BEH. 410 - 04/07/2003
L # 15

Last time: viscoelasticity, lumped to continuum
Today: poroelastic behavior of tissues
- role of fluid flow
- continuum model
- at first, ignore charge or osmotic effects

- Zeman et al. (Sackmann's group) ⟷ $F(t)$, $x(t)$

$$\begin{cases} F(t) = 6\pi a \left(G' x + \mu \dfrac{dx}{dt} \right) \\ \text{sum = Voigt-like assumption} \end{cases}$$

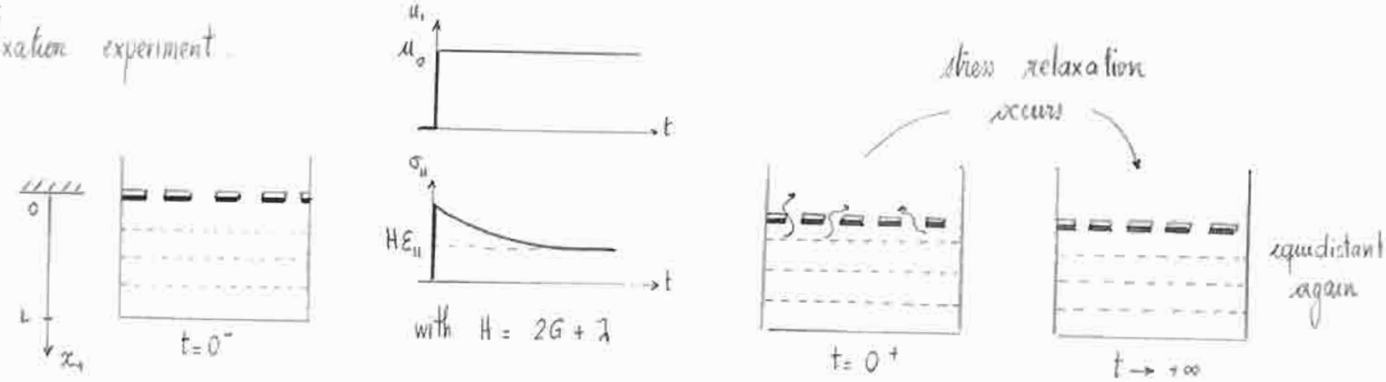
(G': elasticity; μ: Newtonian viscosity)

$$G' = \frac{F_0 \cos\phi}{6\pi a\, x_0}, \quad G'' = \mu\omega = \frac{F_0 \sin\phi}{6\pi a\, x_0}$$

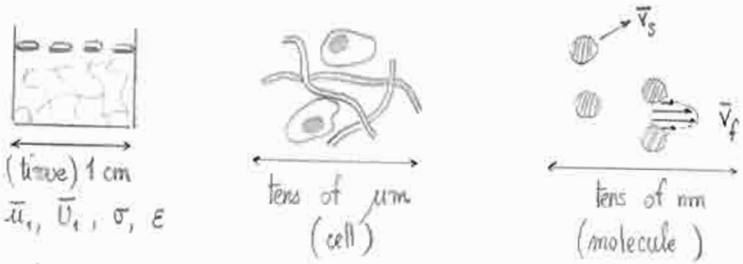
So far: creep, stress relaxation, dynamic compression

- <u>Poroelasticity</u>

- stress relaxation experiment



impose step displacement u_0 at $t = 0$, and find $u_1(x_1, t)$, $p(x_1, t)$, $U_1(x_1, t)$ mean fluid velocity



- A simple <u>continuum poroelastic</u> model

solid: Hookean (linear elastic), homogeneous, isotropic, (or viscoelastic)
liquid: inviscid (liquid - liquid), liquid - solid friction (drag)
} both are incompressible

<u>Literature</u>: soils, rocks, geology, oil extraction — 1940s Biot
gels ($\tau_{swelling} \sim \dfrac{a_i^2}{\text{"}D\text{"}}$, a_i initial radius) 1970s Tanaka at MIT
↳ effect of elasticity & porosity / friction
cartilage — 1980s on

- Governing equations
1. constitutive equation (σ_{ij} & ε_{ij}): Biot
2. fluid - solid viscous interactions: Darcy's law
3. conservation of mass
4. conservation of momentum

- *Constitutive relation*

$$\boxed{\sigma_{ij}^{TOT} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} - p\,\delta_{ij}} \quad (1)$$

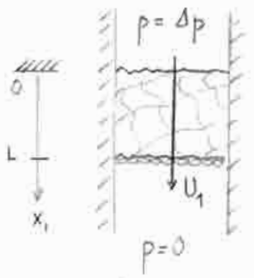
{ total measured stress σ_{ij}^{TOT} ; hydrostatic pressure p — $-p\,\delta_{ij}$ ⟶ fluid contribution

still $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

in 1D $\quad \boxed{\sigma_{11} = \underbrace{(2G+\lambda)}_{H}\,\varepsilon_{11} - p} \quad (1a)$

alternatively, mixture theory $\underline{\underline{\sigma}}^{TOT} = \underline{\underline{\sigma}}^{s} + \underline{\underline{\sigma}}^{f}$

- Fluid - structure interactions : *Darcy's law*



apply Δp, measure $U_1 = \frac{\text{flow rate}}{\text{total area}}$

$U_1 = k\,\frac{\Delta p}{L}$

k = hydraulic permeability $\quad [k] = m^4 \cdot N^{-1} \cdot s^{-1}$

experiment with

$\Delta p = 10 \text{ cm } H_2O = 10^3 \text{ Pa}$

$L = 0.02 \text{ m}$

$U = \frac{\text{volume}}{(\text{area})\,\Delta t} = \frac{250 \text{ mL}}{10 \text{ cm}^2 \quad 100 \text{ s}} = 2.5 \cdot 10^{-3} \text{ m}^3 \cdot s^{-1}$

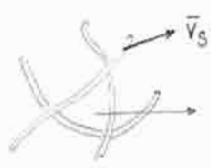
} household sponge $k = \frac{LU}{\Delta p} = 5 \cdot 10^{-4} \frac{m^4}{Ns}$

cortical bone $k \sim 10^{-14} - 10^{-15} \frac{m^4}{Ns}$

$$\boxed{\begin{aligned} \bar{U} &= -k\,\bar{\nabla} p \quad &(2)\\ U_1 &= -k\,\frac{\partial p}{\partial x_1} \quad &(2a)\end{aligned}}$$

see homework # 6

- *Conservation of mass*



$\bar{v}_s$ local solid velocity

$\bar{v}_f$ local fluid velocity

and define

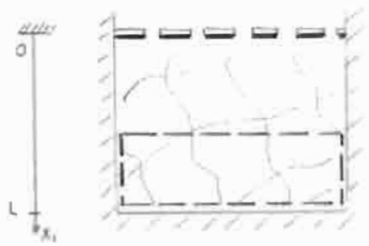
{ U fluid velocity relative to solid, averaged over total area

ϕ porosity $= \frac{\forall_f}{\forall_f + \forall_s} = \frac{A_f}{A_f + A_s} = \frac{A_f}{A_{TOT}}$

relative volume flow $U A_{tot} = A_f\,(v_f - v_s) = A_f\,v_{rel}$

$$\boxed{\bar{U} = \phi\,(\bar{v}_f - \bar{v}_s) = \phi\,\bar{v}_{rel}}$$

a) first approach :



v_f

$v_s = \frac{\partial u_1}{\partial t}$

in fixed control volume

flow through top = flow through bottom

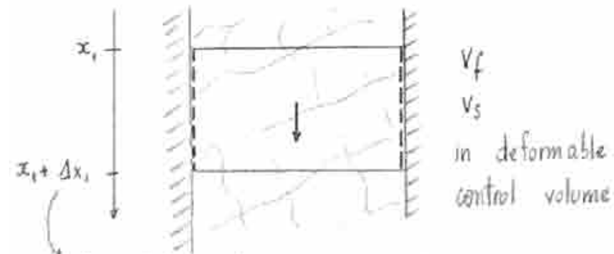
$A_f\,v_f + A_s\,v_s = 0$

$v_s\,\frac{A_s}{A_{TOT}} = -v_f\,\frac{A_f}{A_{TOT}}$ ↙ ϕ

$v_s = -v_f\,\frac{\phi}{1-\phi}$

$$U = \phi\,(v_f - v_s) = \phi\,v_s\left(-\frac{1-\phi}{\phi} - 1\right) = -v_s = -\frac{\partial u_1}{\partial t}$$

b) general (but still in 1D)



after deformation $x_1 + \Delta x_1 (1 + \varepsilon_{11})$

net flux $= A_f (v_f - v_s)\big|_{x_1} - A_f (v_f - v_s)\big|_{x_1 + \Delta x_1}$

change in volume in $\Delta t = A_{TOT} (1 + \varepsilon_{11})\big|_{t+\Delta t} - A_{TOT} (1 + \varepsilon_{11})\big|_t$

$$U_1 = - \frac{\partial u_1}{\partial t} + U_0 \qquad (3a)$$

$$\nabla . \bar{U} = - \nabla . \bar{v}_s = - \bar{\nabla} \cdot \left[\phi \left(\bar{v}_f - \bar{v}_s \right) \right] \qquad (3)$$