frac{1}{2}}$,

N_o = steady state electron density .

Modal Decomposition

From Equations 1.1 and 1.3, elimination of $\overline{V}$ yields

$$\overline{E} = \frac{1}{i\omega\,\epsilon_o\,\epsilon_1}\left[\nabla \times \overline{H} - \overline{J}^s - \frac{ie\,v_o^2}{\omega}\nabla n\right]. \qquad (1.6)$$

Likewise for the velocity vector,

$$\overline{V} = \frac{(\epsilon_1 - 1)}{N_o\,e\,\epsilon_o\,\epsilon_1}\left[\nabla \times \overline{H} - \overline{J}^s - \frac{i\omega\,\epsilon_o\,mv_o^2}{N_o\,e}\nabla n\right], \qquad (1.7)$$

where

$$\epsilon_1 = 1 - \frac{N_o\,e^2}{\omega^2\,m\,\epsilon_o},$$

or

$$\epsilon_1 = 1 - \frac{\omega_p^2}{\omega^2},$$

and ω_p is the plasma resonant frequency. Upon examining Equations 1.6 and 1.7, we find that the velocity and electric fields are produced by two modes, an optical or electromagnetic mode and a plasma mode. Thus we define the following modal decomposition after Cohen [1961]:

$$\begin{aligned} \overline{E} &= \overline{E}_o + \overline{E}_p , \\ \overline{V} &= \overline{V}_o + \overline{V}_p , \end{aligned} \qquad (1.8)$$

where the subscript o stands for the optical mode, and the subscript p stands for the plasma mode.

Thus from Equations 1.6, 1.7, and 1.8, mode field components may be defined as:

$$\bar{E}_o = \frac{1}{i\omega\ \epsilon_o\ \epsilon_1}\ [\nabla \times \bar{H} - \bar{J}^s\ ]\ , \tag{1.9}$$

$$\bar{E}_p = \frac{(\epsilon_1 - 1)\ m\ v_o^2}{N_o\ e\ \epsilon_1}\ \nabla n\ , \tag{1.10}$$

$$\bar{V}_o = \frac{\epsilon_1 - 1}{N_o\ e\ \epsilon_1}\ (\nabla \times \bar{H} - \bar{J}^s) = \frac{e}{i\omega m}\ \bar{E}_o\ , \tag{1.11}$$

$$\bar{V}_p = \frac{v_o^2}{-i\omega\ N_o\ \epsilon_1}\ \nabla n\ . \tag{1.12}$$

If we are in source-free regions, Helmholtz's Principle [Moon, 1961], which states that any vector field well behaved at infinity can be completely specified as the sum of an irrotational vector field and a solenoidal vector field, may be applied to justify the modal decomposition. The optical mode would then be solenoidal, and the plasma mode would be irrotational. It is fairly obvious that in the far-field region the two modes "uncoupled"; however, there will be boundary coupling between the modes if there is some discontinuity in the electron density such as would exist at a metal boundary.

Noting that from Equations 1.9 and 1.4,

$$\nabla \cdot \bar{E}_o = \frac{\rho^s}{\epsilon_o\ \epsilon_1}\ , \tag{1.13}$$

and that from Equations 1.13 and 1.11,

$$\nabla \cdot \bar{V}_o = \frac{e\ \rho^s}{i\omega m\ \epsilon_o\ \epsilon_1}\ . \tag{1.14}$$

Substitution of Equations 1.14 and 1.8 into Equation 1.4 gives

$$\nabla \cdot \overline{V}_p = \frac{-N_o \; e \; \rho^s}{i\omega m \; \epsilon_o \; \epsilon_1} - i\omega n \; , \tag{1.15}$$

The wave equation for n can be obtained by using Equations 1.15 and 1.12. The result is

$$\nabla^2 n + k_p^2 \, n = \frac{\omega_p^2 \; s}{e \; v_o^2} \; , \tag{1.16}$$

where

$$k_p^2 = \frac{\omega^2}{v_o^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right) \; .$$

Equation 1.16 tells us that the plasma mode is excited only by ρ^s.

Potential Functions for the Optical and Plasma Modes

Since the magnetic field is always solenoidal, we can express

$$\overline{H} = \frac{1}{\mu_o} \nabla \times \overline{A} \; , \tag{1.17}$$

where $\overline{A}$ is some vector potential. Then from Equation 1.2

$$\overline{E}_o = - i\omega \overline{A} - \nabla \Phi \; , \tag{1.18}$$

where Φ is the scalar potential. Utilizing the Lorentz gauge that

$$\nabla \cdot \overline{A} + i\omega \, \mu_o \, \epsilon_o \, \epsilon_1 \, \Phi = 0 \; , \tag{1.19}$$

the vector wave equation for $\overline{A}$ can be derived from Equations 1.2, 1.9, 1.17, 1.18, and 1.19.

$$\nabla^2 \overline{A} + k_o^2 \overline{A} = - \mu_o \overline{J}^s \; , \tag{1.20}$$

where

$$k_o^2 = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right) ,$$

and $c = 3 \times 10^8$ meters/sec.

The plasma mode, described by $\bar{E}_p$ and $\bar{V}_p$, is represented in terms of the gradient of the particle density. Therefore, the potential function for the plasma mode is scalar and equal to the density of the particles.

CHAPTER III

THE PROBLEM

Formulation of the Problem

We now turn our attention to the problem of waves propagating down an infinite metal cylinder which is rigid and perfectly conducting. The region internal to the cylindrical surface will be void of any fields, and the medium external to the cylinder will be homogeneous and free of any finite sources. Hence, the modal decomposition procedure may be applied to the field structure external to the cylinder. The cylinder is assumed aligned with the z-axis of a right-handed rectangular coordinate system. The geometry of the problem is described in Figure 1.

Plasma Mode

If it is assumed that the density function is monochromatic and propagating in the positive z-direction, the electron density will have an exp $i(\omega t - kz)$ variation, where k is the longitudinal wave number. If we place the constraint on our fields that they be independent of θ, Equation 1.16 becomes, in cylindrical coordinates:

$$\frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + \lambda_1^2 n = 0 , \qquad (2.1)$$

where $\lambda_1^2 = k_p^2 - k^2$. Solutions of the above wave equation must satisfy the radiation condition at infinity. Imposing this constraint, solutions of Equation 2.1 must have the form,

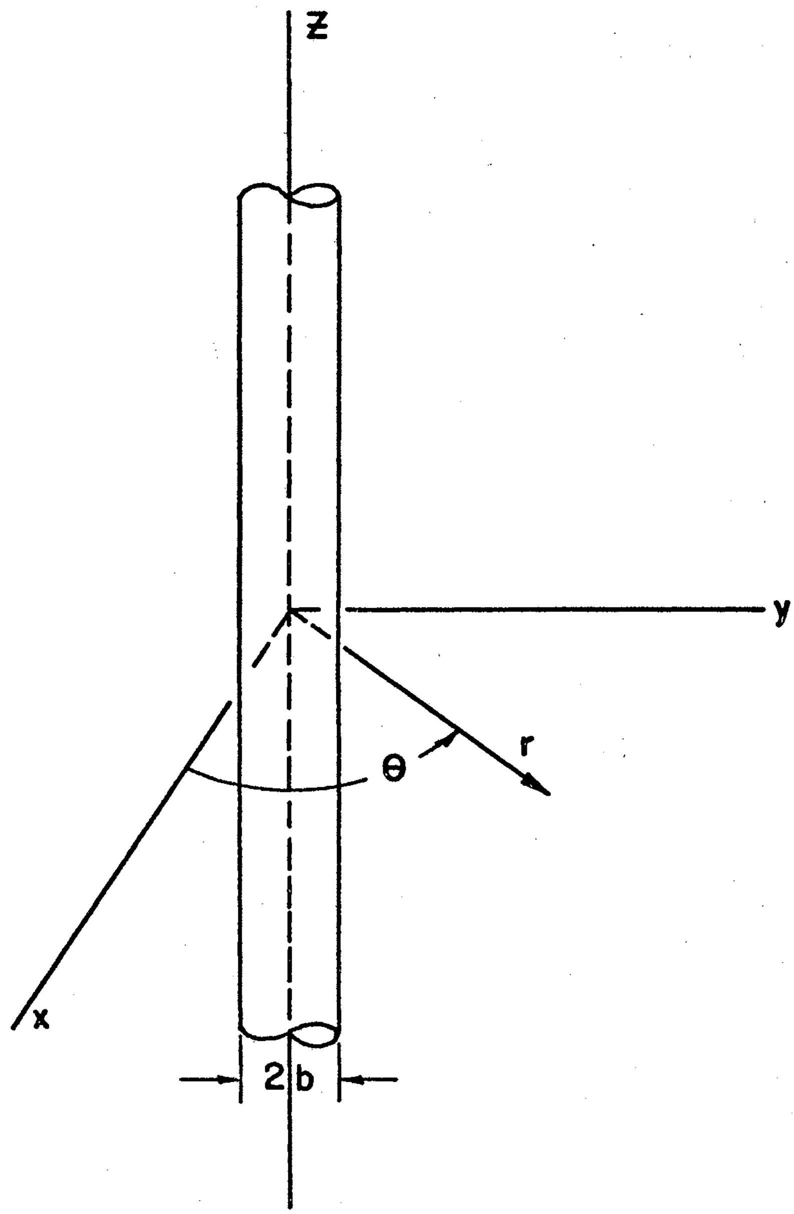

Fig. 1 - CYLINDRICAL COORDINATE SYSTEM.

$$n = A\, H_o^{(2)}(\lambda_1\, r)\, e^{i(\omega t - kz)} , \qquad (2.2)$$

where $H_o^{(2)}(\lambda_1 r)$ is the Hankel function of the second kind and zero order.

The electric field and velocity components associated with the plasma mode, as defined by Equations 1.9 through 1.12, are:

$$E_{rp} = -\frac{(\epsilon_1 - 1)\,\lambda_1\, m\, v_o^2\, A\, H_1^{(2)}\,(\lambda_1\, r)}{N_o\, e\, \epsilon_1} , \qquad (2.3)$$

$$E_{zp} = -\frac{i\,(\epsilon_1 - 1)\, km\, v_o^2\, A\, H_o^{(2)}\,(\lambda_1\, r)}{N_o\, e\, \epsilon_1} ,$$

$$V_{rp} = \frac{\lambda_1\, v_o^2\, A\, H_1^{(2)}\,(\lambda_1 r)}{i\omega\, N_o\, \epsilon_1} ,$$

$$V_{zp} = \frac{k\, v_o^2\, A\, H_o^{(2)}\,(\lambda_1 r)}{\omega\, N_o\, \epsilon_1} ,$$

where $$\epsilon_1 = 1 - \omega_p^2/\omega^2 .$$

Optical Mode

Due to the boundary conditions at the surface of the cylinder, it is easily shown that the optical mode structure will be TM to the z-direction.

For the TM optical mode, we may set

$$\overline{A} = \psi \overline{a}_z , \qquad (2.4)$$

where $\overline{a}_z$ is a unit vector in the z-direction and ψ is the magnitude of the magnetic vector potential. If ψ has an $e^{i(\omega t - kz)}$ variation along the cylinder, substitution of Equation 2.4 into 1.20 yields

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$$\nabla_T^2 \psi + \lambda_2^2 \psi = 0 , \tag{2.5}$$

where $\lambda_2^2 = k_o^2 - k^2$. Solutions of Equation 2.5 in cylindrical coordinates which obey the radiation condition and have no azimuthal variation about the cylinder are of the form,

$$\psi = B\, H_o^{(2)} (\lambda_2 r)\, e^{i(\omega t - kz)} . \tag{2.6}$$

The optical mode electric, magnetic, and velocity fields, as defined by Equation 1.17 and 1.18 are:

$$\begin{aligned} E_{r_o} &= \frac{\lambda_2 k}{\omega \epsilon_o \epsilon_1} B\, H_1^{(2)} (\lambda_2 r) , \\ E_{z_o} &= \frac{\lambda_2^2}{i\omega \epsilon_o \epsilon_1} B\, H_o^{(2)} (\lambda_2 r) , \end{aligned} \tag{2.7}$$

$$H_{\theta_o} = \lambda_2 B\, H_1^{(2)} (\lambda_2 r) ,$$

$$\begin{aligned} V_{r_o} &= \frac{e}{i\omega m} \cdot \frac{\lambda_2 k}{\omega \epsilon_o \epsilon_1} B\, H_1^{(2)} (\lambda_2 r) , \\ V_{z_o} &= \frac{e}{i\omega m} \cdot \frac{\lambda_2^2}{i\omega \epsilon_o \epsilon_1} B\, H_o^{(2)} (\lambda_2 r) . \end{aligned} \tag{2.8}$$

Boundary Conditions

From Equations 2.3, 2.7, and 2.8, the total $\overline{a}_z$-component of the electric field and the total radial-component of the electron velocity are

$$E_z = \frac{\lambda_2^2}{i\, \omega\, m\, \epsilon_o\, \epsilon_1} B\, H_o^{(2)} (\lambda_2 r) - \frac{i\, (\epsilon_1 - 1)\, k\, m\, v_o^2}{N_o\, e\, \epsilon_1} A\, H_o^{(2)} (\lambda_1 r),$$

$$V_r = \frac{e\, \lambda_2\, k}{i\, \omega^2\, m\, \epsilon_o\, \epsilon_1}\, B\, H_1^{(2)}\, (\lambda_2\, r) + \frac{\lambda_1\, v_1^2}{i\, \omega\, N_o\, \epsilon_1}\, A\, H^{(2)}\, (\lambda_1\, r). \qquad (2.9)$$

Applying the boundary conditions that the tangential electric field and the normal velocity must vanish at $r = b$, which is a good approximation, we obtain the following system of equations from Equation 2.9.

$$\begin{bmatrix} \dfrac{\lambda_2^2}{i\, \omega\, \epsilon_o\, \epsilon_1}\, H_o^{(2)}\, (\lambda_2\, b) & \dfrac{-i\, (\epsilon_1 - 1)\, k\, m\, v_o^2}{N_o\, e\, \epsilon_1}\, H_o^{(2)}\, (\lambda_1\, b) \\ \dfrac{e\, \lambda_2\, k}{i\, \omega^2\, m\, \epsilon_o\, \epsilon_1}\, H_1^{(2)}\, (\lambda_2\, b) & \dfrac{\lambda_1\, v_o^2}{i\, \omega\, N_o\, \epsilon_1}\, H_1^{(2)}\, (\lambda_1\, b) \end{bmatrix} \begin{bmatrix} B \\ A \end{bmatrix} = 0 \qquad (2.10)$$

For nontrivial solutions of Equation 2.10, the determinant of the coefficient matrix must vanish which yields the following determinantal equation for the wave number k.

$$\frac{-k^2 \left(\dfrac{\omega^2}{\omega^2}\right)}{\lambda_1\, \lambda_2}\, \frac{H_o^{(2)}\, (\lambda_1\, b)}{H_1^{(2)}\, (\lambda_1\, b)} = \frac{H_o^{(2)}\, (\lambda_2\, b)}{H_1^{(2)}\, (\lambda_2\, b)} \qquad (2.11)$$

CHAPTER IV

THE DETERMINANTAL EQUATION

Solution of Determinantal Equation

We will restrict this paper to the spectrum for k^2 real. If we first assume that the transverse wave numbers λ_1, λ_2 are real, the Hankel function ratios are then complex. From Equation 2.11, one can conclude that k^2 is also complex for the determinantal equation to be satisfied, thus violating our original restriction on the spectrum of k^2. If we now assume that the transverse wave numbers are negative imaginary, the Hankel function ratios become positive imaginary, and from Equation 2.11 k^2 is now positive real. For the transverse wave numbers to be negative imaginary, the following constraints must be placed on k^2.

$$\omega_p > \omega \qquad 0 < k^2 < \infty$$

$$\omega_p < \omega \qquad k_p^2 \leq k^2 < \infty$$

Discussion of the Curves

The eigenvalues k were determined,* using the Hankel function ratio approximation, as derived in Appendix B, on an IBM 1410 digital computer. The dispersion curves (Figures 2, 3, and 4) are plotted for variations in cylinder radii and in several plasma parameters, based upon those of the ionosphere taken from Johnson (1961).

The dispersion curves diaplay hybrid optical-plasma characteristics with the optical mode predominating for $\omega \ll \omega_p$ and the plasma mode predominating for $\omega \gg \omega_p$. At the plasma frequency computor results show that both Hankel function ratios approach one. Thus from Equation 2.11, the following condition must be satisfied:

$$\frac{k^2\left(\frac{\omega_p}{\omega}\right)^2}{\lambda_1 \lambda_2} = 1 ,$$

and solving for k^2 yields

$$k^2 = \frac{\omega^2}{2 v_o^2} \left(1 + \frac{v_o^2}{c^2}\right) ,$$

or

$$k \overset{\circ}{=} \frac{\omega}{\sqrt{2}\, v_o} .$$

If the $x \ln \frac{2}{\gamma x}$ small argument approximation is used for the Hankel function ratio in Equation 2.11, the result is

$$\frac{k^2 x}{k \lambda_1} \circ \lambda_1 b \ln \frac{2}{\gamma \lambda_1 b} = kb \ln \frac{2}{\gamma k b} ,$$

$$\text{or } x \ln \frac{2}{\gamma \lambda_1 b} = \ln \frac{2}{\gamma k b} , \qquad (3.1)$$

where $x = \left(\frac{\omega_p}{\omega}\right)^2$ and $\lambda_2 \overset{\circ}{=} k$. Taking the derivative of k in respect to x in Equation 3.1, one obtains

$$\frac{dk}{dx} = \frac{-k \left[\lambda_1^2 \ln \frac{\gamma \lambda_1 b}{2} + \frac{k_a^2}{2}\right]}{(x - 1)(k^2 - k_a^2)} , \qquad (3.2)$$

where $k_a = \omega / v_o$. For $x = 1$, $\frac{dk}{dx}$ becomes discontinuous indicating a spline point. Graphically solving for the zeroes of dk/dx about $x = 1$, we find the following to be true.

$$k = 4.2 \qquad \text{for } b = .001\text{m.}$$

$$k = 5.77 \qquad \text{for } b = .01\text{m.}$$

These results agree very closely with the computer data. The zoeroes of the derivative indicate that a relative minimum exists for $x \stackrel{\circ}{=} 1$; in fact computer results show that $x = .999999$.

Figure 4 shows that for normalized squared plasma frequencies below .8, the longitudinal wave number can be approximated very well by k_p. This fact will be used later in the paper for radiation resistance calculations. Seshadri (1961) found that for high frequencies the wave numbers associated with a coupled surface wave along a metal-plasma interface approached

$$k = \frac{\omega}{v_o} (1 + x)^{-\frac{1}{2}} .$$

Therefore, from this example, the behavior of k for $\omega \gg \omega_p$ would not be altogether unexpected.

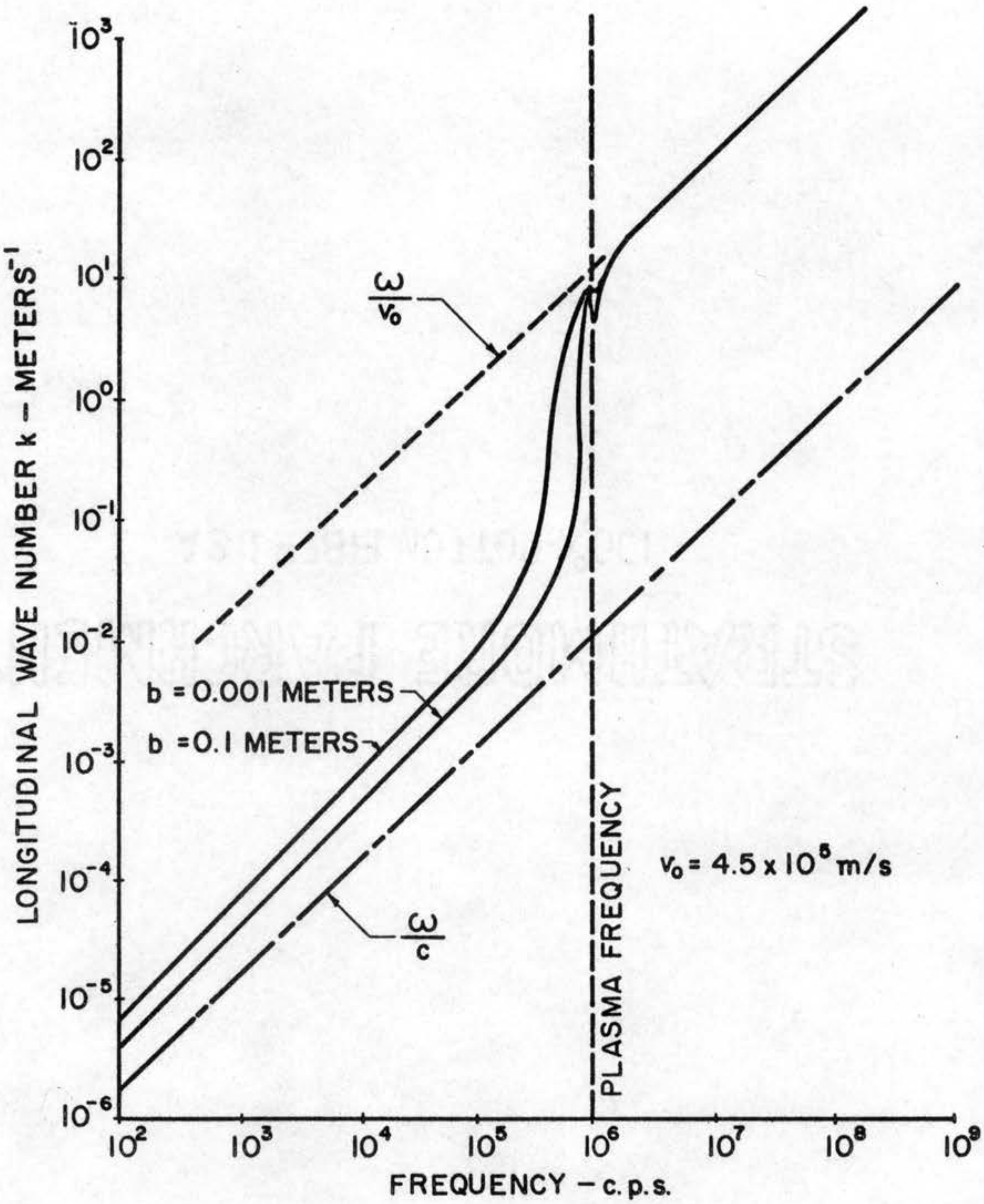

Fig. 2 - LONGITUDINAL WAVE NUMBER k FOR DIFFERING CYLINDER RADII.

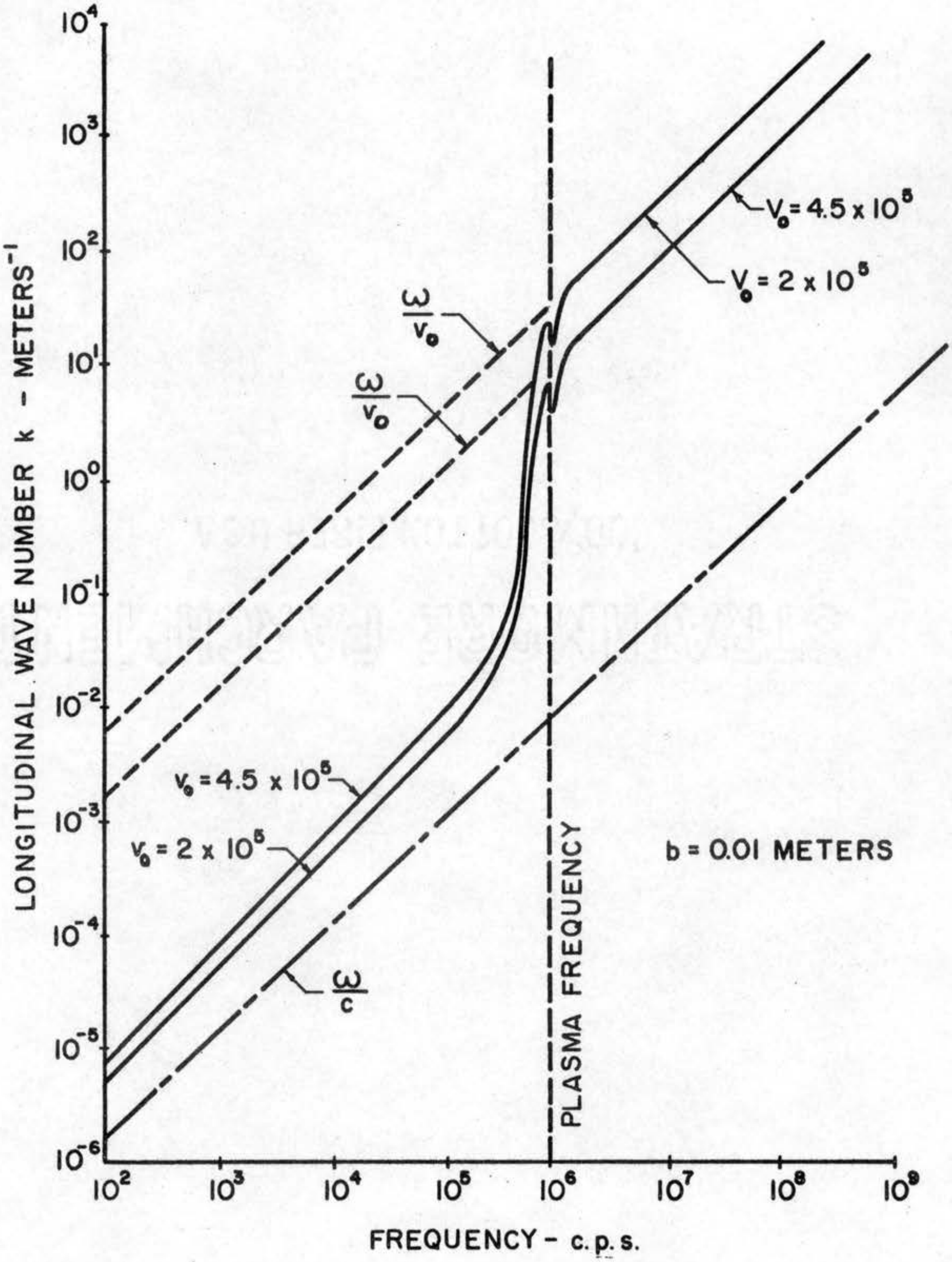

Fig. 3-LONGITUDINAL WAVE NUMBER k FOR DIFFERING ACOUSTICAL VELOCITIES.

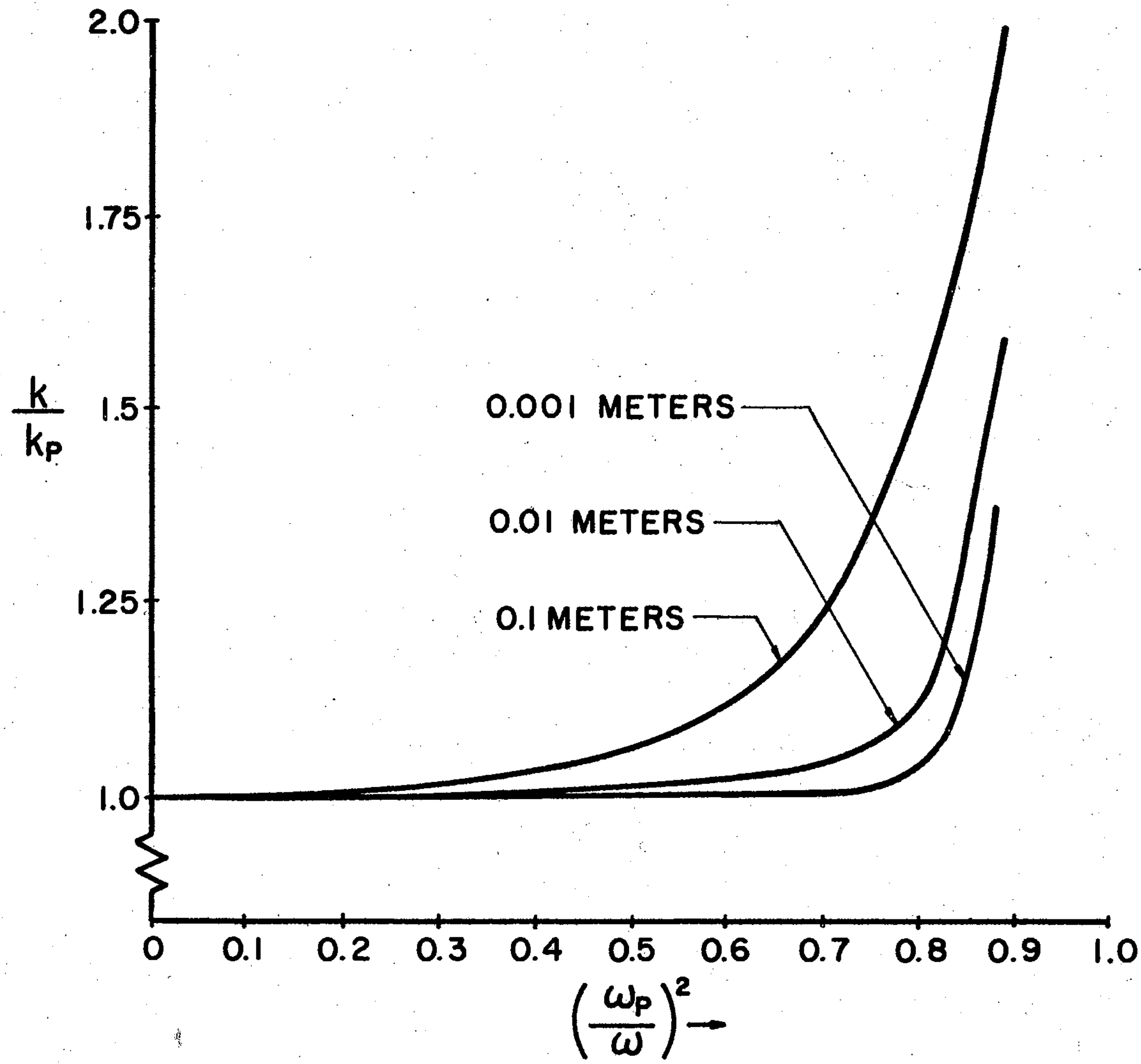

Fig. 4 - VARIATIONS IN THE RATIO OF k TO k_p.

CHAPTER V

RADIATION FROM A CYLINDRICAL DIPOLE

Radiation Zone Fields

Consider a thin cylindrical dipole situated in a compressible plasma as shown in Figure 5. The medium external to the antenna is similar to the plasma described previously. For purposes of calculating the radiation field, we assume that the antenna is a current filament with a distribution having the form,

$$\vec{J}^s = J_o \operatorname{Sin} k \left(h - |z|\right) e^{i\omega t} \vec{a}_z \quad , \tag{4.1}$$

where $\bar{a}_z$ is a unit vector in z-direction, with the related charge distribution,

$$\begin{aligned} \rho^s &= \left(\frac{-ik}{\omega}\right) J_o e^{i\omega t} \operatorname{Cos} k (h - z) \qquad z > 0 \, , \\ &= \left(\frac{ik}{\omega}\right) J_o e^{i\omega t} \operatorname{Cos} k (h + z) \qquad z < 0. \end{aligned} \tag{4.2}$$

The wave number k will be chosen to satisfy the relations previously discussed in the wave analysis.

Solutions for the vector potentials and electron density may now be written as

$$n(\vec{r}) = - \frac{\omega_p^2}{e\, v_o^2\, 4\pi} \int_{v'}^{s} \rho(\vec{r'}) \frac{e^{-ik\, R}}{R} dv' \quad , \tag{4.3}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}^s(\vec{r'}) e^{-ik_o R}}{R} dv' \quad , \tag{4.4}$$

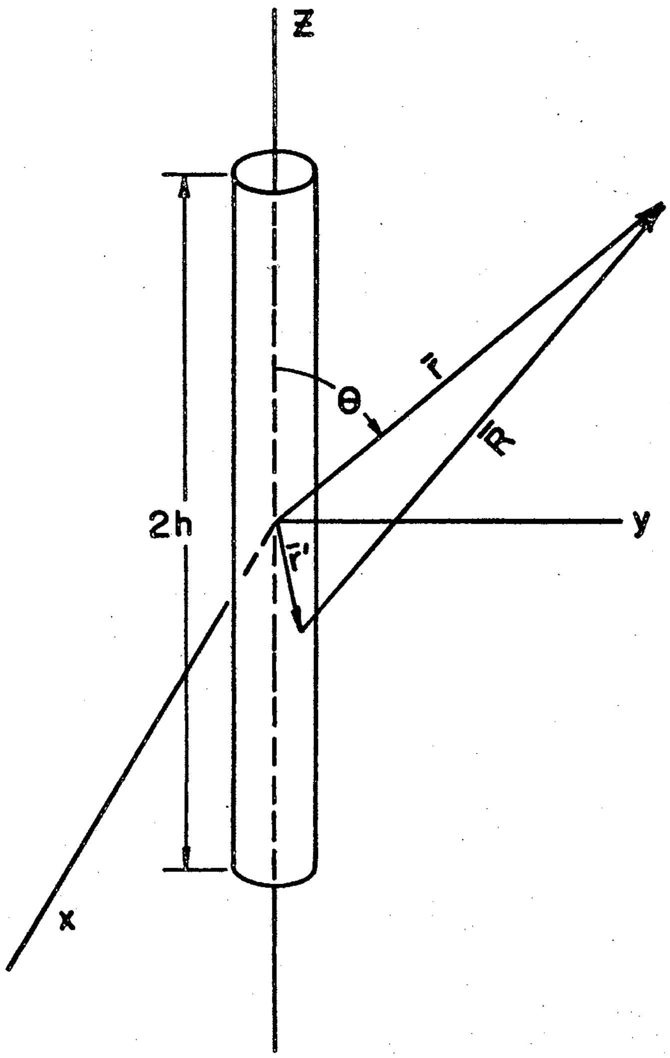

Fig. 5 - SPHERICAL COORDINATE SYSTEM.

where $R = \left|\vec{r} - \vec{r}'\right|$.

Evaluation of the integral in Equation 4.3 and Equation 4.4 for the radiation zone is well known. Therefore, only the results are given below.

$$n(\vec{r}) = -\frac{k_p k \omega^2 I_o}{e v_o^2 2\pi} \frac{e^{i(\omega t - k_p r)}}{r} P(\theta, k_p) \cos\theta , \quad (4.5)$$

$$A_z(\vec{r}) = \frac{-\mu_o k I_o}{2\pi} \frac{e^{i(\omega t - k_o r)}}{r} P(\theta, k_o) , \quad (4.6)$$

where

$$P(\theta, k_p) = \frac{\cos kh - \cos(k_p h \cos\theta}{k^2 - k_p^2 \cos^2\theta} ,$$

$$P(\theta, k_o) = \frac{\cos kh - \cos(k_o h \cos\theta)}{k^2 - k_o^2 \cos^2\theta} ,$$

$$I_o = J_o \pi b^2 .$$

The radius of the cylindrical antenna is such that $b \ll h$.

The radiation fields are then obtained as

$$\vec{E}_p = -\frac{ik k_p^2 (\epsilon_1 - 1) I_o}{\omega \epsilon_o \epsilon_1 2\pi} \frac{e^{i(\omega t - k_p r)}}{r} P(\theta, k_p) \cos\theta \, \vec{a}_r , \quad (4.7)$$

$$\vec{E}_o = -\frac{i\omega \mu_o k I_o}{2\pi} \frac{e^{i(\omega t - k_o r)}}{r} P(\theta, k_o) \sin\theta \, \vec{a}_\theta , \quad (4.8)$$

$$\vec{H}_o = -\frac{ik k_o I_o}{2\pi} \frac{e^{i(\omega t - k_o r)}}{r} P(\theta, k_o) \sin\theta \, \vec{a}_\phi ,$$

where $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{a}_\phi$ are unit vectors in a spherical coordinate system.

Characteristics of the Radiation Fields

Before making calculations pertaining to the radiation field, it is desirable to compare the relative values of the parameters occurring in the field expressions.

From the previous calculation we find

$$\frac{k_p^2}{k_o^2} = \frac{\omega^2/v_o^2 \; (1 - \omega_p^2/\omega^2)}{\omega^2/c^2 \; (1 - \omega_p^2/\omega^2)} = c^2/v_o^2 \gg 1.$$

For $\omega > \omega_p$ we also found that $k^2 \geq k_p^2$, therefore,

$$\omega > \omega_p \quad k^2 \geq k_p^2 \gg k_o^2 \, .$$

However, for $\omega < \omega_p$ we find that both k_p and k_o are negative imaginary and hence no power is radiated by the antenna. The following analysis shall be restricted to the frequency spectra above the plasma frequency.

From the data plotted in Figures 2, 3 and 4, we find that the wave number associated with the current distribution is approximately equal to k_p. Therefore, we have

$$P\,(\theta,\, k_p) \doteq \frac{\text{Cos}\; k_p h - \text{Cos}\; (k_p h\; \text{Cos}\; \theta)}{k_p^2\; \text{Sin}^2\; \theta} \, ,$$

$$P\,(\theta,\, k_o) \doteq \frac{\text{Cos}\; k_p h - \text{Cos}\; (k_o h\; \text{Cos}\; \theta)}{k_p^2}$$

Radiation From a Short Electric Dipole

If we assume $k_p h \ll 1$, the expressions for the radiation fields become:

$$\vec{E}_p = \frac{ik_p^3 (\epsilon_1 - 1) I_o h^2}{4\pi \epsilon_o \epsilon_1} \frac{e^{+i(\omega t - k_p r)}}{r} \cos\theta \, \vec{a}_r , \quad (4.15)$$

$$\vec{E}_o = \frac{i\omega \mu_o k_p I_o h^2}{4\pi} \frac{e^{i(\omega t - k_o r)}}{r} \sin\theta \, \vec{a}_\theta ,$$

$$H_o = \frac{ik_p k_o I_o h^2}{4\pi} \frac{e^{i(\omega t - k_o r)}}{r} \sin\theta \, \vec{a}_\phi .$$

The ratio of the transverse electric field to longitudinal electric field, from Equation 4.15, becomes

$$\frac{E_\theta}{E_r} = \left(\frac{\omega_p v_o}{\omega c}\right)^2 \tan\theta .$$

Since $v_o/c \simeq 10^{-3}$, we see that the radial field is many orders of magnitudes larger than the transverse field for frequencies of the order of the plasma frequency, excluding the plane $\theta = \pi/2$.

Radiation Resistance Expressions

The time-average power density vector, as derived as equation C.14 in Appendix C, is

$$S_r = \tfrac{1}{2} R_e \left[(\bar{E}_o \times \bar{H}) \circ \bar{a}_r + \frac{i v_o^2 \omega m \epsilon_o}{N_o e} n \, \bar{E}_p \circ \bar{a}_r \right]. \quad (4.10)$$

The total power radiated by the antenna will be

$$P_r = \int_o^{2\pi} \int_o^{\pi} S_r r^2 \sin\theta \, d\theta d\phi , \text{ or}$$

$$P_r = \int_o^{2\pi} \int_o^{\pi} S_{or} r^2 \sin\theta \, d\theta d\phi + \int_o^{2\pi} \int_o^{\pi} S_{pr} r^2 \sin\theta \, d\theta d\phi .$$

The radiation resistance of the antenna is defined as

$$\frac{(I_o^{\,2} \sin^2 kh)}{2} R_r = P_r \quad .$$

Therefore R_r can be expressed as

$$R_r = \left(\frac{Z_o}{2\pi \sin^2 kh}\right)\left(\frac{k}{k_o}\right)^2 \int_0^{\pi} P^2 (\theta, k_o) \sin^3 \theta d\theta$$

$$+ \left(\frac{Z_o}{2\pi \sin^2 kh}\right)\left(\frac{k}{k_o}\right)^2 \left(\frac{v_o}{c}\right)\left(\frac{\omega_p}{\omega}\right)^2 \int_0^{\pi} P^2 (\theta', k_p) \cos^2 \theta \sin\theta \, d\theta, \quad (4.11)$$

where $Z_o = \dfrac{120\pi}{\sqrt{1 - (\frac{\omega_p}{\omega})^2}}$.

From Equation 4.11 one can readily see that R_r contains an optical and a plasma component. Therefore we define R_r as

$$R_r = R_{ro} + R_{rp} \quad , \qquad (4.12)$$

R_{ro} = optical component,

R_{rp} = plasma components,

where

$$R_{ro} = \frac{Z_o}{2\pi \sin^2 kh}\left(\frac{k}{k_o}\right)^2 \int_0^{\pi} P^2 (\theta, k_o) \sin^3 \theta d\theta \quad , \qquad (4.13)$$

$$R_{rp} = \frac{Z_o}{2\pi \sin^2 kh}\left(\frac{k}{k_o}\right)^2 \left(\frac{v_o}{c}\right)\left(\frac{\omega_p}{\omega}\right) \int_0^{\pi} P^2 (\theta, k_p) \cos^2 \theta \sin\theta \, d\theta. \quad (4.1l$$

Radiation Resistance of a Dipole

The radiation resistance of the short dipole will consist of two components. These are:

$$R_r = R_{ro} + R_{rp} \ , \tag{4.15}$$

where

$$R_{ro} = \frac{Z_o \left[\frac{v_o}{c}\right]^2 k_p{}^2 h^2}{6\pi} \ , \ \text{(optical)}$$

$$R_{rp} = \frac{Z_o \ (c/v_o) \ (\frac{\omega_p}{\omega})^2 \ k_p{}^2 \ h^2}{12\pi} \ , \ \text{(plasma)}$$

$$Z_o = \frac{120\pi}{\sqrt{1 - \omega_p{}^2/\omega^2}} \ .$$

Calculating the radiation resistance for the half-wave dipole we find:

$$R_r = R_{ro} + R_{rp} \ ,$$

where

$$R_{ro} \doteq \frac{2 \ Z_o}{3\pi} \left(\frac{v_o}{c}\right)^2 = \text{optical component,}$$

$$R_{rp} \doteq \frac{Z_o}{4\pi} \ \frac{c}{v_o} \left(\frac{\omega_p}{\omega}\right)^2 = \text{plasma component,}$$

$$Z_o = 120\pi \ [1 - \frac{\omega_p{}^2}{\omega^2}]^{-\frac{1}{2}} \ .$$

Therefore, we may write:

$$R_{ro} \doteq 80 \ (v_o/c)^2 \ (1 - \omega_p{}^2/\omega^2)^{-\frac{1}{2}} \ , \tag{4.17}$$

$$R_{rp} \doteq 30 \ (c/v_o) \left(\frac{\omega_p}{\omega}\right)^2 \ (1 - \frac{\omega_p{}^2}{\omega^2})^{-\frac{1}{2}} \ . \tag{4.18}$$

As an example of the above results, we consider the case for which $c/v_o = 10^3$. Plots of the radiation resistance component are given in Figure 7. Equation 4.17 and 4.18 are valid for the region

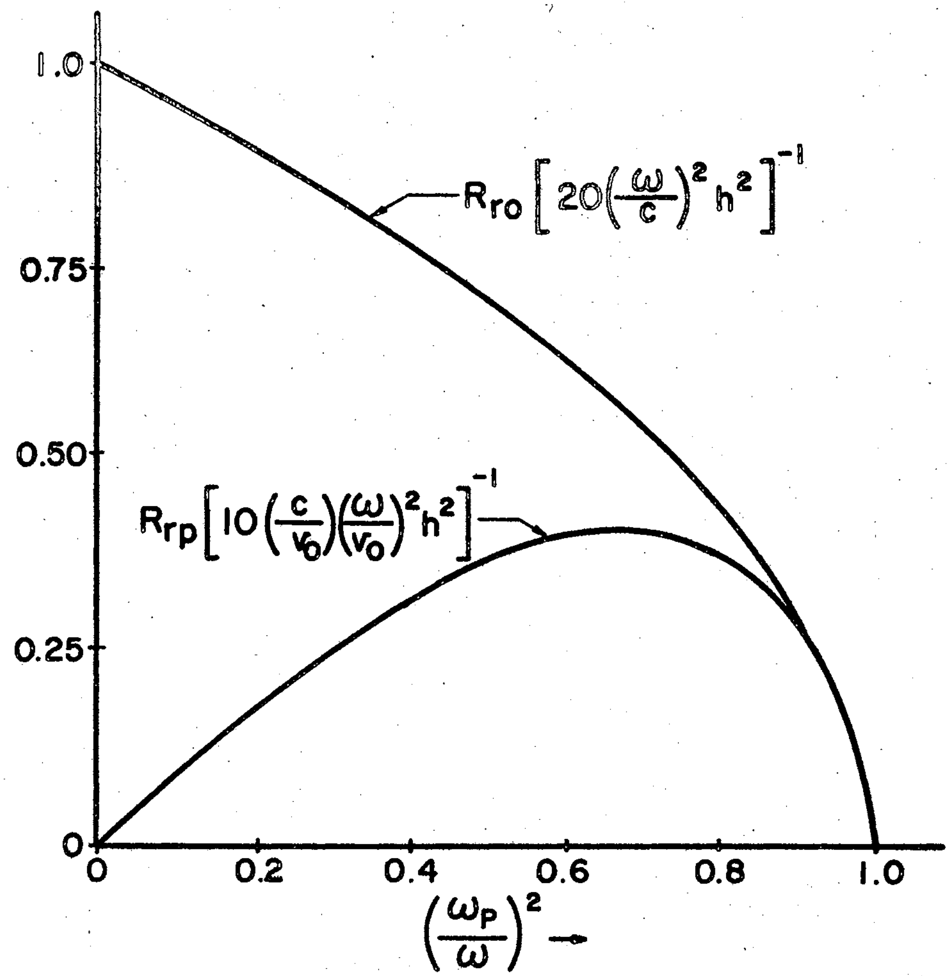

Fig. 6—RADIATION RESISTANCE OF A SHORT DIPOLE.

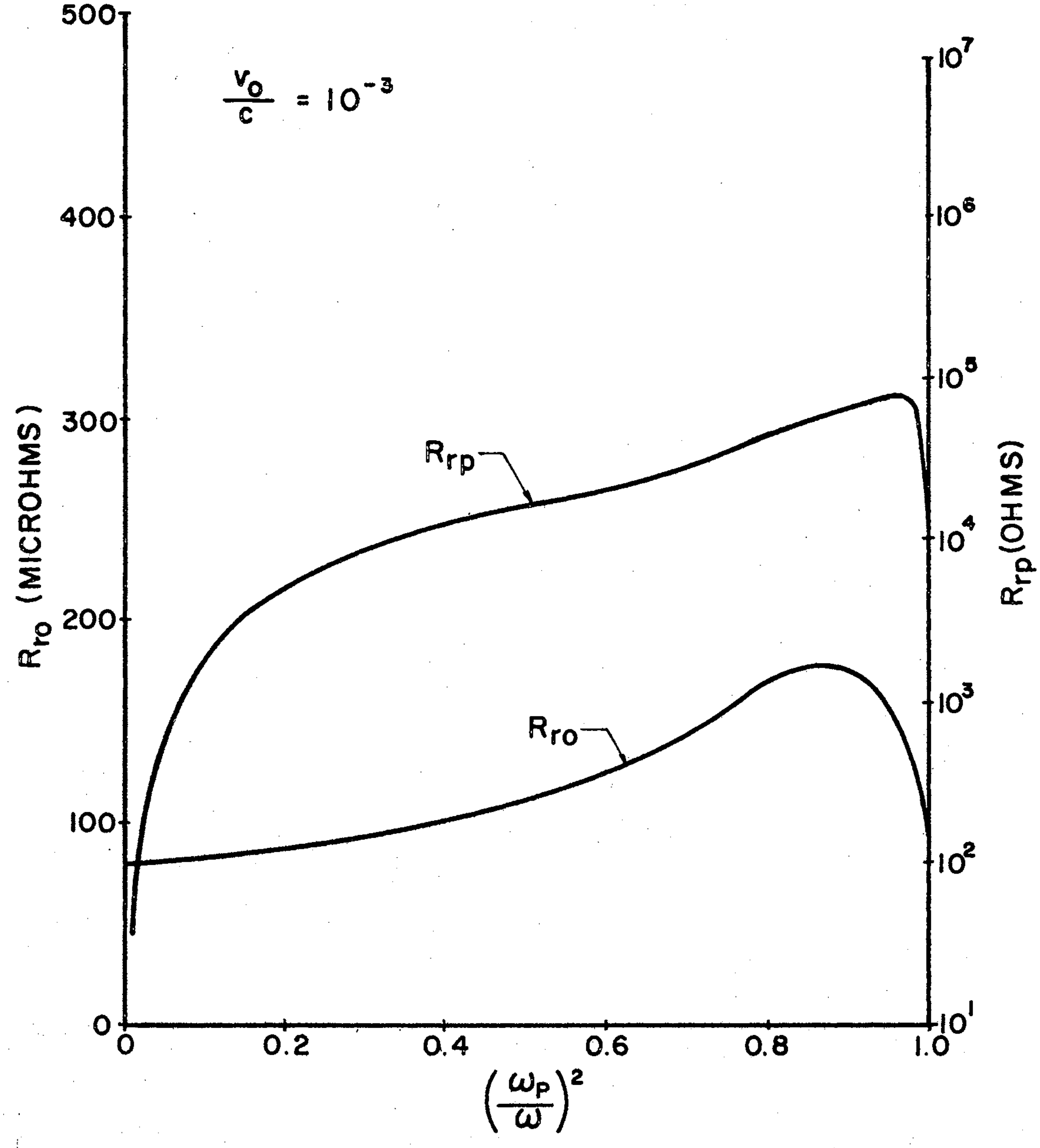

Fig. 7 - RADIATION RESISTANCE OF A HALF-WAVE DIPOLE.

$(\frac{\omega_p}{\omega})^2 \leq 0.8$. As the excitation frequency approaches the plasma frequency, $k \rightarrow \omega_p/\sqrt{2}\, v_o$ as indicated in Figures 2 and 3.

Radiation Resistance for General kh

Because Equations 4.17 and 4.18 are only valid when $k \doteq kp$ or $0 \leq x \leq .8$, it is very desirable to obtain expressions for R_{ro} and R_{rp} for general k and general length. To do this task, one must integrate Equations 4.13 and 4.14 for arbitrary kh. Looking first at R_{rp}, we may make a transformation of variable of $U = \cos\theta$ to produce

$$R_{rp} = \frac{Z_o}{2\pi \sin^2 kh} \left(\frac{k}{k_o}\right)^2 \left(\frac{v_o}{c}\right) \times \int_{-1}^{+1} \left[\frac{\cos kh - \cos(k_p h U)}{U^2 - (k/k_p)^2}\right]^2 U^2 dU .$$

Letting $\dfrac{U^2}{[U^2 - (k/k_p)^2]}$ be expanded in partial fractions, and integrating, we find that R_{rp} may be expressed as

$$R_{rp} = \frac{Z_o}{2\pi \sin^2 kh} \left(\frac{k}{k_o}\right)^2 \left(\frac{v_o}{c}\right) \times I ,$$

where

$$I = \frac{[\cos a - \cos \frac{a}{d}]^2}{d^2 - 1} + [\mathrm{Si}(A) - \mathrm{Si}(B)] [\frac{a}{d} \cos^2 a + \frac{\sin 2a}{2d}]$$

$$+ [\mathrm{Cin}(A) - \mathrm{Cin}(B)] [\frac{a}{2d} \sin 2a - \frac{\cos^2 a}{d}]$$

$$+ [\mathrm{Si}(2A) - \mathrm{Si}(2B)] [\frac{-a}{2d} \cos 2a - \frac{\sin 2a}{4d}]$$

$$+ [\mathrm{Cin}(2A) - \mathrm{Cin}(2B)] [\frac{-a}{2d} \sin 2a + \frac{\cos 2a}{4d}] , \qquad (4.19)$$

$$a = kh, \quad A = a + a/d,$$

$$d = k/k_p, \quad B = a - a/d,$$

and $$Si(A) = \int_o^A \frac{\sin x}{x}\, dx, \quad Cin(A) = \int_o^A \frac{1 - \cos x}{x}\, dx.$$

Si(A) and Cin(A) are tabulated integrals whose values may be found in Jahnke and Emde (1945). The expression for R_{rp} is plotted in Figure 8.

Since $k \gg k_o$, the ratio of $\frac{k}{k_o}$ is very small, and we may apply small argument approximations to $\cos(k_o h \cos\theta)$. Letting

$$\cos(k_o h \cos\theta) \doteq 1 - \frac{(k_o h)^2}{2} \cos^2\theta ,$$

Equation 4.13 becomes

$$R_{ro} = \frac{60}{\sqrt{1 - x}} \frac{1}{\sin^2 a} \left(\frac{v_o}{c}\right)^2 \frac{1}{d^2} \int_o^{\pi} \left[(1 - \cos a) - \frac{(k_o h)^2}{2} \cos^2\theta\right]^2 \sin^3\theta d\theta ,$$

or

$$R_{ro} = \frac{60}{\sqrt{1 - x}} \frac{1}{\sin^2 a} \left(\frac{v_o}{c}\right)^2 \frac{1}{d^2} \left[\frac{4}{3}(1 - \cos a)^2 - \frac{4}{15}(k_o h)^2 (1 - \cos a) + \frac{(k_o h)^4}{35}\right] . \tag{4.20}$$

Equation 4.20 is plotted in Figure 9.

When $kh = \pi/2$, the antenna length becomes one-half wave length, and Equations 4.19 and 4.20 become

$$R_{rp} = \frac{60\, d^2 x}{\sqrt{1 - x}} \left(\frac{c}{v_o}\right) \left[\frac{\cos^2 \frac{\pi}{2d}}{d^2 - 1} \right.$$

$$+ \frac{\pi}{4d} \left[\mathrm{Si}\left(\pi + \frac{\pi}{d}\right) - \mathrm{Si}\left(\pi - \frac{\pi}{d}\right)\right]$$

$$\left. - \frac{1}{4d} \left[\mathrm{Cin}\left(\pi + \frac{\pi}{d}\right) - \mathrm{Cin}\left(\pi - \frac{\pi}{d}\right)\right] \right] , \qquad (4.21)$$

$$R_{ro} = 80 \left(\frac{v_o}{c}\right)^2 \frac{1}{d^2\sqrt{1 - x}} \; . \qquad (4.22)$$

When $k = k_p$, these expressions reduce down to Equations 4.17 and 4.18. It can easily be demonstrated that at $x \doteq 1$, when $a^2 >> 1$, both components of the radiation resistance go to zero as the square root of $(1 - x)$. The radiation resistance for the region $0.8 \leq \left(\frac{\omega_p}{\omega}\right)^2 \leq 1$ is plotted in Figure 7.

The graphs of the plasma and optical components as shown in Figures 8 and 9 demonstrate that there exists a pole at multiples of $kh = \pi$. This behavior is consistant with the linear antenna approximations we have used in this paper. At multiples of π there is a current zero which results in a radiation resistance pole at the input of the antenna, the reference point for impedance calculations. However, a second order theory, such as the King-Middleton (1956) approximation which takes into account gap and end effects, predicts that a pole does not exist for antennas with finite radius to length ratios but approaches a very high value. Further use of the King-Middleton theory for long cylindrical antennas (Gooch, Harrison, King, and Wu, 1963; Iizuka, King and Prasad, 1962) has shown that for free space the maximum in the radiation resistance occur at values of kh which are somewhat less than multiples of π.

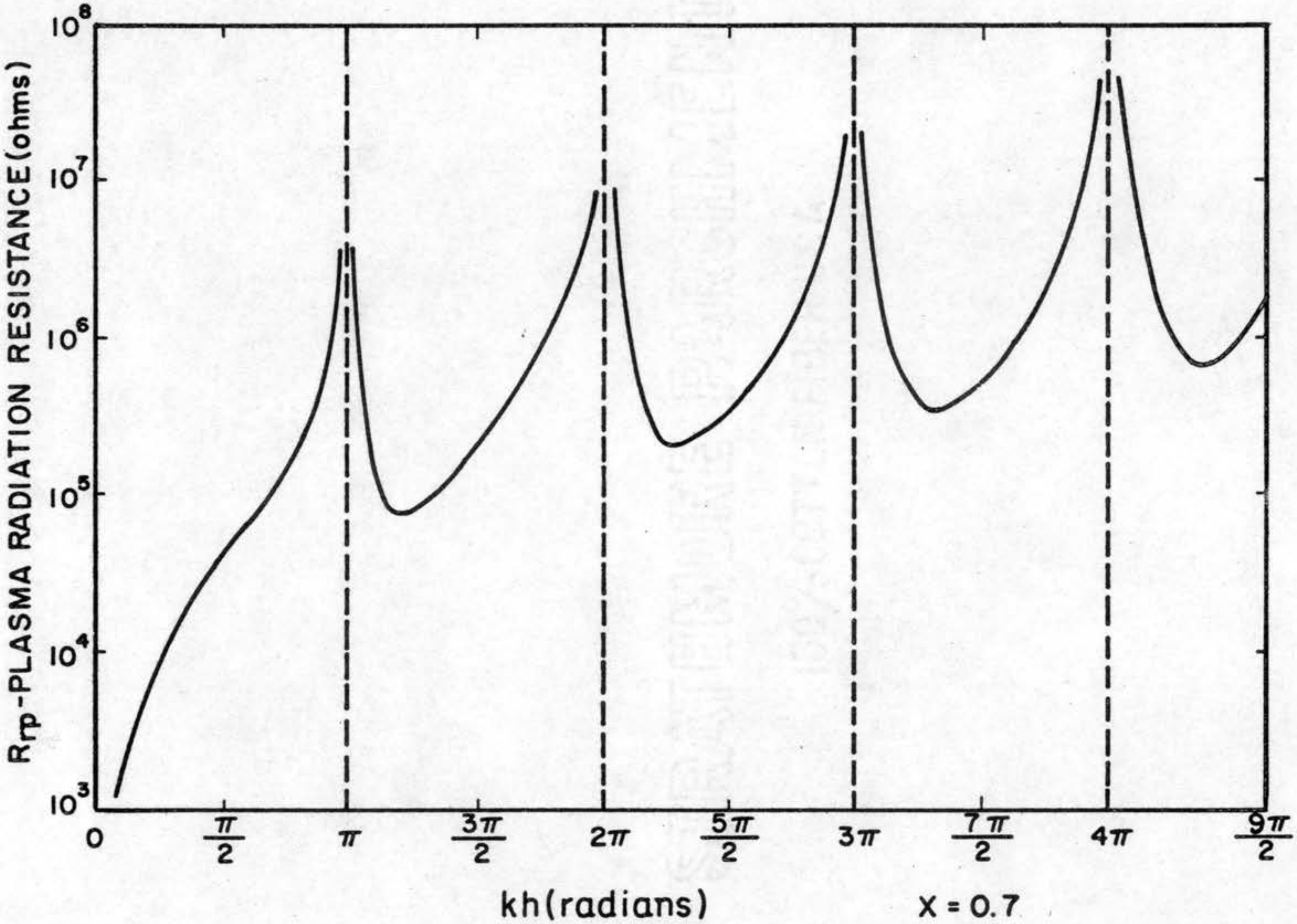

Fig. 8 - PLASMA RADIATION RESISTANCE FOR GENERAL kh.

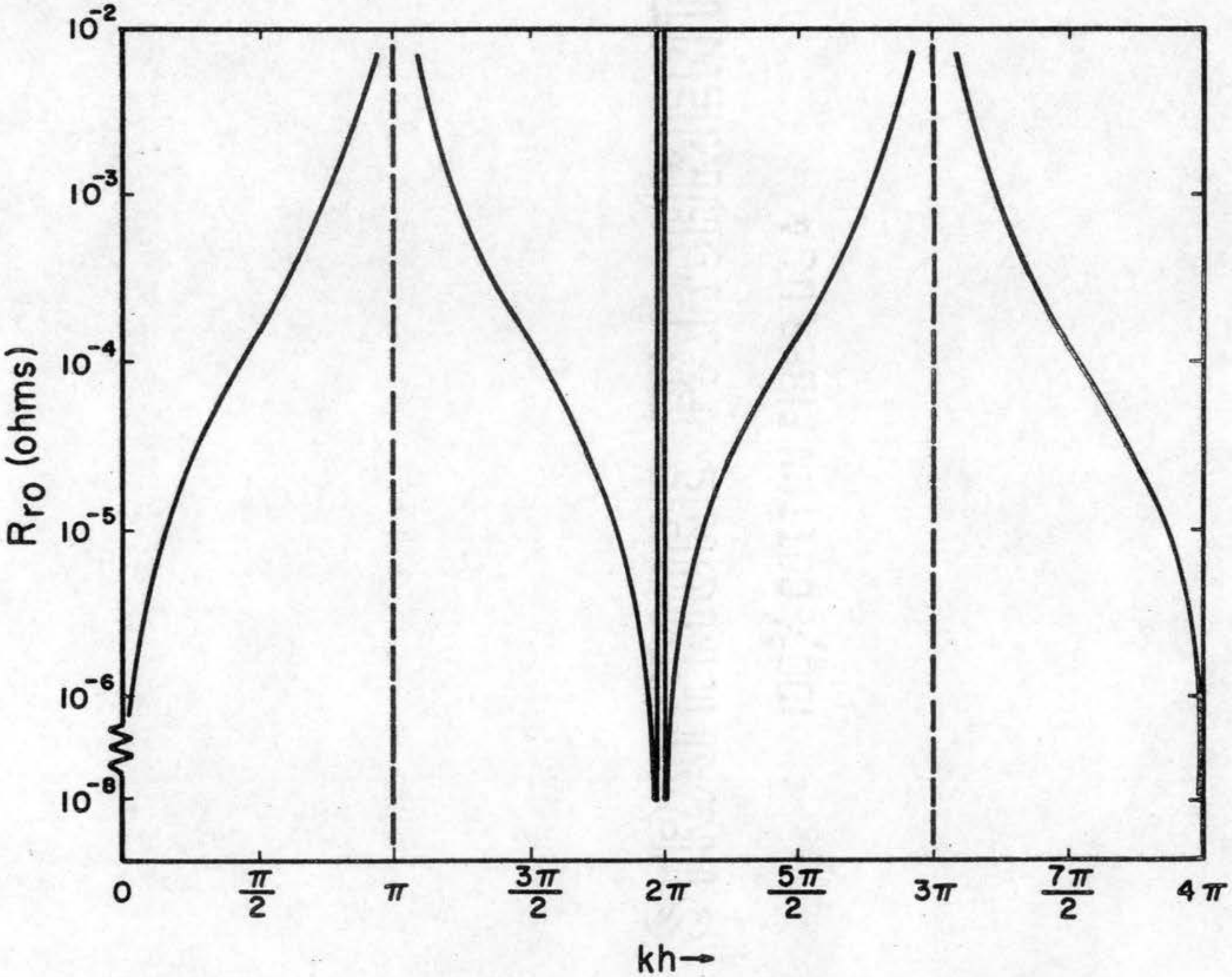

Fig. 9-OPTICAL RADIATION RESISTANCE FOR GENERAL kh.

But the calculations used in this paper for the impedance are still a good approximation neglecting the pole behavior.

Electrical Wavelength

In calculating the electrical length of the antenna, we found that the wavelength measure for the antenna was related to the acoustical wavelength, which is many orders less than the optical wavelength. However, as the exciting frequency approaches the plasma frequency, the phase velocity associated with the current distribution begins to diverge from the plasma plane-wave velocity, as shown in Figures 2, 3 and 4. The plane-wave velocity goes to infinity at the plasma frequency while the current distribution velocity has the value of $\omega/\sqrt{2}\ v_o$ where v_o is the electron acoustic velocity of the plasma. Therefore, as the excitation frequency approaches the plasma frequency one must use the appropriate value of the wave number given in the previous analysis. At frequencies large compared to the plasma frequency, the wavelengths may be approximated as:

$$\lambda_p \doteq \left(\frac{2\pi\, v_o}{\omega}\right) \left(1 + \frac{\omega_p^2}{2\omega^2}\right) , \qquad (4.23)$$

$$\lambda_o \doteq \frac{2\pi\, c}{\omega} \left(1 + \frac{\omega_p^2}{2\omega^2}\right) , \qquad (4.24)$$

provided $\frac{\omega_p}{\omega} << 1$.

Since $c \simeq 10^3\ v_o$ in most applications, the electrical length of a given antenna will be at least three orders of magnitude longer than the same antenna in free space.

CHAPTER VI

CONCLUSIONS AND SUMMARY

The radiation of an electric dipole has been investigated for both the case of a short antenna and a half-wave antenna. However, consideration was given to the current distribution on the antenna, and it was found that for frequencies above the plasma frequency the phase velocity of the current distribution is nearly equal to the electron acoustic velocity. Therefore, the electrical length of the antenna becomes much larger than in free space. Therefore, the radiation characteristics of a given antenna in the compressible plasma is radically different than predicted from incompressible plasma theory.

The results also show that the acoustical electric field is dominant in the radiation zone, except in the region of space broadside to the antenna.

In the introduction, we briefly reviewed Wait's (1965b) paper on the infinitely long cylindrical antenna excited by a circumferential gap. Wait considers the antenna as a boundary value problem and assumes that the fields and the current distribution are produced by a specified voltage at the antenna terminals. However, he found that in evaluating the integrals for the radiation zone fields by saddle point methods the "pole" or electroacoustic waves are not significant at large distances away from the antenna. The electroacoustic waves excited are evanescent in the radial direction, and therefore, Wait concludes that they play no part in the radiation process. Wait also finds that the diameter of the cylinder must be on the order of a Debye length for his analysis to be valid.

In our own analysis we found that the plasma mode radiation fields produced essentially an end fire radiation pattern. As the antenna becomes longer, the majority of the radiation in the plasma mode becomes concentrated in a cone centered along the axis of the antenna. In the limiting case of the infinite antenna, the solid angle of cone would approach zero, and all the energy of the plasma mode would be directed along the cylinder. This reasoning would seem to indicate that radial evanescent character of the plasma mode radiation as found by Wait is correct. Secondly we know that since the finite structure creates standing waves there would be radiation of both modes from the antenna. Thus by these arguments one might conclude that one could not always extrapolate the results of the infinite antenna, as opposed to the infinite cylinder, to the finite cylindrical antenna.

Radiation resistance calculations show that the optical component is so small that it can be ignored in most instances and that the plasma component is several orders of magnitude greater. The large value for the plasma radiation resistance indicates that most of the power is going into the electroacoustic mode.

It is concluded that the compressibility is important for frequency spectrum above the plasma frequency. However, below the plasma frequency, the phase velocity of the current distribution approaches the phase velocity of free space; and hence the impedance characteristics should closely approximate the free space predictions. In contrast, the electrical length may be as much as four orders of magnitudes greater than in free space. Indeed, the radiation characteristics would differ from free space calculations.

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APPENDIX A

LINEARIZATION OF BASIC EQUATIONS

Although the basic set of equations describing wave motion through an isotropic compressible plasma containing ℓ ion species is nonlinear, thereby making the basic set virtually intractable, it may be easily linearized by perturbation techniques. The basic set of equations are:

$$\nabla \times \overline{H} - \epsilon_o \frac{\partial \overline{E}}{\partial t} = \sum_{i=1}^{\ell} n_i \, q_i \, \overline{V}_i \tag{A.1}$$

$$\nabla \times \overline{E} + \mu_o \frac{\partial \overline{H}}{\partial t} = 0 \tag{A.2}$$

$$\frac{\partial N_i}{\partial t} + \nabla \cdot (N_i \cdot \overline{V}_i) = 0 \tag{A.3}$$

$$N_i \, M_i \left[\frac{\partial \overline{V}_i}{\partial t} + (\overline{V}_i \cdot \nabla) \, \overline{V}_i \right] = N_i \, q_i \, [\overline{E} + \mu_o \, (\overline{V}_i \times \overline{H})] - \nabla_{Pi}, \tag{A.4}$$

where

P_i = ith ion specie pressure

N_i = ith ion specie density,

q_i = ith ion specie charge,

$\overline{V}_i$ = ith ion specie velocity,

M_i = ith ion specie mass ,

$\overline{E}, \overline{H}$ = electric and magnetic field .

The perturbation procedure which will be employed in the linearization of (A.1), (A.3) and (A.4) states that the ith ion specie density and velocity may be expressed as the sum of a steady state value and a very small magnitude term which is a function of time. Thus

$$N_i = N_{oi} + n_i \,,$$
$$\overline{V}_i = \overline{V}_{oi} + \overline{V}_{1i} \,, \tag{A.5}$$

where

$$\frac{\partial N_{oi}}{\partial t} = 0 \qquad \frac{\partial \overline{V}_{oi}}{\partial t} = 0, \tag{A.6}$$

$$\left|N_{oi}\right| \gg \left|n_i\right| \qquad \left|\overline{V}_{oi}\right| \gg \left|\overline{V}_{1i}\right| \quad . \tag{A.6}$$

If we assume that the ideal gas laws apply subject to adiabatic gas conditions, then one has

$$P_i = N_i \; KT \qquad T = T_o \left(\frac{N_i}{N_{oi}}\right)^{\gamma - 1} \,, \tag{A.7}$$

where

T = temperature of the plasma,

T_o = steady state plasma temperature,

γ = specific heat ratio,

K = gas constant.

Simplifying A.7 results in

$$P = N_i \; KT_o \left(\frac{N_i}{N_{oi}}\right)^{\quad - 1} \quad . \tag{A.8}$$

Substituting the perturbation relations of Equation A.5 into Equations A.1, A.3, A.4 and A.5 produces, after deleting all nonlinear terms containing products of $\overline{V}_{1i}$ and n_i, the following equations:

$$\nabla \times \overline{H} - \epsilon_p \frac{\partial \overline{E}}{\partial t} = \sum_{i=1}^{\ell} [N_{oi} \; q_i \; V_{1i} + n_i \; q_i \; \overline{V}_{oi}] \quad , \tag{A.9}$$

$$\frac{\partial n_i}{\partial t} + N_{oi} \nabla_o \overline{V}_{1i} = 0 , \tag{A.10}$$

$$N_{oi} M_i \frac{\partial \overline{V}_{1i}}{\partial t} = N_{oi} q_i \overline{E} - \nabla_p , \tag{A.11}$$

$$P_i = N_{oi} KT_o (1 + n_i/N_{oi}) . \tag{A.12}$$

Since by A.6

$$\left| \frac{n_i}{N_{oi}} \right| << 1 ,$$

Equation A.12 may be approximated by

$$P_i \doteq N_{oi} KT_o (1 + \gamma n_i/N_{oi}) ,$$

or

$$P_i = N_{oi} KT_o + \gamma n_i KT_o . \tag{A.13}$$

Assuming that the dynamic perturbation fields have three degrees of freedom, then $\gamma = 3$. If the R.M.S. acoustical velocity for the ith ion species is defined as

$$v_{oi}^2 = \frac{3KT_o}{M_i} , \tag{A.14}$$

then Equation A.13 becomes

$$P_i = N_{oi} KT_o + M_i v_{oi}^2 n_i . \tag{A.15}$$

Since our plasma is described by Maxwell-Boltzmann statistics and hence has no drift velocity, $\overline{V}_{oi} = 0$. Therefore our set of linearized equations is:

$$\nabla \times \overline{H} - \epsilon_o \frac{\partial \overline{E}}{\partial t} = \sum_{i=1}^{\ell} N_{oi} q_i \overline{V}_{1i} ,$$

$$\nabla \times \overline{E} + \mu_o \frac{\partial \overline{H}}{\partial t} = 0 \; ,$$

$$\frac{\partial n_i}{\partial t} + N_{oi} \, \nabla \cdot \overline{V}_{1i} = 0 \; , \qquad \text{(A.16)}$$

$$M_i \, N_{oi} \frac{\partial \overline{V}_{1i}}{\partial t} = N_{oi} \, q_i \, \overline{E} - M_i \, v_{oi}^{\;2} \, \nabla \, n_i \; ,$$

$$P_i = N_{oi} \, KT_o \circ M_i \, v_{oi}^{\;2} \, n_i \; .$$

The first two equations of A.16 are the two Maxwell equations, the third is the continuity equation, the fourth is the force equation, and the last expression is the equation of state.

For a one fluid electron gas, Equations A.16 reduce to:

$$\nabla \times \overline{H} - \epsilon_o \frac{\partial \overline{E}}{\partial t} = N_o \, e \, \overline{V} \; ,$$

$$\nabla \times \overline{E} + \mu_o \frac{\partial \overline{H}}{\partial t} = 0 \; ,$$

$$\frac{\partial n}{\partial t} + N_o \, \nabla \cdot \overline{V} = 0 \; , \qquad \text{(A.17)}$$

$$m \, N_o \frac{\partial \overline{V}}{\partial t} = N_o \, e \, \overline{E} - m \, v_o^{\;2} \, \nabla \, n \; ,$$

$$P = N_o \, KT_o + m \, v_o^{\;2} \, n \; .$$

APPENDIX B

HANKEL RATIO FUNCTION APPROXIMATION

It is well known that the ratio

$$\left| \frac{H_o^{(2)}\ (-\ ix)}{H_1^{(2)}\ (-\ ix)} \right|$$

as demonstrated in Figure 10, for large x approaches 1 and for small x approaches $-x\ \ell n\ \frac{\gamma x}{2}$, where $\gamma = 1.781$. For the region $.5 \leq x \leq 10$, where the two approximations are not valid, other approximations such as an asymptotic series could be used. However in programming a computer solution of the determinantal equation it was found much more advantageous to use a single simple algebraic expression, valid for all arguments, rather than use several different expresions. The following discussion outlines a procedure whereby one may derive such an expression.

When the

$$\mathrm{Log}_{10} \left| \frac{H_o^{(2)}\ (-i\ x)}{H_1^{(2)}\ (-i\ x)} \right|$$

is plotted against x on semi-logarithmic graph paper as shown in Figure 11, the curve looks very much like the type of curves analyzed by Bode plots (Van Valkenburg, 1962). For small arguments of x the curve approached an asymptote of

$$\log_{10} \left(\frac{x}{.4}\right)^{.825},$$

and for large arguments of x the curve approached $\log_{10} 1$ or zero. With these facts in mind, an approximation function was set up to be

$$\log_{10} \left(\frac{x}{.4}\right)^{.825} - \log_{10} \left(\left(\frac{x}{.4}\right)^{R} + 1\right)^{Q} ,$$

where $R \cdot Q$ must equat .825. At the breakpoint, at $x = .4$, the first term is zero, and the second term is $.3 \cdot Q$ and is set equal to the $\log_{10}$ of the Hankel function ratio at $x = .4$. From this procedure R and Q can both be determined, and the final form of approximation function is

$$\left| \frac{H_o^{(2)}\ (-i\ x)}{H_1^{(2)}\ (-i\ x)} \right| = \frac{\left(\frac{x}{.4}\right)^{.825}}{\left[\left(\frac{x}{.4}\right)^{.845} + 1\right]^{.975}} ,$$

which is at most three percent in error.

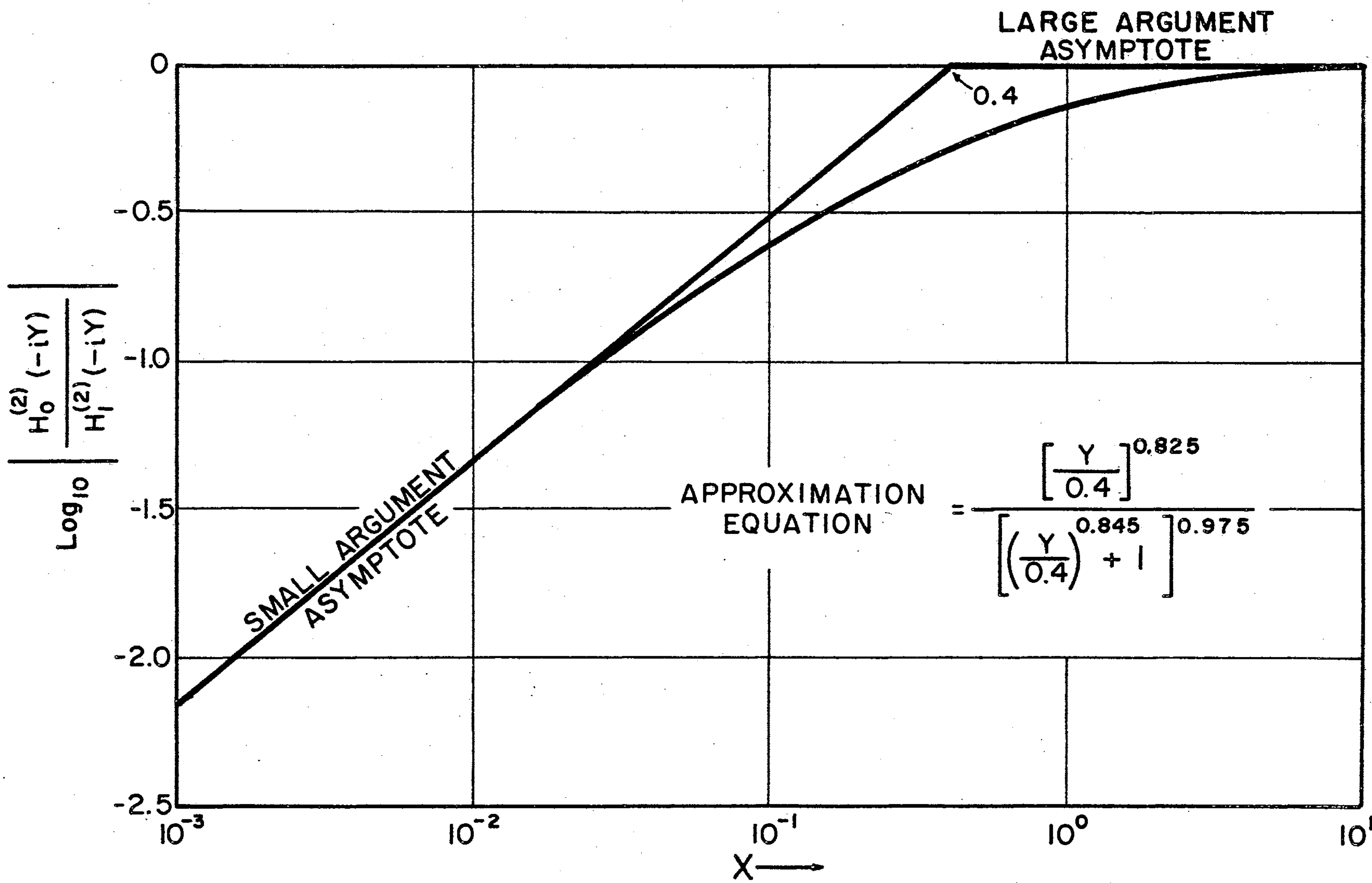

Fig. 10 - BODE PLOT OF THE HANKEL FUNCTION RATIO

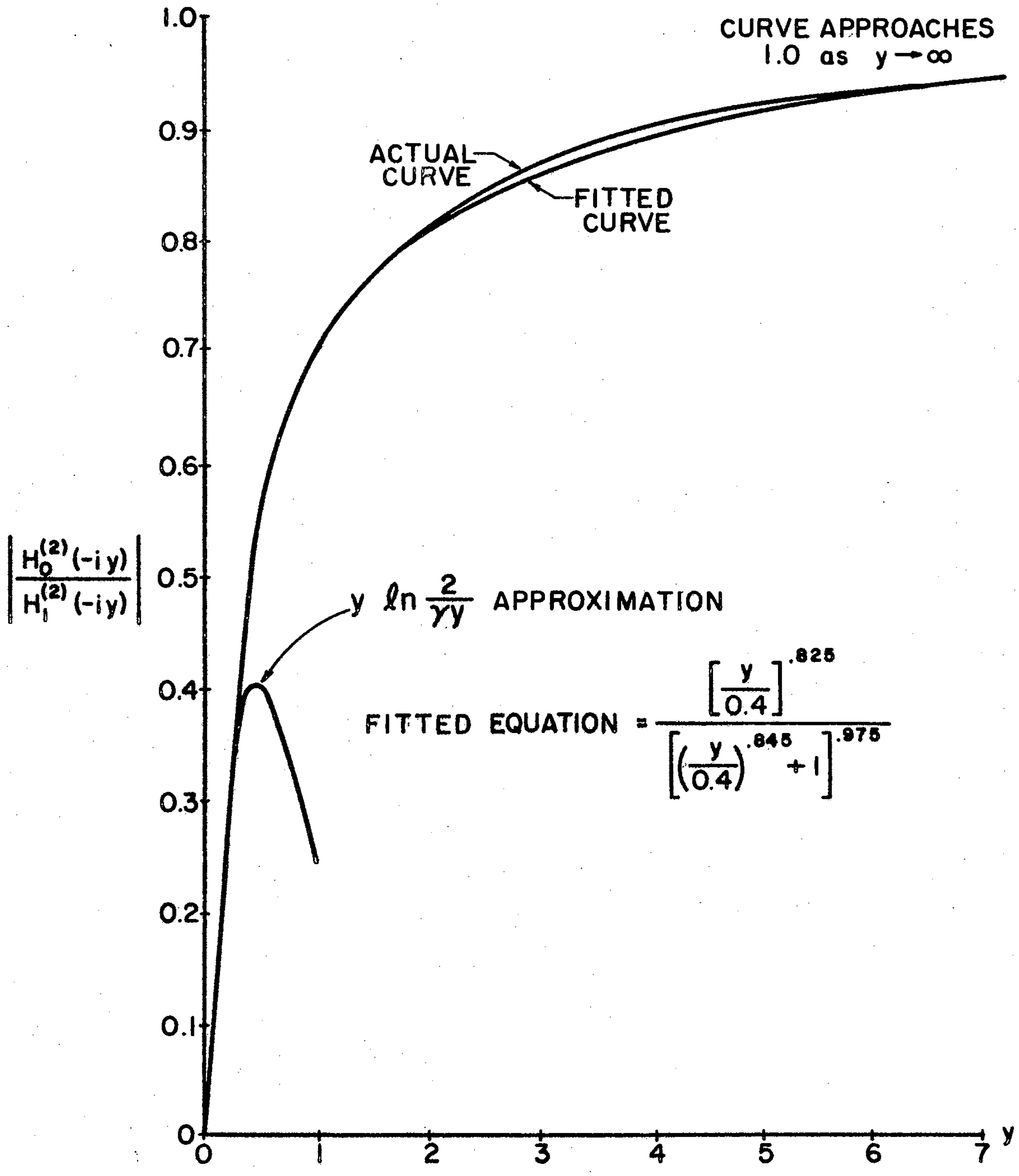

Fig. 11 - HANKEL FUNCTION RATIO APPROXIMATIONS

APPENDIX C

POWER RELATIONSHIPS

Scalar multiplication of Equation 1.2 by $\overline{H}^*$ where $(\ )^*$ denotes the complex conjugate gives, when $\overline{J}^s = 0$,

$$\overline{H}^* \cdot \nabla \times \overline{E} + i\omega\mu_o \overline{H}^* \cdot \overline{H} = 0 , \qquad (C.1)$$

and following the same procedure with $\overline{E}$ upon the conjugate of Equation 1.1 yields

$$\overline{E} \cdot \nabla \times \overline{H}^* - i\omega\epsilon_o \overline{E} \cdot \overline{E}^* = N_o e \overline{V}^* \cdot \overline{E} . \qquad (C.2)$$

Subtracting Equation C.2 from Equation C.1 and applying the identity

$$\nabla \cdot (\overline{E} \times \overline{H}^*) = \overline{H}^* \cdot \nabla \times \overline{E} - \overline{E} \cdot \nabla \times \overline{H}^* ,$$

one obtains

$$\nabla \cdot (\overline{E} \times \overline{H}^*) + N_o e \overline{E} \cdot \overline{V}^* = - i\omega [\mu_o \overline{H} \cdot \overline{H}^* - \epsilon_o \overline{E} \cdot \overline{E}^*] . \qquad (C.3)$$

Noting that from Equation 1.3,

$$N_o e \overline{E} \cdot \overline{V}^* = i\omega m N_o \overline{V} \cdot \overline{V}^* + m v_o^2 \overline{V}^* \nabla n , \qquad (C.4)$$

and that from Equation 1.5,

$$\nabla \cdot \overline{V}^* = + \frac{i\omega n^*}{N_o} , \qquad (C.5)$$

one can substitute Equation C.5 into the identity,

$$\overline{V}^* \cdot \nabla n = \nabla \cdot (n\overline{V}^*) - n \nabla \cdot \overline{V}^* , \qquad (C.6)$$

to produce

$$\overline{V}^* \cdot \nabla n = \nabla \cdot (n\overline{V}^*) \ \frac{i\omega n n^*}{N_o} . \qquad (C.7)$$

With the help of Equation C.7, Equation C.4 becomes

$$N_o\, e\, \overline{E}^* \cdot \overline{V}^* = i\omega m\, N_o\, \overline{V} \cdot \overline{V}^* - \frac{i\omega n n^*}{N_o} + m v_o\, \nabla \cdot (n\overline{V}^*)\ . \qquad (C.8)$$

Eliminating $\overline{E} \cdot \overline{V}^*$ between Equations C.8 and C.3, one finds the following form for Poynting's Theorem for compressible plasmas:

$$\nabla \cdot (\overline{E} \times \overline{H}^* + m v_o^2\, n\overline{V}^*) = -\, i\omega\, (\mu_o\, |\overline{H}|^2 - \epsilon_o\, |\overline{E}|^2 + \frac{|n|^2}{N_o} - m\, N_o\, |\overline{V}|^2), \qquad (C.9)$$

which shows that energy is transported both by electromagnetic and mechanical means. The time averaged representation for the Poynting vector is

$$\overline{S} = \frac{Re}{2}\, [\overline{E} \times \overline{H}^* + m v_o^2\, n\, \overline{V}^*]. \qquad (C.10)$$

But from 1.3

$$\overline{V}^* = \frac{ie}{\omega m}\, \overline{E}^* - i \left(\frac{v_o^2}{\omega N_o}\right) \nabla n^*,$$

therefore

$$\overline{S} = \tfrac{1}{2}\, Re\, [\overline{E} \times \overline{H}^* + i \left(\frac{v_o^2 e n}{\omega}\right) \overline{E}^* - i \left(\frac{m v_o^4}{\omega N_o}\right) n\, \nabla n^*]\ . \qquad (C.11)$$

Under the modal decomposition precedure we have from Chapter II

$$\overline{E}_p = + \frac{m v_o^2}{N_o e}\, \frac{(\epsilon_1 - 1)}{\epsilon_1}\, \nabla n\ ,$$

or

$$\nabla n^* = \frac{N_o e}{m v_o^2}\, \frac{\epsilon_1}{(\epsilon_1 - 1)}\, \overline{E}_p^*\ . \qquad (C.12)$$

Inserting C.12 into C.11, we obtain

$$\bar{S} = 1/2 \text{ Re } [(\bar{E}_o + \bar{E}_p) \times \bar{H}^* + i \left(\frac{v_o^2 en}{\omega}\right) (\bar{E}_o^* + \bar{E}_p^*)$$

$$+ 1/2 \text{ Re } [i \frac{ev_o^2}{\omega} \left(\frac{\epsilon_1}{\epsilon_1 - 1}\right) n\bar{E}_p^*] ,$$

or

$$\bar{S} = 1/2 \text{ Re } [\bar{E}_o \times \bar{H}^* + i \frac{v_o^2 en}{\omega} \bar{E}_o^*]$$

$$+ 1/2 \text{ Re } [\bar{E}_p \times \bar{H}^* + i \frac{v_o^2 e}{\omega} \left(\frac{\omega}{\omega_p}\right)^2 n\bar{E}_p^*]. \qquad (C.13)$$

If we look at the outward power flow from a spherical surface of radius R, the outward power flow density is $\bar{a}_r \cdot \bar{S}$. For the fields in the radiation zone, we know that

$$\bar{a}_r \cdot \bar{E}_o = 0 \qquad \bar{a}_r \times \bar{E}_p = 0 , \qquad \bar{a}_r \cdot \bar{H} = 0 ,$$

and therefore,

$$S_r = 1/2 \text{ Re } [(\bar{E}_o \times \bar{H}^*) \cdot \bar{a}_r] + 1/2 \text{ Re } [i\left(\frac{v_o^2 \omega m \epsilon_o}{N_o e}\right) nE_{pr}^*]. \qquad (C.14)$$

Since

$$\bar{E}_p^* = -i \frac{N_o e}{\omega \epsilon_o} \bar{V}_p^* ,$$

Equation C.14 may be rewritten as

$$S_r = 1/2 \text{ Re } [(\bar{E}_o \times \bar{H}^*) \cdot \bar{a}_r + mv_o^2 n\bar{V}_p^* \cdot \bar{a}_r] , \qquad (C.15)$$

which is very similar in form to C.9.

APPENDIX D

TWO FLUID MODEL

When the compressible plasma under discussion is composed of electrons and singly charged ions, the basic set of linearized equations from Appendix A now becomes for the two fluid model:

$$\nabla \times \bar{H} - i\omega\, \epsilon_o\, \bar{E} = N_o e\, (\bar{V}_i - \bar{V}_e) \,, \tag{D.1}$$

$$\nabla \times \bar{E} + i\omega\, \mu_o\, \bar{H} = 0 \,, \tag{D.2}$$

$$i\, m_i\, \omega\, N_o\, \bar{V}_i = N_o e\, \bar{E} - m_i v_i^{\,2}\, \nabla n_i \,, \tag{D.3}$$

$$i\, m_e\, \omega\, N_o\, \bar{V}_e = -\, N_o e\, \bar{E} - m_e v_e^{\,2}\, \nabla n_e \,, \tag{D.4}$$

$$i\, \omega\, n_i + N_o\, \nabla \cdot \bar{V}_i = 0 \,, \tag{D.5}$$

$$i\, \omega\, n_i + N_o\, \nabla \cdot \bar{V}_e = 0 \,, \tag{D.6}$$

where the subscript e and i refer to the electrons and ions respectively. Samaddar (1964) has devised a modal decomposition procedure which is summarized as follows:

$$\begin{aligned} \bar{E} &= \bar{E}_o + \bar{E}_p \,, \\ \bar{V}_i &= \bar{V}_{oi} + \bar{V}_{pi} \,, \\ \bar{V}_e &= \bar{V}_{oe} + \bar{V}_{pe} \,, \end{aligned} \tag{D.7}$$

where the subscripts o and p signify the optical mode and the plasma mode respectively. Thus

$$\bar{E}_o = \frac{\nabla \times \bar{H}}{i\, \omega\, \epsilon_o\, \epsilon_1} \,, \quad \epsilon_1 = 1 - x_e - x_i \,, \tag{D.8}$$

$$\bar{E}_p = \frac{e}{\omega^2 \epsilon_1 \epsilon_o} [v_e^2 \nabla n_e - v_i^2 \nabla n_i] , \tag{D.9}$$

$$\bar{V}_{pi} = \frac{i}{N_o \omega e} [(1 - x_e) v_i^2 \nabla n_i - x_i v_e^2 \nabla n_e], \tag{D.10}$$

$$\bar{V}_{pe} = \frac{i}{N_o \omega e} [(1 - x_i) v_e^2 \nabla n_e - x_e v_i^2 \nabla n_i], \tag{D.11}$$

$$\bar{V}_{oi} = \frac{e \bar{E}_o}{i \omega m_i} , \quad \bar{V}_{oe} = \frac{-e \bar{E}_o}{i \omega m_e} , \tag{D.12}$$

$$x_i = \frac{N_o e^2}{\omega^2 m_i \epsilon_o} , \quad x_e = \frac{N_o e^2}{\omega^2 m_e \epsilon_o} . \tag{D.13}$$

If we again formulate the problem of waves propagating down an infinite cylinder where in addition to electron density waves we have ion density waves whose normal velocity must vanish at the cylinder wall. But for the two fluid model the plasma wave equations for the ions and the electrons form a coupled pair. Thus, if we require an $e^{i(\omega t - kz)}$ variation along the cylinder, the plasma wave equations become,

$$\nabla_T^2 n_i + (k_i^2 - k^2) n_i + \frac{\omega_{pi}^2}{v_i^2} n_e = 0 , \tag{D.14}$$

$$\nabla_T^2 n_e + (k_e^2 - k^2) n_e + \frac{\omega_{pe}^2}{v_e^2} n_i = 0 ,$$

where $$k_e^2 = \frac{\omega^2}{v_e^2} (1 - x_e) , \quad k_i^2 = \frac{\omega^2}{v_i^2} (1 - x_i) ,$$

and $\nabla_T^2 = \frac{\partial^2}{\partial^2 r} + \frac{1}{r}\frac{\partial}{\partial r}$ in cylindrical coordinates.

The coupled wave equations may be uncoupled to produce the following uncoupled fourth order equations:

$$\nabla_T^{4}\, n_i + (a + d)\, \nabla_T^{2}\, n_i + (da - f^2)\, n_i = 0 \quad ,$$
$$\nabla_T^{4}\, n_e + (a + d)\, \nabla_T^{2}\, n_e + (da - f^2)\, n_i = 0 \quad , \tag{D.15}$$

where
$$a = k_i^{\,2} - k^2 \quad ,$$
$$d = k_e^{\,2} - k^2 \quad ,$$
$$f = \frac{\omega_{pi}^{\,2}}{v_i^{\,2}} = \frac{\omega_{pe}^{\,2}}{v_e^{\,2}} \quad .$$

The uncoupled wave equations may also be expressed as

$$(\nabla_T^{2} + \lambda_1^{\,2}) \cdot (\nabla_T^{2} + \lambda_2^{\,2})\, n_i = 0 \quad , \tag{D.16}$$
$$(\nabla_T^{2} + \lambda_1^{\,2}) \cdot (\nabla_T^{2} + \lambda_2^{\,2})\, n_e = 0 \quad ,$$

where
$$\lambda_{1,2}^{\,2} = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - f^2)}}{2} \quad . \tag{D.17}$$

Solutions for the ion and electron density are then defined as follows:

$$n_e = n_{e1} + n_{e2} \quad ,$$
$$n_i = n_{i1} + n_{i2} \quad , \tag{D.18}$$
$$n_{i1} = h_1\, n_{e2} \;, \quad n_{i2} = h_2\, n_{e2} \quad ,$$
$$h_1 = \frac{\lambda_1^{\,2} - d}{f} \quad , \quad h_2 = \frac{\lambda_2^{\,2} - d}{f} \quad .$$

Then solutions for n_i and n_e which satisfy the radiation condition are of the form

$$n_e = A\, H_o^{(2)}\, (\lambda_1\, r) + B\, H_o^{(2)}\, (\lambda_2\, r) \quad , \tag{D.19}$$

$$n_i = Ah_1 \; H_o^{(2)} \; (\lambda_1 \; r) + Bh_2 \; H_o^{(2)} \; (\lambda_2 \; r) \quad ,$$

From Equations D.9 and D.19 the tangential electric field of the plasma mode is

$$E_{zp} = \frac{-i \; e \; k}{\omega^2 \; \epsilon_o \; \epsilon_1} \; [\; A(v_e{}^2 - v_i{}^2 \; h_1) \; H_o^{(2)} \; (\lambda_1 \; r)$$

$$+ B(v_e{}^2 - v_i{}^2 \; h_2) \; H_o^{(2)} \; (\lambda_2 \; r)] \quad , \tag{D.20}$$

and from Equations D.10, D.11, and D.19 the radial ion and electron velocities are

$$V_{r_{pi}} = \frac{i}{\omega \; N \; \epsilon_1} \; [\lambda_1 \; A \; [x_i v_e{}^2 - v_i{}^2 \; (1 - x_e) \; h_1] \; H_1^{(2)} \; (\lambda_1 \; r)$$

$$+ \lambda_2 \; B[x_i v_e{}^2 - (1 - x_e) \; v_i{}^2 \; h_2] \; H_1^{(2)} \; (\lambda_2 \; r)] \quad , \tag{D.21}$$

$$V_{r_{pe}} = \frac{i}{\omega \; N_o \; \epsilon_1} \; [A\lambda_1 (h_1 \; x_e \; v_i{}^2 - (1 - x_i) \; v_e{}^2) \; H_1^{(2)} \; (\lambda_1 \; r)$$

$$B\lambda_2 (h_2 \; x_e \; v_i{}^2 - (1 - x_i) \; v_e{}^2) \; H_1^{(2)} \; (\lambda_2 \; r)] \quad . \tag{D.22}$$

The optical mode expressions essentially stay the same as the one fluid model with some notation changes. The optical tangential and radial electric and velocity fields are

$$E_{or} = \frac{\lambda_3 \; k}{\omega \; \epsilon_o \; \epsilon_1} \; C \; H_1^{(2)} \; (\lambda_3 \; r) \quad ,$$

$$E_{oz} = \frac{\lambda_3{}^2}{i \; \omega \; \epsilon_o \; \epsilon_1} \; C \; H_o^{(2)} \; (\lambda_3 \; r) \quad , \tag{D.23}$$

$$V_{roi} = \frac{e \; \lambda_3 \; k}{i \; \omega^2 \; m_i \; \epsilon_o \; \epsilon_1} \; C \; H_1^{(2)} \; (\lambda_3 \; r) \quad ,$$

$$V_{roe} = \frac{-e\ \lambda_3\ k}{i\ \omega^2\ m_e\ \epsilon_o\ \epsilon_1}\ C\ H_1^{(2)}\ (\lambda_3\ r) \quad ,$$

where $\lambda_3{}^2 = k_o{}^2 - k^2,\ k_o{}^2 = \frac{\omega^2}{c^2}\ (1 - x_i - x_e).$

For our boundary conditions we shall require the tangential electric field, the normal electron velocity, and normal ion velocity to vanish at r = b. Application of the boundary conditions to Equations D.19, D.20, D.21, D.22 and D.23 requires that the following determinant be zero for non trivial solutions.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = 0 \qquad \text{(D.24)}$$

where

$$A_{11} = \frac{-i\ e\ k}{\omega^2\ \epsilon_o\ \epsilon_1}\ (v_e{}^2 - v_i{}^2\ h_1)\ H_o^{(2)}\ (\lambda_1\ b)\ ,$$

$$A_{12} = \frac{-i\ e\ k}{\omega^2\ \epsilon_o\ \epsilon_1}\ (v_e{}^2 - v_i{}^2\ h_2)\ H_o^{(2)}\ (\lambda_2\ b)\ ,$$

$$A_{13} = \frac{\lambda_3{}^2}{i\ \omega\ \epsilon_o\ \epsilon_1}\ H_o^{(2)}\ (\lambda_3\ b) \quad ,$$

$$A_{21} = \frac{i\ \lambda_1}{\omega\ N_o\ \epsilon_1}\ (x_i\ v_e{}^2 - (1 - x_e) v_i{}^2\ h_1)\ H_1^{(2)}\ (\lambda_1\ b) \quad ,$$

$$A_{22} = \frac{i\ \lambda_2}{\omega\ N_o\ \epsilon_1}\ (x_i\ v_e{}^2 - (1 - x_e) v_i{}^2\ h_2)\ H_1^{(2)}\ (\lambda_2\ b) \quad ,$$

$$A_{23} = \frac{e\ \lambda_3\ k}{i\ \omega^2\ m_i\ \epsilon_o\ \epsilon_1}\ H_1^{(2)}\ (\lambda_3\ b) \quad ,$$

$$A_{31} = \frac{i\ \lambda_1}{\omega\ N_o\ \epsilon_1} (h_i\ x_e\ v_i^{\ 2} - (1 - x_i)v_e^{\ 2})\ H_1^{(2)}\ (\lambda_1\ b) \quad ,$$

$$A_{32} = \frac{i\ \lambda_2}{\omega\ N_o\ \epsilon_1} (h_2\ x_e\ v_i^{\ 2} - (1 - x_i)v_e^{\ 2})\ H_1^{(2)}\ (\lambda_2\ b) \quad ,$$

$$A_{33} = \frac{-e\ \lambda_3\ k}{i\ \omega^2\ m_e\ \epsilon_o\ \epsilon_1}\ H_1^{(2)}\ (\lambda_3\ b) \quad .$$

The determinantal equation, resulting from setting determinant of D.24 to zero, is after some manupulation:

$$\frac{k^2\ (v_e^{\ 2} - h_1\ v_i^{\ 2})(v_e^{\ 2}\ x_i + h_2\ x_e\ v_i^{\ 2})}{\lambda_1\ \lambda_3}\ \frac{H_o^{(2)}\ (\lambda_1\ b)}{H_1^{(2)}\ (\lambda_1\ b)}$$

$$-k^2\ (v_e^{\ 2} - h_2\ v_i^{\ 2})(v_e^{\ 2}\ x_i + h_1\ x_e^{\ 2}\ v_i^{\ 2})\ \frac{H_o^{(2)}\ (\lambda_2\ b)}{H_1^{(2)}\ (\lambda_2\ b)}$$

$$+ (h_1 - h_2)(v_e^{\ 2}\ v_i^{\ 2})\ \frac{H_o^{(2)}\ (\lambda_3\ b)}{H_o^{(2)}\ (\lambda_3\ b)} = 0 \quad , \qquad (D.24)$$

where
$$h_{1,2} = \frac{\lambda_{1,2}^{\ 2} - k_e^{\ 2} + k^2}{\omega_{pi}^{\ 2}/v_i^{\ 2}} \quad ,$$

$$\lambda_{1,2}^{\ 2} = \frac{(k_e^{\ 2} + k_i^{\ 2} - k^2) \pm \sqrt{(k_e^{\ 2} - k_i^{\ 2})^2 + 4\omega_{pi}^{\ 4}/v_i^{\ 4}}}{2} \quad , \qquad (D.25)$$

$$\lambda_3^{\ 2} = \frac{\omega^2}{c^2}\ \epsilon_1 - k^2 \quad .$$

Examining D.24 for the spectrum of k^2 real, we find first that if one assumes that the transverse wavenumbers are real then k must be complex by D.24. But if we assume that the transverse wavenumbers are negative imaginary, then all quantities are either positive or negative real numbers, and a solution might exist.

Since the discriminant of D.25 is always positive and for λ_1^2 to be negative, the following must be true for all frequencies from D.25:

$$(k_i^2 + k_e^2 - 2k^2) \leq 0, \tag{D.26}$$

$$\left|(k_i^2 + k_e^3 - 2k^2)\right| \geq \left|\sqrt{(k_e^2 - k_i^2)^2 + 4\omega_{pi}^4/v_i^4}\right| \tag{D.27}$$

Inequality D.26 reduces to

$$k^2 \geq \frac{k_i^2 + k_e^2}{2} \quad , \tag{D.28}$$

whereas D.27 produces the inequality

$$k^2 \geq \frac{(k_e^2 + k_i^2) \pm \sqrt{(k_e^2 - k_i)^2 + 4\omega_{pi}^4/v_i^4}}{2} \quad , \tag{D.29}$$

which is a stronger condition than D.28. Therefore, for all frequencies, the constraint,

$$k^2 \geq \frac{(k_e^2 + k_i^2) + \sqrt{(k_e^2 - k_i^2)^2 + 4\omega_{pi}^4/v_i^4}}{2} \quad ,$$

must hold.

Although only the formulation of the two fluid model is considered and a computer solution has not been attempted, one might predict the behavior of k from purely physical reasoning. For $\omega < \omega_{pi}$, the ions should dominate because of their large mass. Between ω_{pi} and ω_{pe}, a hybrid ion-electron behavior should take place. For $\omega > \omega_{pe}$, the electrons will dominate producing a type of behavior similar to the one fluid model.

VITA

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