THEORY OF BENDING OF MULTI-LAYER SANDWICH PLATES

By

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THEORY OF BENDING OF MULTI-LAYER
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Boen-Dar Liaw

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NOMENCLATURE

The following symbols have been adopted for use in this thesis:

a	length of plate in x-direction;
b	length of plate in y-direction;
c_1, c_2, c_3, c_4	some constants defined by the geometrical and material properties;
C, C_x, C_y	transverse shear rigidities of plate;
D, D_{xy}	bending and torsional rigidities of plate;
E_1, E_2, E_3	Young's modulus of elasticity of facing membrane;
$G_{1xz}, G_{1yz}, G_{2xz}, G_{2yz}$	shear modulus of elasticity of core;
h_1, h_2	thickness of core;
i	index, designates ith membrane;
j	index, designates jth core;
K_1, K_2	constant defined by ν, C, C_x and C_y;
M_x, M_y, M_{xy}	moment, twisting moment per unit width of plate;
N_x, N_y, N_{xy}	stress resultant per unit width of plate;
p	loading function normal to the plate;
Q_x, Q_y	shear force per unit width of plate;
t_1, t_2, t_3	thickness of facing membrane;
U	strain energy;

V^*, V^{**}	complementary energy, auxiliary functional;
w	Lagrangian multiplier, transverse deflection of the plate;
W	work done by edge forces and moments;
z_1, z_2, z_3, z_o	distance measured from xy-plane to the middle plane of first, second, third membrane, to neutral surface;
α	Lagrangian multiplier, slope;
β	Lagrangian multiplier, slope;
$\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$	generalized displacement at boundary;
$\lambda_1, \lambda_2, \ldots, \lambda_8$	Lagrangian multiplier;
$\nu_1, \nu_2, \nu_3, \nu_o, \nu$	Poisson's ratio of membrane, equivalent Poisson's ratio, common Poisson's ratio;
ξ_1, ξ_2, ξ_3	constant defined on c_1, c_2, c_3 and c_4;
σ_{ix}, σ_{iy}	normal stress in ith membrane;
τ_{ixy}	shearing stress in ith membrane;
∇^2	Laplacian operator;
∇^4	$\nabla^2 \nabla^2$ and
∇^6	$\nabla^2 \nabla^2 \nabla^2$.

Additional symbols used in the example problem and in the Appendix are defined when they appear and are not listed.

CHAPTER I

INTRODUCTION

1.1 General

A small-deflection theory for multi-core sandwich plates is developed by means of variational principles. A set of differential equations governing the deflection, moments and transverse shear forces of the plate is derived.

The sandwich constructions are characterized by the relatively low-stiffness core materials between facing membranes with high moduli of rigidities. In this investigation, the facing materials are considered to be isotropic and homogeneous. However, the different cores possess different elastic properties and, in general, are assumed to be orthotropic. The reduction to the case of isotropic cores is also shown. All deflections are defined on neutral surface, and considered that the transverse deflection of the upper and lower surfaces, at any location on the plate, are the same as that of neutral surface.

In addition to above general description concerning the property of this type of construction, the following assumptions are essentially necessary for this analysis:

(a) The total thickness is still small in comparison with the dimensions in other directions, i.e., the plates are considered to be "thin" plates.

(b) Under all kinds of loadings, all bonds between each layer are considered strong enough so that no bond failure may occur and stresses can be transmitted without discontinuity.

(c) Non-homogeneity of the core cell is neglected.

(d) The transverse rigidities of the core materials are relatively high compared to the facing materials, i.e., transverse shear forces are completely taken by cores.

(e) The core stiffnesses associated with plane stress components in the plane of structure are neglected.

(f) The deformations due to temperature change are not taken into account in this presentation.

The development of the theory falls mainly on the formulation of the complementary energy functional, minimizing process and the elimination of the additional unknowns of Lagrangian multipliers. This portion of analysis forms the content of the second chapter of this thesis. The necessary constant quantities describing the cross-sectional property of the sandwich plate are also defined. Once the set of Euler equations is obtained, the eliminating process can proceed to obtain the differential equations governing the transverse shear forces, bending and torsional moments and the transverse deflection of the plate. Reductions to the particular case of single core sandwich plate and to the ordinary isotropic homogeneous plate equation are also shown in this third chapter. In the fourth chapter, a bending problem of this type of construction with particular edge conditions subjected to a general system of loads is solved to illustrate the application of the developed theory. Summary and conclusions of this study, as well as the desirable extension, are included in the final chapter.

The letter symbols adopted for use in this thesis are defined where they first appear and are listed in the Nomenclature.

1.2 Historical Notes

The analytic study of the sandwich construction becomes increasingly important with the development of new and high strength materials and the complexity of the aeronautic structures. The vast majority of past effort connected with this study has been confined to a single-core construction with two either identical or different facing plates.

The first analytic investigation which appeared in the literature associated with this problem was done by E. Reissner(1) in 1947.* He considered a plate consisting of a core layer with two facing membranes identical both in thickness and elastic properties, and assumed that the face-parallel stresses in the core and the variation of the face stresses over the thickness of the face layers are negligible. The same assumptions were also made by N. J. Hoff(2), but in a more general form for solving the buckling problem. This Reissner-Hoff assumption is one of the main approaches in analyzing this type of construction, and will be adopted as the basic assumption of this dissertation. For non-isotropic sandwich plates, C. Libove and S. B. Batdorf(3) considered the sandwich plate approximately as a non-isotropic thick plate

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* Numbers in parenthesises refer to references in Bibliography.

and extended the classical thin plate theory to sandwich panels by introducing the effective bending and shearing rigidities, and taking the shear deformation into account. In 1951, A. C. Eringen[4] extended this theory to include the flexural rigidity of core in his investigation. Since 1959, in a series of publications, Y. Y. Yu[5],[6],[7] presented a flexural theory, for the isotropic case, to include the shear deformations in the facing materials. Theoretically, his investigation has generalized the Reissner-Hoff's "membrane facings" to the "plate-facings" theory.

In the period between E. Reissner and Y. Y. Yu, a great deal of works had been done by many investigators. However, most of these studies are limited to particular problems which are still based on "membrane facings" theory, and also are confined in single core sandwich constructions. Their contributions are not in the development of the theory, but in the techniques of solving the problems. For instance, S. Cheng[8] modified Reissner's problem for orthotropic cores and related the solution of the sandwich plate equation to the solutions of the biharmonic equation of classical plate theory. In 1960, C. C. Chang and I. K. Ebicoglu[9] presented their studies on the elastic instability of rectangular sandwich panels with orthotropic cores and different face thicknesses and materials.

CHAPTER II

GENERAL ANALYSIS

2.1 Statement of the Problem

A rectangular sandwich plate consisting of two cores of thicknesses h_1 and h_2 and three facing membranes of thicknesses t_1, t_2 and t_3 is considered (Fig. 1). Let the xy-plane be a plane parallel to the undeformed surface of the plate with z-axis along the normal to this plane. Also, let z_1, z_2 and z_3 be the distances measured from the xy-plane to the middle plane of each membrane respectively. Each facing is assumed to be isotropic and homogeneous and possessing different elastic properties, while the cores are both assumed to be orthotropic.

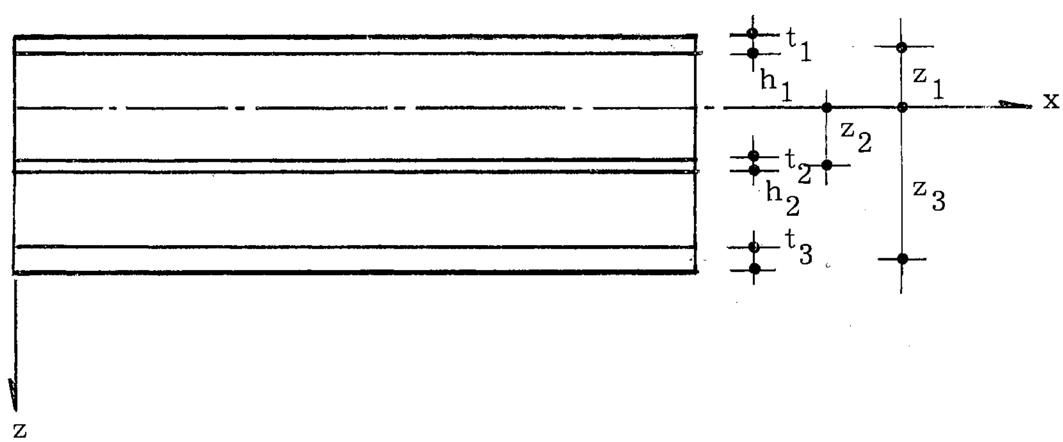

FIG. 1 A CROSS SECTION OF PLATE

The problem to be solved is, then, to develop a theory defining the bending behavior of the sandwich plate due to a general type of externally applied load which is normal to the plate.

The formulation which will be presented in the following sections is also good for a plate with many cores. However, for simplicity in presentation, only a plate with two cores as described above will be considered.

2.2 Stress Resultants and Equilibrium Equations

After the assumptions made previously that the face-parallel stresses of cores and the variations of the stresses of facing membranes are negligible, the stress resultants and stress couples may be defined as follow:

$$M_x = t_1 z_1 \sigma_{1x} + t_2 z_2 \sigma_{2x} + t_3 z_3 \sigma_{3x} \tag{1}$$

$$M_y = t_1 z_1 \sigma_{1y} + t_2 z_2 \sigma_{2y} + t_3 z_3 \sigma_{3y} \tag{2}$$

$$M_{xy} = t_1 z_1 \tau_{1xy} + t_2 z_2 \tau_{2xy} + t_3 z_3 \tau_{3xy} \tag{3}$$

$$Q_x = h_1 \tau_{1xz} + h_2 \tau_{2xz} \tag{4}$$

$$Q_y = h_1 \tau_{1yz} + h_2 \tau_{2yz} \tag{5}$$

$$N_x = t_1 \sigma_{1x} + t_2 \sigma_{2x} + t_3 \sigma_{3x} = 0 \tag{6}$$

$$N_y = t_1 \sigma_{1y} + t_2 \sigma_{2y} + t_3 \sigma_{3y} = 0 \tag{7}$$

$$N_{xy} = t_1\tau_{1xy} + t_2\tau_{2xy} + t_3\tau_{3xy} = 0 \qquad (8)$$

where $M_x\,(M_y)$ designates the bending moment about y(x)-axis, $M_{xy}\,(=M_{yx})$ the twisting moment about y(x)-axis, $Q_x\,(Q_y)$ the transverse shear force on the face normal to x(y)-axis, $N_x\,(N_y)$ the normal force on the face normal to x(y)-axis, $N_{xy}\,(=N_{yx})$ the shear force parallel to the plane of structure and existing on the face normal to x(y)-direction, $\sigma_{ix}(\sigma_{iy})$ the normal stress on ith membrane in the direction of x(y)-axis, τ_{ixy} the shearing stress in ith membrane parallel to xy-plane and $\tau_{jxz}\,(\tau_{jyz})$ the shearing stress on the face normal to x(y)-axis in z-direction of jth core. For a problem of considering bending only, the normal forces N_x and N_y and the shear force N_{xy} are not taken into account, i.e., the stress resultants are considered to be zero.

The moments and transverse shear forces are defined on the plane of structure, as shown in Fig. 2. Summations of moments and transverse forces acting on a differential plane element dx dy of the plate yield the following equilibrium equations:

$$M_{x,x} + M_{xy,y} - Q_x = 0 \qquad (9)$$

$$M_{y,y} + M_{xy,x} - Q_y = 0 \qquad (10)$$

$$Q_{x,x} + Q_{y,y} + p = 0 \qquad (11)$$

where p is the transverse load applied on the differential element of plate.

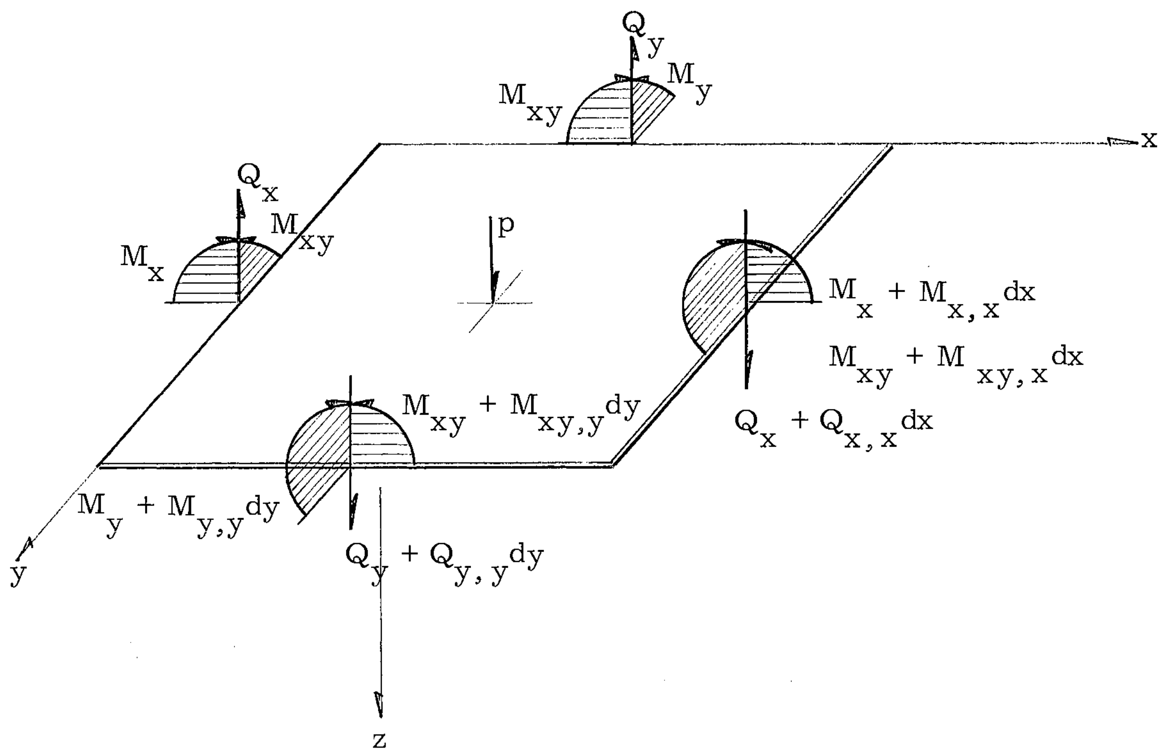

FIG. 2 A DIFFERENTIAL PLATE ELEMENT

2.3 Equations of Compatibility

Considering a rectangular plate of the dimension a by b, the strain energy stored in the system after deformation is expressed in terms of stresses such as:

$$U = \frac{1}{2}\int_0^b\int_0^a\Big[\frac{t_1}{E_1}(\sigma_{1x}^2 + \sigma_{1y}^2 - 2\nu_1\sigma_{1x}\sigma_{1y}) + \frac{2t_1(1+\nu_1)}{E_1}\tau_{1xy}^2$$

$$+ \frac{t_2}{E_2}(\sigma_{2x}^2 + \sigma_{2y}^2 - 2\nu_2\sigma_{2x}\sigma_{2y}) + \frac{2t_2(1+\nu_2)}{E_2}\tau_{2xy}^2$$

$$+ \frac{t_3}{E_3}(\sigma_{3x}^2 + \sigma_{3y}^2 - 2\nu_3\sigma_{3x}\sigma_{3y}) + \frac{2t_3(1+\nu_3)}{E_3}\tau_{3xy}^2\Big]dxdy$$

$$+ \frac{1}{2} \int_{o_c}^{b} \int_{o}^{a} \left[\frac{h_1}{G_{1xz}} \tau_{1xz}^2 + \frac{h_1}{G_{1yz}} \tau_{1yz}^2 + \frac{h_2}{G_{2xz}} \tau_{2xz}^2 + \frac{h_2}{G_{2yz}} \tau_{2yz}^2 \right] dxdy \qquad (12)$$

where the subscript f refers to the facing membranes and c the cores, ν_i the Poisson's ratio of ith facing membrane and G_{ixz} the shear modulus of rigidity of xz-plane of the ith core. Let ζ_1, ζ_2, ζ_3, ζ_4, ζ_5 and ζ_6 be the generalized displacements of the plate prescribed on boundaries, the work done by the boundary forces and moments is

$$W = \int_o^b (Q_x \zeta_1 + M_{xy} \zeta_2 + M_x \zeta_3)_{\substack{x=o \\ x=a}} dy$$

$$+ \int_o^a (Q_y \zeta_4 + M_{xy} \zeta_5 + M_y \zeta_6)_{\substack{y=o \\ y=b}} dx \qquad (13)$$

Then, the complementary energy of the system is defined as[1][10]

$$V^* = U - W \qquad (14)$$

Thus, the problem becomes one of finding the conditions for extremum of the functional V^* subjected to the constraint conditions of equations (1), (2), (3), (4), (5), (6), (7), (8), (9), (10) and (11).

Introducing a set of Lagrangian multipliers λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8, α, β and w, the auxiliary functional can be formulated as follows:

$$V^{**} = \int_o^b \int_o^a \Big\{ \frac{1}{2} \Big[\frac{t_1}{E_1} (\sigma_{1x}^2 + \sigma_{1y}^2 - 2\nu_1 \sigma_{1x} \sigma_{1y}) + \frac{2t_1(1 + \nu_1)}{E_1} \tau_{1xy}^2$$

$$+ \frac{t_2}{E_2} (\sigma_{2x}^2 + \sigma_{2y}^2 - 2\nu_2 \sigma_{2x} \sigma_{2y}) + \frac{2t_2(1 + \nu_2)}{E_2} \tau_{2xy}^2$$

$$+ \frac{t_3}{E_3} (\sigma_{3x}^2 + \sigma_{3y}^2 - 2\nu_3 \sigma_{3x} \sigma_{3y}) + \frac{2t_3(1 + \nu_3)}{E_3} \tau_{3xy}^2$$

$$+ \frac{h_1}{G_{1xz}} \tau_{1xz}^2 + \frac{h_1}{G_{1yz}} \tau_{1yz}^2 + \frac{h_2}{G_{2xz}} \tau_{2xz}^2 + \frac{h_2}{G_{2yz}} \tau_{2yz}^2 \Big]$$

$$+ \lambda_1 (M_x - t_1 z_1 \sigma_{1x} - t_2 z_2 \sigma_{2x} - t_3 z_3 \sigma_{3x})$$

$$+ \lambda_2 (M_y - t_1 z_1 \sigma_{1y} - t_2 z_2 \sigma_{2y} - t_3 z_3 \sigma_{3y})$$

$$+ \lambda_3 (M_{xy} - t_1 z_1 \tau_{1xy} - t_2 z_2 \tau_{2xy} - t_3 z_3 \tau_{3xy})$$

$$+ \lambda_4 (Q_x - h_1 \tau_{1xz} - h_2 \tau_{2xz})$$

$$+ \lambda_5 (Q_y - h_1 \tau_{1yz} - h_2 \tau_{2yz})$$

$$+ \lambda_6 (t_1 \sigma_{1x} + t_2 \sigma_{2x} + t_3 \sigma_{3x})$$

$$+ \lambda_7 (t_1 \sigma_{1y} + t_2 \sigma_{2y} + t_3 \sigma_{3y})$$

$$+ \lambda_8 (t_1 \tau_{1xy} + t_2 \tau_{2xy} + t_3 \tau_{3xy})$$

$$+ \alpha (M_{x,x} + M_{xy,y} - Q_x)$$

$$+ \beta(M_{y,y} + M_{xy,y} - Q_y)$$

$$+ w(Q_{x,x} + Q_{y,y} + p)\Big\} dxdy$$

$$- \int_o^b (Q_x \zeta_1 + M_{xy}\zeta_2 + M_x \zeta_3)_{\substack{x=o \\ x=a}} dy$$

$$- \int_o^a (Q_y \zeta_4 + M_{xy}\zeta_5 + M_y \zeta_6)_{\substack{y=o \\ y=b}} dx \tag{15}$$

It is obvious that the Lagrangian multipliers α, β and w have important physical meanings, i.e., $\alpha(\beta)$ the rotation of the face normal to the x(y)-axis and w the transverse displacement of the sandwich plate. The rest of the set of Lagrangian multipliers are not of primary interest and can be eliminated by using the constraint conditions of equations (1) through (8).

In order to have extrema for the functional V^* subjected to those constraint conditions, the first variation of the auxiliary functional must vanish, i.e., $\delta V^{**} = 0$. For a system of stresses in static equilibrium, it can be proven(10) that this is a minimum of V^*, and that the condition $\delta V^{**} = 0$ gives the set of compatibility conditions of deformations. Carrying out the first variation of V^{**}, integrating by parts and transforming the surface integrals to line integrals by Green's theorem, the set of Euler's equations is obtained as follow:

$$\frac{t_1}{E_1}(\sigma_{1x} - \nu_1 \sigma_{1y}) - t_1 z_1 \lambda_1 + t_1 \lambda_6 = 0 \tag{16}$$

$$\frac{t_1}{E_1}(\sigma_{1y} - \nu_1\sigma_{1x}) - t_1 z_1 \lambda_2 + t_1\lambda_7 = 0 \quad (17)$$

$$\frac{t_2}{E_2}(\sigma_{2x} - \nu_2\sigma_{2y}) - t_2 z_2 \lambda_1 + t_2\lambda_6 = 0 \quad (18)$$

$$\frac{t_2}{E_2}(\sigma_{2y} - \nu_1\sigma_{2x}) - t_2 z_2 \lambda_2 + t_2\lambda_7 = 0 \quad (19)$$

$$\frac{t_3}{E_3}(\sigma_{3x} - \nu_3\sigma_{3y}) - t_3 z_3 \lambda_1 + t_3\lambda_6 = 0 \quad (20)$$

$$\frac{t_3}{E_3}(\sigma_{3y} - \nu_3\sigma_{3x}) - t_3 z_3 \lambda_2 + t_3\lambda_7 = 0 \quad (21)$$

$$\frac{2t_1(1 + \nu_1)}{E_1}\tau_{1xy} - t_1 z_1 \lambda_3 + t_1\lambda_8 = 0 \quad (22)$$

$$\frac{2t_2(1 + \nu_2)}{E_2}\tau_{2xy} - t_2 z_2 \lambda_3 + t_2\lambda_8 = 0 \quad (23)$$

$$\frac{2t_3(1 + \nu_3)}{E_3}\tau_{3xy} - t_3 z_3 \lambda_3 + t_3\lambda_8 = 0 \quad (24)$$

$$\frac{h_1}{G_{1xz}}\tau_{1xz} - h_1\lambda_4 = 0 \quad (25)$$

$$\frac{h_1}{G_{1yz}}\tau_{1yz} - h_1\lambda_5 = 0 \quad (26)$$

$$\frac{h_2}{G_{2xz}} \tau_{2xz} - h_2\lambda_4 = 0 \tag{27}$$

$$\frac{h_2}{G_{2yz}} \tau_{2yz} - h_2\lambda_5 = 0 \tag{28}$$

and

$$\lambda_1 - \alpha_{,x} = 0 \tag{29}$$

$$\lambda_2 - \beta_{,y} = 0 \tag{30}$$

$$\lambda_3 - (\alpha_{,y} + \beta_{,x}) = 0 \tag{31}$$

$$\lambda_4 - (\alpha + w_{,x}) = 0 \tag{32}$$

$$\lambda_5 - (\beta + w_{,y}) = 0 \tag{33}$$

This set of eighteen Euler's equations and those eleven constraint conditions constitute totally a system of twenty-nine equations with twenty-nine unknown quantities, i.e., eleven Lagrangian multipliers, thirteen stresses and five stress resultants and couples.

2.4 Boundary Conditions

In addition to the Euler's equations, a set of physical boundary conditions is also obtained by considering the independent vanishing of each term of the two line integrals in equation (15):

At x = o and x = a,

$$\zeta_1 = w$$

$$\zeta_2 = \beta \tag{34}$$

$$\zeta_3 = \alpha$$

At y = o and y = b,

$$\zeta_4 = w$$

$$\zeta_5 = \alpha \tag{35}$$

$$\zeta_6 = \beta$$

2.5 Stresses, Moments and Shear Forces

Solving for the Lagrangian multipliers λ_1, λ_2, λ_3, λ_4 and λ_5 from equations (29), (30), (31), (32) and (33) in terms of the other set of Lagrangian multipliers λ_6, λ_7, α, β and w and their first partial derivatives such that

$$\lambda_1 = \alpha_{,x} \tag{36}$$

$$\lambda_2 = \beta_{,y} \tag{37}$$

$$\lambda_3 = \alpha_{,y} + \beta_{,x} \tag{38}$$

$$\lambda_4 = \alpha + w_{,x} \tag{39}$$

$$\lambda_5 = \beta + w_{,y} \tag{40}$$

and substituting into equations (16) through (28), the stresses may be expressed as follow:

$$\sigma_{1x} = \frac{E_1 z_1}{1 - \nu_1^2} (\alpha_{,x} + \nu_1 \beta_{,y}) - \frac{E_1}{1 - \nu_1^2} (\lambda_6 + \nu_1 \lambda_7) \tag{41}$$

$$\sigma_{1y} = \frac{E_1 z_1}{1 - \nu_1^2} (\beta_{,y} + \nu_1 \alpha_{,x}) - \frac{E_1}{1 - \nu_1^2} (\lambda_7 + \nu_1 \lambda_6) \tag{42}$$

$$\sigma_{2x} = \frac{E_2 z_2}{1 - \nu_2^2} (\alpha_{,x} + \nu_2 \beta_{,y}) - \frac{E_2}{1 - \nu_2^2} (\lambda_6 + \nu_2 \lambda_7) \tag{43}$$

$$\sigma_{2y} = \frac{E_2 z_2}{1 - \nu_2^2} (\beta_{,y} + \nu_2 \alpha_{,x}) - \frac{E_2}{1 - \nu_2^2} (\lambda_7 + \nu_2 \lambda_6) \tag{44}$$

$$\sigma_{3x} = \frac{E_3 z_3}{1 - \nu_3^2} (\alpha_{,x} + \nu_3 \beta_{,y}) - \frac{E_3}{1 - \nu_3^2} (\lambda_6 + \nu_3 \lambda_7) \tag{45}$$

$$\sigma_{3y} = \frac{E_3 z_3}{1 - \nu_3^2} (\beta_{,y} + \nu_3 \alpha_{,x}) - \frac{E_3}{1 - \nu_3^2} (\lambda_7 + \nu_3 \lambda_6) \tag{46}$$

$$\tau_{1xy} = \frac{E_1 z_1}{2(1 + \nu_1)} (\alpha_{,y} + \beta_{,x}) - \frac{E_1}{2(1 + \nu_1)} \lambda_8 \tag{47}$$

$$\tau_{2xy} = \frac{E_2 z_2}{2(1 + \nu_2)} (\alpha_{,y} + \beta_{,x}) - \frac{E_2}{2(1 + \nu_2)} \lambda_8 \tag{48}$$

$$\tau_{3xy} = \frac{E_3 z_3}{2(1 + \nu_3)} (\alpha_{,y} + \beta_{,x}) - \frac{E_3}{2(1 + \nu_3)} \lambda_8 \tag{49}$$

and

$$\tau_{1xz} = G_{1xz}(\alpha + w_{,x}) \tag{50}$$

$$\tau_{1yz} = G_{1yz}(\beta + w_{,y}) \tag{51}$$

$$\tau_{2xz} = G_{2xz}(\alpha + w_{,x}) \tag{52}$$

$$\tau_{2yz} = G_{2yz}(\beta + w_{,y}) \tag{53}$$

Making use of equations (41) through (49), those constraint conditions of equations (6), (7), and (8), which characterize the problem of bending only, may be written as follow:

$$c_1\lambda_6 + c_2\lambda_7 = c_3\alpha_{,x} + c_4\beta_{,y} \tag{6a}$$

$$c_2\lambda_6 + c_1\lambda_7 = c_4\alpha_{,x} + c_3\beta_{,y} \tag{7a}$$

$$(c_1 - c_2)\lambda_8 = (c_3 - c_4)(\alpha_{,y} + \beta_{,x}) \tag{8a}$$

where c_1, c_2, c_3 and c_4 are merely some constants of geometrical and material properties and are defined as

$$c_1 = \sum_i \frac{E_i t_i}{1 - \nu_i^2}$$

$$c_2 = \sum_i \frac{E_i t_i \nu_i}{1 - \nu_i^2}$$

$$c_3 = \sum_i \frac{E_i t_i z_i}{1 - \nu_i^2}$$

$$c_4 = \sum_i \frac{E_i t_i z_i \nu_i}{1 - \nu_i^2} \quad , \quad i = 1,\ 2,\ 3$$

Solving for λ_6, λ_7 and λ_8 from equations (6a), (7a) and (8a) such that

$$\lambda_6 = \xi_1 \alpha_{,x} + \xi_2 \beta_{,y} \tag{54}$$

$$\lambda_7 = \xi_2 \alpha_{,x} + \xi_1 \beta_{,y} \tag{55}$$

$$\lambda_8 = \xi_3 (\alpha_{,y} + \beta_{,x}) \tag{56}$$

in which ξ_1, ξ_2 and ξ_3 are defined as

$$\xi_1 = \frac{c_1 c_3 - c_2 c_4}{c_1^2 - c_2^2}$$

$$\xi_2 = \frac{c_1 c_4 - c_2 c_3}{c_1^2 - c_2^2}$$

$$\xi_3 = \frac{c_3 - c_4}{c_1 - c_2}$$

and substituting into equations (41) through (49), the stresses in all membranes can be obtained as follow:

$$\sigma_{1x} = \frac{E_1(z_1 - \xi_1 - \nu_1\xi_2)}{1 - \nu_1^2}\alpha_{,x} + \frac{E_1(\nu_1 z_1 - \nu_1\xi_1 - \xi_2)}{1 - \nu_1^2}\beta_{,y} \tag{41a}$$

$$\sigma_{1y} = \frac{E_1(z_1 - \xi_1 - \nu_1\xi_2)}{1 - \nu_1^2}\beta_{,y} + \frac{E_1(\nu_1 z_1 - \nu_1\xi_1 - \xi_2)}{1 - \nu_1^2}\alpha_{,x} \tag{42a}$$

$$\sigma_{2x} = \frac{E_2(z_2 - \xi_1 - \nu_2\xi_2)}{1 - \nu_2^2}\alpha_{,x} + \frac{E_2(\nu_2 z_2 - \nu_2\xi_1 - \xi_2)}{1 - \nu_2^2}\beta_{,y} \tag{43a}$$

$$\sigma_{2y} = \frac{E_2(z_2 - \xi_1 - \nu_2\xi_2)}{1 - \nu_2^2}\beta_{,y} + \frac{E_2(\nu_2 z_2 - \nu_2\xi_1 - \xi_2)}{1 - \nu_2^2}\alpha_{,x} \tag{44a}$$

$$\sigma_{3x} = \frac{E_3(z_3 - \xi_1 - \nu_3\xi_2)}{1 - \nu_3^2}\alpha_{,x} + \frac{E_3(\nu_3 z_3 - \nu_3\xi_1 - \xi_2)}{1 - \nu_3^2}\beta_{,y} \tag{45a}$$

$$\sigma_{3y} = \frac{E_3(z_3 - \xi_1 - \nu_3\xi_2)}{1 - \nu_3^2}\beta_{,y} + \frac{E_3(\nu_3 z_3 - \nu_3\xi_1 - \xi_2)}{1 - \nu_3^2}\alpha_{,x} \tag{46a}$$

$$\tau_{1xy} = \frac{E_1(z_1 - \xi_3)}{2(1 + \nu_1)}(\alpha_{,y} + \beta_{,x}) \tag{47a}$$

$$\tau_{2xy} = \frac{E_2(z_2 - \xi_3)}{2(1 + \nu_2)}(\alpha_{,y} + \beta_{,x}) \tag{48a}$$

$$\tau_{3xy} = \frac{E_3(z_3 - \xi_3)}{2(1 + \nu_3)}(\alpha_{,y} + \beta_{,x}) \tag{49a}$$

Making substitution of these stresses into equations (1), (2), (3), (4) and (5), the moments and transverse shear forces can be expressed in terms of α, β, w and their partial derivatives[8]

$$M_x = D(\alpha_{,x} + \nu_o \beta_{,y}) \tag{57}$$

$$M_y = D(\beta_{,y} + \nu_o \alpha_{,x}) \tag{58}$$

$$M_{xy} = D_{xy}(\alpha_{,y} + \beta_{,x}) \tag{59}$$

$$Q_x = C_x(\alpha + w_{,x}) \tag{60}$$

$$Q_y = C_y(\beta + w_{,y}) \tag{61}$$

where D is the flexural rigidity of the sandwich plate and is defined by the following expression:

$$D = \sum_i \frac{E_i t_i z_i (z_i - \xi_1 - \nu_i \xi_2)}{1 - \nu_i^2} \quad , \quad i = 1,\ 2,\ 3$$

in which

$$\nu_o = \frac{1}{D} \sum_i \frac{E_i t_i z_i (\nu_i z_i - \nu_i \xi_1 - \xi_2)}{1 - \nu_i^2} \quad , \quad i = 1,\ 2,\ 3$$

D_{xy} the torsional rigidity of the sandwich plate and is defined as

$$D_{xy} = \sum_i \frac{E_i t_i z_i (z_i - \xi_3)}{2(1 + \nu_i)} \quad , \quad i = 1,\ 2,\ 3$$

and

$$C_x = \sum_j h_j G_{jxz} \quad , \quad j = 1,\ 2$$

$$C_y = \sum_j h_j G_{jyz} \quad , \quad j = 1,\ 2$$

the shear rigidities of the sandwich plate.

2.6 Specialization for Constant Poisson's Ratio

The theory developed so far is of a general case of non-equal Poisson's ratio for each facing membrane. It has made the problem complicated not only in defining those plate constants but also in locating the so-called "neutral surface" of the deformed structure.

In many instances, the values of Poisson's ratio may be very closely constant for materials with appreciably different moduli of elasticity. Assuming this to be the case, then, starting with the simplification of constants ξ_1, ξ_2 and ξ_3 in such a manner that

$$\xi_2 = 0$$

$$\xi_1 = \xi_3 = z_o = \frac{\sum_i E_i t_i z_i}{\sum_i E_i t_i} \quad , \quad i = 1,\ 2,\ 3$$

and

$$D = \frac{1}{1 - \nu^2} \sum_i E_i t_i z_i (z_i - z_o)$$

$$\nu_o = \frac{\nu D}{D} = \nu \quad ,$$ the common value of Poisson's ratio,

$$D_{xy} = \frac{1 - \nu}{2} D$$

the stresses of membranes may be written as

$$\sigma_{1x} = \frac{E_1(z_1 - z_o)}{1 - \nu^2} (\alpha_{,x} + \nu \beta_{,y}) \tag{41b}$$

$$\sigma_{1y} = \frac{E_1(z_1 - z_o)}{1 - \nu^2} (\beta_{,y} + \nu \alpha_{,x}) \tag{42b}$$

$$\sigma_{2x} = \frac{E_2(z_2 - z_o)}{1 - \nu^2} (\alpha_{,x} + \nu \beta_{,y}) \tag{43b}$$

$$\sigma_{2y} = \frac{E_2(z_2 - z_o)}{1 - \nu^2} (\beta_{,y} + \nu \alpha_{,x}) \tag{44b}$$

$$\sigma_{3x} = \frac{E_3(z_3 - z_o)}{1 - \nu^2} (\alpha_{,x} + \nu \beta_{,y}) \tag{45b}$$

$$\sigma_{3y} = \frac{E_3(z_3 - z_o)}{1 - \nu^2} (\beta_{,y} + \nu \alpha_{,x}) \tag{46b}$$

$$\tau_{1xy} = \frac{E_1(z_1 - z_o)}{2(1 + \nu)} (\alpha_{,y} + \beta_{,x}) \tag{47b}$$

$$\tau_{2xy} = \frac{E_2(z_2 - z_o)}{2(1 + \nu)} (\alpha_{,y} + \beta_{,x}) \tag{48b}$$

$$\tau_{3xy} = \frac{E_3(z_3 - z_o)}{2(1 + \nu)} (\alpha_{,y} + \beta_{,x}) \tag{49b}$$

where $(z_i - z_o)$ can be defined as the distance measured from the neutral surface to the middle plane of ith membrane. The bending and twisting moments are written as

$$M_x = D(\alpha_{,x} + \nu\beta_{,y}) \tag{57a}$$

$$M_y = D(\beta_{,y} + \nu\alpha_{,x}) \tag{58a}$$

$$M_{xy} = \frac{1}{2}(1 - \nu) D (\alpha_{,y} + \beta_{,x}) \tag{59a}$$

The system of twenty-nine equations with twenty-nine unknowns is reduced to that of three equations, i.e., equations (9), (10) and (11), with three unknowns α, β and w. Further reduction for finding the governing differential equations defining the bending behavior of the structure will be performed in the following chapter of this thesis.

CHAPTER III

DERIVATION OF GOVERNING DIFFERENTIAL EQUATIONS

3.1 Shear Forces

After the specialization for constant Poisson's ratio, the cross-sectional elements, moments and shear forces, are expressed in fairly simple forms, which have been shown by S. Cheng[8] for a special case of sandwich plate construction with a single core and two identical facing membranes. It is understood that this is a reasonable assumption in practical purpose. Thus, in the following derivation, the Poisson's ratios of all membranes are specified to be constant, although the modulus of elasticity of each membrane may not be the same.

From equations (60) and (61), the Lagrangian multipliers α and β may be found in terms of the transverse shear forces Q_x and Q_y such that

$$\alpha = \frac{Q_x}{C_x} - w_{,x} \tag{62}$$

$$\beta = \frac{Q_y}{C_y} - w_{,y} \tag{63}$$

Then, making use of these two expressions, equations (57), (58) and (59) can be written as follow:

$$M_x = D\left[\left(\frac{1}{C_x} Q_{x,x} + \frac{\nu_o}{C_y} Q_{y,y}\right) - \left(w_{,xx} + \nu_o w_{,yy}\right)\right] \tag{64}$$

$$M_y = D\left[\left(\frac{1}{C_y} Q_{y,y} + \frac{\nu_o}{C_x} Q_{x,x}\right) - \left(w_{,yy} + \nu_o w_{,xx}\right)\right] \tag{65}$$

$$M_{xy} = D_{xy}\left[\left(\frac{1}{C_y} Q_{x,y} + \frac{1}{C_y} Q_{y,x}\right) - 2\,w_{,xy}\right] \tag{66}$$

Utilizing equations (64), (65) and (66), equations (9) and (10) of moment equilibrium become

$$Q_x = \frac{1}{C_x}\left(D Q_{x,xx} + D_{xy} Q_{x,yy}\right) + \frac{1}{C_y}\left(\nu_o D + D_{xy}\right) Q_{y,xy} - \left[D w_{,xxx} + \left(\nu_o D + 2 D_{xy}\right) w_{,xyy}\right] \tag{67}$$

$$Q_y = \frac{1}{C_y}\left(D Q_{y,yy} + D_{xy} Q_{y,xx}\right) + \frac{1}{C_x}\left(\nu_o D + D_{xy}\right) Q_{x,xy} - \left[D w_{,yyy} + \left(\nu_o D + 2 D_{xy}\right) w_{,xxy}\right] \tag{68}$$

For the case of specifying constant Poisson's ratio, these two expressions may be written as follow:

$$Q_x = \frac{D}{C_x}\left[Q_{x,xx} + \left(\frac{1-\nu}{2}\right) Q_{x,yy}\right] + \frac{D}{C_y}\left(\frac{1+\nu}{2}\right) Q_{y,xy} - D\nabla^2 w_{,x} \tag{67a}$$

$$Q_y = \frac{D}{C_y}\left[Q_{y,yy} + \left(\frac{1-\nu}{2}\right) Q_{y,xx}\right] + \frac{D}{C_x}\left(\frac{1+\nu}{2}\right) Q_{x,xy} - D\nabla^2 w_{,y} \tag{68a}$$

Differentiating equation (67a) with respect to y and equation (68a) with respect to x, and subtracting yield

$$Q_{x,y} - Q_{y,x} = \frac{D}{C_x}\left[Q_{x,xxy} + (\frac{1-\nu}{2})Q_{x,yyy}\right] + \frac{D}{C_y}(\frac{1+\nu}{2})Q_{y,xyy}$$

$$- \frac{D}{C_y}\left[Q_{y,xyy} + (\frac{1-\nu}{2})Q_{y,xxx}\right] - \frac{D}{C_x}(\frac{1+\nu}{2})Q_{x,xxy} \qquad (69)$$

Differentiating equation (69) with respect to y, then,

$$Q_{x,yy} - Q_{y,xy} = \frac{D}{C_x}\left[Q_{x,xxyy} + (\frac{1-\nu}{2})Q_{x,yyyy}\right] + \frac{D}{C_y}(\frac{1+\nu}{2})Q_{y,xyyy}$$

$$- \frac{D}{C_y}\left[Q_{y,xyyy} + (\frac{1-\nu}{2})Q_{y,xxxy}\right] - \frac{D}{C_x}(\frac{1+\nu}{2})Q_{x,xxyy} \qquad (70)$$

Let equation (11) of the force equilibrium be written in a form such that

$$Q_{y,y} = -(Q_{x,x} + p) \qquad (11a)$$

then, by substitution, equation (70) becomes

$$K_1 Q_{x,xxxx} + (K_1 + K_2)Q_{x,xxyy} + K_2 Q_{x,yyyy} - \nabla^2 Q_x$$

$$= p_{,x} - K_1 \nabla^2 p_{,x} \qquad (71)$$

where K_1 and K_2 are some constants defined by the following expressions:

$$K_1 = \frac{(1-\nu)D}{2C_y} \tag{72}$$

$$K_2 = \frac{(1-\nu)D}{2C_x} \tag{73}$$

By the same procedure, the differential equation of Q_y can be obtained as follows:

$$K_1 Q_{y,xxxx} + (K_1 + K_2) Q_{y,xxyy} + K_2 Q_{y,yyyy} - \nabla^2 Q_y$$

$$= p_{,y} - K_2 \nabla^2 p_{,y} \tag{74}$$

Equation (71) and (74) may also be written in more compact forms such that

$$K_2 \nabla^4 Q_x + (K_1 - K_2)\nabla^2 Q_{x,xx} - \nabla^2 Q_x = p_{,x} - K_1 \nabla^2 p_{,x} \tag{71a}$$

$$K_1 \nabla^4 Q_y + (K_2 - K_1)\nabla^2 Q_{y,yy} - \nabla^2 Q_y = p_{,y} - K_2 \nabla^2 p_{,y} \tag{74a}$$

where ∇^4 is the biharmonic operator defined as

$$\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

for cartesian coordinates.

3.2 Deflection Surface

A governing differential equation defining the deflection surface $w(x, y)$ may be obtained by eliminating the shear forces from the force equilibrium equation. Differentiating equation (68a) with respect to y and making use of equation (11a) yield

$$Q_{x,x} = \left[\frac{D}{C_y} - \frac{D}{C_x}\left(\frac{1+\nu}{2}\right)\right] Q_{x,xyy} + \frac{D}{C_y}\left(\frac{1-\nu}{2}\right) Q_{x,xxx}$$

$$+ \frac{D}{C_y}\left[p_{,yy} + \left(\frac{1-\nu}{2}\right) p_{,xx}\right] - p + D\nabla^2 w_{,yy} \qquad (75)$$

Differentiating equation (67a) and also making use of equation (11a) give

$$Q_{x,x} = \frac{D}{C_x}\left(\frac{1-\nu}{2}\right) Q_{x,xyy} + \left[\frac{D}{C_x} - \frac{D}{C_y}\left(\frac{1+\nu}{2}\right)\right] Q_{x,xxx}$$

$$- \frac{D}{C_y}\left(\frac{1+\nu}{2}\right) p_{,xx} - D\nabla^2 w_{,xx} \qquad (76)$$

Subtracting equation (76) from equation (75) and making use of equations (72) and (73), a differential equation of shear force Q_x may be obtained as follows:

$$\nabla^2 Q_{x,x} = \frac{D(1-\nu)}{2(K_2 - K_1)}\left(\nabla^4 w - \frac{p}{D}\right) + \frac{K_1}{K_2 - K_1}\nabla^2 p \qquad (77)$$

By the same way, the differential equation of Q_y may be obtained as follows:

$$\nabla^2 Q_{y,y} = \frac{D(1-\nu)}{2(K_1 - K_2)}\left(\nabla^4 w - \frac{p}{D}\right) + \frac{K_2}{K_1 - K_2}\nabla^2 p \qquad (78)$$

Differentiating equation (71a) and making use of equation (77), a differential equation defining the transverse deflection of the plate is obtained;

$$K_2 \nabla^6 w + (K_1 - K_2) \nabla^4 w_{,xx} - \nabla^4 w$$

$$= \frac{1}{D} \Big[- \frac{2K_1K_2}{1-\nu} \nabla^4 p + (K_2 + \frac{2}{1-\nu} K_1) \nabla^2 p$$

$$+ \frac{1+\nu}{1-\nu} (K_2 - K_1) p_{,xx} - p \Big] \tag{79}$$

where ∇^6 is a differential operator defined as

$$\nabla^6 = \nabla^2 \nabla^2 \nabla^2$$

This equation may also be written in a form as follows:

$$K_1 \nabla^4 w_{,xx} + K_2 \nabla^4 w_{,yy} - \nabla^4 w$$

$$= \frac{1}{D} \Big[- \frac{2K_1K_2}{1-\nu} \nabla^4 p + (K_2 + \frac{2}{1-\nu} K_1) \nabla^2 p$$

$$+ \frac{1+\nu}{1-\nu} (K_2 - K_1) p_{,xx} - p \Big] \tag{80}$$

or in a form, which can be compared with the homogeneous plate equation, such that

$$(1 - K_1 \frac{\partial^2}{\partial x^2} - K_2 \frac{\partial^2}{\partial y^2}) \nabla^4 w = \Big[1 - (K_1 + \frac{2K_2}{1-\nu}) \frac{\partial^2}{\partial x^2}$$

$$- (K_2 + \frac{2K_1}{1-\nu}) \frac{\partial^2}{\partial y^2} + \frac{2K_1K_2}{1-\nu} \nabla^4 \Big] \frac{p}{D} \tag{81}$$

This is a sixth-order linear partial differential equation defining the transverse deflection of the sandwich plate for the case of specializing the Poisson's ratios of all membranes to be constant.

3.3 Reduction to the Case of Isotropic Cores

For the case that all core materials are isotropic, the constants K_1 and K_2 are identical, and are defined as

$$K_1 = K_2 = K = \frac{(1 - \nu)D}{2C} \tag{82}$$

where

$$C = \sum_j h_j G_{jxz} = \sum_j h_j G_{jyz} \tag{83}$$

in which j designates the number of core. The differential equations (71a) and (74a) of shears become

$$\nabla^2 Q_x = - p_{,x} \tag{84}$$

$$\nabla^2 Q_y = - p_{,y} \tag{85}$$

The sixth-order differential equation (81) defining the transverse deflection w reduces to a fourth-order linear differential equation as follows:

$$\nabla^4 w = \frac{p}{D} - \frac{1}{C} \nabla^2 p \tag{86}$$

For those loading conditions which make the second term of the right-hand side of equation (86) vanish, the equation becomes exactly the one of homogeneous plate. This result has been shown previously by E. Reissner[11] and S. Cheng[8] for a particular case of plate with a single core and two identical facing membranes.

CHAPTER IV

A SIMPLY SUPPORTED RECTANGULAR PLATE WITH ORTHOTROPIC CORES

The problem of bending of a simply supported rectangular sandwich plate with orthotropic cores subjected to a system of uniformly distributed loads is considered (Fig. 3). The plate is assumed to be constructed in such a manner that all the membranes are of the same material and cores are arbitrary. Thus, those constants D, C_x and C_y can be calculated by those equations defined in this thesis. Since the load is constant over the entire structure, the differential equation (81) defining the transverse deflection becomes

$$\left(1 - K_1 \frac{\partial^2}{\partial x^2} - K_2 \frac{\partial^2}{\partial y^2}\right) \nabla^4 w = \frac{p}{D} \tag{87}$$

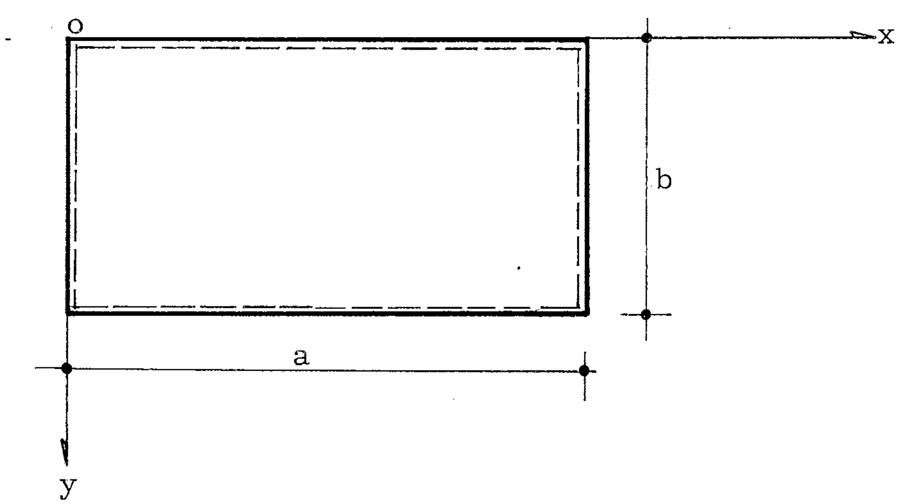

FIG. 3 A SIMPLY SUPPORTED RECTANGULAR PLATE

For a Navier's type of solution, the deflection surface may be assumed in such a form that

$$w = \sum_m \sum_n W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} , \quad \begin{matrix} m = 1, 2, 3, \ldots, \infty \\ n = 1, 2, 3, \ldots, \infty \end{matrix} \tag{88}$$

Then, the loading function p(x, y) may be represented by the shape of deflection surface such that

$$p(x, y) = \sum_m \sum_n P_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} , \quad \begin{matrix} m = 1, 2, 3, \ldots, \infty \\ n = 1, 2, 3, \ldots, \infty \end{matrix} \tag{89}$$

where P_{mn} can be evaluated as follows:

$$P_{mn} = \frac{4}{ab} \int_o^a \int_o^b p(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy \tag{90}$$

which becomes

$$P_m = \frac{16\, p}{mn\pi^2} , \quad \begin{matrix} m = 1, 3, 5, \ldots, \infty \\ n = 1, 3, 5, \ldots, \infty \end{matrix} \tag{91}$$

for constant p over the entire region. Making substitution of equation (88) into equation (87), the coefficient W_{mn} may be solved in terms of the known coefficient of loading function P_{mn} such that

$$W_{mn} = \frac{P_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 \left[1 + K_1 \left(\frac{m\pi}{a}\right)^2 + K_2 \left(\frac{n\pi}{b}\right)^2\right]} \tag{92}$$

or, in a symbolic form:

$$W_{mn} = W^*_{mn} \frac{P_{mn}}{D} \tag{92a}$$

where

$$W^*_{mn} = \frac{1}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 \left[1 + K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2\right]} \tag{93}$$

Thus, the deflection surface becomes

$$w = \frac{1}{D} \sum_m \sum_n W^*_{mn} P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{94}$$

which may be written in a form that can be compared with the Navier's solution of the homogeneous plate subjected to uniformly distributed loads:

$$w = \frac{16\,p}{\pi^6 D} \sum_m \sum_n \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \left[1 + K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2\right]} \tag{94a}$$

where m and n are positive odd integers.

Using equations (77), (78) and (88), the shear forces Q_x and Q_y may be assumed to have forms as follow:

$$Q_x = \sum_m \sum_n A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{95}$$

$$Q_y = \sum_m \sum_n B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}, \quad m = 1, 2, \ldots, \infty \quad n = 1, 2, \ldots, \infty \tag{96}$$

Substituting equations (88) and (95) into equation (77), and equations (88) and (96) into equation (78), the series coefficients A_{mn} and B_{mn} are obtained:

$$A_{mn} = \left(\frac{a}{m\pi}\right)\left[\frac{1-\nu}{2(K_2 - K_1)}\right]\gamma_{mn}P_{mn} \tag{97}$$

$$B_{mn} = -\left(\frac{b}{n\pi}\right)\left[\frac{1-\nu}{2(K_2 - K_1)}\right]\gamma_{mn}P_{mn} \tag{98}$$

where

$$\gamma_{mn} = \frac{1}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\left[\frac{K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2}{1 + K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2}\right] \tag{99}$$

Rewriting equations (62) and (63) in terms of constants K_1 and K_2:

$$\alpha = \frac{2K_2}{D(1-\nu)}Q_x - w_{,x} \tag{62a}$$

$$\beta = \frac{2K_1}{D(1-v)}Q_y - w_{,y} \tag{63a}$$

and making use of equations (88), (95) and (96), the Lagrangian multipliers α and β are obtained:

$$\alpha = \sum_m \sum_n \alpha^*_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{100}$$

$$\beta = \sum_m \sum_n \beta^*_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{101}$$

where α^*_{mn} and β^*_{mn} are determined by those known coefficients A_{mn}, B_{mn} and W_{mn} by the equations as follow:

$$\alpha^*_{mn} = \frac{2K_2}{D(1-\nu)} A_{mn} - (\frac{m\pi}{a}) W_{mn} \quad (102)$$

$$\beta^*_{mn} = \frac{2K_1}{D(1-\nu)} B_{mn} - (\frac{n\pi}{b}) W_{mn} \quad (103)$$

or, in terms of P_{mn} such that

$$\alpha^*_{mn} = \left[(\frac{a}{m\pi}) (\frac{K_2}{K_2 - K_1}) \gamma_{mn} - (\frac{m\pi}{a}) W^*_{mn} \right] \frac{P_{mn}}{D} \quad (102a)$$

$$\beta^*_{mn} = -\left[(\frac{b}{n\pi}) (\frac{K_1}{K_2 - K_1}) \gamma_{mn} + (\frac{n\pi}{b}) W^*_{mn} \right] \frac{P_{mn}}{D} \quad (103a)$$

In these expressions, the constants K_1 and K_2 cannot be the same, that is due to the fact that the differential equations for orthotropic case and isotropic case are distinct. These solutions do satisfy the boundary conditions of a simply supported plate, i.e., $w = o$, $M_x = o$ and $\beta = o$ at the edges $x = o$ and $x = a$ and $w = o$, $M_y = o$ and $\alpha = o$ at the edges $y = o$ and $y = b$.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this thesis is to develop a theory defining the bending behavior of the multi-layer sandwich plates, which are constructed in such a manner that those assumptions stated previously are satisfied, due to a general type of externally applied load normal to the plane of structures. The development of the theory falls mainly upon the formulation of the functional of complementary energy of the system, minimizing process and the elimination of some additional Lagrangian multipliers introduced for the constraint conditions. The problem is formulated in a complete Lagrange form with all stresses as dependent variables, which are functions of two independent variables locating the position on the plane of structure. This type of formulation has not been shown previously.

It is found in this investigation that a "neutral surface" cannot be properly defined for the case that the facing membranes have completely different elastic properties. However, it is understood that the difference of values of Poission's ratio for different materials with appreciably different moduli of elasticity is small. Thus, an additional assumption specifying the constant Poisson's ratio is made in this investigation.

After the specialization for the constant Poisson's ratio, a sixth-order partial differential equation governing the transverse deflection and two supplementary equations governing the transverse shear forces are obtained. This result has generalized the "membrane facings" theory of sandwich construction which was originated by E. Reissner[(1), (11), (12)].

A simple example of a sandwich plate with simply supported edges is solved by the Navier's approach to illustrate the application of the presented theory.

The governing differential equation defining the transverse vibration, without rotatory effect, can be directly obtained by introducing the inertia force to replace the static transverse load p. The presented formulation may also be extended to develop the theory defining the buckling of the multi-layer sandwich plates.

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APPENDIX

AN APPROXIMATE THEORY

A-1 Derivation

By the principle of superposition for small deformations, it is possible to separate the transverse deflection into two parts: one is due to the bending and the other due to the shear deformation. Observing equations (57), (58) and (59), the deflection due to bending may be introduced in such a manner that

$$\alpha = -w_{b,x} \tag{A1}$$

$$\beta = -w_{b,y} \tag{A2}$$

where w_b designates the flexural part of the transverse deflection. Then, equations (57), (58), (59), (60) and (61) can be written in the following forms:

$$M_x = -D(w_{b,xx} + \nu_o w_{b,yy}) \tag{A3}$$

$$M_y = -D(w_{b,yy} + \nu_o w_{b,xx}) \tag{A4}$$

$$M_{xy} = -2D_{xy} w_{b,xy} \tag{A5}$$

$$Q_x = C_x(w_{,x} - w_{b,x}) \tag{A6}$$

$$Q_y = C_y(w_{,y} - w_{b,y}) \tag{A7}$$

Substitution of equations (A6) and (A7) into equation (11) of the force equilibrium yields

$$K_1 w_{s,xx} + K_2 w_{s,yy} = -\left(\frac{2K_1K_2}{1-\nu}\right)\frac{p}{D} \qquad \text{(A8)}$$

where K_1 and K_2 are defined by equations (72) and (73), and w_s is the part of deflection due to shear and is defined as follows:

$$w_s = w - w_b \qquad \text{(A9)}$$

Making use of equations (A3), (A4) and (A5), equations (9) and (10) of the moment equilibrium may be written as

$$C_x w_{s,x} = -D(w_{b,xxx} + \nu_o w_{b,xyy}) - 2D_{xy} w_{b,xyy} \qquad \text{(A10)}$$

$$C_y w_{s,y} = -D(w_{b,yyy} + \nu_o w_{b,xxy}) - 2D_{xy} w_{b,xxy} \qquad \text{(A11)}$$

Differentiating equations (A10) with respect to x and (A11) with respect to y and adding together yield

$$D w_{b,xxxx} + 2(2D_{xy} + \nu_o D) w_{b,xxyy} + D w_{b,yyyy}$$

$$= -(C_x w_{s,xx} + C_y w_{s,yy}) \qquad \text{(A12)}$$

Replacing C_x and C_y by K_1 and K_2 and using equation (A8), equation (A12) becomes

$$D w_{b,xxxx} + 2(2D_{xy} + \nu_o D) w_{b,xxyy} + D w_{b,yyyy} = p \qquad \text{(A13)}$$

which may be written as

$$\nabla^4 w_b = \frac{p}{D} \tag{A14}$$

for the case of specifying the constant Poisson's ratio for all membranes. For the case that all core materials are isotropic, equation (A8) is reduced to a Poisson's equation:

$$\nabla^2 w_s = - \frac{p}{C} \tag{A15}$$

where C is the constant defined by equation (83). Making operation on equation (A15) by the Laplacian operator and adding to equation (A14) yield

$$\nabla^4 w = \frac{p}{D} - \frac{1}{C} \nabla^2 p \tag{A16}$$

which has been shown previously in Chapter III as equation (86) of this thesis.

For the case with orthotropic cores, the sixth-order equation governing the transverse deflection now is approximated by equations (A8) and (A14). It is also shown that, for the case with isotropic cores, this approximate theory becomes exact in the sense of previous derivation.

A-2 Comparison of Solutions

The problem considered previously in Chapter IV of a simply supported rectangular sandwich plate is solved by the approximate theory derived in this appendix. The solutions for the w_b and w_s are assumed to have the forms as follow:

$$w_b = \sum_m \sum_n W_{bmn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A17}$$

$$w_s = \sum_m \sum_n W_{smn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A18}$$

The loading function $p(x, y)$ is also represented by the shape of the deflection surface that

$$p(x, y) = \sum_m \sum_n P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A19}$$

where P_{mn} is evaluated by the same way described previously in Chapter IV. Making substitutions of equation (A17) into (A14) and (A18) into (A8) respectively, yield

$$W_{bmn} = \frac{P_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \tag{A20}$$

$$W_{smn} = \frac{\frac{2K_1K_2}{1-\nu}}{K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2} \left(\frac{P_{mn}}{D}\right) \tag{A21}$$

Thus, the solution of the total deflection is

$$w = \frac{1}{D} \sum_m \sum_n W^{**}_{mn} P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A22}$$

where W^{**}_{mn} is defined as follows:

$$W^{**}_{mn} = \frac{1}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} + \frac{\frac{2K_1K_2}{1-\nu}}{K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2} \tag{A23}$$

Designating the ratio between the constants K_1 and K_2 such that

$$\eta = \frac{K_2}{K_1} = \frac{C_y}{C_x} \tag{A24}$$

then, W^{**}_{mn} becomes

$$W^{**}_{mn} = \frac{1}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} + \frac{\frac{2\eta K_1}{1-\nu}}{\left(\frac{m\pi}{a}\right)^2 + \eta\left(\frac{n\pi}{b}\right)^2} \tag{A25}$$

The Lagrangian multipliers used to calculate the moments and the shear forces are obtained by substituting equations (A17) and (A18) into equations (A1), (A2), (A6) and (A7):

$$\alpha = -\frac{1}{D}\sum_m\sum_n \frac{\left(\frac{m\pi}{a}\right) P_{mn}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A26}$$

$$\beta = -\frac{1}{D}\sum_m\sum_n \frac{\left(\frac{n\pi}{b}\right) P_{mn}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{A27}$$

$$Q_x = \sum_m\sum_n \frac{\left(\frac{m\pi}{a}\right) K_1}{K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A28}$$

and

$$Q_y = \sum_m\sum_n \frac{\left(\frac{n\pi}{b}\right) K_2}{K_1\left(\frac{m\pi}{a}\right)^2 + K_2\left(\frac{n\pi}{b}\right)^2} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{A29}$$

Solutions of these forms do satisfy the boundary conditions described in Chapter IV.

Comparison between the solutions of the approximate theory and the "exact" theory shows that the approximate theory is valid only for the case of isotropic cores or in case the deformations due to shear are negligibly small. This conclusion proves that the assumptions made by T. E. Falgout* are not true, in general.

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*T. E. Falgout, "A Differential Equation of Free Transverse Vibrations of Isotropic Sandwich Plates," Developments in Mechanics, Vol. 1, Proceedings of the Seventh Midwestern Mechanics Conference, 1961.

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