ANALYSIS OF ONE-BAY, MULTI-STORY, RECTANGULAR FRAMES BY MODIFIED MOMENT DISTRIBUTION

By

CHARLES O. HELLER

Bachelor of Science
Oklahoma State University
Stillwater, Oklahoma
1959

Submitted to the Faculty of the Graduate School
of the Oklahoma State University
in partial fulfillment of
the requirements for
the degree of
MASTER OF SCIENCE
January, 1960

ANALYSIS OF ONE-BAY, MULTI-STORY,
RECTANGULAR FRAMES BY MODIFIED
MOMENT DISTRIBUTION

Thesis Approved:

Jan J. Tuma

Thesis Adviser

Roger L. Flanders

Robert MacVicar

Dean of the Graduate School

PREFACE

This study is an extension of a discussion submitted to the American Society of Civil Engineers (1) in October, 1959. It is the out-growth of the instruction of structural analysis by Professor Jan J. Tuma.

I wish to express my indebtedness to Professor Tuma not only for his invaluable assistance and constructive criticism in the preparation of this thesis, but also for his kind guidance throughout my years at college.

I also wish to thank the staff of the School of Civil Engineering for the valuable instruction given me and for giving me the opportunity to work with them during my graduate study.

I express my most sincere gratitude to my parents, Ilona and Rudolph Heller, and to my wife, Susan Elizabeth, for their kindness and faith in me during the less successful phase of my college career.

I wish to thank Mrs. Willie Bernardi for her careful typing of the manuscript.

C. O. H.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

h_j	Height of column $\overline{ij}$
$i, j, k, \ldots i', j', k', \ldots$	Letters designating joints of frame
p	Maximum intensity of triangular load
w	Intensity of uniformly distributed load
C_{ij}	Carry-over factor
$C_{ij}^{(I)}, C_{ij}^{(II)}$	New carry-over factors
$C_{ij}^{(II)} K_{ij}^{(II)}$	New carry-over stiffness factor
D_{ij}	Distribution factor
FM_{ij}	Total fixed end moment
FV_{ij}	Fixed end shear
GM_{ij}	Guided moment
K_{ij}	Stiffness factor
$K_{ij}^{(I)}, K_{ij}^{(II)}$	New stiffness factors
K'_{ij}	Modified stiffness factor for Case I $(K_{ij} - K_{ij}C_{ij})$
L	Length of bay
$M_{ij}^{(I)}, M_{ji}^{(II)}$	End moments
P_j	Transverse external load
S_{ij}	Sidesway factor
V_{ij}	Horizontal shear due to loads

θ_j' . . . . Slope of member at j - Case I

θ_j'' . . . . Slope of member at j - Case II

Δ_j . . . . Relative displacement of j

Σ . . . . Summation

ψ_j . . . . $\frac{\Delta_j}{h_j}$

PART I

INTRODUCTION

A simple method for the analysis of symmetrical, one-bay, multi-story, rectangular towers is presented. This presentation is an extension of a discussion presented by the writer to the American Society of Civil Engineers in December 1959 (1). The discussion deals with the analysis of towers containing prismatic members, whereas, in this thesis, the completely general case of members of any cross-sectional area is considered.

The study is restricted to coplanar systems and the customary assumptions of structural analysis are made.

There exists a long line of investigators who paid considerable attention to the special nature of an unsymmetrically loaded frame of the above type, including many who took advantage of the symmetry and antisymmetry of the loaded structure. The idea of resolving a symmetrical, one-bay, multi-story frame, unsymmetrically loaded, into a symmetrical and an antisymmetrical system was shown by Andrée (2), Newell (3), Bayer (4), Naylor (5), Pei (6), and others. The modified method of moment distribution for analyzing such a frame was originally introduced by Perri (7), Hadley (8), and Kavanagh (9). Similar material was later discussed under a new title by Grinter and Tsao (10). The application of this approach was then fully explained by Parcel and Moorman (11) through a numerical example. Later, this method was also shown by Kupferschmid (12) and Kazda (13).

Recently, Goldberg (14) suggested an analysis of one-bay, multi-story, rectangular frames by means of three-slope equations. Modifications of Goldberg's approach and some additional possibilities were then demonstrated by Nubar (15), Sobotka (16), Chang (17), and Cooke (18).

In this study, a general derivation of this modified method of moment distribution is developed. A new concept, the guided moment, is introduced. This concept brings about a short and simple solution since all translations are treated in a single moment distribution. This accomplishes the elimination of as many unknown translations as there are stories, and the problem is greatly simplified.

The writer was first introduced to the analysis of one-bay, multi-story, rectangular towers by Professor Jan J. Tuma (19) in Courses CE 4B4 (Theory of Structures II) and CE 620 (Seminar in Carry-Over Procedures in Structural Analysis), taught at the Oklahoma State University in the spring semesters of 1958 and 1959, respectively.

The subsequent discussion is divided into six parts. The first part contains a statement of the problem. The second deals with the case of the symmetrical frame acted upon by a symmetrical system of loads. The third considers the case of the same symmetrical frame acted upon by an antisymmetrical system of loads. The fourth part states the procedure of analysis, while the fifth demonstrates this procedure through a numerical example. Finally, the results are discussed and a conclusion is drawn.

PART II

DEFINITION OF THE PROBLEM

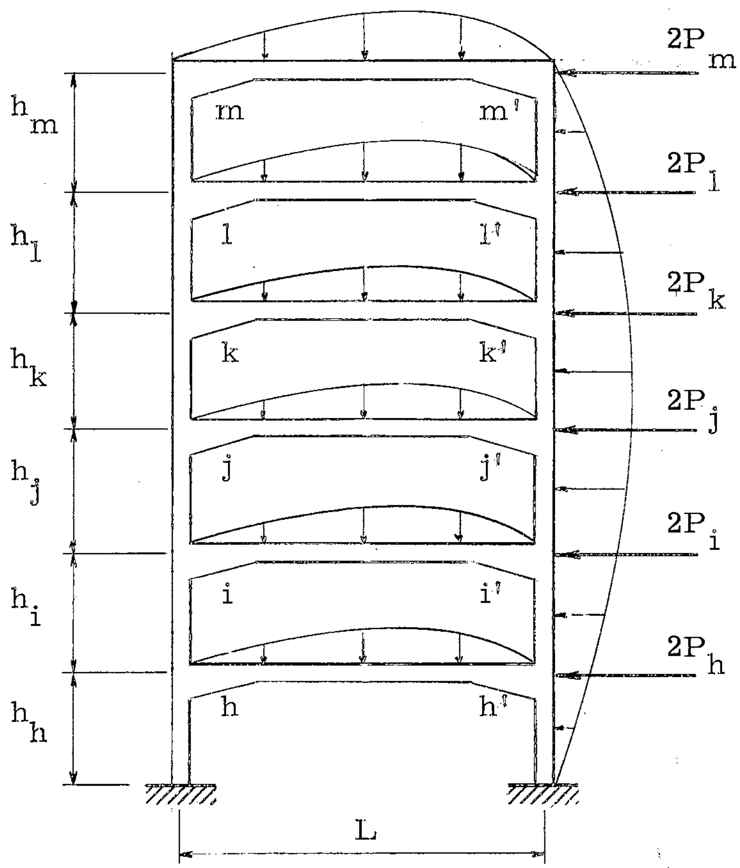


Fig. 2-1
Unsymmetrically Loaded Frame

A multi-story two-column symmetrical bent, acted upon by a general system of loads, is considered (Fig. 2-1). The cross-sections of both columns and girders are variable in a given span.

Since the structure is symmetrical and unsymmetrically loaded, the resolution of the system into a symmetrical (Case I) and antisymmetrical (Case II) condition offers many advantages, as has been shown by Andrée (2), Newell (3), Bayer (4), Naylor (5), and Pei (6).

PART III

SYMMETRICAL CONDITION - CASE I

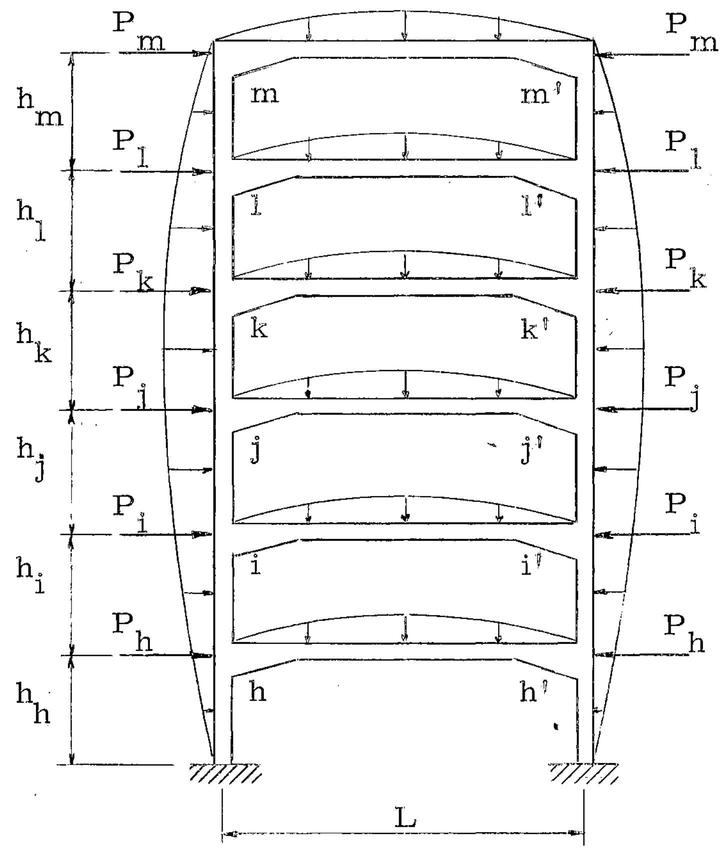


Fig. 3-1
Symmetrically Loaded Frame

The deformation curve of the frame due to the Case I loading (Fig. 3-1) is symmetrical with respect to the vertical axis of the structure. Thus, the joint rotations of the left side are symmetrical with their counterparts of the right side and no translation takes place.

The end moment equations of a typical portion of this frame (Fig. 3-2) are:

$$\left.\begin{aligned}
M_{kj}^{(I)} &= K_{kj}\theta'_{k} + C_{jk}K_{jk}\theta'_{j} + FM_{kj}^{(I)} \\
M_{jk}^{(I)} &= K_{jk}\theta'_{j} + C_{kj}K_{kj}\theta'_{k} + FM_{jk}^{(I)} \\
M_{jj}^{(I)} &= K_{jj}(1-C_{jj})\theta'_{j} + FM_{jj}^{(I)} \\
M_{ji}^{(I)} &= K_{ji}\theta'_{j} + C_{ij}K_{ij}\theta'_{i} + FM_{ji}^{(I)} \\
M_{ij}^{(I)} &= K_{ij}\theta'_{i} + C_{ji}K_{ji}\theta'_{j} + FM_{ij}^{(I)} ,
\end{aligned}\right\} \quad (3\text{-}1)$$

in which

M = Final end moment

(I) = Case I denotation

K = Stiffness factor

θ' = Slope of beam due to Case I loading

C = Carry-over factor

FM = Fixed-end moment

Subscripts: First letter denotes near end, second letter denotes far end.

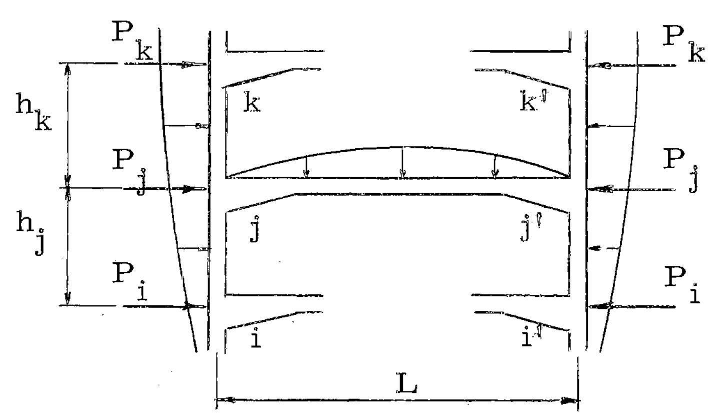


Fig. 3-2
Typical Portion of Frame with Case I Loading

From the equilibrium of joint j (any joint),

$$M_{jk}^{(I)} + M_{jj}^{(I)} + M_{ji}^{(I)} = 0 , \qquad (3\text{-}2a)$$

or in terms of Eqs. (3-1),

$$\left.\begin{aligned} C_{kj}K_{kj}\theta'_{k} + \left[K_{jk} + K_{jj}(1-C_{jj}) + K_{ji} \right] \theta'_{j} + C_{ij}K_{ij}\theta'_{i} &= \\ = - FM_{jk}^{(I)} - FM_{jj}^{(I)} - FM_{ji}^{(I)} &. \end{aligned}\right\} \qquad (3\text{-}2b)$$

With notations

$$C_{kj}K_{kj} = C_{kj}^{(I)} K_{kj}^{(I)} \qquad \Big| \qquad C_{ij}K_{ij} = C_{ij}^{(I)} K_{ij}^{(I)}$$

$$K_{jk} + K_{jj}(1-C_{jj}) + K_{ji} = \Sigma K_{j}^{(I)}$$

$$- FM_{jk}^{(I)} - FM_{jj}^{(I)} - FM_{ji}^{(I)} = - \Sigma FM_{j}^{(I)} ,$$

Eq. (3-2b) becomes

$$C_{kj}^{(I)} K_{kj}^{(I)} \theta'_{k} + \Sigma K_{j}^{(I)} \theta'_{j} + C_{ij}^{(I)} K_{ij}^{(I)} \theta'_{i} = - \Sigma FM_{j}^{(I)} . \qquad (3\text{-}2)$$

If θ'_{k} and θ'_{i} are assumed to be temporarily zero,

$$\theta'_{j} = - \frac{\Sigma FM_{j}^{(I)}}{\Sigma K_{j}^{(I)}} . \qquad (3\text{-}3)$$

Then, the end moments at j, Eqs. (3-1), reduce to

$$\left.\begin{aligned} M_{jk}^{(I)} &= - D_{jk}^{(I)} \Sigma FM_{j}^{(I)} + FM_{jk}^{(I)} \\ M_{jj}^{(I)} &= - D_{jj}^{(I)} \Sigma FM_{j}^{(I)} + FM_{jj}^{(I)} \\ M_{ji}^{(I)} &= - D_{ji}^{(I)} \Sigma FM_{j}^{(I)} + FM_{ji}^{(I)} . \end{aligned}\right\} \qquad (3\text{-}4)$$

The symbols

$$D_{jk}^{(I)} = \frac{K_{jk}^{(I)}}{\Sigma K_{j}^{(I)}} \qquad \Big| \qquad D_{jj}^{(I)} = \frac{K_{jj}^{(I)}}{\Sigma K_{j}^{(I)}} \qquad \Big| \qquad D_{ji}^{(I)} = \frac{K_{ji}^{(I)}}{\Sigma K_{j}^{(I)}} \qquad (3\text{-}5)$$

are the distribution factors for joint j.

Similarly, the end moments at the far ends k and i are:

$$\left.\begin{aligned} M_{kj}^{(I)} &= - C_{jk}^{(I)} D_{jk}^{(I)} \Sigma FM_j^{(I)} + FM_{kj}^{(I)} \\ M_{ij}^{(I)} &= - C_{ji}^{(I)} D_{ji}^{(I)} \Sigma FM_j^{(I)} + FM_{ij}^{(I)} . \end{aligned}\right\} \quad (3\text{-}6)$$

Since θ'_j is eliminated due to the symmetry of deformation, no carry-over occurs in the jj' direction.

The formulas (3-4) and (3-6) are perfectly general and can be applied to any joint of Fig. 3-1. Using the constants and fixed end moments from these equations, the moment distribution can be readily applied.

PART IV

ANTISYMMETRICAL CONDITION - CASE II

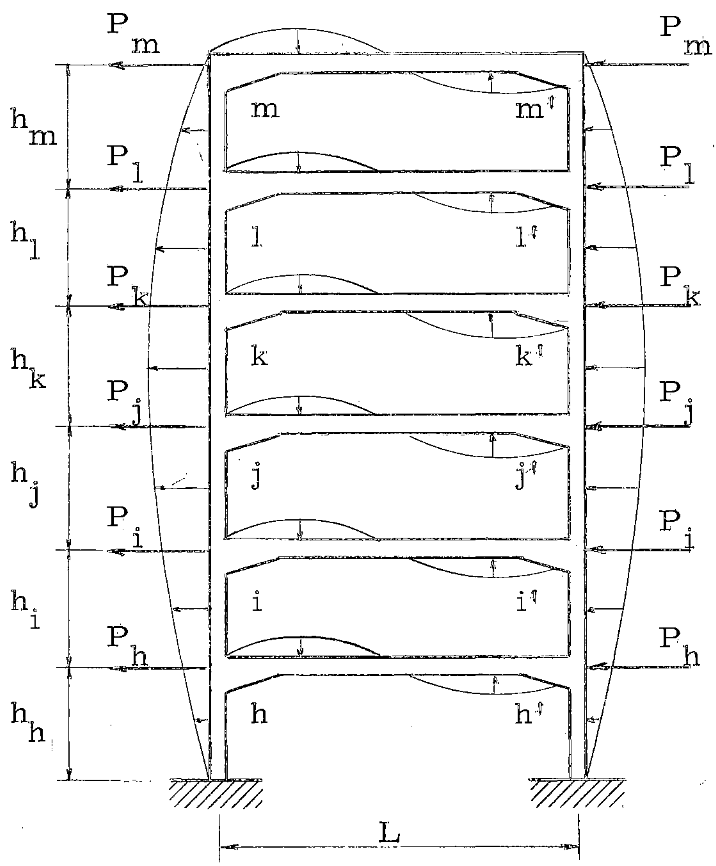


Fig. 4-1

Antisymmetrically Loaded Frame

The deformation curve of this frame (Fig. 4-1) is antisymmetrical with respect to the vertical axis of the structure. Therefore, the joint rotations of the left side are antisymmetrical with their counterparts on the right side and each story translates an amount Δ in the direction of the loads.

1. Slope Deflection Equations

The end moment equations of a typical portion of this frame are:

$$\left.\begin{aligned}
M_{kj}^{(II)} &= K_{kj}\theta_k'' + C_{jk}K_{jk}\theta_j'' + S_{kj}\psi_k + FM_{kj}^{(II)} \\
M_{jk}^{(II)} &= K_{jk}\theta_j'' + C_{kj}K_{kj}\theta_k'' + S_{jk}\psi_k + FM_{jk}^{(II)} \\
M_{jj}^{(II)} &= K_{jj}\,(1 + C_{jj})\,\theta_j'' + FM_{jj}^{(II)} \\
M_{ji}^{(II)} &= K_{ji}\theta_j'' + C_{ij}K_{ij}\theta_i'' + S_{ji}\psi_j + FM_{ji}^{(II)} \\
M_{ij}^{(II)} &= K_{ij}\theta_i'' + C_{ji}K_{ji}\theta_j'' + S_{ij}\psi_j + FM_{ij}^{(II)} ,
\end{aligned}\right\} \quad (4\text{-}1)$$

where

$$\psi_k = \frac{\Delta_k}{h_k} \qquad\qquad \psi_j = \frac{\Delta_j}{h_j} ,$$

$$S_{kj} = K_{kj} + C_{jk}K_{jk} = K_{kj} + C_{kj}K_{kj} \;\Big|\; S_{jk} = K_{jk} + C_{kj}K_{kj} = K_{jk} + C_{jk}K_{jk},$$

and

Δ_k = relative displacement of the joint k,

Δ_j = relative displacement of the joint j,

S_{kj}, S_{jk}, S_{ji}, S_{ij} = sidesway factors.

The other terms of Eqs.(4-1) are similar to those which were explained for Eqs. (3-1).

2. Shear Equations

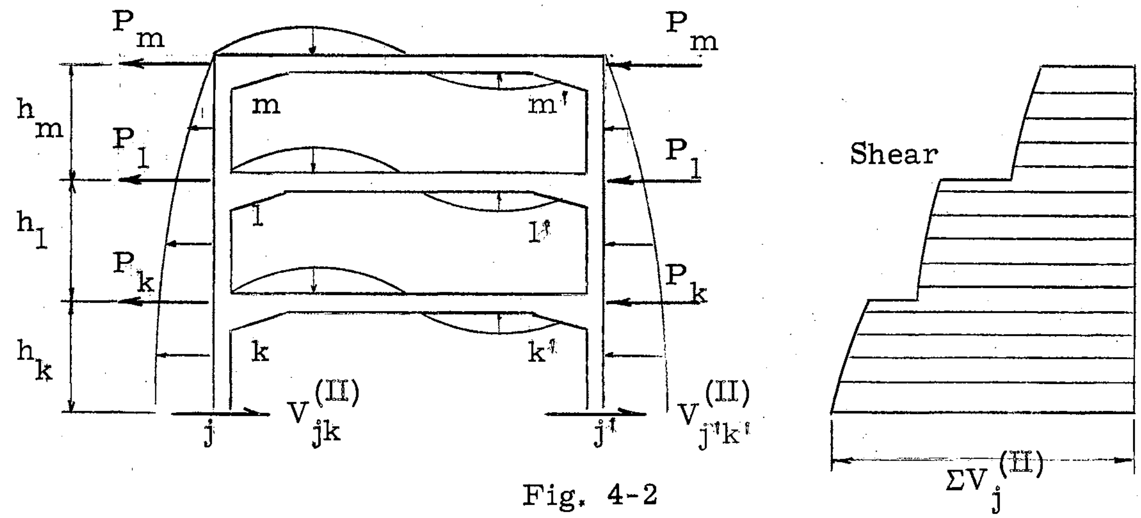


Fig. 4-2

Portion of Frame Above Girder $\overline{jj'}$

From the equilibrium of horizontal forces at jj' (Fig. 4-2),

$$\overrightarrow{V_{jk}^{(II)}} + \overrightarrow{V_{j'k'}^{(II)}} - \overleftarrow{\Sigma V_j^{(II)}} = 0, \tag{4-2}$$

where

$V_{jk}^{(II)}$ ($V_{j'k'}^{(II)}$) = horizontal shear of the member $\overline{jk}$ ($\overline{j'k'}$) due to loads,

$\Sigma V_j^{(II)}$ = summation of all horizontal loads above $\overline{jj'}$.

Since

$$V_{jk}^{(II)} = \frac{M_{jk}^{(II)} + M_{kj}^{(II)}}{h_k} + BV_{jk}^{(II)}, \tag{4-3}$$

the equilibrium equation becomes

$$S_{kj}\theta_k'' + S_{jk}\theta_j'' + (S_{kj} + S_{jk})\psi_k + h_k FV_{jk}^{(II)} - \frac{h_k}{2}\Sigma V_j^{(II)} = 0, \tag{4-4}$$

where

$FV_{jk}^{(II)}$ = horizontal shear of the fixed-end member $\overline{jk}$ at j due to loads (Fig. 4-3),

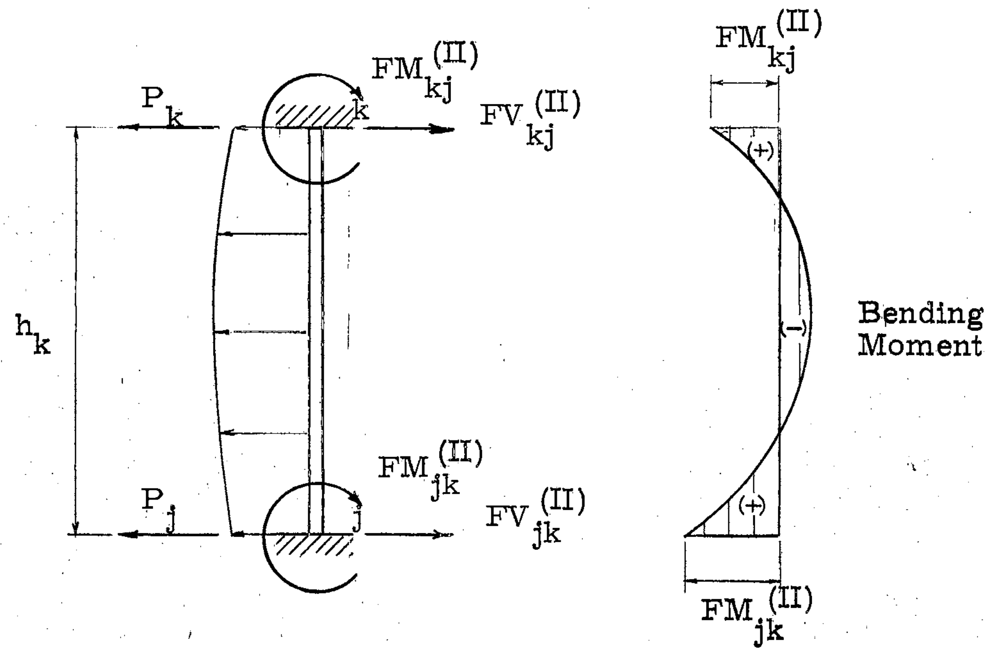


Fig. 4-3

Fixed-end Member $\overline{jk}$

and $\quad BV_{jk}^{(II)}$ = horizontal shear of the simply supported member $\overline{jk}$.

From Eq. (4-4),

$$\left.\begin{aligned}\psi_k = & -\frac{S_{kj}}{S_{kj}+S_{jk}}\,\theta''_k - \frac{S_{jk}}{S_{kj}+S_{jk}}\,\theta''_j - \frac{h_k}{S_{kj}+S_{jk}}\,FV_{jk}^{(II)} + \\ & + \frac{h_k}{2(S_{kj}+S_{jk})}\,\Sigma V_j^{(II)} = 0,\end{aligned}\right\}\text{(4-5a)}$$

and similarly,

$$\left.\begin{aligned}\psi_j = & -\frac{S_{ji}}{S_{ji}+S_{ij}}\,\theta''_j - \frac{S_{ij}}{S_{ji}+S_{ij}}\,\theta''_i - \frac{h_j}{S_{ji}+S_{ij}}\,FV_{ij}^{(II)} + \\ & + \frac{h_j}{2(S_{ji}+S_{ij})}\,\Sigma V_i^{(II)} = 0.\end{aligned}\right\}\text{(4-5b)}$$

In terms of Eqs. (4-5), the end moment equations at j become:

$$\left.\begin{aligned} M_{jk}^{(II)} &= K_{jk}^{(II)}\theta_j'' + C_{kj}^{(II)}K_{kj}^{(II)}\theta_k'' + GM_{jk}^{(II)} \\ M_{jj}^{(II)} &= K_{jj}^{(II)}\theta_j'' + FM_{jj}^{(II)} \\ M_{ji}^{(II)} &= K_{ji}^{(II)}\theta_j'' + C_{ij}^{(II)}K_{ij}^{(II)}\theta_i'' + GM_{ji}^{(II)}, \end{aligned}\right\} \quad (4\text{-}6)$$

where

$K_{jk}^{(II)}$ = the new stiffness factor of the member $\overline{jk}$,

$C_{kj}^{(II)}$ = the new carry-over factor of the member $\overline{kj}$,

$GM_{jk}^{(II)}$ = the fixed end moment of the guided member $\overline{jk}$, (Guided Moment).

3. New Functions

The new terms of Eqs. (4-6) must next be explained. These are:

a. New stiffness factor

b. New carry-over stiffness factor

c. New carry-over factor

d. Guided moment.

a. New Stiffness Factor

The new column stiffness factor $K_{ji}^{(II)}$ is the moment M_{ji} required at the free end j of the cantilever column $\overline{ij}$ (Fig. 4-4) to produce a rotation of one radian at that end:

$$K_{ji}^{(II)} = K_{ji} - \frac{S_{ji}^{\;2}}{S_{ji} + S_{ij}} . \qquad (4\text{-}7a)$$

The new stiffness factor $K_{jj}^{(II)}$ of the girder $\overline{jj'}$ is equal to the sidesway factor S_{jj}:

$$K_{jj}^{(II)} = S_{jj} = K_{jj} + K_{jj}C_{jj} . \qquad (4\text{-}7b)$$

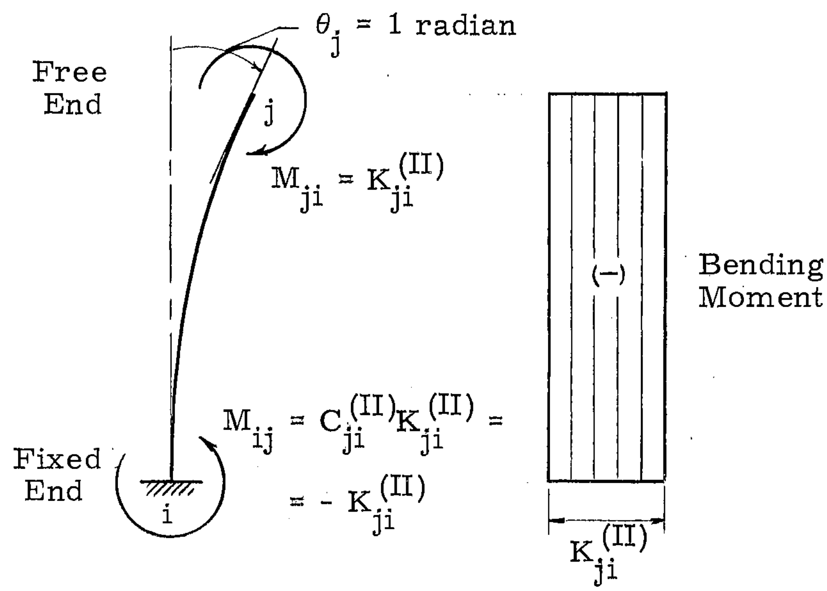


Fig. 4-4

Cantilever Column $\overline{ij}$

b. New Carry-Over Stiffness Factor

The new carry-over stiffness factor $C_{ji}^{(II)} K_{ji}^{(II)}$ is the moment M_{ij} produced at the fixed end i of the cantilever column $\overline{ij}$ (Fig. 4-4) by the rotation of one radian at the free end j:

$$C_{ji}^{(II)}K_{ji}^{(II)} = C_{ji}K_{ji} - \frac{S_{ji}S_{ij}}{S_{ji} + S_{ij}} . \qquad (4\text{-}8)$$

c. New Carry-Over Factor

Since there is no rotation of joint i (Fig. 4-4), the slope deflection equation for M_{ij} becomes

$$M_{ij} = C_{ji}^{(II)} K_{ji}^{(II)} \theta_j = C_{ji}^{(II)} K_{ji}^{(II)} , \qquad (4\text{-}9)$$

and, since M_{ij} and M_{ji} must be equal in magnitude but opposite in sense,

$$C_{ji}^{(II)} = - \frac{K_{ji}^{(II)}}{K_{ji}^{(II)}} = - 1. \qquad (4\text{-}10)$$

Eq. (4-10) is the new carry-over factor for all columns, prismatic and nonprismatic, with a Case II loading. The new carry-over factor for the girder $\overline{jj'}$ (any girder) is zero.

d. Guided Moment

The guided moment $GM_{ji}^{(II)}$ ($GM_{ij}^{(II)}$) is the moment required at the guided (fixed) end j (i) of column $\overline{ij}$ (Fig. 4-5) to prevent rotation at the guided end while permitting it to translate horizontally:

$$\left.\begin{aligned} GM_{ji}^{(II)} &= FM_{ji}^{(II)} - \frac{S_{ji}}{S_{ji} + S_{ij}} h_j FV_{ij}^{(II)} + \frac{S_{ji}}{2(S_{ji} + S_{ij})} h_j \Sigma V_i^{(II)} \\ GM_{ij}^{(II)} &= FM_{ij}^{(II)} - \frac{S_{ij}}{S_{ji} + S_{ij}} h_j FV_{ij}^{(II)} + \frac{S_{ij}}{2(S_{ji} + S_{ij})} h_j \Sigma V_i^{(II)} . \end{aligned}\right\} (4\text{-}11)$$

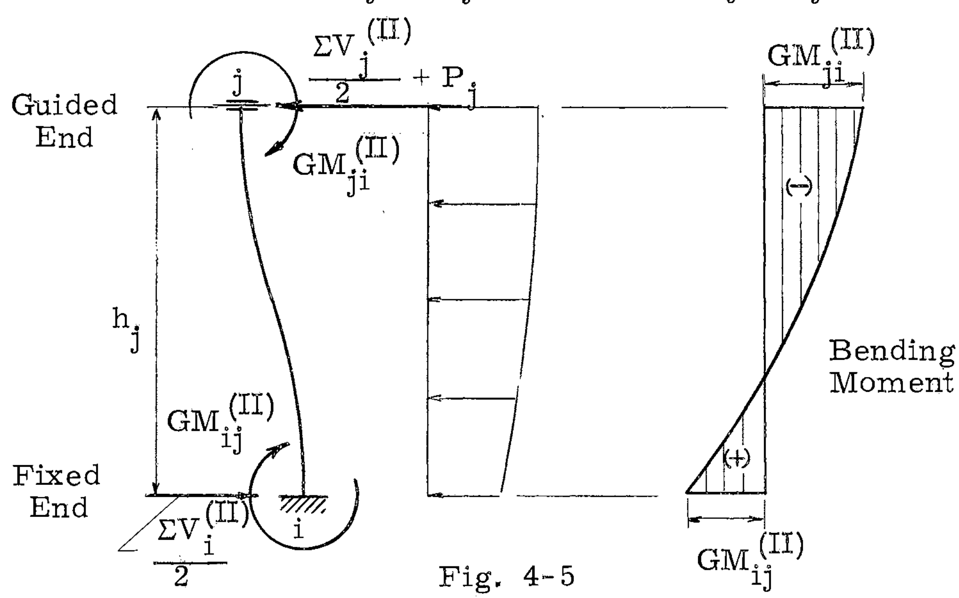


Fig. 4-5
Guided Column $\overline{ij}$

It is obvious that, in the case of a prismatic column, the evaluation of the three terms of the equation (Eqs. 4-11) for the guided moment at i ($GM_{ij}^{(II)}$) will yield equal absolute values to that of their counterparts at j ($GM_{ji}^{(II)}$). However, the first terms of the equations ($FM_{ij}^{(II)}$ and $FM_{ji}^{(II)}$) will be opposite in sense.

The values for guided moments for the most common loading conditions are shown in Table I. These values are for prismatic columns only and can be derived by either Eqs. (4-11) or the area-moment method. The effect of load on stories above j must be considered in evaluating the formulas in Table I.

4. End Moment Equations

From the equilibrium of moments at joint j (any joint),

$$\left.\begin{aligned} \theta_k'' (C_{kj}^{(II)}K_{kj}^{(II)}) + \theta_j'' (K_{jk}^{(II)} + K_{jj}^{(II)} + K_{ji}^{(II)}) + \theta_i'' (C_{ij}^{(II)}K_{ij}^{(II)}) = \\ = -GM_{jk}^{(II)} - FM_{jj}^{(II)} - GM_{ji}^{(II)} . \end{aligned}\right\} \quad (4\text{-}12)$$

Since

$$K_{jk}^{(II)} + K_{jj}^{(II)} + K_{ji}^{(II)} = \Sigma K_j^{(II)}$$

$$GM_{jk}^{(II)} + FM_{jj}^{(II)} + GM_{ji}^{(II)} = \Sigma FM_j^{(II)} ,$$

the equilibrium, or three-slope, equation becomes

$$\theta_k'' C_{kj}^{(II)}K_{kj}^{(II)} + \theta_j'' \Sigma K_j^{(II)} + \theta_i'' C_{ij}^{(II)}K_{ij}^{(II)} = - \Sigma FM_j . \quad (4\text{-}13)$$

If θ_k'' and θ_i'' are assumed to be temporarily zero,

$$\theta_j'' = - \frac{\Sigma FM_j^{(II)}}{\Sigma K_j^{(II)}} , \quad (4\text{-}14)$$

and the expressions for end moments at j (Eqs. 4-6) reduce to

$$\left.\begin{aligned} M_{jk}^{(II)} &= - D_{jk}^{(II)} \Sigma FM_j^{(II)} + GM_{jk}^{(II)} \\ M_{jj}^{(II)} &= - D_{jj}^{(II)} \Sigma FM_j^{(II)} + FM_{jj}^{(II)} \\ M_{ji}^{(II)} &= - D_{ji}^{(II)} \Sigma FM_j^{(II)} + GM_{ji}^{(II)} \quad . \end{aligned}\right\} \quad (4\text{-}15)$$

The symbols

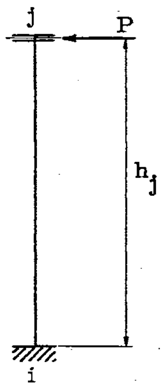
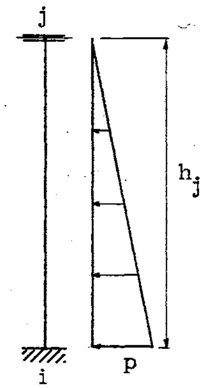
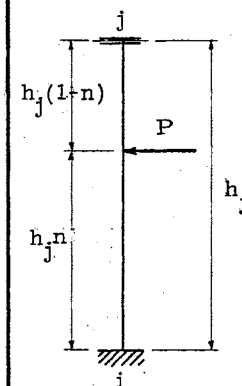
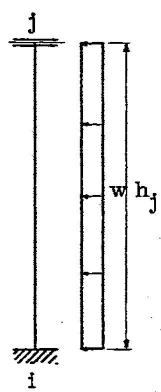
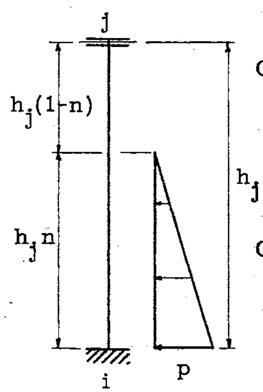
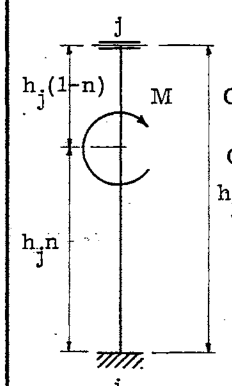
$$D_{jk}^{(II)} = \frac{K_{jk}^{(II)}}{\Sigma K_j^{(II)}} \quad \Bigg| \quad D_{jj}^{(II)} = \frac{K_{jj}^{(II)}}{\Sigma K_j^{(II)}} \quad \Bigg| \quad D_{ji}^{(II)} = \frac{K_{ji}^{(II)}}{\Sigma K_j^{(II)}} \quad (4\text{-}16)$$

are the new distribution factors for joint j (any joint).

Taking advantage of the above relationships and Eq. (4-10), the end moments at the far ends k and i become

$$\left.\begin{aligned} M_{kj}^{(II)} &= - D_{kj}^{(II)} \Sigma FM_j^{(II)} + GM_{kj}^{(II)} \\ M_{ij}^{(II)} &= - D_{ij}^{(II)} \Sigma FM_j^{(II)} + GM_{ij}^{(II)} \quad . \end{aligned}\right\} \quad (4\text{-}17)$$

TABLE I — GUIDED MOMENTS FOR COMMON LOADING CONDITIONS

 $GM_{ij}^{(II)} = \frac{Ph_j}{2}$ $GM_{ji}^{(II)} = \frac{Ph_j}{2}$	 $GM_{ij}^{(II)} = \frac{ph_j^2}{8}$ $GM_{ji}^{(II)} = \frac{ph_j^2}{24}$	 $GM_{ij}^{(II)} = Ph_j n \left(1 - \frac{n}{2}\right)$ $GM_{ji}^{(II)} = \frac{Ph_j n^2}{2}$
 $GM_{ij}^{(II)} = \frac{wh_j^2}{3}$ $GM_{ji}^{(II)} = \frac{wh_j^2}{6}$	 $GM_{ij}^{(II)} = \frac{ph_j^2 n^2}{24}(4 - n)$ $GM_{ji}^{(II)} = \frac{ph_j^2 n^3}{24}$	 $GM_{ij}^{(II)} = M(n - 1)$ $GM_{ji}^{(II)} = - Mn$

PART V

PROCEDURE

The procedure of analysis of one-bay, multi-story, rectangular frames may be summarized in the following steps:

A. Divide problem into two cases of loading, symmetrical and antisymmetrical, Case I and II, respectively.

B. Case I - Symmetrical Case:

1. Calculate the elastic constants: stiffness factors (K), carry-over factors (C), modified stiffness factors (K'), new stiffness factors ($K^{(I)}$), and new distribution factors ($D^{(I)}$).

2. Calculate the load functions: fixed end moments due to dead load, live load and wind load. Sum the fixed end moments.

3. Perform moment distribution to obtain end moments due to the Case I loading.

C. Case II - Antisymmetrical Case

1. Calculate the elastic constants: stiffness factors (K), carry-over factors (C), sidesway factors (S), new stiffness factors ($K^{(II)}$), new carry-over stiffness factors ($C^{(II)}K^{(II)}$), new carry-over factors ($C^{(II)}$), and new distribution factors ($D^{(II)}$).

2. Calculate the guided moments.

3. Perform moment distribution to obtain end moments due to Case II loading.

D. Final End Moments: Calculate the final end moments by adding the results of Parts B and C.

E. Numerical Control: Perform numerical control of Part D by means of shear equations.

This procedure is followed in the numerical example set forth in the following part of this study.

PART VI

NUMERICAL EXAMPLE

A. Statement of the Problem

A typical multi-story frame (Fig. 6-1) is to be analyzed by the moment distribution procedure set forth in the preceeding parts of this study.

It is assumed that concrete weighs 125 lbs./cu. ft., and that all members are one foot wide and have equal moduli of elasticity. All values are given in kips, feet, or kip-feet.

Due to the symmetry of the structure, the problem may be divided into two cases of loading, symmetrical and antisymmetrical, as shown in Figs. 6-2. Only one distribution is then necessary for each case and the superposition of the two results provides the final end moments.

B. Case I - Symmetrical Case

1. Elastic Constants

The elastic constants for haunched beams were taken from Reference (11).

a. Stiffness Factors

$$K_{01} = K_{10} = K_{12} = K_{21} = K_{23} = K_{32} = 13.50$$

$$K_{11} = K_{22} = K_{33} = 71.67$$

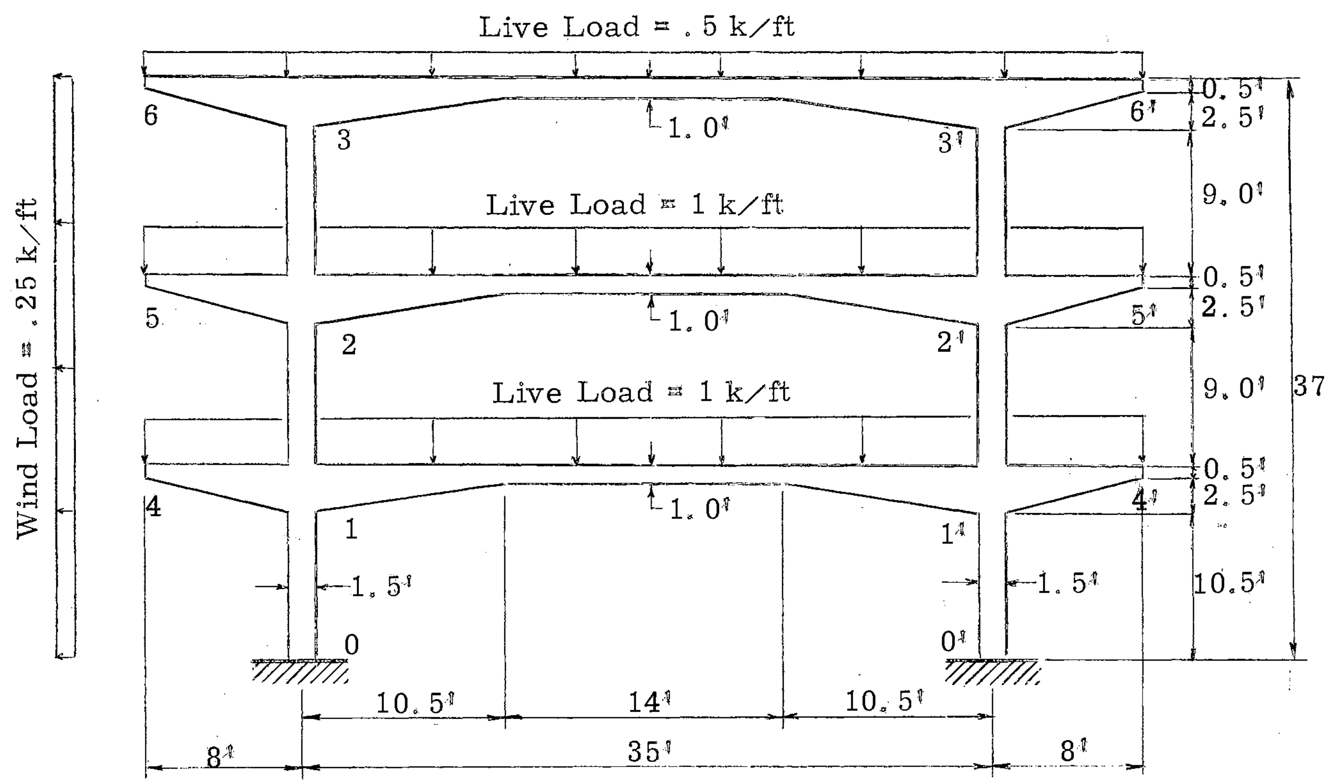
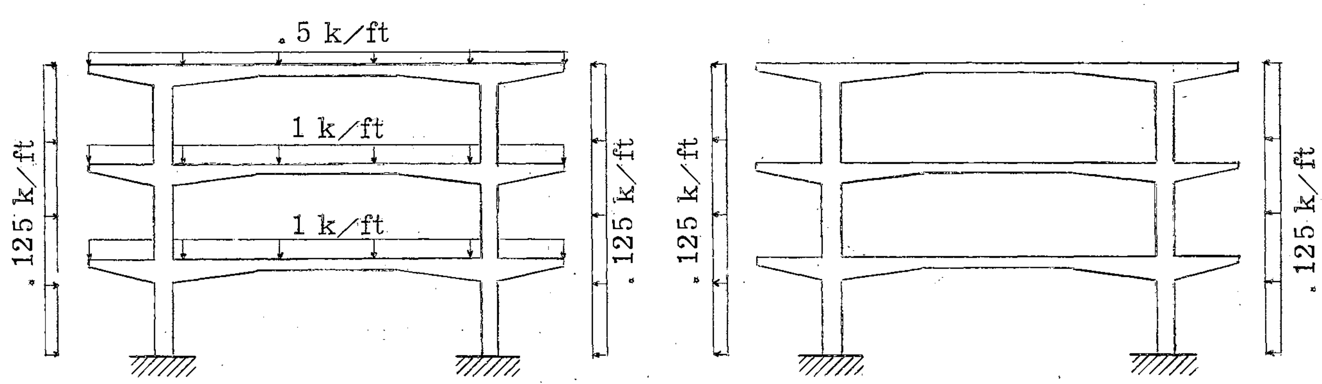


Fig. 6-1

One-bay, Multi-story Frame



a) Case I Loading

b) Case II Loading

Fig. 6-2

Modifications of Loading

b. Carry-over Factors

$$C_{01} = 0$$

$$C_{10} = C_{12} = C_{21} = C_{23} = C_{32} = .5$$

$$C_{11} = C_{22} = C_{33} = .785$$

c. Modified Stiffness Factors

$$K'_{11} = K'_{22} = K'_{33} = 71.67\,(1-.785) = 15.42$$

d. New Stiffness Factors

$$K_{01}^{(I)} = K_{10}^{(I)} = K_{12}^{(I)} = K_{21}^{(I)} = K_{23}^{(I)} = K_{32}^{(I)} = 13.50$$

$$K_{11}^{(I)} = K_{22}^{(I)} = K_{33}^{(I)} = 15.42$$

e. New Distribution Factors

$$D_{01}^{(I)} = D_{14}^{(I)} = D_{25}^{(I)} = D_{36}^{(I)} = 0$$

$$D_{10}^{(I)} = D_{12}^{(I)} = D_{21}^{(I)} = D_{23}^{(I)} = \frac{K_{10}^{(I)}}{\Sigma K_1^{(I)}} = \frac{13.50}{42.42} = .318$$

$$D_{32}^{(I)} = \frac{K_{32}^{(I)}}{\Sigma K_3^{(I)}} = \frac{13.50}{28.92} = .467$$

$$D_{11}^{(I)} = D_{22}^{(I)} = \frac{K_{11}^{(I)}}{\Sigma K_1^{(I)}} = \frac{15.42}{42.42} = .364$$

$$D_{33}^{(I)} = \frac{K_{33}^{(I)}}{\Sigma K_3^{(I)}} = \frac{15.42}{28.92} = .533$$

2. Load Functions

a. Fixed End Moments Due to Dead Load

$$FM_{14}^{(DL)} = FM_{25}^{(DL)} = FM_{36}^{(DL)} = (8)(.5)(1)(.125) = .500$$

$$FM_{11}^{(DL)} = FM_{22}^{(DL)} = FM_{33}^{(DL)} =$$

$$= - (.1099)(1)(1)(.125)(35)^2 -$$

$$- (.0147)(1)(2)(.125)(85)^2 = - 21.330$$

b. Fixed End Moments Due to Live Load

$$FM_{14}^{(LL)} = FM_{25}^{(LL)} = (.5)(1)(8)^2 = 32.000$$

$$FM_{36}^{(LL)} = (.5)(.5)(8)^2 = 16.000$$

$$FM_{11}^{(LL)} = FM_{22}^{(LL)} = - (.1099)(1)(35)^2 = - 134.628$$

$$FM_{33}^{(LL)} = - (.1099)(.5)(35)^2 = - 67.314$$

c. Fixed End Moments Due to Wind Load

$$FM_{01}^{(WL)} = - FM_{10}^{(WL)} = FM_{12}^{(WL)} = - FM_{21}^{(WL)} =$$

$$= FM_{23}^{(WL)} = - FM_{32}^{(WL)} = - \frac{(.125)(12)^2}{12} = -1.500$$

d. Total Fixed End Moments

$$FM_{14}^{(I)} = FM_{25}^{(I)} = 32.500$$

$$FM_{36}^{(I)} = 16.500$$

$$FM_{01}^{(I)} = - FM_{10}^{(I)} = FM_{12}^{(I)} = - FM_{21}^{(I)} = FM_{23}^{(I)} = - FM_{32}^{(I)} =$$

$$= - 1.500$$

$$FM_{11}^{(I)} = FM_{22}^{(I)} = - 155.958$$

$$FM_{33}^{(I)} = - 88.644$$

3. Distribution of Moments

The distribution of moments for Case I is carried out in Table II. The moments of the right side of the structure are opposite in sense to the final end moments obtained by this distribution.

C. Case II - Antisymmetrical Case

1. Elastic Constants

a. Stiffness Factors

$$K_{01} = K_{10} = K_{12} = K_{21} = K_{23} = K_{32} = 13.50$$

$$K_{11} = K_{22} = K_{33} = 71.67$$

b. Carry-over Factors

$$C_{01} = C_{10} = C_{12} = C_{21} = C_{23} = C_{32} = .5$$

$$C_{11} = C_{22} = C_{33} = .785$$

c. Sidesway Factors (Eqs. 4-1)

$$S_{01} = S_{10} = S_{12} = S_{21} = S_{23} = S_{32} =$$

$$= K_{10}\ (1 + C_{10}) = 13.50\ (1 + .5) = 20.25$$

$$S_{11} = S_{22} = S_{33} =$$

$$= K_{11}\ (1 + C_{11}) = 71.67\ (1 + .785) = 127.93$$

d. New Stiffness Factors (Eqs. 4-7)

$$K_{01}^{(II)} = K_{10}^{(II)} = K_{12}^{(II)} = K_{21}^{(II)} = K_{23}^{(II)} = K_{32}^{(II)} =$$

$$= K_{10} - \frac{S_{10}^{\ 2}}{S_{10} + S_{01}} = 13.50 - 10.125 = 3.375$$

$$K_{11}^{(II)} = K_{22}^{(II)} = K_{33}^{(II)} = 20.25$$

e. New Carry-over Stiffness Factors (Eq. 4-8)

$$C_{01}^{(II)}K_{01}^{(II)} = C_{10}^{(II)}K_{10}^{(II)} = C_{12}^{(II)}K_{12}^{(II)} = C_{21}^{(II)}K_{21}^{(II)} =$$

$$C_{23}^{(II)}K_{23}^{(II)} = C_{32}^{(II)}K_{32}^{(II)} = C_{10}K_{10} - \frac{S_{10}\,S_{01}}{S_{10} + S_{01}} =$$

$$= 6.75 - 10.125 = -3.375$$

$$C_{11}^{(II)}K_{11}^{(II)} = C_{22}^{(II)}K_{22}^{(II)} = C_{33}^{(II)}K_{33}^{(II)} = 0$$

f. New Carry-over Factors (Eq. 4-10)

$$C_{01}^{(II)} = 0$$

$$C_{10}^{(II)} = C_{12}^{(II)} = C_{21}^{(II)} = C_{23}^{(II)} = C_{32}^{(II)} = -1$$

g. New Distribution Factors (Eqs. 4-16)

$$D_{10}^{(II)} = D_{12}^{(II)} = D_{21}^{(II)} = D_{23}^{(II)} =$$

$$= \frac{K_{10}^{(II)}}{\Sigma K_1^{(II)}} = \frac{3.375}{3.375 + 20.250 + 3.375} = .125$$

$$D_{11}^{(II)} = D_{22}^{(II)} = \frac{K_{11}^{(II)}}{\Sigma K_1^{(II)}} = \frac{3.375}{27.000} = .750$$

$$D_{32}^{(II)} = \frac{K_{32}^{(II)}}{\Sigma K_3^{(II)}} = \frac{3.375}{3.375 + 20.250} = .143$$

$$D_{33}^{(II)} = \frac{K_{33}^{(II)}}{\Sigma K_3^{(II)}} = \frac{20.250}{23.625} = .857$$

TABLE II				DISTRIBUTION OF MOMENTS							CASE I
MEMBER	01	10	14	11'	12	21	25	22'	23	32	36	33'
$D^{(I)}$	0	- .318	0	- .364	- .318	- .318	0	- .364	- .318	- .467	0	- .533
$C^{(I)}$	0	.5	0	0	.5	.5	0	0	.5	.5	0	0
$FM^{(I)}$	+ 1.500	- 1.500	+32.500	-155.958	+ 1.500	- 1.500	+32.500	-155.958	+ 1.500	- 1.500	+16.500	-88.644
1D		+39.260		+ 44.938	+39.260	+39.260		+ 44.938	+39.260	+34.392		+39.252
1C	+19.630				+19.630	+19.630			+17.196	+19.630		
2D		- 6.242		- 7.146	- 6.242	-11.711		- 13.404	-11.711	- 9.167		-10.463
2C	- 3.121				- 5.856	- 3.121			- 4.584	- 5.856		
3D		+ 1.862		+ 2.132	+ 1.862	+ 2.450		- 2.805	+ 2.450	+ 2.735		+ 3.121
Final Moments	+18.009	+33.380	+32.500	-116.034	+50.154	+45.008	+32.500	-121.619	+44.111	+40.234	+16.500	-56.734

TABLE III				DISTRIBUTION OF MOMENTS							CASE II
MEMBER	01	10	14	11'	12	21	25	22'	23	32	36	33'
$D^{(II)}$	0	- .125	0	- .750	- .125	- .125	0	- .750	- .125	- .143	0	- .857
$C^{(II)}$	0	-1	0	0	-1	-1	0	0	-1	-1	0	0
$GM^{(II)}$	+24.000	+21.000			+15.000	+12.000			+ 6.000	+ 3.000		
1D		- 4.500		-27.000	- 4.500	- 2.250		-13.500	- 2.250	- .429		-2.571
1C	+ 4.500				+ 2.250	+ 4.500			+ .429	+ 2.250		
2D		- .281		- 1.688	- .281	- .616		- 3.697	- .616	- .322		-1.928
2C	+ .281				+ .616	+ .281			+ .322	+ .616		
3D		- .077		- .462	- .077	- .075		- .453	- .075	- .088		- .528
Final Moments	+28.781	+16.142	0	-29.150	+13.008	+13.840	0	-17.650	+ 3.810	+ 5.027	0	-5.027

$$D_{01}^{(II)} = D_{14}^{(II)} = D_{25}^{(II)} = D_{36}^{(II)} = 0$$

2. Guided Moments (Eqs. 4-11)

$$GM_{01}^{(II)} = FM_{01}^{(II)} - \frac{1}{2} h_1 \ FV_{01}^{(II)} + \frac{1}{4} h_1 \ \Sigma V_0^{(II)}$$

$$= \frac{(.125)(12)^2}{12} - \frac{1}{2} \ \frac{(12)(12)(.125)}{2} + \frac{1}{4} \ (12)(36)(.250)$$

$$= +1.500 - 4.500 + 27.000 = 24.000$$

$$GM_{10}^{(II)} = -1.500 - 4.500 + 27.000 = 21.000$$

$$GM_{12}^{(II)} = +1.500 - 4.500 + 18.000 = 15.000$$

$$GM_{21}^{(II)} = -1.500 - 4.500 + 18.000 = 12.000$$

$$GM_{23}^{(II)} = +1.500 - 4.500 + 9.000 = 6.000$$

$$GM_{32}^{(II)} = -1.500 - 4.500 + 9.000 = 3.000$$

3. Distribution of Moments

The distribution of moments for Case II is carried out in Table II. The moments of the right side of the structure are equal in value and sense to those obtained by this distribution.

D. Final End Moments

The final end moments are obtained by summing the final moments of Cases I and II:

$$M_{01} = +18.009 + 28.781 = +46.790$$

$$M_{10} = +33.380 + 16.142 = +49.522$$

$$M_{14} = +32.500 + 0 = +32.500$$

$$M_{11'} = -116.034 - 29.150 = -145.184$$

$$M_{12} = +50.154 + 13.008 = +63.162$$

$$M_{21} = +45.008 + 13.840 = +58.848$$

$$M_{25} = +32.500 + 0 = +32.500$$

$$M_{22'} = -121.619 - 17.650 = -139.269$$

$$M_{23} = +44.111 + 3.810 = +47.921$$

$$M_{32} = +40.234 + 5.027 = +45.261$$

$$M_{36} = +16.500 + 0 = +16.500$$

$$M_{33'} = -56.734 - 5.027 = -61.761$$

$$M_{0'1'} = -18.009 + 28.781 = +10.772$$

$$M_{1'0'} = -33.380 + 16.142 = -17.238$$

$$M_{1'4'} = -32.500 + 0 = -32.500$$

$$M_{1'1} = +116.034 - 29.150 = +86.884$$

$$M_{1'2'} = -50.154 + 13.008 = -37.146$$

$$M_{2'1'} = -45.008 + 13.840 = -31.168$$

$$M_{2'5'} = -32.500 + 0 = -32.500$$

$$M_{2'2} = +121.619 - 17.650 = +103.969$$

$$M_{2'3'} = -44.111 + 3.810 = -40.301$$

$$M_{3'2'} = -40.234 + 5.027 = -35.207$$

$$M_{3'6'} = -16.500 + 0 = -16.500$$

$$M_{3'3} = +56.734 - 5.027 = +51.707$$

E. Numerical Control

Numerical control is performed by the use of shear equations.

First, the structure is cut immediately above the supports (0 and 0') and the equilibrium equation is written:

$$\frac{M_{01} + M_{10}}{h_1} + \frac{M_{0'1'} + M_{1'0'}}{h_1} + \frac{wh}{2} - w\,(3h) = 0$$

$$+ \frac{46.790 + 49.522}{12} + \frac{10.772 - 17.238}{12} + \frac{(.250)(12)}{2} - (.250)(36) \doteq 0$$

$$+ 8.026 - .539 + 1.500 - 9.000 \doteq 0$$

Second, the structure is cut immediately above girder $\overline{11'}$ and the equilibrium equation is written:

$$\frac{M_{12} + M_{21}}{h_2} + \frac{M_{1'2'} + M_{2'1'}}{h_2} + \frac{wh}{2} - w\,(2h) = 0$$

$$+ \frac{63.162 + 58.848}{12} + \frac{-37.146 - 31.168}{12} + \frac{(.250)(12)}{2} - (.250)(24) \doteq 0$$

$$+ 10.168 - 5.693 + 1.500 - 6.000 \doteq 0$$

Third, the structure is cut immediately above girder $\overline{22'}$ and the equilibrium equation is written:

$$\frac{M_{23} + M_{32}}{h_3} + \frac{M_{2'3'} + M_{3'2'}}{h_3} + \frac{wh}{2} - wh = 0$$

$$+ \frac{47.921 + 45.261}{12} + \frac{-40.301 - 35.207}{12} + \frac{(.250)(12)}{12} - (2.50)(12) \doteq 0$$

$$+ 7.765 - 6.292 + 1.500 - 3.000 \doteq 0$$

The reason for the slight errors in the check is the fact that the moment distribution procedure was stopped after three cycles. It can be carried out to any desired degree of accuracy.

PART VII

SUMMARY AND CONCLUSIONS

The primary objective of this study is to develop a simplified method of moment distribution for one-bay, multi-story, rectangular frames consisting of members of any cross-section. The principles of such a method were originally presented by Professor Tuma in Reference (19).

The introduction of the guided moments for the Case II loading offers a fast and accurate method for computing the end moments. Regardless of the number of stories, the analysis requires only two distributions of moments, since all unknown displacements are eliminated in the derivation of equations in Part IV.

All equations presented in this study are perfectly general and can be used in the analysis of all one-bay, multi-story, rectangular frames.

A SELECTED BIBLIOGRAPHY

1. Discussion by Heller, Charles O., to "Analysis of Two-Column Symmetrical Frames," Proceedings, ASCE, vol. 85, ST10, December, 1959, p. 2037-125.

2. Andrée, W. L., Das B = U Verfahren, R. Oldenburg, Munich, 1919.

3. Newell, J. S., "Symmetric and Anti-Symmetric Loadings," Civil Engineering, April, 1939, pp. 249-257.

4. Bayer, K., Die Statik im Stahlbetonbau, Springer, Berlin, 1948, 2nd. Ed., pp. 457-480.

5. Naylor, N., "Sidesway in Symmetrical Building Frames," The Structural Engineer, April 1950, p. 99.

6. Discussion by Pei, Ming-Lung, to "Joint Translation by Cantilever Moment Distribution," Transactions, ASCE, vol. 119, 1954, p. 1208.

7. Perri, Joseph G., "Modified Method of Moment Distribution for Analyzing Compression Members as Parts of Rigid Frames," Ph. D. Thesis, New York University, June 1948.

8. Hadley, Richard H., "Sidesway Analysis of Rectangular Rigid Frames by Component Moment Methods," M. S. Thesis, University of Washington, 1949.

9. Kavanagh, T. C., "Analysis and Design of Columns in Frames Subject to Translation," Column Research Council Report, 1950.

10. Grinter, L. E. and Tsao, C. H., "Joint Translation by Cantilever Moment Distribution," Transactions, ASCE, vol. 119, 1954, p. 1195.

11. Parcel, J. I. and Moorman, R. B. B., Analysis of Statically Indeterminate Structures, John Wiley and Sons, Inc., New York, N. Y., 1955, pp. 263, 279-285.

12. Kupferschmid, Victor, Ebene und Räumliche Rahmentragwerke, Springer-Verlag, Vienna, 1952, p. 177.

13. Kazda, Jaromír, Rozvod Momentů, S. N. T. L., Prague, 1959, pp. 37, 318-327, 363-374.

14. Goldberg, John E., "Lateral Load Analysis of Two-Column Bents," Proceedings, ASCE, vol. 84, No. ST3, Paper 1638, May, 1958.

15. Discussion by Nubar, Yves, to "Lateral Load Analysis of Two-Column Bents," Proceedings, ASCE, vol. 84, ST7, November, 1958, p. 1857-43.

16. Discussion by Sobotka, Zdeněk, to "Lateral Load Analysis of Two-Column Bents," Proceedings, ASCE, vol. 84, ST7, November, 1958, p. 1857-48.

17. Discussion by Chang, I. Chen, to "Lateral Load Analysis of Two-Column Bents," Proceedings, ASCE, vol. 84, ST7, November, 1958, p. 1857-56.

18. Discussion by Cooke, B. R., to "Lateral Load Analysis of Two-Column Bents," Proceedings, ASCE, vol. 84, ST8, December, 1958, p. 1882-13.

19. Lecture by Tuma, Jan J., Civil Engineering 620 - Seminar in Carry-Over Methods, Oklahoma State University, April, 1959.

VITA

Charles O. Heller

Candidate for the Degree of

Master of Science

Thesis: ANALYSIS OF ONE-BAY, MULTI-STORY, RECTANGULAR FRAMES BY MODIFIED MOMENT DISTRIBUTION

Major Field: Civil Engineering

Biographical:

Personal Data: Born in Prague, Czechoslovakia, on January 25, 1936, the son of Rudolph and Ilona Heller.

Education: Attended grade schools in Prague, Czechoslovakia, and Morristown, New Jersey. Graduated from Morristown High School in June, 1954. Attended the Oklahoma State University from September, 1954 to January, 1960, having spent one intervening year at Rutgers University and one summer at Stevens Institute of Technology. Completed the requirements for the degree of Bachelor of Science in Civil Engineering at the Oklahoma State University in January, 1959 and received the degree in May, 1959. Completed the requirements for the degree of Master of Science at the Oklahoma State University in January, 1960.

Professional Experience: Employed as a draftsman by the Bell Telephone Laboratories in Whippany, New Jersey, during the summers of 1955 and 1956. Employed by the School of Civil Engineering at the Oklahoma State University as a Student Assistant (Research and Grading) from September, 1958 to May, 1959, and as a Graduate Assistant (Research and Teaching) from June, 1959 until January, 1960. Author of seven articles in the Oklahoma State Engineer Magazine and one in the Proceedings of the American Society of Civil Engineers. Presently, an Associate Member of the American Society of Civil Engineers and a Junior Member of the Oklahoma Society of Professional Engineers and the National Society of Professional Engineers.